Introduction to Cartan geometry

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Contents

1 Introduction

Cartan geometries are a solution to the very general question: *what is a geometric structure?* Riemannian geometry, conformal geometry and projective geometry are examples of geometric situations.

The mindset is the following. A Cartan geometry should first be a manifold with an homogenous space attached to each point. For instance in Riemannian geometry each point has an attached Euclidean space by equipping the tangent space with the Riemannian metric. This data is then equipped with a Cartan connection explaining how the homogeneous spaces are infinitesimally *connected*.

When one has two different Cartan geometries, one can ask if they are equivalent. For instance, when are two Riemannian manifold isometric or at least locally isometric? This is a deep question known under the general name of the *equivalence problem*. In Riemannian geometry, the differential system $g = \sum \mathrm{d}x_i^2$ asks wether the space is locally euclidean. It is the case if, and only if, a curvature tensor vanishes. Cartan geometries give a similar procedure for all the geometries: a curvature tensor vanishes if, and only if, the space is locally homogeneous.

But when the curvature is not zero, the equivalence problem is harder to solve. What is the meaning of two curvature on two different spaces being equal? Cartan's method for the equivalence problem is a general procedure to study and solve this problem in many situations. An important example is given by the class of the symmetric spaces: those are the Riemannian spaces that are not flat but have a parallel curvature tensor. With Cartan's method one can verify when two spaces with this property are locally equivalent or not.

In this course, we will describe Cartan geometries and introduce the local equivalence problem between geometric structures. The main global problem we will deal with is the classification of smooth Anosov flows on a compact three manifold and, more generally, of non-compact automorphisms groups acting on a compact manifold preserving a contact distribution and two transverse lines contained in the contact plane at each point of the manifold.

2 Pfaff equations and Cartan's method

One of the basic problems in geometry is to understand the equivalence between geometric objects. For instance, given two Riemannian manifolds when are they locally or globally isometric? The main idea of Cartan's method is to associate to a manifold with a geometric structure another manifold (with higher dimension) where the geometric structure is given by a parallelism of its cotangent bundle. The parallelism can be defined in very general situations but when the geometric structure is simple enough one can describe it by an important mathematical object which will be introduced later: Cartan connections on principal bundles.

2.1 Frobenius theorem

A reference for this section is [**Wa**]. The basic theorem which is the foundation of the theory is the existence of a local flow defined by a vector field. It is a natural generalization of the following example.

Example 1 With mild regularity conditions (for instance C^1) a vector field on the real line can be locally integrated. Let $X = \alpha(t)\frac{\partial}{\partial t}$ such a vector field. A solution of the Cauchy problem $f'(t) = X(f(t))$ with initial condition $f(0) = x$ has a maximal solution defined on an open interval (a_x, b_x) where a_x or b_x could be infinite.

Theorem 2.1. *(**local flow**) Let X be a C^1 vector field on a manifold M. There exists an open set $A = \{ (t,x) \mid a_x < t < b_x \} \subset \mathbf{R} \times M$ and a function $\phi : A \to M$ (we write $\phi(t,x) = \phi_t(x)$), such that*

1. *$\phi_0 = Id$ (so, in particular, $a_x < 0 < b_x$).*
2. *$\frac{d\phi_t(x)}{dt} = X(\phi_t(x))$.*
3. *$\phi_t(x)$, for $t \in (a_x, b_x)$, is a maximal solution of the equation $\frac{d\gamma(t)}{dt} = X(\gamma(t))$ with initial condition $\gamma(0) = x$.*

We will also use the time-dependent version of the local flow. That is, for a vector field $X_t(x)$ which depends on time defined on an open subset $\Omega \subset \mathbf{R} \times M$ there exists a local solution $\phi_t(t_0, x_0)$ to the equation

$$\frac{d\phi_t(t_0, x_0)}{dt} = X_t(\phi_t(t_0, x_0))$$

with initial condition $\phi_{t_0}(t_0, x_0) = x_0$.

The flow box theorem gives a local normal form for a vector field on a manifold:

Theorem 2.2. *(**flow box theorem**) Let X be a C^1 vector field on a manifold M. For each $x \in M$ there exists an open set $U \subset M$ containing x and a chart $\phi : U \to \mathbf{R}^n = \{(x_1, \cdots, x_n) | x_i \in \mathbf{R}\}$ such that $\phi(x) = 0$ and $\phi_*(X) = \frac{\partial}{\partial x_1}$.*

Proof. The idea is to follow the flow starting from a hypersurface transverse to the vector field at the point x. The time will be the first coordinate of a chart.

One can always choose a chart $\psi : V \to \psi(V)$ on a neighborhood V of x so that $\psi(x) = 0$ and $\psi_*(X(x)) = \frac{\partial}{\partial x_1}$. Consider the hypersurface containing x defined by $\psi^{-1}((0, x_2, \cdots, x_n))$ with $(0, x_2, \cdots, x_n) \in \psi(V)$. The existence of the flow implies that for a relatively compact $U \subset V$, there exists $\varepsilon > 0$ such that the flow is defined on $(-\varepsilon, \varepsilon) \times U$. Define then $\sigma(x_1, x_2, \cdots, x_n) = \phi_{x_1}(\psi^{-1}((0, x_2, \cdots, x_n)))$, the flow at time x_1 starting at the point $\psi^{-1}((0, x_2, \cdots, x_n))$. On a perhaps smaller neighborhood one can invert σ to obtain a chart satisfying the condition of the theorem. Indeed

$$\sigma_*(\frac{\partial}{\partial x_1}(x_1, \cdots x_n) = \frac{d}{dt}\phi_{x_1}(\psi^{-1}(0, x_2, \cdots, x_n)) = X(\phi_{x_1}(\psi^{-1}(0, x_2, \cdots, x_n)) = X \circ \sigma.$$

□

In two real dimensions, one can improve the flow box theorem to obtain that two given vector fields can be normalized to be along coordinates of a chart:

Proposition 2.3. *Let X_1 and X_2 be C^1 vector fields on a two dimensional manifold M which are linearly independent at every point. For each $x \in M$ there exists an open set $U \subset M$ containing x and a chart $\phi : U \to \mathbf{R}^2 = \{(x_1, x_2) | x_i \in \mathbf{R}\}$ such that $\phi(x) = 0$ and $\phi_*(X_1) \in \langle \frac{\partial}{\partial x_1} \rangle$ and $\phi_*(X_2) \in \langle \frac{\partial}{\partial x_2} \rangle$.*

Proof. We may suppose that there is a chart $\psi : V \to \psi(V)$ on a neighborhood V of x so that $\psi(x) = 0$ and $\psi_*(X_1(x)) = \frac{\partial}{\partial x_1}$ and $\psi_*(X_2(x)) = \frac{\partial}{\partial x_2}$. The proof of the previous theorem shows that there exists a neighborhood U of x such that each point $y \in U$ is in a unique integral line of X_1 passing through a point $\psi^{-1}(0, x_2(y))$ and a unique integral line of X_2 passing through a point $\psi^{-1}(x_1(y), 0)$. The map $\phi : U \to \mathbf{R}^2$ defined by $y \to (x_1(y), x_2(y))$ is C^1 with $d\phi(x) = Id$. This defines a coordinate chart in perhaps a smaller neighborhood. □

Distributions on a manifold, that is, subbundles of the tangent bundle are examples of geometric structures. In the following, for simplicity sake, we assume that a distribution $D \subset TM$ is of constant rank.

Definition 2.4. *Let D be a distribution on a manifold M. We say that a submanifold $\phi : N \to M$ is an integral manifold of D if $d\phi(T_x N) \subset D(\phi(x))$ for all $x \in N$.*

An important problem is to give conditions so that the dimension of the integral manifold coincides with the rank of the distribution. Essentially, the condition says that the vector fields in the distribution form a Lie algebra:

Definition 2.5. *We say a distribution D generated by vector fields $\{X_1, \cdots, X_n\}$ defined on an open set U of a manifold is involutive if for all i and j, $[X_i, X_j]$ is a vector field in the distribution.*

We can state now the main theorem of this section.

Theorem 2.6. *Let M be an m-dimensional manifold and D a C^1 distribution of rank n. Then D is involutive if and only if for every $x \in M$ there exists a coordinate chart $(x_1, \cdots, x_m)$ such that D is generated by $\frac{\partial}{\partial x_i}$, for $1 \le i \le n$.*

Proof. The case $n = 1$ is precisely the content of the flow-box theorem. The idea of the proof for $n > 1$ is to linearize one of the generating vector fields around x and then chose a hyperplane transversal to this field at x to obtain a distribution of rank $n-1$ on it and then use induction.

Let us start with generating vector fields $(\frac{\partial}{\partial x_1}, X_2, \cdots, X_n)$ where we linearized the first field in a coordinate system $(x_1, y_2, \cdots y_m)$ which we can suppose centred at 0. Here, in order to simplify notations we write $\frac{\partial}{\partial x_1}$ for the vector field on the manifold defined by the corresponding vector field in the chart. The distribution D induces a distribution D' of rank $n-1$ on the codimension one submanifold N passing through 0 defined by $x_1 = 0$: the distribution D' is generated by

$$X_i' = X_i - X_i(x_1)\frac{\partial}{\partial x_1}$$

for $2 \le i \le n$. Indeed, these vectors are tangent to the transverse submanifold because $X_i'(x_1) = 0$. One proves that this distribution is an involutive distribution (exercise). Here, for simplicity, we suppose that $n = 2$ and therefore the induced distribution is generated by a vector field in N. Using the flow-box theorem again, there exists a neighborhood of 0 in N with coordinates $(w_2, \cdots, w_m)$ such that $X_2' = \frac{\partial}{\partial w_2}$. We claim the adapted coordinates on a neighborhood of 0 in M are

$$(x_1, \cdots, x_m) = (x_1, w_2 \circ \pi, \cdots, w_m \circ \pi)$$

where π is the projection to N along the orbits of $\frac{\partial}{\partial x_1}$ (in coordinates we have $\pi(x_1, y_2, \cdots y_m) = (y_2, \cdots y_m)$). First observe that, for $i > 1$, $X_2'(x_i) = X_2'(w_i \circ \pi(x_1, y_2, \cdots y_m)) = X_2'(w_i(y_2, \cdots y_m))$ and therefore by definition of the coordinate chart in N, at points in N we have $X_2'(x_i) = 0$ for $i > 2$ along N. We need to show that $X_2'(x_i) = 0$, for $i > 2$, at all points in a whole neighborhood of the origin. For that sake we compute

$$\frac{\partial}{\partial x_1} X_2'(x_i) = X_2' \frac{\partial x_i}{\partial x_1} + [\frac{\partial}{\partial x_1}, X_2'](x_i)$$

which, because the distribution is involutive, can be written as

$$\frac{\partial}{\partial x_1} X_2'(x_i) = X_2' \frac{\partial x_i}{\partial x_1} + a_1 \frac{\partial}{\partial x_1}(x_i) + a_2 X_2'(x_i),$$

for two functions a_1 and a_2. The first two terms in the right side are clearly null. We obtain then the differential equation

$$\frac{\partial}{\partial x_1} X_2'(x_i) = a_2 X_2'(x_i).$$

For each $i > 2$, this is a first order ordinary differential equation with initial condition $X_2'(x_i) = 0$ at a point $(0, x_2, \cdots x_m)$. By unicity, $X_2'(x_i) = 0$ for all $(x_1, x_2, \cdots x_m)$ in a neighborhood of the origin.

□

We proved then that a distribution is involutive if and only if for each $y \in M$ there exists an integral manifold of maximal dimension equal to the rank of the distribution passing through y. In local coordinates defined by Frobenius theorem the integral manifolds are given locally by $(x_1, \cdots, x_n) \to (x_1, \cdots, x_n, x^0_{n+1}, \cdots x^0_m)$, where x^0_i, for $n < i \leq m$, are constants. In fact, one can prove that there exists a unique maximal connected integral manifold passing through y (see [**Wa**]).

2.2 Differential ideals and the equivalence problem

2.2.1 Differential ideals and Frobenius theorem

We will work with smooth forms. Let M be an n-dimensional manifold and $\Omega^*(M)$ be the set of sections of the space $\Lambda T^* M$, the graded algebra of the exterior powers of the cotangent bundle. The space $\Omega^*(M)$ is the space of all the differential forms of M.

Definition 2.7. *A differential ideal $I \subset \Omega^*(M)$ is an homogeneous ideal for the exterior algebra which is closed under exterior derivative.*

Here, homogeneous ideal means that if $\alpha \in I$ and $\alpha = \alpha^0 + \cdots + \alpha^p$ is its decomposition with $\alpha^i \in \Omega^i(M)$ for $0 \leq i \leq p$ then $\alpha^i \in I$ for all i.

The (algebraic) ideal generated in $\Omega^*(M)$ generated by a 1-form θ is given by all multiples of this form by functions on the manifold. The differential ideal generated by a 1-form θ consists of all combinations of θ and $\mathrm{d}\theta$. Ideals of this type are studied in Pfaff's problem. A simple case is the ideal generated by a unique closed form. A particular description of this ideal, which is simply all multiples of the closed form, is obtained invoking Poincaré's lemma.

Lemma 2.8. *For any closed $(p+1)$-form α there exits locally a p-form β such that*

$$\alpha = \mathrm{d}\beta.$$

Definition 2.9. *If I is a differential ideal, an integral submanifold is an immersion $\phi \colon N \to M$ such that $\phi^*\omega = 0$ for any $\omega \in I$.*

Note If α is a 1-form that annihilates a distribution then since

$$\mathrm{d}\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \tag{1}$$

the ideal generated by such α is closed under the exterior derivative if, and only if, $[X, Y]$ belongs also to the distribution.

The most natural example of ideals in $\Omega^*(M)$ arises as the ideal I_D of forms which annihilate a distribution D.

There is a correspondence between the distribution D and the ideal I_D. If the distribution is given by k fields, we chose a coordinate system such that at a fixed point $dx^1, \cdots, dx^k$ restricted to the distribution are independent. They will be clearly independent on a neighborhood. One can write, restricted to the distribution, for $k+1 \leq j \leq n$, $dx^j = \sum_{i=1}^k c^j_i dx^i$. Therefore one gets $n-k$ independent forms $dx^j - \sum_{i=1}^k c^j_i dx^i$ vanishing on the distribution.

The ideal I_D is a differential ideal if, and only, if the distribution is involutive and Frobenius theorem is stated in this language as the following.[1]

[1] See F. Warner, *Foundations of differentiable manifolds and Lie groups.*

Theorem 2.10 (Frobenius). *Let I be a differential ideal locally (algebraically) generated by $(n-p)$ independent 1-forms. Then, for each $x \in M$, there exists a unique maximal (of dimension p) connected integral manifold of I passing through x.*

In fact, it suffices that the 1-forms in the statement be of regularity C^1.

Example 1 If the ideal is generated by a single 1-form θ, then being a differential ideal means that $d\theta = \theta \wedge \omega$, for ω a 1-form. (Hence $d\theta \wedge \theta = 0$.) The extreme opposite is the case of a contact form θ which satisfies $d\theta \wedge \theta \neq 0$ at every point.

Exercise Prove that if $\theta(x) \neq 0$ and $\theta \wedge d\theta = 0$ then, at a neighborhood of x, there exists a 1-form such that $d\theta = \theta \wedge \alpha$.

Example 2 If the ideal is generated by the 1-form $dy - p\,dx$ and $dp - F(x, y, p)\,dx$ in $\mathbf{R}^3$ we obtain one dimensional integral submanifolds which correspond to solutions of a second order differential equation.

Example 3 A partial differential equation of the form

$$F(x_i, u, \frac{\partial}{\partial x_i}) = 0$$

with certain regularity conditions, can be translated into the problem of finding integral submanifolds to the ideal generated by $du - p_i\,dx_i$ restricted to the submanifold defined by the function $F(x_i, u, p_i) = 0$ in $\mathbf{R}^{2n+1}$.

Exercise Consider $M = \mathbf{R}^n \times \mathbf{R}^m$ with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_m)$ and a differential ideal I generated by

$$\omega^i = \sum_j a^i_j(x_1, \cdots, y_m)dx_j \text{ (for } 1 \leq i \leq N),\ dy_1, \cdots, dy_m,$$

where a^i_j are functions on M. Show that the differential ideal is algebraically generated by these 1-forms if and only if, for each fixed $(y_1, \cdots, y_m) \in \mathbf{R}^m$, the differential ideal generated by the forms ω^i restricted to $\mathbf{R}^n \times \{(y_1, \cdots, y_m)\}$ is algebraically generated.

An important situation which gives rise to involutive distributions is given in the following definition.

Definition 2.11. *Let I be a differential ideal. The characteristic distribution is defined by*

$$D_I(x) = \{\ v \in T_xM \mid \iota_v I_x \subset I_x\ \}.$$

We say that the differential ideal is non-singular if the distribution is of constant rank.

Here I_x is the ideal in $\Lambda^*_x M$ obtained by evaluating all elements of I at x.

Example In the particular case of a differential ideal generated by only one closed 2-form α it is given by

$$D = \{\, v \in TM \mid \iota_v \alpha = 0 \,\}.$$

We can now state Cartan's result on the integrability of the characteristic distributions.

Lemma 2.12. *The characteristic distribution of a non-singular differential ideal is an involutive distribution.*

Proof. Let I be a differential ideal. Clearly, from Cartan's formula

$$L_X = d \circ \iota_X + \iota_X \circ d,$$

we obtain that if X is characteristic then $L_X I \subset I$. Suppose now that X and Y are two vector fields. We have (see Proposition 3.10, pg. 35, in [**KoN**])

$$L_X \iota_Y - \iota_Y L_X = \iota_{[X,Y]}.$$

Again, this formula clearly shows that if X and Y are characteristic then $[X, Y]$ is characteristic.

□

2.2.2 The equivalence problem

The equivalence problem in its simplest form is the following. Let M_1 and M_2 be manifolds of the same dimension n and $\{\omega_1^i\}$ and $\{\omega_2^i\}$ be coframe sections, that is, n independent 1-forms (at every point of the manifold). Does there exist a diffeomorphism

$$\psi\colon M_1 \to M_2 \text{ such that } \psi^* \omega_2^i = \omega_1^i\ ? \tag{2}$$

To answer to that question Cartan used the graph method. The idea is to find the map ψ by its graph in $M_1 \times M_2$. The graph is obtained as an integral submanifold of a differential ideal.

Theorem 2.13. *Let M_1 and M_2 be manifolds and π_1, π_2 the projections of $M_1 \times M_2$ onto M_1 and M_2 respectively. Let $(\omega_2^i)_{1\le i\le n}$ be a basis of 1-forms of M_2 and $(\omega_1^i)_{1\le i\le n}$ be a family of forms M_1 respectively. If the ideal of forms on $M_1 \times M_2$ generated by*

$$\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i) \tag{3}$$

is a differential ideal then, for each pair $(x, y) \in M_1 \times M_2$, there exists a map $\phi : U \to M_2$, defined on a neighborhood of x, such that $\phi(x) = y$ and

$$\phi^*(\omega_2^i) = \omega_1^i. \tag{4}$$

Proof. The generating 1-forms are linearly independent because ω_2^i are linearly independent. By Frobenius theorem, there exists a unique maximal submanifold G of dimension n containing a point $(x, y) \in M_1 \times M_2$ which is an integral submanifold of the differential ideal.

We show now that the submanifold is locally a graph. Consider a vector $(v_1, v_2) \in \mathrm{T}G \subset \mathrm{T}M_1 \times \mathrm{T}M_2$. If $(\pi_1)_*(v_1, v_2) = 0$ then $v_1 = 0$ and therefore

$$\pi_1^*(\omega_1^i)(v_1, v_2) = \omega_1^i((\pi_1)_*(v_1, v_2)) = 0$$

which implies (because G is an integral submanifold of the ideal) that $\pi_2^*\omega_2^i(v_1, v_2) = 0$. We conclude that $v_2 = 0$. Therefore $\mathrm{T}_{(x,y)}G$ is isomorphic to $\mathrm{T}_{m_1}M_1$ and π_1 is a local diffeomorphism.

Let $F\colon U \to G$ be a local inverse of π_1. We have that $F(m) = (m, \phi(m))$ for a certain function $\phi\colon U \to M_2$(that is $\phi = \pi_2 \circ F$). Moreover, as $\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i) = 0$ on G, we obtain $F^*(\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i)) = 0$ and therefore $\omega_1^i = \phi^*(\omega_2^i)$. □

Remark In the theorem, if $(\omega_1^i)_{1\leq i\leq n}$ generates T^*M_1 then ϕ is an immersion. If furthermore the dimension of M_1 is n then the map ϕ is a local diffeomorphism.

Example One special case occurs if we suppose that the coframes in M_1 and M_2 both verify the same differential equation with constant coefficients:

$$\mathrm{d}\omega^i = c^i_{jk}\omega^j \wedge \omega^k, \tag{5}$$

with c^i_{jk} constant numbers shared by both M_1 and M_2. Here we use Einstein convention of sum of repeated indices. Then, observe that

$$\begin{aligned}
\mathrm{d}\left(\pi_1^*\omega_1^i - \pi_2^*\omega_2^i\right) &= \pi_1^*(\mathrm{d}\omega_1^i) - \pi_2^*(\mathrm{d}\omega_2^i) &(6)\\
&= \pi_1^*\left(c^i_{jk}\omega_1^j \wedge \omega_1^k\right) - \pi_2^*\left(c^i_{jk}\omega_2^j \wedge \omega_2^k\right) &(7)\\
&= c^i_{jk}\left(\pi_1^*(\omega_1^j \wedge \omega_1^k) - \pi_2^*(\omega_2^j \wedge \omega_2^k)\right) &(8)\\
&= c^i_{jk}\left(\left(\pi_1^*\omega_1^j - \pi_2^*\omega_2^j\right) \wedge \pi_1^*\omega_1^k - \pi_2^*\omega_2^j \wedge \left(\pi_2^*\omega_2^k - \pi_1^*\omega_1^k\right)\right) &(9)
\end{aligned}$$

so that the ideal is differential and M_1 and M_2 are hence locally equivalent.

The case of Lie groups is particularly important. With any left-invariant frame (X_i) and its coframe (ω^i) we get structure constants c^i_{jk} verifying the preceding condition:

$$\mathrm{d}\omega^i = c^i_{jk}\omega^j \wedge \omega^k. \tag{10}$$

A basis of 1-forms (ω^i) on a manifold M is called a parallelism of M. An automorphism of a parallelism (ω^i) defined over a manifold M is a diffeomorphism $\phi\colon M \to M$ such that $\phi^*\omega^i = \omega^i$. From unicity in the theorem above we obtain the following corollary.

Corollary 2.14. *Any automorphism of a parallelism with a fixed point is the identity.*

In particular this gives credit to the fact that the dimension of the group of automorphisms, if it is a Lie group, is at most the dimension of the manifold. We will prove latter that the automorphism group is a Lie group for many geometric structures and this gives a way to compute the maximal dimension of the automorphism group of a geometry. The idea is to construct, from the geometric data, another manifold with a canonical parallelism. The dimension of that manifold gives the dimension of the group of automorphisms. In Cartan geometries, this canonical parallelism is called a Cartan connexion.

Observe that an automorphism of a parallelism is an isometry of the manifold equipped with the Riemannian metric defined by imposing that the coframe (ω^i) is orthonormal.

A parallelism on M defined by a coframe (ω^i) can also be described by a map $\omega : \mathrm{T}M \to \mathbf{R}^n$ which is an isomorphism restricted to the tangent space at any point. We note then (M,ω) a manifold equipped with an $\mathbf{R}^n$-valued 1-form defining a parallelism. One can define a 'constant' vector field associated to $X \in \mathbf{R}^n$ as the vector field on M $\tilde{X}(x) = \omega^{-1}(X)$. For each sufficiently small $X \in \mathbf{R}^n$ we define an exponential map

$$exp(x, X) = \phi_1(x),$$

where $\phi_1(x)$ is the flow of $\tilde{X}$ computed at the time 1. The differential of the exponential map at the origin is the identity and therefore at each point $x \in M$, $exp(x,\cdot) : U \to M$ is a diffeomorphism between a neighborhood of the origin and its image.

Exercise Let $g_k \in \mathrm{Aut}(M,\omega)$ be a sequence of automorphisms of M equipped with a parallelism $\omega : TM \to \mathbf{R}^n$ such that there exists $x \in M$ such that $g_k(x)$ converges. Then g_k converges to an automorphism in the compact-open topology.

Definition 2.15. *A Killing field of (M,ω) is a vector field X on M such that its flow consists of elements of the automorphism group.*

The definition is equivalent to the condition that $L_X\omega^i = 0$ for all i.

2.3 Pfaff problem

Consider a differential ideal on a manifold generated by a 1-form, say θ. One is interested in giving a normal form for θ by choosing appropriate coordinates.

Pfaff's problem is the problem of finding integral manifolds of a system $\theta = 0$ where θ is a 1-form. Here one can multiply the 1-form by a nowhere zero function and the solutions will be the same. In other terms, one is interested in finding a coordinate chart where the form has a simple normal form up to a scalar function. The classification of normal forms is simpler if we impose a constant rank condition on $\mathrm{d}\theta$.

Definition 2.16. *We say a 2-form α is of rank p at $x \in M$ if $\alpha^p(x) = 0$ and $\alpha^{p+1}(x) = 0$.*

We recall also that the rank of a skew-symmetric bilinear form Ω defined on a vector space V is given by half the dimension of the subspace $\{\, \iota_v\Omega \,\} \subset V^*$. It has a normal form given by

$$e_1 \wedge \mathrm{e}_2 + \cdots + e_{2p-1} \wedge e_{2p},$$

in a particular basis $(e_1, \cdots, e_n)$ of V.

Theorem 2.17. *If ω is a 1-form such that $\mathrm{d}\omega$ is of constant rank p around a point $x \in M$ one can find coordinates $(x_1, \cdots, x_{n-p}, y_1, \cdots y_p)$ and a function S on a perhaps smaller neighborhood, such that*

$$\omega = \mathrm{d}S + x_1\,\mathrm{d}y_1 + x_2\,\mathrm{d}y_2 + \cdots x_p\,\mathrm{d}y_p.$$

Proof. We define the characteristic distribution of the differential ideal I generated by $\mathrm{d}\omega$ with constant rank p. It is given by

$$D = \{\, v \in TM \mid \iota_v\,\mathrm{d}\omega = 0 \,\}.$$

Observe also that, in this case, the characteristic distribution has dimension $n - 2p$, where n is the dimension of the manifold M. By Frobenius theorem, on a neighborhood of each point, there exists a coordinate chart $u_1, \cdots, u_{n-1}, y_1$ such that the integral manifolds are given locally by $(u_1, \cdots, u_{n-2p}) \to (u_1, \cdots, u_{n-2p}, u_{n-2p+1}, \cdots, u_n, y_1)$ (with fixed last coordinates). Therefore, by the definition of the characteristic distribution, the 2-form $\mathrm{d}\omega$ may be written in terms of the $2p$ independent generators $\mathrm{d}x_{n-2p+1}, \cdots, \mathrm{d}y_1$. Indeed, the vector fields $\frac{\partial}{\partial x_i}$ for $1 \leq i \leq n-2p$ are in the kernel of $\mathrm{d}\omega$. We want a coordinate system which simplifies the expression of the form $\mathrm{d}\omega$.

We will first find a foliation of codimension p of a coordinate chart such that the restriction of $\mathrm{d}\omega$ to each leaf is null proving the following lemma.

Lemma 2.18. *There exists a coordinate system* $(z_1, \cdots, z_{n-p}, y_1, \cdots, y_p)$ *such that* $\phi^*(\mathrm{d}\omega) = 0$ *for each embedding* $\phi : (z_1, \cdots, z_{n-p}) \to (z_1, \cdots, z_{n-p}, y_1, \cdots, y_p)$ *with fixed* $y_1, \cdots, y_p$.

Proof. Consider the embedding $\iota : (u_1, \cdots, u_{n-1}) \to (u_1, \cdots, u_{n-1}, y_1)$, where y_1 is fixed. The pullback of $\mathrm{d}\omega$, $\iota^*(\mathrm{d}\omega)$, has rank $p-1$ as the last coordinate is fixed. The characteristic distribution defined by the differential system $\iota^*(\mathrm{d}\omega)$ is involutive. By a previous exercise, the characteristic distribution of the differential system generated by $\iota^*(\mathrm{d}\omega)$ (viewed in the neighborhood by taking its pullback by the projection map) and $\mathrm{d}y_1$ is also involutive. The distribution has dimension $n-2p+1$. Therefore one can find coordinates $(w_1, \cdots, w_{n-2p}, w_{n-2p+1}, \cdots, w_{n-2}, y_2, y_1)$ such that $\iota^*(\mathrm{d}\omega)$ is expressed in terms of the differentials of $w_{n-2p+2}, \cdots, w_{n-2}, y_2, y_1$. One can repeat this argument until fixing exactly p coordinates to obtain $(z_1, \cdots, z_{n-p}, y_1, \cdots, y_p)$ such that $\phi^*(\mathrm{d}\omega) = 0$ for each embedding $\phi : (z_1, \cdots, z_{n-p}) \to (z_1, \cdots, z_{n-p}, y_1, \cdots, y_p)$. □

We also want a coordinate system which simplifies the expression of the form ω. In order to do so we will use Poincaré's lemma to pass from $\mathrm{d}\omega$ to ω. We obtain that $\iota^*(\omega) = \mathrm{d}f$ which can be written as $\iota^*(\omega - \mathrm{d}F) = 0$ for a function F defined on a neighborhood of the origin. In other words, one can write

$$\omega - dF = f_1 \,\mathrm{d}y_1 + \cdots + f_p \,\mathrm{d}y_p,$$

where f_i are functions on the neighborhood.

Now, from $(\mathrm{d}\omega)^p \neq 0$ we obtain that $\mathrm{d}f_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}f_p \wedge \mathrm{d}y_p \neq 0$. This implies that one can choose a coordinate system $(x_1, \cdots, x_{n-p}, y_1, \cdots, y_p)$ such that

$$\omega = dF + x_1 \,\mathrm{d}y_1 + \cdots + x_p \,\mathrm{d}y_p.$$

□

An imediate consequence of this result is the normal form for symplectic forms.

Theorem 2.19. *Le* Ω *be a closed two form of constant rank* p. *Then there exists local coordinates such that*

$$\Omega = dx_1 \wedge dy_1 + \cdots + dx_p \wedge dy_p.$$

Proof. By Poincaré's theorem one can write locally $\Omega = d\omega$. We apply then the previous theorem to ω and differentiate back. □

The final normal form result known as Darboux's theorem finds normal coordinates for a 1-form satisfying a regularity condition.

Theorem 2.20. 1. *Suppose* θ *is a 1-form such that* $d\theta$ *has constant rank* r *at each point and such that* $\theta \wedge (d\theta)^r = 0$. *Then, there exists local coordinates* $(x_1, \cdots, x_{n-r}, y_1, \cdots, y_r)$ *such that*

$$\theta = x_1 dy_1 + \cdots x_r dy_r.$$

2. *Suppose θ is a 1-form such that $d\theta$ has constant rank r at each point and such that $\theta\wedge(d\theta)^r \neq 0$ at every point. Then, there exists local coordinates $(x_1,\cdots,x_{n-r},y_1,\cdots,y_r)$ such that*
$$\theta = x_1 dy_1 + \cdots x_r dy_r + dx_{r+1}.$$

Example 1 If θ is a contact form, that is $\theta\wedge(d\theta)^n \neq 0$ at every point (where the dimension of M is $2n+1$) then one can write locally (in coordinates $(x_1,\cdots,x_{n+1},y_1,\cdots,y_n)$)

$$\theta = x_1 dy_1 + \cdots x_n dy_n + dx_{n+1}.$$

2.4 Global problems

Let M be a closed manifold and let ξ be a contact distribution. Darboux's theorem says that there are no local invariants of that structure. We will prove a more general form of Darboux's theorem in the case of contact structures and that any deformation of the contact structure is equivalent to itself. This is a rigidity theorem of contact structures and shows that different contact structures on a given manifold are far apart. Two manifolds equipped with contact structures are called contactomorphic if there exists a diffeomorphism between them which sends one distribution to the other.

Let ψ_t be an isotopy (a differentiable family of diffeomorphisms with $\psi_0 = Id$) of a manifold M and let X_t be the time-dependent vector field on M defined by $X_t\circ\psi_t = \dot\psi(t)$. That means that ψ_t is the flow of X_t.

The fundamental theorem for global results is the completeness theorem of flows on a compact manifold:

Theorem 2.21. *On a closed manifold the flow of a vector field (time-dependent or not) exists for all times.*

Recall the definition of the Lie derivative $L_X\omega = \frac{d}{dt}\psi_t^*\omega_{|t=0}$ (where ψ_t is the flow generated by X, that is, $X = \dot\psi_{|t=0}$ and $\psi_0 = Id$) and Cartan's formula

$$L_X\omega = \iota(X)d\omega + d\iota(X)\omega.$$

Lemma 2.22. *Let ω_t be a time-dependent family of differential forms on M. Then*

$$\frac{d}{dt}\left(\psi_t^*\omega_t\right) = \psi_t^*\left(\dot\omega_t + L_{X_t}\omega_t\right).$$

Proof. If ω_t is a function then the formula is valid:

$$\frac{d}{dt}\left(\psi_t^*\omega_t\right) = \frac{d}{dt}\left(\omega_t(\psi_t)\right) = \dot\omega_t(\psi_t) + \omega_t(\dot\psi_t) = \psi_t^*\left(\dot\omega_t + \mathfrak{L}_{X_t}\omega_t\right).$$

If ω_t is a 1-form then

$$\frac{d}{dt}\left(\psi_t^*\omega_t\right) = \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_t^*\omega_t}{h}$$
$$= \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_{t+h}^*\omega_t + \psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h}$$
$$= \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_{t+h}^*\omega_t}{h} + \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h} = \psi_t^*\left(\dot\omega_t + \mathfrak{L}_{X_t}\omega_t\right)$$

□

The following theorem contains, as a special case, Darboux's local form theorem for contact structures.

Theorem 2.23 (Local structure around a compact)**.** *Let M be a manifold and $N \subset M$ a smooth compact submanifold. Suppose ξ_0 and ξ_1 are (co-oriented) contact structures on M which coincide on N (or more generaly $\xi_0 \cap TN = \xi_1 \cap TN$). Then there exists a neighborhood of N and an isotopy ψ_t defined over that neighborhood such that $\psi_0 = Id$ and $\psi_1(\xi_0) = \xi_1$ with $\psi_{t_{|N}} = Id$.*

Proof. Suppose ξ_0 and ξ_1 are given by the 1-forms α_0 and α_1 respectively which we assume to coincide on N. A weaker condition is that $\alpha_{0_{|TM}} = \alpha_{1_{|TM}}$. Define the 1-form

$$\alpha_t = (1-t)\alpha_0 + t\alpha_1$$

which is clearly contact in a neighborhood of N by compactness. Moreover, at every point of N, $\alpha_t = \alpha_0$ when restricted to TN.

We define the isotopy as the flow defined by the time-dependent vector field $v_t = h_t R_t + y_t$ where y_t is horizontal with respect to α_t, that is $\alpha_t(y_t) = 0$.

We need $\psi_t^* \alpha_t = f_t \alpha_0$ for all $t \in [0,1]$. By the lemma

$$\frac{d}{dt}\left(\psi_t^* \alpha_t\right) = \psi_t^* \left(\dot\alpha_t + \iota(v_t) d\alpha_t + d\iota(v_t)\alpha_t\right).$$

The equation is satisfied if and only if

$$\dot\alpha_t + \iota(v_t) d\alpha_t + d\iota(v_t)\alpha_t = \frac{\dot f_t}{f_t} \circ \psi_t^{-1}.\alpha_t.$$

Evaluating at R_t we obtain

$$\dot\alpha_t(R_t) + dh_t(R_t) = \frac{\dot f_t}{f_t} \circ \psi_t^{-1} = \mu_t.$$

We have for every t, $\dot\alpha_{t_{|TN}} = 0$. For a given function h_t μ_t is determined and by the previous equation $d\iota(v_t)\alpha_t$ is determined which in turn determines y_t.

We want $v_t = 0$ on N. For that sake we impose the condition

$$\dot\alpha_t + dh_t = 0$$

along N. As $\dot\alpha_{t_{|TN}} = 0$ we can also impose $h_t = 0$ on N and that condition is compatible with the previous equation. □

Theorem 2.24 (Gray)**.** *Let ξ_t be a smooth family of contact structures on a closed manifold. Then there exists an isotopy ψ_t such that $\psi_0 = Id$ and $\psi_1(\xi_0) = \xi_1$.*

Proof. Let α_t be a smooth family of forms corresponding to ξ_t. We need to find a family of of diffeomorphisms ψ_t such that $\psi_t^* \alpha_t = f_t \alpha_0$. Let v_t be the vector field generating the isotopy. By Lemma 2.4, this is equivalent to

$$\frac{d}{dt}\left(\psi_t^* \alpha_t\right) = \dot f_t \alpha_0 = \frac{\dot f_t}{f_t}\psi_t^* \alpha_t = \psi_t^* \left(\dot\alpha_t + \iota(v_t) d\alpha_t + d\iota(v_t)\alpha_t\right).$$

So that a necessary and sufficient condition for the existence of the isotopy is that

$$\dot\alpha_t + \iota(v_t) d\alpha_t + d\iota(v_t)\alpha_t = \frac{\dot f_t}{f_t} \circ \psi_t^{-1}.\alpha_t$$

We impose that v_t is horizontal, that is, $\alpha_t(v_t) = 0$. We obtain the condition

$$\dot{\alpha}_t + \iota(v_t)d\alpha_t = \frac{\dot{f}_t}{f_t} \circ \psi_t^{-1}.\alpha_t. \tag{11}$$

If R_t is the Reeb vector field for α_t we have

$$\dot{\alpha}_t(R_t) = \frac{\dot{f}_t}{f_t} \circ \psi_t^{-1},$$

Therefore the function $\frac{\dot{f}_t}{f_t} \circ \psi_t^{-1}$ is determined by the family α_t. Going back to equation 11 the vector v_t is determined as the form $d\alpha_t$, restricted to the distribution, is non-degenerate. As the manifold is closed the vector field v_t can be integrated to obtain an isotopy ψ_t. □

3 Lie groups and homogenous spaces

3.1 Lie groups and Lie algebras

We start with the definition of a Lie group. General references for this section are [**Wa**; **Kn**; **Il**; **Sharpe**].

Definition 3.1. *A Lie group is a group G that is also a differential manifold and such that the operations of multiplication and inversion are smooth. That is, the maps $G \times G \to G$ and $G \to G$ given by $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are smooth.*

Definition 3.2. *A homomorphism $H \to G$ of Lie groups is a group homomorphism which is a smooth map. The automorphism group of H is the group of bijective homomorphisms of H into H.*

Note that if we ignore continuity in the definition of homomorphisms of Lie groups one might obtain a much larger set.

To each Lie group is associated a Lie algebra which can be thought as the space of tangent vectors at the identity of the group.

Definition 3.3. *A Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ is a real vector space of finite dimension equipped with a bilinear map*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \tag{12}$$

satisfying, for any $x, y, z \in \mathfrak{g}$ the anti-commutativity property $[x, y] = -[y, x]$ and the Jacobi identity:

$$[z, [x, y]] = [[z, x], y][x, [z, y]]. \tag{13}$$

Definition 3.4. *A homomorphism $\alpha \colon \mathfrak{h} \to \mathfrak{g}$ between Lie algebras is a homomorphism of vector spaces preserving the Lie bracket, that is, $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$ for all $X, Y \in \mathfrak{h}$. The automorphism group of $\mathfrak{h}$ is the group of bijective homomorphisms of $\mathfrak{h}$ into $\mathfrak{h}$.*

Let G be a Lie group. If $a \in G$ is fixed, then one can consider the translations $L_a(g) = ag$ and $R_a(g) = ga$ called left and right multiplication respectively.

Definition 3.5. *A vector field X on a Lie group G is* left invariant *if, for any $a \in G$, $(L_a)_*(X) = X$. Similarly, it is* right invariant *if $(R_a)_*(X) = X$.*

Note that this condition means $(L_a)_*(X(g)) = X(ag)$.

An important consequence of this definition is that left (or right) invariant vector fields are determined by their value at the identity of the group and the Lie bracket of two invariant vector fields is again invariant. Therefore the set of left invariant vector fields forms a Lie algebra that can be identified to the tangent space of the group at the identity.

Definition 3.6. *The Lie algebra of a Lie group G is the set*

$$\mathfrak{g} = \left\{X \in C^{\infty}(\mathrm{T}G) \,\middle|\, \forall a \in G,\ (L_a)_*(X) = X\right\} \tag{14}$$

of left invariant vector fields on G equipped with the bilinear map given by the bracket between vector fields.

A subgroup $H \subset G$ which is a Lie group and such that the inclusion map is smooth is a called a Lie subgroup. Imposing that the inclusion is an embedding is equivalent to assuming that the subgroup is closed as a subspace of G (this result is called the closed-subgroup theorem or Cartan theorem).

The relation between Lie algebra homomorphisms and Lie group homomorphisms is described by the following Theorem. Its proof is an application of Cartan's method.

Theorem 3.7. *Let H and G be Lie groups and* $\phi\colon H \to G$ *a smooth homomorphism. Then* $\mathrm{d}\phi_e\colon \mathfrak{h} \to \mathfrak{g}$ *is a homomorphism. Conversely, if* $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ *is a homomorphism and H is simply connected, then there exists a unique smooth homomorphism* $\phi\colon H \to G$ *such that* $\alpha = \mathrm{d}\phi_e$.

Corollary 3.8. *The automorphism group of a simply connected Lie group is isomorphic to the automorphism group of its Lie algebra.*

Exercice What is the group of automorphism of $\mathbf{R}$? One has to distinguish the automorphisms of Lie group from the automorphisms of the group without the differential structure.

Examples

1. The additive group $\mathbf{R}^n$. The automorphism group coincides with linear isomorphisms of $\mathbf{R}^n$, that is to say $\mathrm{GL}(n,\mathbf{R})$. But note that the full group of group automorphisms (not necessarily continuous) of the group $\mathbf{R}^n$ contains non-linear maps.

2. The set of matrices with determinant one $\mathrm{SL}(n,\mathbf{R})$ and the usual product of matrices as group law.

3. Let G be a Lie group, $N \subset G$ be a normal subgroup and $K \subset G$ a subgroup satisfying $N \cap K = \{e\}$ and $G = NK$. (This last condition means that $g \in G$ can always be written as nk with $n \in N$ and $k \in K$.) With these conditions, we say that G is the semidirect product of K and N and write $G = N \rtimes K$. Observe that if $g_1 = n_1 k_1$ and $g_2 = n_2 k_2$ then $g_1 g_2 = n_1(k_1 n_2 k_1^{-1}) k_1 k_2$.

   An example is given by the affine linear group $\mathrm{Aff}(\mathbf{R}^n) = \mathbf{R}^n \rtimes \mathrm{GL}(n,\mathbf{R})$. Given an affine transformation T acting on the affine plane $\mathbf{R}^n$, the choice of a base point $0 \in \mathbf{R}^n$ allows to write

$$T(x) = c + f(x) \tag{15}$$

with $c \in \mathbf{R}^n$ and $f \in \mathrm{GL}(n,\mathbf{R})$. This decomposition is unique. Hence $\mathrm{Aff}(\mathbf{R}^n) = \mathbf{R}^n\,\mathrm{GL}(n,\mathbf{R})$. Note that the change of the base point from $0 \in \mathbf{R}^n$ to $\zeta \in \mathbf{R}^n$ translates to:

$$\zeta + T(x-\zeta) = \zeta + (c - f(\zeta)) + f(x) \tag{16}$$

therefore the linear part f of T is independent of the choice of the base point, but the translational part depends on it.

The composition of two transformations T_1, T_2 is given by:

$$T_1(T_2(x)) = c_1 + f_1(c_2 + f_2(x)) = (c_1 + f_1(c_2)) + f_1 f_2(x) \tag{17}$$

and it proves that $\mathrm{Aff}(\mathbf{R}^n)$ is indeed the semidirect product $\mathbf{R}^n \rtimes \mathrm{GL}(n,\mathbf{R})$.

Note that a convenient representation of the affine group into $\mathrm{GL}(n+1,\mathbf{R})$ is given by

$$(c,f) \mapsto \begin{pmatrix} f & c \\ 0 & 1 \end{pmatrix}. \tag{18}$$

4. Semidirect products $G = N \rtimes K$ are in correspondance with split exact sequences

$$1 \to N \to G \to K \to 1 \tag{19}$$

and in the case of the affine group, we have indeed

$$0 \to \mathbf{R}^n \to \mathrm{Aff}(\mathbf{R}^n) \to \mathrm{GL}(n,\mathbf{R}) \to 1 \tag{20}$$

with the last morphism being independent of the choice of a base point and therefore is indeed restricted to the identity on $\mathrm{GL}(n,\mathbf{R})$.

5. The three dimensional Heisenberg group Heis(3) is defined as

$$\mathrm{Heis}(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| (x,y,z) \in \mathbf{R}^3 \right\} \tag{21}$$

The group law is again the matrix product and is described by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & z+z'+x\cdot y' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

Another description of the same group is given by $\mathbf{C} \times \mathbf{R}$ with the group law

$$(x + \boldsymbol{i}y, z)\cdot(x' + \boldsymbol{i}y', z') = \left((x+x') + \boldsymbol{i}(y+y'), z + z' + \frac{1}{2}(xy' - yx')\right). \tag{23}$$

Both descriptions are compatible. One can start with the Lie algebra:

$$\mathfrak{heis}(3) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{24}$$

The exponential of an element is

$$\exp\left(\begin{pmatrix}0 & x & z\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix}\right) = \begin{pmatrix}1 & x & z+\frac{1}{2}xy\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}. \tag{25}$$

Therefore $\exp\colon \mathfrak{heis}(3) \to \mathrm{Heis}(3)$ is a diffeomorphism. The group law defines a group structure on the Lie algebra by taking the logarithm: For $X, Y \in \mathfrak{heis}(3)$ define

$$X \cdot Y = \log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] \tag{26}$$

and this law on $\mathfrak{heis}(3)$:

$$\begin{pmatrix}0 & x & z\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix} \cdot \begin{pmatrix}0 & x' & z'\\ 0 & 0 & y'\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & x+x' & z+z'+\frac{1}{2}(xy'-yx')\\ 0 & 0 & y+y'\\ 0 & 0 & 0\end{pmatrix} \tag{27}$$

gives the second description.

In the case of the Heisenberg group (which is diffeomorphic to $\mathbf{R}^3$) one can use the group operation on the Lie algebra to determine the automorphisms.

Proposition 3.9. *The automorphism group of* $\mathrm{Heis}(3)$ *(described by coordinates* $(x+\boldsymbol{i}y, t) = (z, t) \in \mathbf{C}\times\mathbf{R}$*) is generated by the following transformations.*

(a) Transformations $(z, t) \mapsto (A(z), t)$ *where* $A\colon \mathbf{C}\to\mathbf{C}$ *is symplectic with respect to the form* $\mathrm{Im}(z\overline{z'}) = xy' - yx'$.

(b) Dilations $(z, t) \mapsto (az, a^2 t)$*, with* $a \in \mathbf{R}_+^*$.

(c) Conjugations by a translation $(a+\boldsymbol{i}b, c) \in \mathrm{Heis}(3)$*:* $(x+\boldsymbol{i}y, t) \mapsto (x+\boldsymbol{i}y, t+ay-bx)$.

(d) The inversion map $(z, t) \mapsto (\overline{z}, -t)$.

Proof. We decompose an automorphism $\phi\colon \mathrm{Heis}(3) \to \mathrm{Heis}(3)$ by decomposing its derivative $\mathrm{d}\phi_e\colon \mathfrak{heis}(3) \to \mathfrak{heis}(3)$. With a linear automorphism $\mathrm{d}\phi_e$, we can write $\mathrm{d}\phi_e(x+\boldsymbol{i}y, t) = (A(x, y, t), at+bx+cy)$, where A a linear transformation and a, b, c three real numbers.

We note that an automorphism has to preserve the center of the group: if ζ is in the center, then $0 = \mathrm{d}\phi_e[\zeta, \cdot] = [\mathrm{d}\phi_e\zeta, \mathrm{d}\phi_e\cdot] = [\mathrm{d}\phi_e\zeta, \cdot]$. Therefore A can not depend on t. (The center of $\mathfrak{heis}(3)$ is exactly $(0, t)$.)

From $(A(x, y), at+bx+cy)$ one can compose with the conjugation by a translation such that $\mathrm{d}\phi_e$ becomes $(A(x, y), at)$. (Choose the translation $(-c+\boldsymbol{i}b, 0)$.)

Next, if a is negative then we compose with an inversion. We obtain $(A'(x, y), |a|t)$ with A' that is either A or $\overline{A}$. Then we can compose by a dilatation by $\lambda = \sqrt{|a|}^{-1}$ so that we obtain $(\lambda A'(x, y), t)$.

Now, because t is fixed, $\lambda A'$ must be a symplectic transformation of $\mathbf{C}$. □

Note Hilbert's 5th problem deals with the question of to what extent a topological group has a differential structure. This problem has many interpretations. One of the most important of them was solved by Gleason, Montgomery-Zippin and Yamabe among other contributions: *every connected locally compact topological group without small subgroups (a neighborhood of the identity does not contain a subgroup other than the trivial subgroup) is a Lie group.*

3.1.1 The Maurer-Cartan form

Given a Lie group G and its Lie algebra $\mathfrak{g}$, one might wonder how $\mathfrak{g}$ controls the full tangent space $\mathrm{T}G$. Since G is a group, we can always translate T_eG to any T_gG by doing a left translation L_g or a right translation R_g. We choose to identify any tangent space T_gG with the left translation $(L_g)_*\mathrm{T}_eG$. This identification defines a map $\mathrm{T}G \to G \times \mathfrak{g}$ which is encoded by the Maurer-Cartan form.

Definition 3.10. *The (left) Maurer-Cartan form on a Lie group G is the $\mathfrak{g}$-valued 1-form θ defined by*

$$\forall X_g \in \mathrm{T}_gG,\ \theta(X_g) = (L_g)_*^{-1}(X_g) \in \mathfrak{g}. \tag{28}$$

Note Let X be a vector field on G, then $\theta(X) = v$ is constant, if and only if, X is left-invariant and $X(g) = (L_g)_* v$. Choosing a basis of $\mathfrak{g}$ defines a parallelism of G.

Cartan's formula is also valid for vector valued 1-forms. That is, for any 1-form $\alpha : TM \to V$ with values on a vector space V, we have

$$\mathrm{d}\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]). \tag{29}$$

Proposition 3.11 (Structural equation)**.** *For any $X, Y \in \mathrm{T}_gG$,*

$$\mathrm{d}\theta(X,Y) + [\theta(X),\theta(Y)] = 0. \tag{30}$$

Proof. We can evaluate $\mathrm{d}\theta(X,Y)$ by assuming that X, Y are extended by left-invariant vector fields X^* and Y^*. For any left-invariant vector field X^*, the image by the Maurer-Cartan form is constant on $X^*(g)$ for any $g \in G$. Therefore $X^*(\theta(Y^*))$ and $Y^*(\theta(X^*))$ are both zero. Moreover, since X^*, Y^* are left-invariant, so is $[X^*, Y^*]$ and therefore $\theta([X^*,Y^*]) = [\theta(X),\theta(Y)]$. □

Maurer-Cartan form in coordinates The choice of a basis $(e_1,\ldots,e_n)$ of $\mathfrak{g}$ allows us to write $\theta = (\theta^1,\ldots,\theta^n)$ by duality. With X_i the left-invariant vector field verifying $\theta(X_i) = e_i$, we can determine the *structure coefficients*:

$$[X_i, X_j] = \sum_k c_{ij}^k X_k. \tag{31}$$

The structural equation becomes:

$$\mathrm{d}\theta^k(X,Y) = -\sum_{i<j} c_{ij}^k \theta^i \wedge \theta^j. \tag{32}$$

The Maurer Cartan form is then

$$\theta = \sum_i \theta^i e_i.$$

Note Here we use a convention which might be different in some cases (see [**KoN**] pg. 28) and is sometimes the cause of a factor of $\frac{1}{2}$ in the formula. In fact we define

$$\theta^1 \wedge \theta^2(X,Y) = \theta^1(X) \otimes \theta^2(Y) - \theta^1(Y) \otimes \theta^2(X) \tag{33}$$

in contrast with

$$\theta^1 \wedge \theta^2(X,Y) = \frac{1}{2}\left(\theta^1(X) \otimes \theta^2(Y) - \theta^1(Y) \otimes \theta^2(X)\right). \tag{34}$$

Example Consider the group $SO(2) \subset GL(2,\mathbf{R})$. This group is parametrized as follows:

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{35}$$

In that coordinate, we obtain

$$dg_\phi = \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} d\phi \tag{36}$$

The Lie algebra is one dimensional and is generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{37}$$

The Maurer-Cartan form translates dg_ϕ for any ϕ to dg_0 by a left translation. Therefore it is given by

$$\theta_\phi = g(\phi)^{-1}\, dg_\phi \tag{38}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^{-1} \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} d\phi \tag{39}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} d\phi. \tag{40}$$

Matrix groups If $G \subset GL(n,\mathbf{R})$ is a matrix group with Lie algebra $\mathfrak{g} \subset M_{n\times n}$ one can write the Maurer-Cartan form at $g \in G$ and it is given by $\theta_g = g^{-1}\, dg$.

Here we interpret dg as the differential of the embedding of G into the space of matrices $M_{n\times n}$. In coordinates, if g_{ij} is the embedding, one has $\theta_g = g^{-1}_{ik}\, dg_{kj}$, which is a $\mathfrak{g}$-valued 1-form.

Vector space valued forms The Maurer-Cartan form is an example of vector space valued form. We define the wedge product of a V_1-valued 1-form θ_1 and a V_2-valued 1-form θ_2 to be the $V_1 \otimes V_2$-valued form

$$\theta_1 \wedge \theta_2(X,Y) = \theta_1(X) \otimes \theta_2(Y) - \theta_1(Y) \otimes \theta_2(X). \tag{41}$$

If there exists a bilinear map $[\cdot,\cdot] : V \times V \to V$ we note the composition of $\wedge$ (for 1-forms) and $[\cdot,\cdot]$ by

$$[\theta_1 \wedge \theta_2](X,Y) := [\theta_1(X), \theta_2(Y)] - [\theta_1(Y), \theta_2(X)]. \tag{42}$$

Observe then that $[\theta(X), \theta(Y)] = \frac{1}{2}[\theta \wedge \theta](X,Y)$.

Exercice ($\mathfrak{g}$-valued n-forms) Writing, in general, θ_n for a $\mathfrak{g}$-valued n-form we may define the exterior derivative and the product of two forms accordingly. Prove the following formulae:

1. $[\theta_p \wedge \theta_q] = (-1)^{pq}[\theta_q \wedge \theta_p]$,
2. $(-1)^{pr}[[\theta_p \wedge \theta_q] \wedge \theta_r] + (-1)^{qr}[[\theta_r \wedge \theta_p] \wedge \theta_q] + (-1)^{qp}[[\theta_q \wedge \theta_r] \wedge \theta_p]$.

Moreover,

$$d[\theta_p \wedge \theta_q] = [d\theta_p \wedge \theta_q] + (-1)^{pq+1}[\theta_p \wedge d\theta_q]. \tag{43}$$

Darboux derivatives

A Maurer-Cartan form allows the computation of Darboux derivatives.

Definition 3.12. *If $f\colon M \to G$ is smooth and if θ is the Maurer-Cartan form of G then the Darboux derivative of f is:*

$$f^*\theta = \theta \circ f_*. \tag{44}$$

Example In $\mathbf{R}^n$ the Darboux derivative is in a sense closer to the usual derivative than the differential. Indeed, recall that if $f\colon \mathbf{R}^p \to \mathbf{R}^n$ is smooth, then

$$\forall (x,v) \in \mathrm{T}\mathbf{R}^n,\ f_*(x,v) = (f(x), \mathrm{d}f_x(v)). \tag{45}$$

The maps f_* and $\mathrm{d}f$ depend on the base point. But with the Darboux derivative one identifies all tangent spaces to the tangent space at the origin:

$$f^*\theta(x,v) = \theta(f(x), \mathrm{d}f_x(v)) = T_{-f(x)}{}_*(\mathrm{d}f_x(v)) \in \mathrm{T}_0(\mathbf{R}^n) \tag{46}$$

where $T_{-f(x)}$ is the translation $T_{-f(x)}(z) = z - f(x)$.

Theorem 3.13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and M a submanifold of G. Suppose there exists a $\mathfrak{g}$-valued 1-form ϕ defined on M satisfying the Maurer-Cartan formula $d\phi + \frac{1}{2}[\phi \wedge \phi] = 0$. Then for any $m \in M$ there exists a map $f : U \to G$ defined on a neighbourhood of m such that $\phi = f^*\theta$ where θ is the Maurer-Cartan form of G. Moreover if $f' : U \to G$ is another map satisfying this condition $f' = L_h \circ f$ for a certain $h \in G$.*

Proof. We consider, in the product $M \times G$, the Lie algebra valued form

$$\omega = \pi_1^*(\phi) - \pi_2^*(\theta),$$

where π_1 and π_2 are the projections of the product on each of the factors. Let I be the ideal generated by the components ω^i_j of ω. This is a differential ideal because

$$\begin{aligned} 2d\omega &= 2(\pi_1^*(d\phi) - \pi_2^*(d\theta)) = -\pi_1^*([\phi \wedge \phi]) + \pi_2^*([\theta \wedge \theta]) \\ &= -[(\pi_1^*\phi - \pi_2^*\theta) \wedge \pi_1^*\phi] - [\pi_2^*\theta \wedge (\pi_1^*\phi - \pi_2^*\theta)] \end{aligned}$$

and we invoke the previous theorem to conclude the existence of the map $f : U \to G$.

A submanifold passing through another point (m_0, hg) is clearly given by $(m, hf(m))$ and by unicity this implies that $f' = L_h \circ f$. □

The exponential map

One parameter subgroups of a group G are defined by elements of the Lie algebra. For any $X \in \mathfrak{g}$ one defines a homomorphism

$$\exp_X\colon \mathbf{R} \to G, \tag{47}$$

which is the unique homomorphism satisfying $\exp_X^*\theta = X$.

Definition 3.14. *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is defined by*

$$\exp(X) = \exp_X(1). \tag{48}$$

Although exp has several properties analogous to the real exponential, due to the non-commutativity, one has a more complicated formula for the product of two exponentials (it is the Baker-Campbell-Hausdorff formula which is only valid locally):

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \cdots\right). \tag{49}$$

If $\phi\colon H \to G$ is a group homomorphism one has

$$\exp \circ \mathrm{d}\phi_e = \phi \circ \exp_e. \tag{50}$$

Lemma 3.15. *Let X^* be a left-invariant vector field corresponding to an element $X \in \mathfrak{g}$. Then its flow is given as the right multiplication by the exponential map $R_{\exp(tX)}$.*

Proof. Since X^* is left-invariant, so must be its flow. Therefore the integral curve at $g \in G$ is given by $L_g \exp(tX) = R_{\exp(tX)} g$. Hence the flow is given by $R_{\exp(tX)}$. □

3.1.2 The adjoint representation

An action of a Lie group G on a manifold induces a representation of the group on the automorphism group of the tangent space of a fixed point of the action. For, let $\phi\colon G \times M \to M$ be an action with a fixed point $G \cdot p = p$ at $p \in M$. Then for every $g \in G$, define $\phi_g : M \to M$ ($\phi_g(x) = \phi(g,x)$) and then the automorphism $\rho(g) = \phi_{g_{|p}} : \mathrm{T}_p M \to \mathrm{T}_p$. One then verifies that the map $\rho : G \to Aut(\mathrm{T}_p M)$ defined by $\rho(g) = \rho_g$ is a representation.

In particular the adjoint action $G \times G \to G$ defined by $(g,h) \mapsto ghg^{-1}$ induces the representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{T}_e G)$ (observe that $\mathrm{Aut}(\mathrm{T}_e G)$ is isomorphic to $\mathrm{GL}(n,\mathbf{R})$ with $n = \dim_{\mathbf{R}} G$). For $g \in G$, Ad_g is the automorphism

$$\mathrm{Ad}_g(X) = \mathrm{d}(h \mapsto ghg^{-1})_e(X) = (L_g)_*(R_{g^{-1}})_* X \tag{51}$$

The adjoint representation is also exactly what we need to compare the Maurer-Cartan form θ defined by left-invariance with the action by right translations.

Proposition 3.16. *For any $g \in G$, the Maurer-Cartan form θ verifies*

$$R_g^*\theta(X) = \mathrm{Ad}_g^{-1}(\theta(X)). \tag{52}$$

Proof. Assume that $X = (L_x)_* v$. By the preceding definition, we have:

$$\begin{aligned} R_g^*\theta(X) &= \theta((R_g)_* X) && (53)\\ &= \theta((R_g)_*(L_x)_* v) && (54)\\ &= \theta((L_x)_*(R_g)_* v) && (55)\\ &= \theta((R_g)_* v) && (56)\\ &= (L_g)_*^{-1}(R_g)_* v = \mathrm{Ad}_g^{-1} v. && (57)\end{aligned}$$

□

The differential of Ad_g at the origin $g = e$ is denoted by $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathrm{T}_e G)$:

$$\mathrm{ad}_X = \mathrm{dAd}_e(X). \tag{58}$$

It is in fact given by the bracket of the Lie algebra.

Lemma 3.17. *Let $X, Y \in \mathfrak{g} \cong \mathrm{T}_e G$. Then*

$$\mathrm{dAd}_e(X)(Y) = \mathrm{ad}_X(Y) = [X, Y]. \tag{59}$$

The adjoint automorphism by $g \in G$ fits in the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \\ \big\downarrow \exp & & \big\downarrow \exp \\ G & \xrightarrow{g(\cdot)g} & G \end{array} \tag{60}$$

and the adjoint representation satisfies

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \\ \big\downarrow \exp & & \big\downarrow \exp \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathfrak{g}) \end{array} \tag{61}$$

More generally, we have:

Proposition 3.18. *The differential of the representation* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{T}_e G)$ *at $g \in G$ computed at the vector $X^* = (L_g)_* X \in \mathrm{T}_g G$ is*

$$\mathrm{dAd}_g(X)(Y) = \mathrm{Ad}_g(\mathrm{ad}_X(Y)). \tag{62}$$

Proof. Writing a path through g as $L_g\gamma(t)$ with $\gamma(0) = e$ and $\dot{\gamma}(0) = X$ we have $\mathrm{Ad}_{L_g\gamma(t)}(Y) = \mathrm{Ad}_g \circ \mathrm{Ad}_{\gamma(t)}(Y)$. Therefore

$$(\mathrm{dAd}_g(X))(Y) = \left.\frac{\mathrm{dAd}_g \circ \mathrm{Ad}_{\gamma(t)}}{\mathrm{d}t}\right|_{t=0}(Y) = \mathrm{Ad}_g \circ \mathrm{ad}_X(Y). \tag{63}$$

□

Proposition 3.19. *If θ_G is the Maurer-Cartan form, then for any function ψ with values in G and any 1-form α with values in $\mathfrak{g}$,*

$$\mathrm{Ad}_\psi \psi^* \theta_G = -{\psi^{-1}}^* \theta_G, \tag{64}$$

$$\mathrm{d}\big(\mathrm{Ad}_\psi(\alpha)\big) = \Big[-{\psi^{-1}}^* \theta_G \wedge \mathrm{Ad}_\psi(\alpha)\Big] + \mathrm{Ad}_\psi \, \mathrm{d}\alpha \tag{65}$$

$$= \mathrm{Ad}_\psi \left(\left[\psi^* \theta_G \wedge \alpha\right] + \mathrm{d}\alpha\right). \tag{66}$$

3.2 Homogeneous spaces

Homogeneous spaces will be the flat model geometries. They appear naturally when there exists a transitive action. Indeed, if $G \times M \to M$ is a transitive action one can identify M with the quotient ${}^{G}\!/_{H_x}$ where H_x is the isotropy subgroup of a chosen element $x \in M$. A different choice $gx \in M$ gives rise to the isotropy $H_{gx} = gH_x g^{-1}$.

Definition 3.20. *A homogeneous space is a differential manifold obtained by the quotient of a Lie group G by a closed Lie subgroup $H \subset G$. We note the set of left cosets gH by ${}^{G}\!/_{H}$.*

The group G acts transitively on the homogeneous space ${}^{G}\!/_{H}$ by left translations, the isotropy subgroup at the identity being H.

Note If H were not closed then the quotient ${}^{G}\!/_{H}$ would not separated with the quotient topology.

Examples

1. *The Euclidean space.*

   The group of the isometries of the Euclidean space is $\mathrm{Eucl} = \mathbf{R}^n \rtimes \mathrm{O}(n)$. It acts on $\mathbf{R}^n$ with isotropy $\mathrm{O}(n)$. Therefore $\mathbf{R}^n = {}^{\mathrm{Eucl}}\!/_{\mathrm{O}(n)}$ as homogeneous space.

2. *The hyperbolic space.*

   Hyperbolic space is the simply connected complete constant negative sectional curvature Riemannian space. Its connected isometry group is $\mathrm{SO}(n,1)$ with isotropy $\mathrm{SO}(n)$. Here $\mathrm{SO}(n,1)$ is the group preserving the quadratic form

$$\begin{pmatrix} \mathrm{id}_{\mathbf{R}^n} & 0 \\ 0 & -1 \end{pmatrix}. \tag{67}$$

3. *The similarity group acting on* $\mathbf{R}^n$.

   The connected similarity group is the group $\mathrm{Sim}(\mathbf{R}^n) = \mathbf{R}^n \rtimes (\mathbf{R}_+^* \times \mathrm{O}(n))$. It is a subgroup of the affine group $\mathrm{Aff}(\mathbf{R}^n)$. Transformations of $\mathbf{R}_+^* \times \mathrm{O}(n)$ are of the form $\lambda P(x)$ with $\lambda > 0$ and P an orthogonal transformation.

   The similarity group is the conformal group acting on $\mathbf{R}^n$. (Each conformal transformation has to be defined on the full space $\mathbf{R}^n$.) Therefore, it consists of the transformations of $\mathbf{R}^n$ which preserve angles. The isotropy at the origin is $\mathbf{R}_+^* \times \mathrm{O}(n)$.

4. *The conformal sphere.*

   There are more conformal transformations than just $\mathrm{Sim}(\mathbf{R}^n)$. But those are not defined strictly on $\mathbf{R}^n$ but rather on the one-point compactification S^n. The conformal sphere is the homogeneous space $\mathrm{PO}(n+1,1) / \mathrm{Sim}(\mathbf{R}^n)$.

5. *The projective space.*

   The projective space $\mathbf{RP}^n$ is the homogenous space $\mathrm{GL}(n+1,\mathbf{R}) / H$ where

$$H = \left\{ \begin{pmatrix} \star & \star \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{GL}(n,\mathbf{R}) \right\}. \tag{68}$$

6. *Flag spaces.*

   The projective space is an example of flag spaces. A flag is a sequence $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbf{F}^n$ for any field $\mathbf{F}$. For instance, the projective space $\mathbf{FP}^n$ is the set of lines in $\mathbf{F}^{n+1}$.

   A complete flag is a flag with $\dim V_i = i$. They are maximal in length. When $\mathbf{F} = \mathbf{C}$ we get an homogeneous space structure with the quotient

$${}^{\mathrm{SU}(n)}\!/_{\mathrm{S}(\mathrm{U}(1) \times \cdots \times \mathrm{U}(1))}. \tag{69}$$

7. *Stiefel manifolds.*

   The space of orthonormal k-frames in $\mathbf{R}^n$ (with $0 < k < n$) is the Stiefel manifold $S(k,n)$. It is possible to show that

$$S(k,n) = {}^{\mathrm{SO}(n)}\!/_{\mathrm{SO}(n-k)}. \tag{70}$$

8. *Every manifold is a homogeneous space.*

   The full group of the diffeomorphisms of a manifold is not a Lie group but might be described by an analogous structure with infinite dimension.

   The easiest situation is for a compact manifold, say M. The smooth diffeomorphism group $\mathrm{Diff}^\infty(M)$ has a structure of a Fréchet Lie group which is homeomorphic to the space of smooth vector fields. The group $\mathrm{Diff}^\infty(M)$ acts transitively on M. Therefore, any manifold can be considered as a homogeneous space $\mathrm{Diff}^\infty(M)/H$, where H is the isotropy at a point in M, that is to say, the set of diffeomorphisms fixing the point. We will not deal with infinite dimension Lie groups.

Construction à la Cartan We can reproduce how Cartan described the construction of the Maurer-Cartan form at the early stages of the theory. In fact, we here describe the main technique of the moving frame (*repère mobile*) that Cartan attributes to Darboux.

Consider the affine space $\mathbf{R}^3$. At any point $m \in \mathbf{R}^3$, associate a frame (e_1, e_2, e_3) base at m. The map (e_1, e_2, e_3) should be smooth depending on m.

The infinitesimal change of m by δm can be expressed by:

$$\delta m = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3. \tag{71}$$

It gives a 1-form with values in $\mathbf{R}^3$.

The infinitesimal change of a base vector e_i by δe_i can be described by the image of an infinitesimal matrix acting on (e_1, e_2, e_3):

$$\delta e_i = \omega_i^1 e_1 + \omega_i^2 e_2 + \omega_i^3 e_3 \tag{72}$$

and this furnishes a 1-form with values in $\mathfrak{gl}(3)$.

Those four 1-forms $\theta = (\delta e_1, \delta e_2, \delta e_3, \delta m)$ compose the Maurer-Cartan form of the affine space.

3.2.1 The tangent space

With a homogeneous space G/H the tangent space can be described infinitesimally and the action of G (on the left) can be measured.

At eH, the tangent space is naturally isomorphic to $\mathfrak{g}/\mathfrak{h}$ as linear spaces. Therefore, the tangent bundle of the homogenous spaces $\mathrm{T}G/H$ can be seen as a quotient of the trivial bundle

$$G \times_H \mathfrak{g}/\mathfrak{h}. \tag{73}$$

The quotient will be by the right action of H:

$$(g, v) \cdot h \sim (gh, \mathrm{Ad}(h)^{-1} v). \tag{74}$$

Note that at the isotropy $H \subset G$, the action of $h \in H$ on a point pH is $hpH = hph^{-1}H$ and therefore H acts on $\mathrm{T}_{eH} G/H$ by $\mathrm{Ad}(h)$.

Proposition 3.21. *There exists a canonical isomorphism*

$$\mathrm{T}G/H \cong G \times_H \mathfrak{g}/\mathfrak{h}. \tag{75}$$

Proof. Let $\pi\colon G \to {}^{G}\!/_{H}$ be the quotient map. Let $\phi\colon G \times {}^{\mathfrak{g}}\!/_{\mathfrak{h}} \to \mathrm{T}{}^{G}\!/_{H}$ be defined by

$$\phi(g, v) = (gH, \pi_*(L_g)_* v). \tag{76}$$

We prove that this map is well defined in the quotient by the right action of H. Note that $\pi_*(R_h)_* = \pi$ since $\pi \circ R_h = \pi$ and $\pi_*(L_g)_* = (L_g)_*\pi_*$.

$$\begin{aligned}
\phi((g, v)\cdot h) &= \phi(gh, \mathrm{Ad}(h)^{-1} v) && (77)\\
&= (ghH, \pi_*(L_{gh})_* \mathrm{Ad}(h)^{-1} v) && (78)\\
&= (gH, (L_g)_*\pi_*(R_h)_* v) && (79)\\
&= (gH, (L_g)_*\pi_* v) = \phi(g, v) && (80)
\end{aligned}$$

We can check that this morphism is injective at every point. If $\phi(g, v) = (gH, 0)$ then $\pi_* v = 0$ and therefore $v \in \mathfrak{h}$. It is surjective by dimensionality. □

3.2.2 Effective pairs

It is important to keep track of both groups G and H and not only their quotient space. On the other hand it is reasonable to consider only connected quotients ${}^{G}\!/_{H}$.

Definition 3.22. *We will refer as a Klein geometry a pair (G, H) such that the homogeneous space ${}^{G}\!/_{H}$ is connected.*

There are two conditions which one can add without much loss of generality, namely, that the action of G be effective and that G be connected.

Note that if $g \in G$ acts trivially on ${}^{G}\!/_{H}$ then $geH = eH$ and therefore $g \in H$. Let $h \in H$ be acting trivially. For any $g \in G$ and any coset pH we would have that $ghg^{-1}pH = g(h(g^{-1}pH))$ is equal to $g(g^{-1}pH)$ since h acts trivially on $g^{-1}pH$ and therefore $ghg^{-1}pH = pH$. So if h acts trivially, then ghg^{-1} does too.

Definition 3.23. *We say that a maximal subgroup $K \subset H$ which is normal in G is the kernel of a Klein geometry. The action of K is trivial and we say that the geometry is effective if $K = \{e\}$.*

If K is the maximal normal subgroup in H (the definition implies that K is a closed subgroup of G) one can consider the effective geometry $({}^{G}\!/_{K}, {}^{H}\!/_{K})$ which describes the same homogeneous space as ${}^{({}^{G}\!/_{K})}\!/_{({}^{H}\!/_{K})}$. It is diffeomorphic to ${}^{G}\!/_{H}$ with an equivariant action by ${}^{G}\!/_{K}$.

Sometimes one might consider non-effective Klein geometries. For instance, ${}^{\mathrm{SL}(2,\mathbf{R})}\!/_{\mathrm{SO}(2)}$ corresponds to the hyperbolic geometry but the subgroup $\mathbf{Z}_2 \subset \mathrm{SL}(2,\mathbf{R})$ generated by $-\mathrm{id}$ is a maximal normal subgroup contained in $\mathrm{SO}(2)$. Nonetheless, this subgroup is discrete and is does not intervene infinitesimally.

If G is not connected one can consider the connected component containing the identity $G_e \subset G$ and we obtain that ${}^{G}\!/_{H}$ is diffeomorphic to ${}^{G_e}\!/_{(H \cap G_e)}$ with an equivariant action by G_e. This follows since if ${}^{G}\!/_{H}$ is connected, one has $G = G_eH$. On the other hand, one can prove that if H is connected then G is also connected.

Lemma 3.24. *Let $N \subset G$ be a normal subgroup with corresponding algebras $\mathfrak{n} \subset \mathfrak{g}$. Then for all $v \in \mathfrak{g}$ and $n \in N$,*

$$\mathrm{Ad}_n(v) - v \in \mathfrak{n}. \tag{81}$$

Proof. Since N is normal, for any $g \in G$ and any $n \in N$ we have $ngn^{-1}g^{-1} \in N$. Let $g(t) = \exp(tv)$. We have:

$$(L_n L_{g(t)} R_{n^{-1}}) g(-t) \in N \tag{82}$$

and by derivation at $t = 0$:

$$\mathrm{Ad}_n(v) - v \in \mathfrak{n}. \tag{83}$$

□

Reciprocally, this condition implies, by differentiation along a path in N, that $[\mathfrak{n}, \mathfrak{g}] \subset \mathfrak{n}$ so $\mathfrak{n}$ is an ideal of G.

We will need to identify maximal normal subgroups of G contained in $H \subset G$. The goal is to obtain properties for effective Klein geometries. The easiest way to start is with a normal subgroup N of H ($N = H$ is the most natural choice) so that its Lie algebra $\mathfrak{n}$ is an ideal of $\mathfrak{h}$. According to the preceding lemma, a candidate for a normal subgroup of G contained in $N \subset H$ is

$$N' = \left\{ n \in N \,|\, \forall v \in \mathfrak{g}, \mathrm{Ad}_n v - v \in \mathfrak{n} \right\}. \tag{84}$$

The subgroup N' might be much smaller that N. At least, it is still normal in H:

$$\mathrm{Ad}_{hnh^{-1}}(v) - v = \mathrm{Ad}_h(\mathrm{Ad}_n \mathrm{Ad}_{h^{-1}}(v) - \mathrm{Ad}_{h^{-1}}(v)) \in \mathrm{Ad}_h(\mathfrak{n}) \subset \mathfrak{n}. \tag{85}$$

The greatest normal subgroup of G which is contained in H is obtained by the following procedure.

Proposition 3.25. *Suppose G is connected and $H \subset G$ a closed Lie subgroup. Define the decreasing sequence of subgroups of H:*

$$N_0 = H, \tag{86}$$

$$\forall i \geq 0,\ N_{i+1} = \left\{ n \in H \,|\, \mathrm{Ad}_n v - v \in \mathfrak{n}_i,\ \forall v \in \mathfrak{g} \right\}. \tag{87}$$

Then, each $N_i \subset H$ is a closed normal subgroup of H and the intersection

$$N_\infty = \bigcap_i N_i \subset H \tag{88}$$

is the largest normal subgroup of G contained in H.

Proof. The fact that N_i and N_∞ are normal will depend on the following computation, related to the preceding paragraph. Let $n \in G$, $g \in G$ and $k \geq 0$. Assume that $\mathrm{Ad}_n v = v + w(v)$ for any $v \in \mathfrak{g}$, with a corresponding $w(v) \in \mathfrak{n}_k$. Then

$$\mathrm{Ad}_{gng^{-1}} v = \mathrm{Ad}_g \mathrm{Ad}_n(\mathrm{Ad}_{g^{-1}} v) \tag{89}$$

$$= \mathrm{Ad}_g \left(\mathrm{Ad}_{g^{-1}} v + w(\mathrm{Ad}_{g^{-1}}(v)) \right) \tag{90}$$

$$= v + \mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))). \tag{91}$$

Now, to see that each group N_i is normal in H, note that if $n \in N_i$ and $g \in H$ then the preceding computation shows that gng^{-1} belongs to N_i if, and only if, $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_{i-1}$. By hypothesis, $w(\mathrm{Ad}_{g^{-1}}(v)) \in \mathfrak{n}_{i-1}$. By recurrence, $\mathrm{Ad}_g(\mathfrak{n}_{i-1}) \subset \mathfrak{n}_{i-1}$, showing that we have indeed $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_{i-1}$.

It is clear that N_∞ is well defined and is normal in H. We have to show it is also normal in G. First, $\mathfrak{n}_\infty \subset \mathfrak{g}$ is an ideal. Indeed, by differentiation of $\mathrm{Ad}_n(v) = v + w(v)$ along a path $n(t)$ we have $[n, v] = w'(v)$ and it belongs to to $\mathfrak{n}_\infty$ since $w(v)$ does.

Since $\mathfrak{n}_\infty \subset \mathfrak{g}$ is an ideal and G is connected, the component of the identity of N_∞ is normal in G. But then it implies $\mathrm{Ad}_{g^{-1}} \mathfrak{n}_\infty = \mathfrak{n}_\infty$. By the preceding computation it implies $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_\infty$ and therefore that N_∞ is indeed normal.

To complete the proof, we show that for a normal subgroup $N \subset G$ contained in H, $N \subset N_\infty$: by induction, $N \subset H$ and if $N \subset N_i$ so $\mathfrak{n} \subset \mathfrak{n}_i$ and therefore $N \subset \{n \in H \,|\, \mathrm{Ad}_n v - v \in \mathfrak{n}_i,\ \forall v \in \mathfrak{g}\} = N_{i+1}$. □