Measures of Transverse Paths in Sub-Riemannian Geometry

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Abstract

We define a class of lengths of paths in a sub-Riemannian manifold. It includes the length of horizontal paths but it also measures the length of transverse paths. It is obtained by integrating an infinitesimal measure which generalizes the norm on the tangent space. This requires the definition and the study of the metric tangent space (in Gromov sense). As an example we compute those measures in the case of contact sub-Riemannian manifolds.

1 Introduction

Let $(M, \mathcal{D}, g)$ be a C^∞ sub-Riemannian manifold: M is a C^∞ manifold, $\mathcal{D} \subset TM$ a C^∞ distribution on M and g a C^∞ Riemannian metric on $\mathcal{D}$ (such manifolds are also called Carnot-Caratheodory spaces).

We assume that Chow's Condition is satisfied: let $\mathcal{D}^s$ denote the $\mathbb{R}$-linear span of brackets of degree $\leq s$ of vector fields tangent to $\mathcal{D}^1 = \mathcal{D}$; then, at every $p \in M$, there exists an integer $r = r(p)$ such that $\mathcal{D}^r(p) = T_pM$.

Horizontal paths are those paths which are always tangent to $\mathcal{D}$. The length of horizontal paths is then obtained as in Riemannian geometry integrating the norm of their tangent vectors. Chow's condition implies that one can join any two points of the manifold by a horizontal path and therefore a distance d can be defined.

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The purpose of this paper is to introduce a length or a measure for a transverse path, that is, a path whose tangent vector does not belong to the distribution. In [7] and [8] we consider and estimate global measures of paths, such as entropy, Hausdorff measure and complexity. Here we want a definition modeled after the definition of the length: the measure should appear as the integral of an infinitesimal measure, which generalizes the norm of the tangent vector. Defining such an infinitesimal measure of transverse directions requires a careful study of the metric structure of the tangent space.

The notion of metric tangent space, called tangent cone here, has been defined by Gromov [4]. In sub-Riemannian geometry, the tangent cone is itself a sub-Riemannian manifold and has moreover an algebraic structure: a group at a regular point (Mitchell [12]) and a homogeneous space in general (Bellaïche [3]). Recall that a point is regular if all $\mathcal{D}^s$, $s \geq 1$, have constant rank around this point. However these last two papers only describe the tangent cone up to isomorphisms (the privileged coordinates in [3]). Using Mitchell's description, Margulis and Mostow [10, 11] give an intrinsic definition of the tangent cone which holds at regular points. In this work we need to consider non-regular points. Indeed, to be included in the regular locus is not a generic property of paths (though a point is generically regular).

In Section 2, following the ideas of Margulis-Mostow on one hand and of Bellaïche on the other hand, we propose an intrinsic definition of the tangent cone at any point. Furthermore we introduce a filtration of the tangent cone and relate it to the natural filtration of the tangent space defined by the filtration.

In Section 3, we define a class of lengths of paths obtained as an integration of infinitesimal measures on the tangent cone. One of those lengths coincides with the usual length.

Finally, in Section 4 we compute the measures for contact and Martinet sub-Riemannian manifolds, the latter being an important example of non-regular sub-Riemannian manifold (see [1, 13]).

2 Tangent cone

2.1 Privileged coordinates

In this section we recall some results about sub-Riemannian geometry. For a general introduction with references see [3].

Theorem 1 (Ball-Box Theorem [3],[5]). *There exist integers* $1 = w_1 \leq \cdots \leq$

$w_n = r$, positive constants K_1, K_2 and a system of C^∞ local coordinates $(x_1, \ldots, x_n)$ centered at p (called privileged coordinates at p) such that, for any x in M close enough to p,

$$K_1 \left(|x_1|^{1/w_1} + \cdots + |x_n|^{1/w_n}\right) \leq d(p, x) \leq K_2 \left(|x_1|^{1/w_1} + \cdots + |x_n|^{1/w_n}\right).$$

Moreover, the privileged coordinates are adapted to the filtration

$$\{0\} \subset \mathcal{D}^1(p) \subset \mathcal{D}^2(p) \subset \cdots \subset \mathcal{D}^r(p) = T_pM$$

in the sense that dx_j vanishes on $\mathcal{D}^{w_j-1}(p)$ and does not vanish identically on $\mathcal{D}^{w_j}(p)$.

Given privileged coordinates $x = (x_1, \ldots, x_n)$ at p, we identify a neighborhood of p in M with a neighborhood of 0 in $\mathbb{R}^n$. Defining a dilation by $\delta_s x = (s^{w_1} x_1, \ldots, s^{w_n} x_n)$, there exists a sub-Riemannian distance $\hat{d}$ on $\mathbb{R}^n$ which is homogeneous under the dilation, that is, $s\hat{d}(x, x') = \hat{d}(\delta_s x, \delta_s x')$. Moreover there exists a constant C such that for q and q' in a neighborhood of p ([3, p. 69])

$$-C\hat{d}(p, q) d(q, q')^{1/r} \leq d(q, q') - \hat{d}(q, q') \leq C\hat{d}(p, q) d(q, q')^{1/r}. \tag{1}$$

As a consequence of this inequality, the metric tangent space (in Gromov sense) at p to the sub-Riemannian manifold (M, d) is isometric to $\mathbb{R}^n$ endowed with the sub-Riemannian distance $\hat{d}$ [3, p. 71].

2.2 Rectifiability

We consider the set $\mathcal{C}_p$ of absolutely continuous parametrized curves $c(s)$ in M, defined on some interval $0 \leq s \leq \varepsilon$, differentiable at 0 and with $c(0) = p$.

Definition. Two curves $c_1(s)$ and $c_2(s)$ are called *equivalent* if

$$\lim_{s \to 0} \frac{1}{s} d\big(c_1(s), c_2(s)\big) = 0.$$

Definition. A curve $c(s) \in \mathcal{C}_p$ is called *rectifiable* at $s = 0$ if $d\big(p, c(s)\big) \leq Cs$ as $s \to 0$ and it is equivalent to a C^∞ curve in $\mathcal{C}_p$.

Remark. In [11] a rectifiable path is supposed to be C^∞. We will need in the following to work with a more general class of paths.

Definition. The *tangent cone* C_pM to M at p is the set of equivalence classes $[c(s)]$ of rectifiable curves (see Margulis and Mostow [10, 11]). The distance on the tangent cone is

$$\bar{d}\big([c_1(s)], [c_2(s)]\big) = \lim_{s \to 0} \frac{1}{s} d\big(c_1(s), c_2(s)\big).$$

This distance $\bar{d}$ in C_pM allows to define an infinitesimal measure for tangent vectors to $c(s)$ at points where the curve $c(s)$ is rectifiable. But not every curve in $\mathcal{C}_p$ is rectifiable. We introduce below a notion of rectifiability which generalizes the definition above.

Definition. Let $k \geq 1$ be a real number. A curve $c(s) \in \mathcal{C}_p$ is called *k-rectifiable* at $s = 0$ if $c(s^k)$ is rectifiable.

If $c(s)$ is k-rectifiable, then $c(s)$ is k'-rectifiable for any $k' \geq k$.

Definition. For $c \in \mathcal{C}_p$ set $k_0 = \inf\{k \geq 1 \text{ such that } c(s) \text{ is } k\text{-rectifiable}\}$.

Observe that $k_0 = \inf\{k \geq 1 \text{ such that } d(p, c(s^k)) \leq Cs \}$. In fact, if $d(p, c(s^k)) \leq Cs$ then, for $k' > k$, $c(s^{k'})$ is equivalent to the constant C^∞ curve p.

Recall that here $r = r(p)$ is the smallest integer such that $\mathcal{D}^r(p) = T_pM$.

Lemma 2. *For $c(s)$ in $\mathcal{C}_p$, $k_0 \leq r$.*

Proof. Let $c(s) \in \mathcal{C}_p$. We denote by $(x_1(s), \ldots, x_n(s))$ the coordinates of $c(s)$ for $s \geq 0$ small enough. As $x_i(s)$ is a differentiable function with $x_i(0) = 0$ we have $|x_i(s)| \leq Cs$. Now $d\big(p, c(s^r)\big) \leq C' \sum_i s^{r/w_i} \leq C''s$. This shows that $k_0 \leq r$. □

Remark that the curve might not be k_0-rectifiable at 0 because $c(s^{k_0})$ might not be equivalent to a C^∞ curve.

Example. Consider the Engel group, $\mathbb{R}^4$ with a distribution $\mathcal{D}$ generated by the vector fields $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + x^2\frac{\partial}{\partial w}$. Define the metric g on $\mathcal{D}$ such that X_1 and X_2 are orthonormal. The only non-vanishing Lie brackets are those obtained from $X_3 = [X_1, X_2]$ and $X_4 = [X_1, [X_1, X_2]]$ by permutations. The exponential coordinates $(x, y, z, w) = \exp(x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4)$ satisfy the Ball-Box Theorem with $w_1 = w_2 = 1$, $w_3 = 2$ and $w_4 = 3$.

Consider the differentiable curve

$$c(s) = (x_1(s), x_2(s), x_3(s), x_4(s)) = (s, s, s^2, s^3 \sin^6(1/s)).$$

We have $d(0, c(s)) \leq K_2(|x_1(s)| + |x_2(s)| + |x_3(s)|^{1/2} + |x_4(s)|^{1/3}) \leq 4K_2s$. But it is not equivalent to a C^∞ curve. In fact, one can show using Campbell-Hausdorff formula, that if $c'(s)$ is equivalent to $c(s)$, then

$$c'(s) = (s + o(s), s + o(s), s^2 + o(s^2), s^3 \sin^6(1/s) + o(s^3))$$

which is clearly not C^∞ at 0.

Next, using the dilations in a system of privileged coordinates we give a precise characterization of rectifiability.

Lemma 3. *A curve $c(s)$ is rectifiable if and only if the limit $\lim_{s\to 0}\delta_{s^{-1}}c(s) = x$ exists. In that case $c(s)$ is equivalent to $\delta_s x$.*

Proof. Suppose $c(s)$ is rectifiable. It is then equivalent to a C^∞ curve $c'(s)$. The definition of equivalence and (1) implies that $\frac{1}{s}\hat{d}(c(s), c'(s))$ tends to zero as $s \to 0$. Since $\hat{d}$ is homogeneous, $\hat{d}(\delta_{s^{-1}}c(s), \delta_{s^{-1}}c'(s))$ tends to zero.

We denote by $(y_1(s), \ldots, y_n(s))$ the coordinates of $c'(s)$ for $s \geq 0$ small enough. Since $c'(s)$ is rectifiable, each $y_i(s)$ can be written as $s^{w_i}x_i(s)$, where $x_i(s)$ is C^∞. Thus $\delta_{s^{-1}}c'(s) = (x_1(s), \ldots, x_n(s))$ tends to $x = (x_1(0), \ldots, x_n(0))$ and so does $\delta_{s^{-1}}c(s)$.

We can prove in the same way that, if $\lim_{s\to 0}\delta_{s^{-1}}c(s) = x$ exists, then $c(s)$ is rectifiable and equivalent to $\delta_s x$. □

Each class in the tangent cone has then a representative $c(s) = \delta_s x$. Moreover, using (1), we have $\bar{d}\big([c_1(s)], [c_2(s)]\big) = \hat{d}(x_1, x_2)$. This shows that our tangent cone corresponds to Gromov's definition:

Corollary 4. *The tangent cone $(C_pM, \bar{d})$ is isometric to $(\mathbb{R}^n, \hat{d})$ and so to the metric tangent space (in Gromov sense) to (M, d) at p.*

Another consequence of Lemma 3 is the following.

Corollary 5. *Every curve $c(s)$ in $\mathcal{C}_p$ is r-rectifiable.*

Proof. We denote as in the previous lemma by $(y_1(s), \ldots, y_n(s))$ the coordinates of $c(s)$ for $s \geq 0$ small enough. Since $c(s)$ is differentiable at 0, each $y_i(s)$ equals $sx_i(s)$, where the function $x_i(s)$ has a limit at 0. Then $\delta_{s^{-1}}c(s^r) = (s^{r-w_1}x_1(s), \ldots, s^{r-w_n}x_n(s))$ has a limit and $c(s)$ is r-rectifiable. □

Remark. Observe that, if $c(s)$ is equivalent to a C^∞ curve, then it is k_0-rectifiable (the proof uses the the same arguments as above). Compare with the example in the Engel group, where the curve is not equivalent to a C^∞ one and $k_0 = 1$ is not equal to $r = 3$.

2.3 Tangent space versus Tangent cone

Definition. Two k-rectifiable curves $c_1(s)$ and $c_2(s)$ in $\mathcal{C}_p$ are called *k-equivalent* if

$$\lim_{s\to 0}\frac{1}{s}d\big(c_1(s), c_2(s)\big)^k = 0,$$

that is if $c_1(s^k)$ and $c_2(s^k)$ are equivalent.

Definition. We define $C_p^k M \subset C_p M$ as the set of equivalence classes $[c(s^k)]$ of rectifiable curves $c(s^k)$.

A k-rectifiable curve $c(s)$ is k'-equivalent to the constant curve p for every $k' > k$. In particular, for $k > r$, every curve in $\mathcal{C}_p$ is k-equivalent to p.

As a consequence of Lemma 3 one can effectively describe $C_p^k M$.

Lemma 6. *$C_p^k M$ is the set of equivalence classes of $\delta_s(0, \ldots, 0, x_l, \ldots, x_n)$ where $w_i < k$ for $i < l$.*

Proof. Let $[c(s^k)] \in C_p^k M$. We write $c(s)$ as $(y_1(s), \ldots, y_n(s))$ in privileged coordinates. By Lemma 3, for every i $\lim_{s\to 0} s^{-w_i} y_i(s^k) = x_i$ exists and $\delta_s(x_1, \ldots, x_n)$ is equivalent to $c(s^k)$. Moreover, for $w_i < k$ we have $x_i = 0$ because $y_i(s)$ is differentiable. □

Remark. This lemma shows that if $m \leq k < m+1$ for an integer m, then $C_p^k M = C_p^m M$. But, on the other hand, observe that k_0 might not be an integer.

We have the following filtration of the tangent cone:

$$C_p M = C_p^1 M \supset C_p^2 M \supset \cdots \supset C_p^r M \supset C_p^{r+1} M = [p].$$

In the following we will relate that filtration to the filtration of the tangent space

$$\{0\} \subset \mathcal{D}^1(p) \subset \mathcal{D}^2(p) \subset \cdots \subset \mathcal{D}^r(p) = T_p M. \tag{2}$$

Lemma 7. *If $\dot{c}(0) = \left.\frac{dc(s)}{ds}\right|_{s=0}$ does not belong to $\mathcal{D}^k(p)$, then $k_0 \geq k+1$.*

Proof. As in the proof of Corollary 5 we write $c(s) = (y_1(s), \ldots, y_n(s))$ in privileged coordinates and each $y_j(s)$ as $s x_j(s)$, where the function $x_j(s)$ has a limit x_j at 0. As privileged coordinates are adapted to the filtration of the tangent space, there exists a nonzero x_i of weight $w_i \geq k+1$. Thus, for any $l < k+1$, $s^{-w_i} y_i(s^l) = s^{l-w_i} x_i(s^l)$ tends to infinity as $s \to 0$. This implies that $s^{-1} d(p, c(s^l)) \to \infty$ for any $l < k+1$, and so that $k_0 \geq k+1$. □

Corollary 8. *If, for an integer k, $c(s)$ is k-rectifiable then $\dot{c}(0)$ belongs to $\mathcal{D}^k(p)$.*

Lemma 9. *If $c(s)$ is k-equivalent to $c'(s)$, for an integer k, then $\dot{c}(0) - \dot{c}'(0) \in \mathcal{D}^{k-1}(p)$.*

Proof. Using the same notation as above, $y_i(s) = s^{w_i/k} x_i + o(s^{w_i/k})$ and $y_i'(s) = s^{w_i/k} x_i' + o(s^{w_i/k})$. As, by definition, $c(s^k)$ is equivalent to $c'(s^k)$ we have $x_i = x_i'$. Observe then that, for $w_i = k$, $\dot{y}_i(0) = \dot{y}_i'(0)$. □

The lemmas above show that the map

$$I_k : C_p^k M \to \mathcal{D}^k(p)/\mathcal{D}^{k-1}(p)$$

defined by $I_k([c(s^k)]) = \dot{c}(0) + \mathcal{D}^{k-1}(p)$ is well defined and onto.

Lemma 10. $\ker I_k = C_p^{k+1} M$.

Proof. By Lemma 6, a class in $C_p^k M$ has a representative $c(s^k) = \delta_s(0, \dots, 0, x_l, \dots, x_n)$ where $w_i < k$ for $i < l$. We observe that $\dot{c}(0) = 0$ if and only if $x_j = 0$ for $w_j = k$. This implies that $[c(s^k)] \in C_p^{k+1} M$. □

Remark that $C_p^k M / C_p^{k+1} M$ is a vector space. In fact this space is represented by equivalence classes of $\delta_s(0, \dots, 0, x_l, \dots, x_j, 0, \dots, 0)$ where $w_l = \cdots = w_j = k$. We conclude with the following theorem which summarizes the previous discussion.

Theorem 11. $\overline{I}_k : C_p^k M / C_p^{k+1} M \to \mathcal{D}^k(p)/\mathcal{D}^{k-1}(p)$ *is an isomorphism of vector spaces.*

We can give another characterization of $C_p^k M / C_p^{k+1} M$ based on class of horizontal and transverse paths.

Lemma 12. *Let $c(s)$ in $\mathcal{C}_p$ be a curve such that $\dot{c}(s)$ belongs to $\mathcal{D}^k\big(c(s)\big)$ at $s = 0$ and almost everywhere on $[0, \varepsilon]$. Then $c(s)$ is k-rectifiable.*

If moreover $\dot{c}(0)$ belongs to $\mathcal{D}^k(p)/\mathcal{D}^{k-1}(p)$, then $c(s^k)$ is equivalent to $\delta_s \dot{c}(0)$ in a particular system of privileged coordinates.

Proof. Let $c(s) \in \mathcal{C}_p$ be a curve such that $\dot{c}(s)$ belongs to $\mathcal{D}^k\big(c(s)\big)$ almost everywhere on $[0, \varepsilon]$ and $\dot{c}(0)$ belongs to $\mathcal{D}^{k'}(p)/\mathcal{D}^{k'-1}(p)$, for $k' \leq k$. Reducing k and k' if needed, we assume $k = w_l$ and $k' = w_{l'}$.

We first choose vector fields $Y_1, \dots, Y_n$ which values at p form a basis of $T_p M$ adapted to the filtration (2) and such that $Y_{l'}(p) = \dot{c}(0)$. There exist measurable functions ε_i on $[0, \varepsilon]$ such that

$$\dot{c}(s) = Y_{l'}\big(c(s)\big) + \sum_{w_i \leq w_l} \varepsilon_i(s) Y_i\big(c(s)\big) \qquad \text{a.e. on } [0, \varepsilon]. \tag{3}$$

Moreover, since $c(s)$ is differentiable at $s = 0$, each function ε_i satisfies

$$\int_0^s \varepsilon_i(\tau) d\tau = o(s).$$

We define now a system of local coordinates near p

$$x \mapsto \exp(x_{l'} Y_{l'}) \circ \exp(x_n Y_n) \circ \cdots \circ \exp(x_1 Y_1)(p).$$

We have $Y_{l'}(x) = \partial/\partial x_{l'}$ and the coordinates are privileged at p (see for instance [6]). Thus the jth component Y_i^j of the vector field $Y_i(x)$ is of weighted order $w_j - w_i$ [3], that is

$$\left|Y_i^j(x)\right| \leq Cte\|x\|^{w_j - w_i},$$

where $\|x\|$ is the pseudo-norm $|x_1|^{1/w_1} + \cdots + |x_n|^{1/w_n}$.

Denote by $x_1(s), \ldots, x_n(s)$ the coordinates of $c(s)$. We write (3) as

$$\dot{x}_j(s) = \delta_{jl'} + \sum_{w_i \leq w_l} \varepsilon_i(s) Y_i^j\big(x(s)\big) \qquad \text{a.e. on } [0, \varepsilon].$$

We take the integral from 0 to s

$$\begin{aligned} x_j(s) &= \delta_{jl'} s + \sum_{w_i \leq w_l} \int_0^s \varepsilon_i(\tau) Y_i^j\big(x(\tau)\big) d\tau, \\ &= \delta_{jl'} s + \sum_{w_i \leq w_l} \left[Y_i^j\big(x(s)\big) \int_0^s \varepsilon_i(\tau) d\tau - \int_0^s \frac{d}{dt}\Big(Y_i^j\big(x(t)\big)\Big)\Big(\int_0^t \varepsilon_i(\tau) d\tau\Big) dt \right]. \end{aligned}$$

We can bound the terms of the sum in function of $\|x(s)\|$. For the first one, using $w_i \leq w_l = k$, we obtain

$$\Big|Y_i^j\big(x(s)\big) \int_0^s \varepsilon_i(\tau)\Big| \leq \|x(s)\|^{w_j - k} o(s).$$

For the second one, we compute first

$$\begin{aligned} \left|\frac{d}{dt}\Big(Y_i^j\big(x(t)\big)\Big)\right| &= \left|\sum_{\alpha=1}^n \frac{\partial Y_i^j}{\partial x_\alpha}\big(x(t)\big) \dot{x}_\alpha(t)\right| \\ &\leq \sum_{\alpha=1}^n Cte\, \|x(t)\|^{w_j - w_i - w_\alpha} \|x(t)\|^{w_\alpha - k}, \end{aligned}$$

and then we obtain

$$\left|\int_0^s \frac{d}{dt}\Big(Y_i^j\big(x(t)\big)\Big)\Big(\int_0^t \varepsilon_i(\tau) d\tau\Big) dt\right| \leq s\|x(s)\|^{w_j - 2k} o(s).$$

Finally we have shown, for $j = 1, \ldots, n$,

$$\frac{1}{s^{w_j}}\left|x_j(s^k) - \delta_{jl'} s^k\right| \leq \left(\frac{\|x(s^k)\|}{s}\right)^{w_j - k} O(s) + \left(\frac{\|x(s^k)\|}{s}\right)^{w_j - 2k} O(s). \tag{4}$$

Assume now that the ratio $\|x(s^k)\|/s$ is not bounded when $s \to 0$. There exists j and a sequence $s_N \to 0$ such that

$$\begin{cases} b_N = \frac{1}{s_N^{w_j}}\left|x_j(s_N^k)\right| \to +\infty, \\ \frac{\|x(s_N^k)\|}{s_N} \leq Cte\, b_N^{1/w_j}. \end{cases}$$

Observe that j has to be greater than k since $x(s)$ is differentiable at $s=0$. The inequality (4) implies then

$$b_N \le b_N \left(b_N^{-k/w_j} O(s_N) + b_N^{-2k/w_j} O(s_N) \right)$$

which contradicts the fact that $b_N \to +\infty$.

Hence the ratio $\|x(s^k)\|/s$ is bounded when $s \to 0$. Using (4), we have, when $s \to 0$

$$\frac{1}{s^{w_{l'}}} x_{l'}(s^k) \sim s^k \qquad \text{and} \qquad \frac{1}{s^{w_j}} x_j(s^k) \to 0 \text{ for } j \neq l'.$$

Thus, in our system of privileged coordinates, $\delta_{s^{-1}} c(s^k)$ has a limit as $s \to 0$ and so $c(s)$ is k-rectifiable by Lemma 3. Moreover, if $\dot{c}(0)$ belongs to $\mathcal{D}^k(p)/\mathcal{D}^{k-1}(p)$ (i.e. $k'=k$), that limit is $\dot{c}(0) = (0, \dots, 1, \dots, 0)$. This concludes the proof. $\square$

It results from this lemma that the set of equivalence classes of horizontal curves $\mathcal{H}or = \{[c(s)], c(s) \text{ horizontal curve}\}$ is included in the tangent cone $C_p^1 M$. Similarly, for an integer k, $1 \le k \le r$, the set $\mathcal{H}or_k = \{[c(s^k)], c(s) \text{ tangent a.e. to } \mathcal{D}^k/\mathcal{D}^{k-1}\}$ is included in $C_p^k M$ (by definition of horizontal curves, $\mathcal{H}or = \mathcal{H}or_1$). As a consequence of Theorem 11 and Lemma 12 we obtain the following.

Corollary 13. *$C_p^k M / C_p^{k+1} M$ is isomorphic to $\mathcal{H}or_k$.*

3 Measures

3.1 Infinitesimal measures

Definition. The *k-dimensional infinitesimal measure* of a k-rectifiable curve $c(s)$ at p is

$$\operatorname{meas}_p^k\big(c(s)\big) = \bar{d}\big([p], [c(s^k)]\big)^k.$$

Remark. Given a system of privileged coordinates, we have $\operatorname{meas}_p^k\big(c(s)\big) = \hat{d}(0,x)^k$ where $x = \lim_{s\to 0} \delta_{s^{-1}} c(s^k)$. Ball-Box Theorem gives then an estimate of the infinitesimal measure

$$K_1' \left(|x_1|^{1/w_1} + \cdots + |x_n|^{1/w_n} \right)^k \le \operatorname{meas}_p^k\big(c(s)\big) \le K_2' \left(|x_1|^{1/w_1} + \cdots + |x_n|^{1/w_n} \right)^k.$$

This infinitesimal measure is actually a measure on $C_p^k M$ and we will use it indiscriminately as a function of $c(s)$ or of $[c(s^k)]$. Remark that, if $k' > k$ and $c(s)$ is k-rectifiable, then $\operatorname{meas}_p^{k'}\big(c(s)\big) = 0$.

Lemma 14. *The measure* meas_p^k *is homogeneous*

- *with respect to the parameterization, i.e., if $c(s)$ is a k-rectifiable curve and $\lambda > 0$ a real number, then*
$$\mathrm{meas}_p^k\big(c(\lambda s)\big) = \lambda\,\mathrm{meas}_p^k\big(c(s)\big);$$

- *with respect to a dilation in privileged coordinates:*
$$\mathrm{meas}_p^k\big(\delta_\lambda c(s)\big) = \lambda\,\mathrm{meas}_p^k\big(c(s)\big).$$

Proof. The second point follows directly from the expression $\mathrm{meas}_p^k\big(c(s)\big) = \hat{d}(0,x)^k$ in the remark above.

In the first point, the curve $c(\lambda s)$ is clearly k-rectifiable and its k-dimensional infinitesimal measure is $\bar{d}\big([p],[c(\lambda s^k)]\big)^k$. The lemma then results from

$$\bar{d}\big([p],[c(\lambda s^k)]\big) = \lim_{s\to 0}\frac{1}{s}d\big(p,c(\lambda s^k)\big) = \lambda^{1/k}\lim_{\sigma\to 0}\frac{1}{\sigma}d\big(p,c(\sigma^k)\big)$$

with $\sigma = \lambda^{1/k}s$. □

Let us compute the one-dimensional infinitesimal measure of an horizontal path $c(s)$. As in Riemannian geometry, the limit $\lim_{s\to 0}\frac{1}{s}d\big(p,c(s)\big)$ is equal to $\sqrt{g(\dot{c}(0))}$. Thus $\mathrm{meas}_p^1 = \sqrt{g}\circ I_1$ on $\mathcal{H}or$.

3.2 Lengths

We define now a class of lengths of paths as integrals of infinitesimal measures.

Definition. Let $\gamma : [a,b] \to M$ be an absolutely continuous path and $k \geq 1$. The path γ is called *k-rectifiable at t_0* if the curve $c(s) = \gamma(t_0+s) \in \mathcal{C}_{\gamma(t_0)}$ and is k-rectifiable. We extend the definition of k-dimensional infinitesimal measure by setting

$$\mathrm{meas}_{t_0}^k\big(\gamma\big) = \begin{cases} \mathrm{meas}_{\gamma(t_0)}^k\big(c(s)\big) & \text{if } \gamma \text{ is } k\text{-rectifiable at } t_0, \\ +\infty & \text{otherwise.} \end{cases}$$

Recall that $\gamma(t+s)$ belongs to $\mathcal{C}_{\gamma(t)}$ if and only if γ is differentiable at t and that γ is differentiable a.e. on $[a,b]$ since it is an absolutely continuous path. On the other hand, if γ is differentiable at a point t, $\mathrm{meas}_t^k\big(\gamma\big) = 0$ for $k > k_0(t)$, with $k_0(t) = \inf\{k \geq 1$ such that γ is k-rectifiable at $t\}$, and $\mathrm{meas}_t^k\big(\gamma\big) = +\infty$ for $k < k_0(t)$. Only the $k_0(t)$-dimensional infinitesimal measure can be finite and nonzero, but of course, the curve may not be $k_0(t)$-rectifiable (see the example in Section 2.2).

Definition. Let $\gamma : [a,b] \to M$ be an absolutely continuous path and $k \geq 1$. The *k-dimensional length* of γ is

$$\mathrm{Length}_k(\gamma) = \int_a^b \mathrm{meas}_t^k(\gamma)\ dt.$$

These lengths can be obtained more explicitely.

Theorem 15. *For any absolutely continuous path $\gamma : [a,b] \to M$ and any $k \geq 1$,*

$$\mathrm{Length}_k(\gamma) = \int_a^b \left[\limsup_{s \to 0} \frac{1}{s} d\big(\gamma(t), \gamma(t+s^k)\big) \right]^k dt.$$

Proof. Let $k \geq 1$. Remark first that, if γ is k-rectifiable at t, then

$$\left[\limsup_{s \to 0} \frac{1}{s} d\big(\gamma(t), \gamma(t+s^k)\big) \right]^k = \mathrm{meas}_t^k\big(\gamma\big). \tag{5}$$

This equality also holds if γ is differentiable at t and $k < k_0(t)$ (in this case both terms equal $+\infty$).

Let k_γ be the smallest integer l such that $\dot\gamma(t)$ belongs to $\mathcal{D}^l\big(\gamma(t)\big)$ a.e. on $[a,b]$ (k_γ exists since $r(\gamma(t))$ is bounded on γ).

- If $k \geq k_\gamma$: by Lemma 12, γ is k-rectifiable a.e. on $[a,b]$. The equality (5) holds then a.e. on γ, which gives the required value for $\mathrm{Length}_k(\gamma)$.

- If $k < k_\gamma$: there is a subset $E \subset [a,b]$ of positive measure such that $\dot\gamma$ exists and does not belong to $\mathcal{D}^{\lfloor k \rfloor}$ on E ($\lfloor k \rfloor$ is the integer part of k). By Lemma 7, $k_0(t) \geq \lfloor k \rfloor + 1 > k$ on E. Thus both members of (5) are infinite on a set of positive measure and their integral on $[a,b]$ are infinite. This ends the proof.

□

A first consequence of this theorem is that Length_k has the invariance property of the usual length.

Lemma 16. *The k-dimensional length of a path does not depend on the parameterization of the path,* i.e. *if $g : [a',b'] \to [a,b]$ is a C^1 homeomorphism, then $\mathrm{Length}_k(\gamma) = \mathrm{Length}_k(\gamma \circ g)$.*

Proof. Changing if necessary g into $-g$, we assume $g'(\tau) > 0$ a.e. on $[a',b']$. Let $\tau \in [a',b']$ such that $g'(\tau) > 0$ and $t = g(\tau)$. For one $c \in [\tau, \tau + \sigma^k]$, we have

$$d\big(\gamma \circ g(\tau), \gamma \circ g(\tau + \sigma^k)\big) = d\big(\gamma(t), \gamma(t + \sigma^k g'(c))\big).$$

Setting $s^k = \sigma^k g'(c)$, we obtain

$$\begin{aligned}\limsup_{\sigma\to 0}\frac{1}{\sigma}d\big(\gamma\circ g(\tau),\gamma\circ g(\tau+\sigma^k)\big) &= \limsup_{\sigma\to 0}\big(g'(c)\big)^{1/k}\frac{1}{s}d\big(\gamma(t),\gamma(t+s^k)\big)\\ &= \big(g'(\tau)\big)^{1/k}\limsup_{s\to 0}\frac{1}{s}d\big(\gamma(t),\gamma(t+s^k)\big).\end{aligned}$$

Thus the k-length of $\gamma\circ g$ is

$$\begin{aligned}\mathrm{Length}_k\big(\gamma\circ g\big) &= \int_{a'}^{b'}\left[\limsup_{\sigma\to 0}\frac{1}{\sigma}d\big(\gamma\circ g(\tau),\gamma\circ g(\tau+\sigma^k)\big)\right]^k\,d\tau\\ &= \int_{a'}^{b'}\left[\limsup_{s\to 0}\frac{1}{s}d\big(\gamma(g(\tau)),\gamma(g(\tau)+s^k)\big)\right]^k g'(\tau)\,d\tau = \mathrm{Length}_k\big(\gamma\big).\end{aligned}$$

□

We have also, as a consequence of the proof of Theorem 15:

Corollary 17. *Let k_γ be the smallest $k \geq 1$ such that γ is k-rectifiable a.e. on $[a,b]$. Then k_γ is an integer and*

- $\mathrm{Length}_k(\gamma) = \begin{cases} 0 & \textit{for } k > k_\gamma \\ +\infty & \textit{for } k < k_\gamma \end{cases}$;
- *k_γ is the smallest integer l such that $\dot\gamma(t)$ belongs to $\mathcal{D}^l\big(\gamma(t)\big)$ a.e. on $[a,b]$.*

Remark.
- A path γ is k_γ-rectifiable a.e. and only the k_γ-dimensional length can be finite and nonzero. However $\mathrm{Length}_{k_\gamma}(\gamma)$ may be infinite when the path contains singular points for the filtration of the tangent space (see [7, 8] and the Martinet case in Section 4).
- The number k_γ defines a dimension of the path, similar to the Hausdorff dimension.

To sum up, the infinitesimal measure can be seen as generalizing the norm and Length_k appears as a class of lengths, the first, Length_1, being the usual length:

Lemma 18. *For any path γ,*

$$\mathrm{Length}_1(\gamma) = \mathrm{length}(\gamma).$$

Moreover, γ is rectifiable a.e. if and only if it is horizontal, that is if and only if $\mathrm{Length}_1(\gamma) < \infty$.

Proof. Recall that the usual length of a path is defined as

$$\text{length}(\gamma) = \begin{cases} \int \sqrt{g(\dot\gamma)} & \text{if } \gamma \text{ is horizontal,} \\ +\infty & \text{otherwise.} \end{cases}$$

By Theorem 15, the one-dimensional length $\text{Length}_1(\gamma)$ is infinite if γ is not a.e. rectifiable. Now, Lemma 7 and Corollary 8 imply that a path is a.e. rectifiable if and only if it is horizontal. Since $\text{meas}_p^1 = \sqrt{g} \circ I_1$ on $\mathcal{H}or$, it shows that the one-dimensional length equals the usual length. □

4 The contact case

4.1 Infinitesimal 2-measure

Let $(M, \mathcal{D}, g)$ be a (2n+1)-dimensional contact sub-Riemannian manifold. That is, $\mathcal{D}$ is a 2n-dimensional contact distribution. We will describe a differential geometric way to compute the length of a path.

Recall that we can fix a unique canonical contact form θ such that $(d\theta_{\mathcal{D}})^n = dv$, where dv is the volume form on $\mathcal{D}$. The form θ defines a vector field ξ satisfying $\theta(\xi) = 1$ and $L_\xi \theta = d\theta(\xi, \cdot) = 0$ (here L_ξ is the Lie derivative along ξ).

Lemma 19. *For $p \in M$ there exists a constant K_p such that for every $c(s) \in \mathcal{C}_p$*

$$\text{meas}_p^2(c(s)) = K_p |\theta(\dot c(0))|$$

Proof. By Theorem 11 $C_p^2 M \simeq \mathcal{D}^2(p)/\mathcal{D}(p)$. Observe that in this case both sides are one dimensional vector spaces. From Lemma 14 meas_p^2 defines a norm on $C_p^2 M$ and it is clear that $|\theta(\cdot)|$ defines a norm on $\mathcal{D}^2(p)/\mathcal{D}(p)$. Therefore the two norms are proportional. □

We will compute the constant K_p for a three dimensional contact manifold and will show that it is independent of p for the above choice of contact form. We use the following standard normal coordinates.

Lemma 20. *Given a sub-Riemannian contact manifold of dimension 3, locally one can define normal coordinates (x, y, t) centered at $(0, 0, 0)$ such that*

1. $\theta = dt + \alpha$ *where* $d\alpha_{\mathcal{D}} = \sqrt{g}\, dx dy_{\mathcal{D}} = dv$
2. $\xi = \frac{\partial}{\partial t}$ *at* $(0, 0, t)$.

To determine the constant K_p it is enough to compute the measure of a vertical curve. Let $c(s) = (0,0,s)$ in the coordinates of the preceding lemma. Recall that $\mathrm{meas}_p^2(c(s)) = \lim_{s\to 0} \frac{1}{s^2} d\big(0, c(s^2)\big)^2$.

Lemma 21. *If the sub-Riemannian metric is invariant then* $d\big(0, c(a)\big)^2 = 4\pi a$.

Proof. Recall that $d\big(0, c(a)\big) = \inf\{\mathrm{length}(\gamma)\}$ taken for horizontal paths γ where $\mathrm{length}(\gamma) = \int \sqrt{g(\dot\gamma)}$.

Using the normal coordinates we define ϖ as the projection on the xy plane. As the sub-Riemannian metric is invariant we observe that $\mathrm{length}(\gamma)$ is equal to the length of $\varpi(\gamma)$ in the xy plane equipped with the Euclidean metric defined by the projection.

An horizontal path satisfies $\theta(\dot\gamma) = 0$. Integrating from 0 to $c(a)$, we obtain that $a = -\int\int \alpha$, that is the area enclosed by the closed curve $\varpi(\gamma)$.

Thus $d\big(0, c(a)\big)$ is the smallest perimeter of a surface of area a in the Euclidean xy plane. The solution to this isoperimetric problem is given by:

$$d\big(0, c(a)\big)^2 = 4\pi a.$$

□

Theorem 22. *Let* $(M, \mathcal{D}, g)$ *be a sub-Riemannian contact manifold of dimension 3 and* θ *the canonical contact form. Then for every* $p \in M$ *and* $c(s) \in \mathcal{C}_p$,

$$\mathrm{meas}_p^2(c(s)) = 4\pi|\theta(\dot c(0))|.$$

Proof. Near $p = (0,0,0)$ in normal coordinates, we write g as $g^0 + xg^x + yg^y + tg^t$, where $g^0(x,y,t) = g(0,0,0)$ is invariant. We denote also respectively by length^0 and d^0 the length of horizontal paths and the sub-Riemannian distance corresponding to the metric g^0. Let γ be an horizontal path connecting 0 to $c(s) = (0,0,s)$. We choose as a parameter of γ the arc-length for the metric g^0. We have

$$\mathrm{length}(\gamma) = \int \sqrt{1 + xg^x(\dot\gamma) + yg^y(\dot\gamma) + tg^t(\dot\gamma)}\, d\tau.$$

Set $\varepsilon(s) = d\big(0, c(s)\big)$. We want to estimate the distance $d\big(0, c(s)\big)$, so we assume that γ belongs to the closed sub-Riemannian ball $B_d(0, \varepsilon(s))$. On the other hand any Riemannian extension g' of the sub-Riemannian metric g yields a Riemannian distance d' satisfying $d' \le d$. The ball $B_d(0, \varepsilon(s))$ is then included in the Riemannian ball $B_{d'}(0, \varepsilon(s))$.

Finally, since g^x, g^y, g^t are smooth, there exist K and $s_0 > 0$ independent of γ such that, if $s < s_0$, then

$$\text{length}^0(\gamma)(1 - K\varepsilon(s)) \leq \text{length}(\gamma) \leq \text{length}^0(\gamma)(1 + K\varepsilon(s)).$$

So the distance $d\big(0, c(s)\big)$ is equivalent to $d^0\big(0, c(s)\big)$, which is given by Lemma 21.

We obtain finally

$$\text{meas}_p^2(c(s)) = \lim_{s\to 0} \frac{1}{s^2} d\big(0, c(s^2)\big)^2 = 4\pi.$$

As $\theta(\dot{c}(0)) = 1$, this concludes the proof. $\square$

To obtain the formula in higher dimensions we introduce invariants of the sub-Riemannian structure.

As, restricted to $\mathcal{D}$, $d\theta$ is skew-symmetric, there exists a skew-symmetric linear operator $H : \mathcal{D} \to \mathcal{D}$ satisfying

$$d\theta(X, Y) = g(HX, Y)$$

for vectors X and Y in $\mathcal{D}$. At each point one can choose an orthonormal basis $\{X_i, Y_i\}$ of $\mathcal{D}$ such that $H(X_i) = \lambda_i Y_i$ and $H(Y_i) = -\lambda_i X_i$, with $\lambda_i > 0$. As $d\theta^n = dv$ we observe that $\prod_{i=1}^n \lambda_i = 1/(2^n n!)$. Only the symmetric functions of the λ_i define invariant functions on the sub-Riemannian manifold. On the other hand, $\lambda_{max} = \max_i \lambda_i$ is also a well defined function.

A model in the case of constant λ_i is the Heisenberg group H^{2n+1} where an orthonormal basis is given by $\{X_i, Y_i\}$ with $[X_i, Y_i] = \lambda_i \xi$. A simple computation (see for instance [2]) shows that the sub-Riemannian distance for a vertical path verifies $d(0, c(a))^2 = 4\pi a/\lambda_{max}$. This is a generalization of Lemma 21 and following the same reasoning as in Theorem 22 we obtain

Theorem 23. *Let $(M, \mathcal{D}, g)$ be a sub-Riemannian contact manifold of dimension $2n+1$ and θ the canonical contact form. Then for every $p \in M$ and $c(s) \in \mathcal{C}_p$,*

$$\text{meas}_p^2(c(s)) = \frac{4\pi}{\lambda_{max}(p)} |\theta(\dot{c}(0))|.$$

Remark. Notice that $\lambda_{max}(p)$ can be seen as the vertical coordinate of the "biggest" bracket. Such coefficients appear naturally in estimates of sub-Riemannian balls [9] and of entropy [8].

4.2 Application to the Martinet case

Consider now the flat Martinet case, that is, the sub-Riemannian manifold $(\mathbb{R}^3, \mathcal{D}, g)$ where $\mathcal{D} = \ker\omega$, $\omega = dz - \frac{y^2}{2}dx$ being the Martinet one-form and $g = dx^2 + dy^2$ is the flat metric on $\mathcal{D}$. This distribution $\mathcal{D}$ is of contact type off of the Martinet plane $y = 0$, since $\omega \wedge d\omega = y dx \wedge dy \wedge dz$ is nonzero when $y \neq 0$. It is easy to verify that the corresponding canonical contact form is $\theta = \frac{1}{y}\omega$. The Martinet plane is the locus of the singular points ($\mathcal{D}^2$ is not a distribution at these points).

We will compute the 2-length of paths close to the Martinet plane. We consider two kind of paths, namely, for a given $\varepsilon \geq 0$:

- $\gamma_\varepsilon(t) = (0, t, t)$, $t \in [\varepsilon, 1]$;
- $\gamma'_\varepsilon(t) = (t, t, 0)$, $t \in [\varepsilon, 1]$.

Theorem 22 gives directly the infinitesimal 2-measure on these paths outside the Martinet plane:

$$\mathrm{meas}_t^2\big(\gamma_\varepsilon\big) = \frac{4\pi}{t} \quad \text{and} \quad \mathrm{meas}_t^2\big(\gamma'_\varepsilon\big) = 2\pi t \quad \text{for } t > 0.$$

We obtain the 2-lengths by integrating:

$$\begin{cases} \mathrm{Length}_2\big(\gamma_\varepsilon\big) = 4\pi \log(\frac{1}{\varepsilon}) & \text{for } \varepsilon > 0 \\ \mathrm{Length}_2\big(\gamma_0\big) = +\infty \end{cases} \quad \text{and} \quad \mathrm{Length}_2\big(\gamma'_\varepsilon\big) = \pi(1 - \varepsilon^2).$$

The path γ_0 gives an example of an almost everywhere 2-rectifiable path ($k_{\gamma_0} = 2$) with infinite 2-length. The reason is that γ_0 is not 2-rectifiable at $t = 0$ (it is 3-rectifiable), $\gamma_0(0)$ being a singular point. On the other hand, the 2-length of γ'_0 is finite because γ'_0 is 2-rectifiable everywhere (and actually rectifiable at $t = 0$).

Note that, in the case of a Martinet sub-Riemannian manifold with a general metric g on $\mathcal{D}$, we will have the same behaviour for $\mathrm{Length}_2\big(\gamma_\varepsilon\big)$ and $\mathrm{Length}_2\big(\gamma'_\varepsilon\big)$ as above.

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