Rigidity and flexibility of triangle groups in complex hyperbolic geometry

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Abstract

We show that the Teichmüller space of the triangle groups of type (p, q, ∞) in the automorphism group of the two dimensional complex hyperbolic space contains open sets of 0, 1 and 2 real dimensions. In particular we identify the Teichmüller space near embeddings of the modular group preserving a complex geodesic.

1 Introduction

Let Γ be the triangular group of type (p, q, ∞), that is, the abstract group presented by

$$< \iota_0, \iota_1, \iota_2 : \iota_0^2 = 1, \iota_1^2 = 1, \iota_2^2 = 1, (\iota_0 \circ \iota_1)^p = 1, (\iota_0 \circ \iota_2)^q = 1 > .$$

The faithful discrete embeddings of Γ in the isometry group of the one dimensional complex disc, that is $\mathbf{P}\widehat{\mathbf{U}(1,1)}$ (containing the holomorphic and the anti-holomorphic transformations),with anti-holomorphic embedding of generators and $\iota_1 \circ \iota_2$ embedded as a parabolic, are rigid.

We consider in this paper faithful discrete embeddings of Γ in $\mathbf{P}\widehat{\mathbf{U}(2,1)}$ (containing the holomorphic and the anti-holomorphic transformations), the isometry group of the two dimensional complex ball (with the natural metric invariant under biholomorphisms). Let the *Teichmüller space* denote the space of faithful discrete embeddings modulo conjugation in $\mathbf{P}\widehat{\mathbf{U}(2,1)}$, with anti-holomorphic generators and such that $\iota_1 \circ \iota_2$ is parabolic. We obtain the following description.

Theorem 1.1 *The Teichmüller space of the triangle groups of type (p, q, ∞), with $2 < p \leq q$, in the automorphism group of the two dimensional complex hyperbolic space contains open sets of 0, 1 and 2 real dimensions.*

Each open set of Teichmüller space in the theorem contains a $\mathbf{C}$-Fuchsian embedding, that is, an embedding which fixes a complex geodesic setwise.

The triangle groups of type $(2,\, p,\, \infty)$ are special. The involution of order 2 cannot be deformed and we loose one parameter in the deformation space. Observing that the index 2 subgroup of holomorphic transformations of the triangle group of type $(2,\, 3,\, \infty)$ is the modular group $\mathbf{SL}(2, \mathbf{Z})$, we obtain the following

Theorem 1.2 *The Teichmüller space of the modular group in the biholomorphic automorphism group of the two dimensional complex hyperbolic space around a representation that fixes a complex geodesic is of dimension 0 or 1.*

Of course we impose the parabolic generator to be represented by parabolics. We will describe explicitly the embeddings in each case of the theorem.

It is important to contrast that result with the rigidity result of ([T], [GM], [G1]). If the group has a **C**-Fuchsian embedding which is cocompact in the fixed complex geodesic, any nearby deformation is **C**-Fuchsian. In our case the volume is finite. It is interesting that both rigidity and flexibility occur and depend on the particular **C**-Fuchsian embedding.

Embeddings of triangle groups of type (∞, ∞, ∞) were previously analyzed in [GuP], [GP] and [FK]. It was shown in [FK] that the Teichmüller space, in that case, contains an open set of real dimension 4. Deformations of the modular group were also obtained by J. Parker [P] independently. The difficulty in the present case is the appearance of elliptic transformations. The proof is based on a Poincaré's polyhedron theorem for complex hyperbolic geometry developed in [FZ]. We construct explicitly the fundamental domains. The idea behind it is that anti-holomorphic reflections which fix real geodesics are the analog of inversions in classical conformal geometry. Polyhedra are constructed having faces, foliated by complex geodesics, invariant under those reflections. They are a generalization of Mostow's bisectors and we will call them **C**-spheres. Edges are complex geodesics in the intersection of two **C**-spheres.

We hope that the methods of this paper will achieve (see Remark 5.5, p. 17), in the future, a precise description of Teichmüller space in the case of embeddings by anti-holomorphic transformations. On the other hand it would be interesting to obtain our results using a modification of the method of Higgs bundles (see [H]) for non-compact surfaces.

An interesting problem would be to understand the behavior of automorphic forms under the deformation of modular group.

2 The complex hyperbolic space and its boundary

In this section and the following we collect general results about the complex hyperbolic space. As a reference we use [G2],[KoR] and [FK].

2.1 $\mathbf{PU}(2,1)$, the Heisenberg group and the Cayley transform

Let $\mathbf{C}^{n+2}$ denote the complex vector space equipped with the Hermitian form

$$b(z, w) = -\overline{z}_1 w_1 + \overline{z}_2 w_2 + \cdots + \overline{z}_{n+2} w_{n+2}.$$

Consider the following subspaces in $\mathbf{C}^{n+2}$:

$$\begin{aligned} V_0 &= \{ z \in \mathbf{C}^{n+2} \ : \ b(z,z) = 0 \ \}, \\ V &= \{ z \in \mathbf{C}^{n+2} \ : \ b(z,z) < 0 \ \}. \end{aligned}$$

Let $P : \mathbf{C}^{n+2} \setminus \{0\} \to \mathbf{C}P^{n+1}$ be the canonical projection onto the complex projective space. Then $H^{n+1}_{\mathbf{C}} = P(V)$ equipped with the Bergman metric is the complex hyperbolic space. The orientation preserving isometry group of $H^{n+1}_{\mathbf{C}}$ is generated by $\mathbf{PU}(n+1,1)$, the unitary group of b and the anti-holomorphic transformations. We denote it by $\widehat{\mathbf{PU}(n,1)}$. Also, $\mathbf{PU}(n+1,1)$ is the group of biholomorphic transformations of $H^{n+1}_{\mathbf{C}}$. Let $S^{2n+1} = P(V_0)$. Then S^{2n+1} is the boundary of $H^{n+1}_{\mathbf{C}}$. We may consider $H^{n+1}_{\mathbf{C}}$ and S^{2n+1} as the unit ball and the unit sphere in $\mathbf{C}^{n+1}$.

We restrict our attention to the two dimensional complex case and in the following we use the conventions of [KoR] (see also [G2]). The mapping

$$C : (w_1, w_2) \mapsto \left(z_1 = \frac{iw_1}{1+w_2}, z_2 = i\frac{1-w_2}{1+w_2} \right)$$

is usually referred to as the *Cayley transform*. It maps the unit ball

$$B = \{ \ w \in \mathbf{C}^2 \ : \ |w_1|^2 + |w_2|^2 < 1 \ \}$$

biholomorphically onto

$$V = \{ \ z \in \mathbf{C}^2 \ : \ \mathrm{Im}\,(z_2) > |z_1|^2 \ \}.$$

The Cayley transform leads to a generalized form of the *stereographic projection*. This mapping $\pi : S^3 \setminus \{-e_2\} \to \mathbf{R}^3$, where $S^3 = \partial B$ and $e_2 = (0,1) \in \mathbf{C}^2$, is defined as the composition of the Cayley transform restricted to $S^3 \setminus \{-e_2\}$ followed by the projection

$$(z_1, z_2) \mapsto (z_1, \mathrm{Re}\,(z_2)).$$

The stereographic projection π can be extended to a mapping from S^3 onto the one-point compactification $\overline{\mathbf{R}}^3$ of $\mathbf{R}^3$.

The *Heisenberg group* $\mathbf{H}$ is the set of pairs $(z,t) \in \mathbf{C} \times \mathbf{R}$ with the product

$$(z,t) \cdot (z',t') = (z+z', t+t' + 2\mathrm{Im}\, z\overline{z}').$$

Using the stereographic projection, we can identify $S^3 \setminus \{-e_2\}$ with $\mathbf{H}$ and S^3 with the one-point compactification $\overline{\mathbf{H}}$ of $\mathbf{H}$. The inverse function of the stereographic projection is given by

$$\pi^{-1}(z,t) = \left(\frac{-2iz}{1+|z|^2 - it}, \frac{1-|z|^2+it}{1+|z|^2-it} \right).$$

Observe that the x-axis in the Heisenberg group corresponds to the intersection of S^3 with the real plane $\mathrm{Re}\,(w_1) = 0, \mathrm{Im}\,(w_2) = 0$. Also, the y-axis corresponds to the intersection of S^3 with the real plane $\mathrm{Im}\,(w_1) = 0, \mathrm{Im}\,(w_2) = 0$.

The Heisenberg group acts on itself by left translations. Heisenberg translations by $(0,\ t)$ for $t \in \mathbf{R}$ are called *vertical translations.*

Positive scalars $\lambda \in \mathbf{R}_+$ act on $\mathbf{H}$ by *Heisenberg dilations*

$$d_\lambda : (z,t) \mapsto (\lambda z, \lambda^2 t).$$

If $\rho \in \mathbf{U}(1)$, then ρ acts on $\mathbf{H}$ by *Heisenberg rotation*

$$\rho : (z,t) \mapsto (\rho z, t).$$

The *Heisenberg complex inversion* of $\mathbf{H}$ is defined on $\mathbf{H} \setminus \{(0,0)\}$ by

$$h : (z,t) \mapsto \left(\frac{-z}{|z|^2 - it}, -\frac{t}{|z|^4 + t^2} \right).$$

Note that $h = \pi \circ j \circ \pi^{-1}$, where j is the involution

$$j : (w_1, w_2) \mapsto (-w_1, -w_2),\ (w_1, w_2) \in \mathbf{C}^2.$$

The map $\hat{m}$ defined by

$$\hat{m} : (z,t) \mapsto (\overline{z}, -t),$$

corresponds to

$$\pi^{-1} \circ \hat{m} \circ \pi(w_1, w_2) = (-\overline{w}_1, \overline{w}_2).$$

All these actions extend trivially to the compactification $\overline{\mathbf{H}}$ of $\mathbf{H}$. It is well known that the group G of transformations of $\overline{\mathbf{H}}$ generated by all Heisenberg translations, dilations, rotations, and h coincides with $\pi^{-1} \circ \mathbf{PU}(2,1) \circ \pi$, and the group $\hat{G} =< G, \hat{m} >$ is the group of all conformal transformations of $\overline{\mathbf{H}}$ (see [KoR] and [G2]).

We need explicit representations of the matrices corresponding to transformations on $\mathbf{SU}(2,1)$.

The transformations $R_\theta : (z,t) \mapsto (\exp(i\theta)z, t)$ and $T_{z,t} : (z', t') \mapsto (z,t).(z',t')$ are represented respectively by the following matrices in $\mathbf{SU}(2,1)$.

$$R_\theta = \begin{pmatrix} \exp(-i\theta/3) & 0 & 0 \\ 0 & \exp(2i\theta/3) & 0 \\ 0 & 0 & \exp(-i\theta/3) \end{pmatrix},$$

$$T_{z,t} = \begin{pmatrix} 1 + |z|^2/2 - it/2 & i\overline{z} & |z|^2/2 - it/2 \\ -iz & 1 & -iz \\ -|z|^2/2 + it/2 & -i\overline{z} & 1 - |z|^2/2 + it/2 \end{pmatrix}.$$

The anti-holomorphic transformations on the ball $(z_1, z_2) \mapsto (\overline{z}_1, -\overline{z}_2)$ (which corresponds to the standard inversion, see the next section) and $\pi^{-1} \circ \hat{m} \circ \pi : (z_1, z_2) \mapsto (-\overline{z}_1, \overline{z}_2)$ correspond respectively to the matrices in $\mathbf{SU}(2,1)$

$$I_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } I_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Their action on homogeneous coordinates should be preceded by the conjugation.

We will use the following proposition that characterizes certain elliptic elements of $\mathbf{PU}(2,1)$. We say that a matrix in $\mathbf{SU}(2,1)$ is elliptic if it is conjugate to an element of $\mathbf{U}(2)$ (see [G2]).

Proposition 2.1 ([G2]) *Let $A_1, A_2 \in \mathbf{SU}(2,1)$ be elliptic elements. Then they are conjugate if and only if* $\mathrm{tr}\,(A_1) = \mathrm{tr}\,(A_2)$.

Observe that this implies that they are conjugate in $\mathbf{PU}(2,1)$ if and only if the cube of their traces are equal.

3 R-circles, C-circles and C-spheres

There are two kinds of totally geodesic submanifolds of real dimension 2 in $H^2_{\mathbf{C}}$: *complex geodesics* (represented by $H^1_{\mathbf{C}} \subset H^2_{\mathbf{C}}$) and *totally real geodesic 2-planes* (represented by $H^2_{\mathbf{R}} \subset H^2_{\mathbf{C}}$). Each of these totally geodesic submanifold is a model of the hyperbolic plane.

Consider the complex hyperbolic space $H^2_{\mathbf{C}}$ and its boundary $\partial H^2_{\mathbf{C}} = S^3$. We will call $\mathbf{C}$-*circles* the intersections of S^3 with the boundaries of totally geodesic complex submanifolds $H^1_{\mathbf{C}}$ in $H^2_{\mathbf{C}}$. Analogously, we will call $\mathbf{R}$-*circles* the intersections of S^3 with the boundaries of totally geodesic totally real submanifolds $H^2_{\mathbf{R}}$ in $H^2_{\mathbf{C}}$.

3.1 R-circles

Definition 3.1 *An inversion on an $\mathbf{R}$-circle is a non-trivial conformal transformation which fixes it pointwise.*

Observe that an inversion has invariant $\mathbf{R}$-circles, one of them being pointwise fixed. Moreover, an $\mathbf{R}$-circle defines a unique inversion. There is, then, a one-to-one correspondence between inversions and $\mathbf{R}$-circles. For instance the transformation $\hat{m}(z,t) = (\overline{z}, -t)$ on the Heisenberg group is the inversion that fixes pointwise the $\mathbf{R}$-circle $\mathrm{Im}\,(z) = 0$.

Proposition 3.2 (FZ) *Let I_1 and I_2 be reflections on the $\mathbf{R}$-circles $\mathbf{R}_1$ and $\mathbf{R}_2$,*

i) $I_1 \circ I_2$ is parabolic if and only if $\mathbf{R}_1$ and $\mathbf{R}_2$ intersect at one point.

ii) $I_1 \circ I_2$ is loxodromic if and only if $\mathbf{R}_1$ and $\mathbf{R}_2$ do not intersect and are not linked.

iii) $I_1 \circ I_2$ is elliptic if and only if $\mathbf{R}_1$ and $\mathbf{R}_2$ are linked or intersect at two points. In the first case there are two (exactly two in most cases) $\mathbf{C}$-circles setwise invariant under both inversions. In the last case there exists one pointwise invariant $\mathbf{C}$-circle under $I_1 \circ I_2$.

Proof. See [FZ] for the proof. Observe that an elliptic element is conjugated to an element of $\mathbf{U}(2)$ which generically has two one dimensional complex eigenspaces where the action is just a rotation. □

Definition 3.3 *Let I_1 and I_2 be reflections on the $\mathbf{R}$-circles $\mathbf{R}_1$ and $\mathbf{R}_2$ linked or intersecting twice. We say that $I_1 \circ I_2$ has* type (φ_1, φ_2) *if it is a rotation of angles φ_1 and φ_2 on the invariant $\mathbf{C}$-circles.*

3.2 C-circles and C-spheres

Proposition 3.4 (see [G2]) *In the Heisenberg model,* **C***-circles are either vertical lines or ellipses, whose projection on the z-plane are circles.*

Definition 3.5 *The* contact plane *at $M = (a, b, c)$ is the plane $P(M) := Z - t + 2aY - 2bX$.*

The circle of **center** $M = (a, b, c)$ and *radius* R is the intersection of the contact plane at M and the cylinder $(X - a)^2 + (Y - b)^2 = R^2$.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two circles of centers (a_1, b_1, c_1), (a_2, b_2, c_2) and *radii* R_1 and R_2. Let d and h be the horizontal and vertical distances between centers

$$d = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}, h = c_2 - c_1, S = \frac{1}{2}(a_1 b_2 - a_2 b_1).$$

Proposition 3.6 (Linking of C-circles [FK]) *The* **C***-circles* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are linked if and only if*

$$\left(d^2 - (R_1 + R_2)^2\right)\left(d^2 - (R_1 - R_2)^2\right) + (h + 4S)^2 = \left(d^2 - (R_2^2 - R_1^2)\right)^2 + (h + 4S)^2 - 4d^2 R_1^2 < 0.$$

Observe that $\mathbf{C}_1$ and $\mathbf{C}_2$ are not linked if their projections are not, that is

$$\left(d^2 - (R_1 + R_2)^2\right)\left(d^2 - (R_1 - R_2)^2\right) > 0$$

or if

$$4d^2 R_1^2 < (h + 4S)^2,$$

that is, $\mathbf{C}_1$ does not intersect the plane defining $\mathbf{C}_2$ (see also Lemma 6.1).

In the following definition we allow a point to be a (degenerate) **C**-circle.

Definition 3.7 *A* **C***-sphere around an* **R***-circle is an union of* **C***-circles invariant under the inversion on the* **R***-circle, which is homeomorphic to a sphere. We will call the axis of the* **C***-sphere the set of centers of these invariant* **C***-circles.*

In particular a **C**-sphere contains two degenerate **C**-circles and its axis has starting and ending points in the **R**-circle. See also [FZ].

Definition 3.8 *The surface of centers of an* **R***-circle is the set of points which are centers of invariant* **C***-circles under the inversion on the* **R***-circle.*

Such **C**-circles have two points in common with the **R**-circle. Observe that for finite **R**-circles this is a 2 dimensional surface but for an infinite **R**-circle this coincides with the **R**-circle.

For a finite **R**-circle the center completely determines the **C**-circle (Proposition 3.9). But, observe that for an infinite **R**-circle, a *radius* should be specified for each center.

A given axis determines a surface obtained by the union of **C**-circles defined by the centers. But that surface might have self-intersections. We will call a *good axis* an axis whose associated surface is homeomorphic to the two dimensional sphere.

We will consider parts of **C**-spheres as faces of polyhedra. By abuse of language we will also refer to them as **C**-spheres. They will be disjoint union of **C**-circles between two fixed **C**-circles in a **C**-sphere. Analogously we will refer to the part of the axis corresponding to that portion of **C**-sphere as the corresponding axis.

3.3 Standard R-circle

Consider the following transformation on the Heisenberg group

$$I_0 = \hat{m} \circ h : (z, t) \mapsto \left(\frac{-\overline{z}}{|z|^2 + it}, \frac{t}{|z|^4 + t^2} \right).$$

which corresponds to

$$\pi^{-1} \circ I \circ \pi(w_1, w_2) = \pi^{-1} \circ \hat{m} \circ h \circ \pi(w_1, w_2) = (\overline{w}_1, -\overline{w}_2).$$

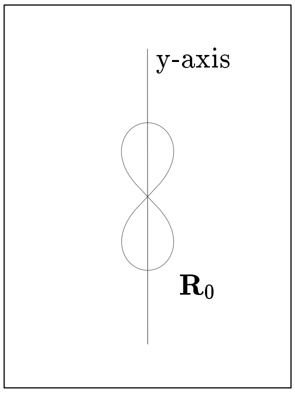


Figure 1: $\mathbf{R}_0$ top view

I_0 leaves pointwise fixed the standard **R**-circle $\mathbf{R}_0$ (see [G2] for details)

$$r^2 + it = -e^{-2i\theta}$$

where $z = re^{i\theta}$. In cylindrical coordinates $\mathbf{R}_0$ is given by

$$r = \sqrt{-\cos(2\theta)}, \quad z = \sin(2\theta).$$

Proposition 3.9 ([FK]) *The surface of centers* $\mathbf{S}_0$ *of the standard* $\mathbf{R}$*-circle is given, in cylindrical coordinates, by* $(r, \theta, \sin(2\theta))$*. If* $r^2 + \cos(2\theta) \geq 0$*, the radius of a* **C***-circle with center coordinates* r, θ *is* $\sqrt{r^2 + \cos(2\theta)}$*.*

From the above parametrization we see that the surface of centers satisfies the algebraic equation

$$Z(X^2 + Y^2) = 2XY.$$

The *radius* of the **C**-circle is then $R^2 = r^2 + \cos(2\theta)$. We obtain the standard **R**-circle for $R = 0$.

The part of the algebraic set (solution of the equation above) with $r^2 + \cos(2\theta) < 0$ will be called *surface of imaginary centers*. Although it does not correspond to any center of an invariant **C**-circle, it is a usefull set.

R. Schwarz (cf. [S]) calls hybrid cones certain surfaces foliated by **R**-circles. Part of our surface of centers is a hybrid cone in his sense. The use we make of them in this work is mainly to parametrize **C**-spheres, while his use of them is as boundaries of fundamental domains. One could imagine that probably a complete description of fundamental domains in complex hyperbolic geometry should take into account both **C**-spheres, foliated by **C**-circles, and surfaces foliated by **R**-circles, as Schwarz's hybrid cones.

3.4 Infinite R-circles

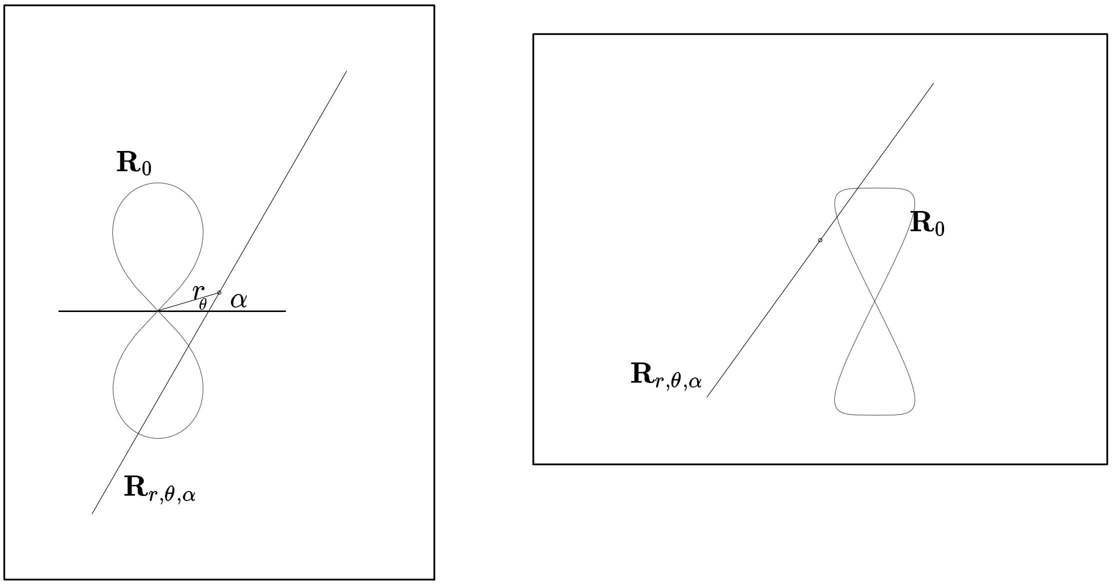


Figure 2: $\mathbf{R}_{r,\theta,\alpha}$; lateral view (right) and top view (left). The distinguished point is an intersection of the infinite **R**-circle with the surface of centers of $\mathbf{R}_0$.

Definition 3.10 $\mathbf{R}_{r,\theta,\alpha}$ *is the infinite* **R**-*circle passing through* $M = (r, \theta, \sin(2\theta))$ *whose projection onto the* z*-plane is the line of slope* $\tan(\alpha)$. *It is given by*

$$P(M) : Z = \sin(2\theta) + 2r(\cos(\theta)Y - \sin(\theta)X), P_\alpha(M) : \sin(\alpha)X - \cos(\alpha)Y = r\sin(\theta - \alpha).$$

Observe that $\mathbf{R}_{r,\theta,\alpha}$ is horizontal if and only if $\alpha = \theta \mod \pi$.

Proposition 3.11 (Linking of R-circles) *If* $R^2 = r^2 + \cos(2\theta) < 0$, *then* $\mathbf{R}_{r,\theta,\alpha}$ *is horizontal and intersects* $\mathbf{R}_0$ *twice or is linked with* $\mathbf{R}_0$. *On the other hand, any* **R**-*circle that is linked with* $\mathbf{R}_0$ *intersects the surface of imaginary centers once.*

Proof. Let $\mathbf{R} = \mathbf{R}_{r,\theta,\alpha}$ and $M = (r, \theta, \sin(2\theta))$. Let us consider the intersection of $\mathbf{R}_0$ with $P(M)$. We obtain points $(\rho = \sqrt{-\cos(2u)}, u, \sin(2u))$, such that $\sin(\theta - u)(\cos(\theta + u) + r\rho) = 0$. We get the two points $(\pm\sqrt{-\cos(2\theta)}, \theta, \sin(2\theta))$ or $(\cos(\theta+u)+r\rho) = 0$, that gives $(r^2+\cos(2\theta))\cos(2u) + \sin(\theta - u)^2 = 0$, which is impossible. Now the plane $P_\alpha(M)$ separates the two points if $\alpha \neq \theta[\pi]$. □

3.5 Infinite R-circles and surface of centers

Let $\mathbf{R}$ be an infinite $\mathbf{R}$-circle passing through $\mathbf{S}_0$ at $M_1 = (r_1, \theta_1, \sin(2\theta_1))$ and $M_2 = (r_2, \theta_2, \sin(2\theta_2))$. Then we have

$$\sin(\theta_1 - \theta_2)(\cos(\theta_1 + \theta_2) + r_1 r_2) = 0.$$

If $\sin(\theta_1 - \theta_2) = 0$, then $\mathbf{R}$ is horizontal and intersects $\mathbf{S}_0$ at $r = 0$ again.

Proposition 3.12 *Let $\mathbf{R}$ be an infinite $\mathbf{R}$-circle intersecting $\mathbf{S}_0$ twice. Then $\mathbf{R}$ intersects $\mathbf{S}_0$ in a third point and we have*

$$\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} = 0, \quad \frac{\sin(2\theta_1)}{R_1^2} + \frac{\sin(2\theta_2)}{R_2^2} + \frac{\sin(2\theta_3)}{R_3^2} = 0,$$

where $M_i = (r_i, \theta_i, \sin(2\theta_i))$ are the intersection points and $R_i^2 = r_i^2 + \cos(2\theta_i)$.

Proof. We consider the polynomial equations

$$\begin{aligned} &Z(X^2 + Y^2) = 2XY, r^2 = (X^2 + Y^2), TX = Y, \\ &Z = \sin(2\theta_1) + 2r_1(\cos(\theta_1)Y - \sin(\theta_1)X), \\ &Z = \sin(2\theta_2) + 2r_2(\cos(\theta_2)Y - \sin(\theta_2)X). \end{aligned}$$

Using different elimination techniques, we get a third-degree polynomial equation that is satisfied by r^2, a third-degree polynomial equation for Z and a third-degree polynomial equation for T. As we know already two roots of these polynomials, we get (using the fact that $r_1r_2+\cos(\theta_1+\theta_2) = 0$)

$$Z_3 = \frac{R_1^2 \sin(2\theta_2) + R_2^2 \sin(2\theta_1)}{R_1^2 + R_2^2}, \; r_3^2 = \frac{\sin(\theta_1 - \theta_2)^2}{R_1^2 + R_2^2}, \; T_3 = \frac{r_1 \cos(\theta_2) - r_2 \cos(\theta_1)}{r_1 \sin(\theta_2) - r_2 \sin(\theta_1)}.$$

We then obtain $R_3^2 = r_3^2 + \dfrac{1 - T_3^2}{1 + T_3^2} = -\dfrac{R_1^2 R_2^2}{R_1^2 + R_2^2}$. □

Observe that in this case one of the intersection points is an imaginary center and that $\mathbf{R}$ is linked with $\mathbf{R}_0$ (Prop. 3.11).

On the other hand we get (see Prop. 3.2)

Proposition 3.13 *Let $\mathbf{R}$ be an infinite non-horizontal $\mathbf{R}$-circle, linked with $\mathbf{R}_0$. Then $\mathbf{R}$ intersects the surface of centers in three points. One is imaginary and the other corresponding $\mathbf{C}$-circles are linked.*

Proof. Let $\mathbf{R} = \mathbf{R}_{r,\theta,\alpha}$. We have $R^2 = r^2 + \cos(2\theta) < 0$. Points of $\mathbf{R}$ are

$$X = r\cos(\theta) - \frac{1}{2}\frac{\cos(\alpha)\,(Z - \sin(2\,\theta))}{r\sin(\alpha - \theta)}, Y = r\sin(\theta) - \frac{1}{2}\frac{\sin(\alpha)\,(Z - \sin(2\,\theta))}{r\sin(\alpha - \theta)}$$

Looking to the intersection of $\mathbf{R}$ and $Z(X^2 + Y^2) = 2XY$, one finds a polynomial P of degree 2 whose discriminant is

$$16\sin(\alpha-\theta)^2\left(-4\,r^4\sin(\alpha-\theta)^2 + \cos(\alpha+\theta)^2 + 4\,r^2\sin(-\alpha+\theta)\sin(\alpha+\theta)\right)$$

and is positive when $0 \le r^2 < -\cos(2\theta)$. We thus obtains two other points M_1 and M_2.

Looking to the corresponding *radii* of $\mathbf{C}$-circles we get $P = (R^2 - R_1^2)(R^2 - R_2^2)$ with $R_1^2R_2^2 > 0$ when $0 \le r^2 < -\cos(2\theta)$. It shows that R_1^2 and R_2^2 are positive because of Proposition 3.12 and $R_3 = r^2 + \cos(2\theta) < 0$.

The evaluation of the linking condition (Prop. 3.6) gives $-2(r_1^2 + \cos(2\theta_1))(r_2^2 + \cos(2\theta_2)) < 0$ so the $\mathbf{C}$-circles at M_1 and M_2 are linked. □

4 Configurations of a standard and an infinite R-circle

Consider the configuration space of an infinite $\mathbf{R}$-circle $\mathbf{R}_1$ and the standard $\mathbf{R}$-circle $\mathbf{R}_0$. Using a description of the configuration space of lines in the plane as a Möbius band (take as coordinates the angle, between 0 and π, from the x-axis and the oriented measure of a segment starting from the origin and arriving perpendicularly to a line) one can clearly obtain, by vertical translations, that the configuration space is $\mathrm{M} \times \mathbf{R}$, where M is the Möbius band. One could further use the dihedral symmetry of the standard $\mathbf{R}$-circle (generated by reflections on the the two horizontal x-axis and y-axis) to reduce the configuration space to angles between 0 and $\pi/2$ and segments of positive measures.

Observe that the configuration space above is not the configuration space of two $\mathbf{R}$-circles up to transformations of $\mathbf{PU}(2,1)$. As one of the $\mathbf{R}$-circles is infinite, the transformations are in the isotropy group $\mathbf{U}(1) \rtimes \mathrm{H}$.

We will deal only with configurations that give elliptic elements. By Proposition 3.2 the infinite $\mathbf{R}$-circles should be linked with the standard $\mathbf{R}$-circle.

It will be important to identify equivalent configurations under $\mathbf{PU}(2,1)$. Before doing that, we single out some special configurations which will represent each equivalent class.

Definition 4.1 *The* $\mathbf{C}$*-standard elliptic* (φ_1, φ_2)*-configuration is the one where* $\mathbf{R}_1 = \mathbf{R}_{r,\theta,\alpha}$ *with* $r = 0, \theta = \pi/4 - \varphi_1/2, \alpha = \pi/4 + \varphi_1/2 - \varphi_2$.

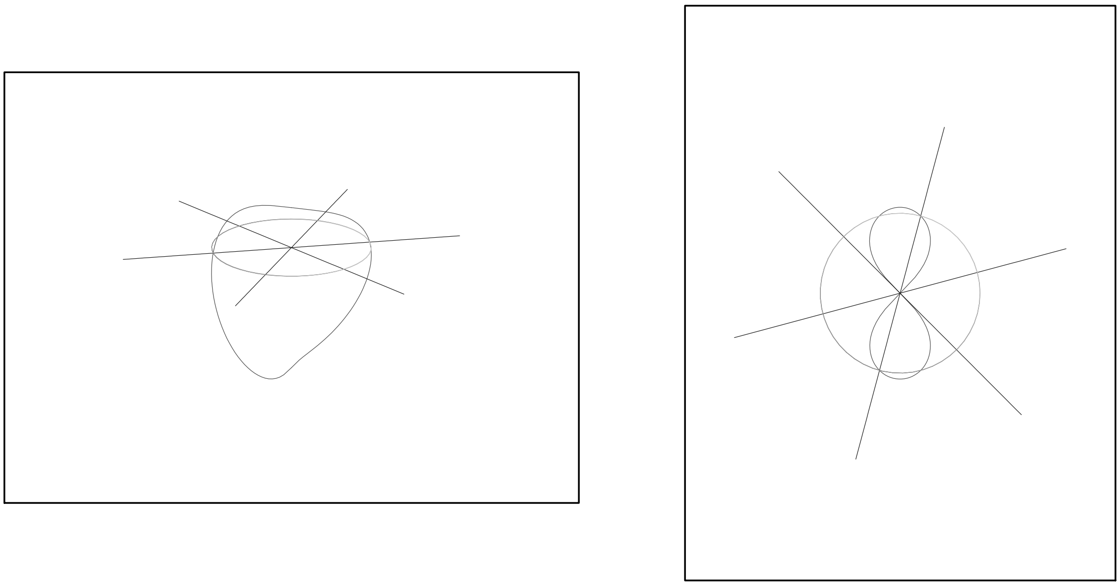


Figure 3: **C**-standard configurations $(\pi/3, n\pi/3)$, $0 \le n \le 2$; two views showing the invariant **C**-circle. The configurations $(\pi/3, 0)$ whose line intersects twice the finite **R**-circle and $(\pi/3, \pi/3)$ are rigid.

Proposition 4.2 *The composition* $I_0 \circ I_{(\varphi_1,\varphi_2)}$ *of the two inversions corresponding to* $\mathbf{R}_0$ *and the* **C***-standard elliptic* (φ_1, φ_2)*-configuration* R_1 *is of type* $(2\varphi_1, 2\varphi_2)$ *(see 3.3).*

In particular the **C**-standard $(\varphi_1, 0)$-configuration corresponds to the horizontal line at height $\sin(\pi/2 - \varphi_1)$ intersecting the standard **R**-circle twice.

The proof of this proposition follows from the following elementary lemma.

Lemma 4.3 *The angle of tangency to the standard* **R***-circle* $\mathbf{R}_0$ *at the point with polar coordinate* θ *is* $3\theta - \pi/2$*. The angle from the ray at angle* θ *to the tangent to the* **R***-circle* $\mathbf{R}_0$ *at the intersection point is* $2\theta - \pi/2$*.*

We will determine those configurations equivalent to a fixed standard one. We start with a rigidity result.

Theorem 4.4 (Rigidity) *The standard* $(\varphi_1, 0)$*-configuration (for* $\pi/4 \le \varphi_1 \le \pi/2$*) is the unique configuration corresponding to its conjugacy class .*

Proof. Observe that, if the conjugacy class of the composition of inversions is determined by the angles φ_1 and 0, the two **R**-circles intersect. A simple computation then shows that the only element in the conjugacy class is the one defined by the standard one. □

To study more general conjugacy classes we consider the subgroup generated by two inversions, that is, I_0 and $M \circ I_x \circ \bar{M}^{-1}$ where $M = T \circ D \circ R$ (R is a rotation, D is a dilation and T is a Heisenberg translation). The corresponding matrix in $\mathbf{SU}(2,1)$ is given by $A = M \circ R_x \circ \overline{M}^{-1} \circ R_0$.

One computes

$$
\begin{aligned}
M &= \frac{\exp(-i\alpha/3)}{\lambda}\begin{pmatrix} 1/2\,(\lambda^2(1+|z|^2-it)+1) & i\lambda\exp(i\alpha)\bar{z} & 1/2\,(\lambda^2(1+|z|^2-it)-1) \\ -i\lambda^2 z & \lambda\exp(i\alpha) & -i\lambda^2 z \\ 1/2\,(\lambda^2(1-|z|^2+it)-1) & -i\lambda\exp(i\alpha)\bar{z} & 1/2\,(\lambda^2(1-|z|^2+it)+1) \end{pmatrix}, \\
A &= \exp(-2/3i\alpha)\begin{pmatrix} 1+|z|^2-it-\exp(2i\alpha)\bar{z}^2 & i(z-\exp(2i\alpha)\bar{z}) & -|z|^2+it+\exp(2i\alpha)\bar{z}^2 \\ -i(z-\exp(2i\alpha)\bar{z}) & -\exp(2i\alpha) & i(z-\exp(2i\alpha)\bar{z}) \\ -|z|^2+it+\exp(2i\alpha)\bar{z}^2 & i(z-\exp(2i\alpha)\bar{z}) & -1+|z|^2-it-\exp(2i\alpha)\bar{z}^2 \end{pmatrix}.
\end{aligned}
$$

When $z = re^{i\theta}$ and $t = \sin(2\theta)$, we get

$$
\begin{aligned}
\mathrm{tr}\,(A) &= e^{-\frac{2}{3}i\alpha}(-2i\sin(2\theta)+2r^2-2r^2e^{-2i(\theta-\alpha)}-e^{2i\alpha}) \\
&= e^{\frac{1}{3}i(\alpha-3\theta)}\left(-4i(r^2+\cos(2\theta))\sin(\alpha-\theta)+e^{i(\alpha-3\theta)}-2e^{-i(\alpha-3\theta)}\right).
\end{aligned}
$$

We thus deduce that

Proposition 4.5 *If $\alpha_0-\theta_0\neq 0$ (which excludes the situation of Theorem 4.4 and the $\mathbf{C}$-standard $(\pi/2,\pi/2)$-configuration), A and A_0 are in the same conjugacy class if*

$$
\alpha-3\theta=\alpha_0-3\theta_0,\quad r^2+\cos(2\theta)=(r_0^2+\cos(2\theta_0))\frac{\sin(\alpha_0-\theta_0)}{\sin(\alpha-\theta)},
$$

or equivalently

$$
\alpha-3\theta=\alpha_0-3\theta_0,\quad r^2=\frac{1}{2}\frac{\sin(\alpha_0+\theta_0)-\sin(\alpha+\theta)}{\sin(\alpha-\theta)}+r_0^2\frac{\sin(\alpha_0-\theta_0)}{\sin(\alpha-\theta)}.
$$

Observe that in this case we have $\mathrm{tr}\,(A)^3=\mathrm{tr}\,(A_0)^3$. We also have other solutions considering that an infinite $\mathbf{R}$-circle $R_{r,\theta,\alpha}$ may be given by $\alpha \mod \pi$ and by three values of (r,θ).

We consider now the family $\mathbf{R}_{r,\theta,\alpha}$ of infinite $\mathbf{R}$-circles defined by

$$
\alpha=3\theta+(\alpha_0-3\theta_0)=3\theta+\pi/4+\varphi_1/2-\varphi_2-3(\pi/4-\varphi_1/2)=3\theta-\pi/2+2\varphi_1-\varphi_2
$$

and

$$
r^2=\frac{1}{2}\frac{\sin(\alpha_0+\theta_0)-\sin(\alpha+\theta)}{\sin(\alpha-\theta)}=-\frac{\cos(2\theta+\varphi_1-\varphi_2)\cos(2\theta+\varphi_1)}{\cos(2\theta+2\varphi_1-\varphi_2)}.
$$

Remark 4.6 *The curve defined above has 3 branches (the 3 intersection points with the surface of centers). The intersection with the surface of imaginary centers is a closed curve. Each component has $\alpha-3\theta$ as invariant. As the sign of r^2 changes at $\theta=\theta_0$ if $r_0=0$ one can take $r=\dfrac{r^2}{\sqrt{|r^2|}}$ as continuous parametrization.*

Observe that when $\theta=\pi/4-\varphi_1/2$ we obtain precisely the $\mathbf{C}$-standard (φ_1,φ_2)-configuration.

Theorem 4.7 (Rigidity) *The $\mathbf{C}$-standard (φ,φ)-configuration is rigid in its conjugacy class.*

Proof. The proof follows from solving the equation $\mathrm{tr}\,(A) = -1$. From the solution above, we obtain that, if $\varphi_1 = \varphi_2$, $r^2 = -\cos(\theta)$. That is, the other solution gives a parabolic configuration. $\square$

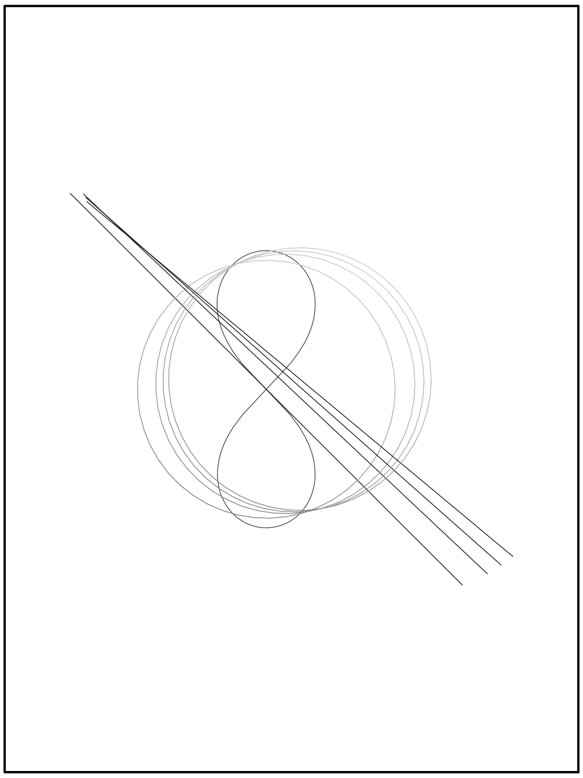


Figure 4: Four configurations in the family $I_{\pi/3,2\pi/3}(\theta)$: top view showing invariant $\mathbf{C}$-circles.

Theorem 4.8 (Flexibility) *The family of compositions $I_0 \circ I_{\varphi_1,\varphi_2}(\theta)$ where $I_{\varphi_1,\varphi_2}(\theta)$ denotes the inversion on the $\mathbf{R}$-circle in the family above, parametrized by θ, is in the conjugacy class defined by the angles $2\varphi_1$ and $2\varphi_2$.*

Proof. Using the Prop. 4.5 we have in $\mathbf{SU}(2,1)$ $\mathrm{tr}\,(I_0 \circ \widetilde{I_{\varphi_1,\varphi_2}}(\theta)) = \mathrm{tr}\,(I_0 \circ \widetilde{I_{\varphi_1,\varphi_2}})$. $\square$

We end this section with an observation concerning $\mathbf{R}$-Fuchsian embeddings, that is, embeddings which fix an $\mathbf{R}$-circle (see also [FZ]).

Definition 4.9 *The $\mathbf{R}$-standard elliptic φ_1-configuration is the one where the infinite $\mathbf{R}$-circle intersects the y-axis perpendicularly at $y = \sin(\pi/2 - \varphi_1)$.*

In this case we have $\mathbf{R}_1 = \mathbf{R}_{r,\theta,\alpha}$ and $r = \sin(\pi/2 - \varphi_1), \theta = \pi/2, \alpha = 0$.

Proposition 4.10 *The $\mathbf{R}$-standard elliptic φ_1-configuration is in the same conjugacy class as the $\mathbf{C}$-standard elliptic $(\varphi_1, -\varphi_1)$-configuration.*

Proof. The only elliptic elements of $\mathbf{PU}(2,1)$ which preserve an $\mathbf{R}$-circle are elements of that form. $\square$

5 Discrete embeddings

Poincaré's polyhedron theorem is the main tool we use to prove discreteness. A general version, without parabolics, was proved in [FZ] and in [FK] we stated a version containing only parabolics. Here we state an appropriate version with both elliptic and parabolic elements.

Let $\{R_i\}$ be a finite collection of finite $\mathbf{R}$-circles and $\{S_i\}$ be a collection of $\mathbf{C}$-spheres around each of them. Suppose that, pairwise, the $\mathbf{R}$-circles *either* intersect at most at one point where the corresponding $\mathbf{C}$-spheres intersect tangentially *or* they are linked or intersect twice and the corresponding $\mathbf{C}$-spheres intersect in one of the invariant $\mathbf{C}$-circles. The intersecting $\mathbf{C}$-circles will be called *edges* and the piece of the $\mathbf{C}$-surface between two consecutive edges is called a *face*.

Theorem 5.1 (Poincaré polyhedron) *Let $\{R_i\}$ be a finite collection of finite $\mathbf{R}$-circles and $\{S_i\}$ be a collection of $\mathbf{C}$-spheres around each of them with the hypothesis as above. Suppose that*

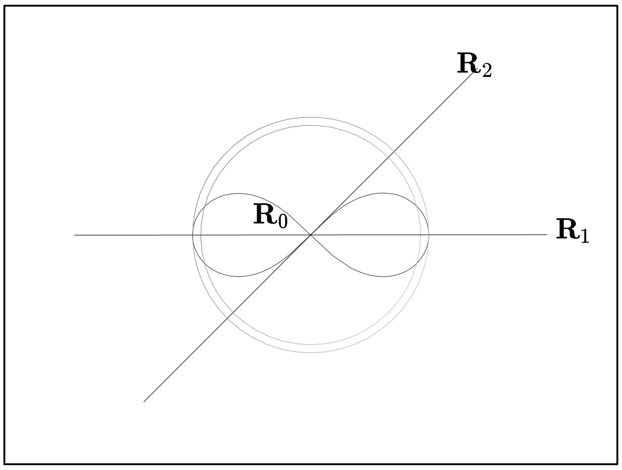
i) at each edge the angles are of type $(\pi/q, n\pi/q)$ where the rotation π/q fixes pointwise the edge,

ii) the closure of the unbounded component of the complement of each $\mathbf{C}$-sphere containing a face contains all other faces.

Then the group generated by inversions on each $\mathbf{R}$-circle is discrete and a fundamental domain is the unbounded component of the complement of the union of all faces.

The fact that the $\mathbf{C}$-surfaces are unions of $\mathbf{C}$-circles implies that one can extend those surfaces canonically as hypersurfaces in the complex hyperbolic space where they define a "polyhedron". See [FZ] and [FK] for more details.

5.1 Configurations of the standard and two infinite R-circles



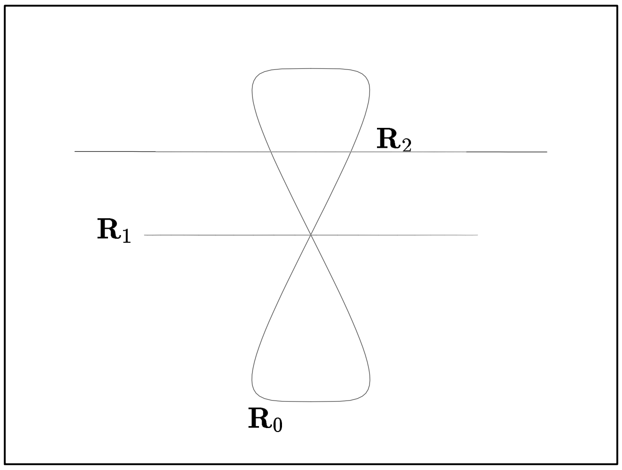


Figure 5: standard configuration $\mathbf{C}(\pi/2, 0, \pi/3, 2\pi/3)$: lateral view (right) and top view (left) with invariant $\mathbf{C}$-circles

Let Γ be the triangular group of type (p, q, ∞), that is, the abstract group presented by

$$\Gamma =< \iota_0, \iota_1, \iota_2 : {\iota_0}^2 = 1, \iota_1^2 = 1, \iota_2^2 = 1, (\iota_0 \circ \iota_1)^p = 1, (\iota_0 \circ \iota_2)^q = 1 > .$$

We want to determine the subspace of $\mathrm{Hom}(\Gamma, \widehat{\mathbf{PU}(2,1)})$ where the image of ι_i are inversions and such that the embedding is injective and discrete. If the image of ι_i are inversions, three $\mathbf{R}$-circles $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_0$ are defined. The relations in the presentation imply that $\mathbf{R}_1, \mathbf{R}_2$ can be considered as non-intersecting infinite $\mathbf{R}$-circles and $\mathbf{R}_0$ a finite one. Moreover, in order to prove discreteness, $\mathbf{R}_1, \mathbf{R}_0$ should be equivalent to the $\mathbf{C}$-standard $(\pi/p, n\pi/p)$-configuration and $\mathbf{R}_2, \mathbf{R}_0$ should be equivalent to the $\mathbf{C}$-standard $(\pi/q, m\pi/q)$-configuration, where n, m are integers.

5.2 Standard embeddings of the (2,3,∞)-triangle group

In this section we give examples of standard embeddings in the case of the triangular group of type $(2, 3, \infty)$. There are 6 $\mathbf{C}$-Fuchsian embeddings and 1 $\mathbf{R}$-Fuchsian embedding.

Observe that in the $\mathbf{C}$-Fuchsian case, $\mathbf{R}_2$ could be embedded as in the $(\pi/3, 0)$-configuration, $(\pi/3, \pi/3)$-configuration or $(\pi/3, 2\pi/3)$-configuration. $\mathbf{R}_1$ could be as in the $(\pi/2, 0)$-configuration or $(\pi/2, \pi/2)$-configuration.

The combination of those cases gives us the 6 $\mathbf{C}$-Fuchsian embeddings. The only $\mathbf{R}$-Fuchsian embedding is given by the $\mathbf{R}$-standard $\pi/3$-configuration together with the $\mathbf{R}$-standard $\pi/2$-configuration.

All those embeddings are discrete and injective. The embeddings will be denoted by their angles. For instance $\mathbf{C}(\varphi_1, \varphi_2, \varphi_1', \varphi_2')$ is the $\mathbf{C}$-standard embedding with $\mathbf{R}_1$ as in (φ_1, φ_2)-configuration and $\mathbf{R}_2$ as in (φ_1', φ_2')-configuration.

5.3 Teichmüller space

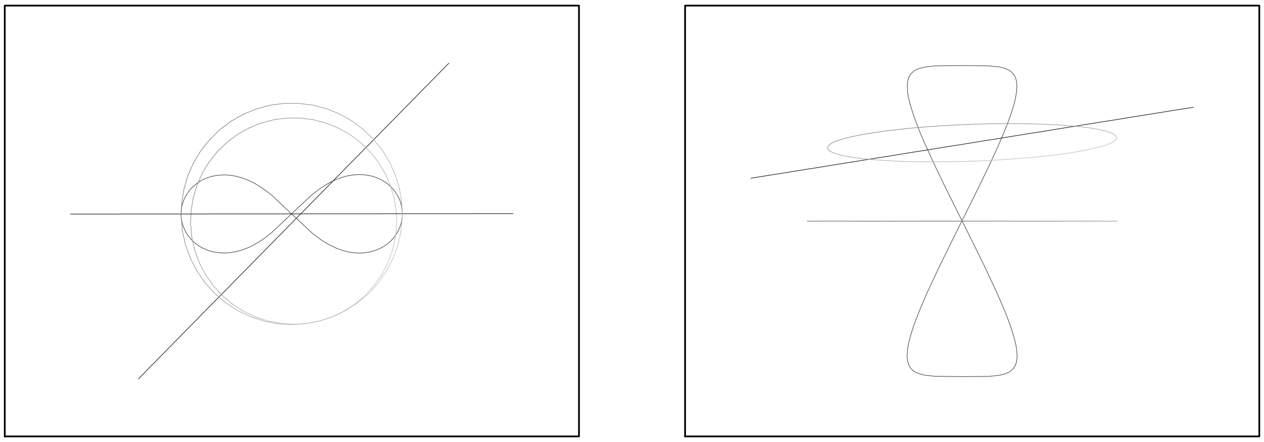


Figure 6: deformed configuration; lateral view (right) and top view (left) with invariant $\mathbf{C}$-circles

Consider the Teichmüller space of the triangle group of type (p, q, ∞) in $\widehat{\mathbf{PU}(2,1)}$.

Theorem 5.2 • *Each embedding* $\mathbf{C}(\pi/p, 0, \pi/q, 0)$, $\mathbf{C}(\pi/p, \pi/p, \pi/q, \pi/q)$ *or* $\mathbf{C}(\pi/p, 0, \pi/q, \pi/q)$ *is isolated in the Teichmüller space.*

- *Each embedding* $\mathbf{C}(\pi/p, 0, \pi/q, n\pi/q)$ *or* $\mathbf{C}(\pi/p, \pi/p, \pi/q, n\pi/q)$ *with* $n > 1$*, is contained in a real one dimensional open set of the Teichmüller space.*

- *Each embedding* $\mathbf{C}(\pi/p, m\pi/p, \pi/q, n\pi/q)$ $(n, m > 1)$ *is contained in a real two dimensional open set of the Teichmüller space.*

Proof. We give the idea of the proof, referring to some technical lemmas in the next section.

- A configuration $\mathbf{C}(\pi/p, 0, \pi/q, 0)$ is isolated in Teichmüller space by applying Theorem 4.4 twice.

- A configuration $\mathbf{C}(\pi/p, \pi/p, \pi/q, n\pi/q)$ can be deformed (if $n > 1$) only because of the one parameter deformation of $\mathbf{R}_2$, $\mathbf{R}_1$ being rigid by Theorem 4.7.

- By Theorem 4.8 we can describe the one parameter family of deformations of $\mathbf{C}(\pi/p, 0, \pi/q, n\pi/q)$ $(n > 1)$ and the two parameter family of deformations of $\mathbf{C}(\pi/p, m\pi/p, \pi/q, n\pi/q)$ $(m, n > 1)$. In the first case, Theorem 4.4 shows rigidity of $\mathbf{R}_1$. In the last case we have

$$\alpha = 3\theta - \pi/2 + 2\varphi_1 - \varphi_2$$

and

$$r^2 = -\frac{\cos(2\theta + \varphi_1 - \varphi_2)\cos(2\theta + \varphi_1)}{\cos(2\theta + 2\varphi_1 - \varphi_2)}$$

where $(\varphi_1, \varphi_2) = (\pi/p, m\pi/p)$ or $(\varphi_1, \varphi_2) = (\pi/q, n\pi/q)$ give the coordinates of the infinite $\mathbf{R}$-circle $\mathbf{R}_1$ and $\mathbf{R}_2$ as $\mathbf{R}_{r(\theta),\theta,\alpha(\theta)}$. Of course, if $p = 2$ we use those formulas only for $\mathbf{R}_2$, $\mathbf{R}_1$ being rigid.

Using Poincaré's theorem we should find three $\mathbf{C}$-spheres S_0, S_1, S_2 invariant respectively by $I_0, I_1(\theta_1), I_2(\theta_2)$, the inversions in the three $\mathbf{R}$-circles such that the intersection of S_1 and S_2 is the point at infinity, the intersection of S_1 and S_0 is one of the invariant $\mathbf{C}$-circles by I_0 and $I_1(\theta_1)$ and the intersection of S_2 and S_0 is one of the invariant $\mathbf{C}$-circles by I_0 and $I_2(\theta_2)$. The surfaces will depend continuously on θ_1 and θ_2, but we will not write explicitly the angles in order to simplify notations.

The $\mathbf{C}$-circle $\mathbf{C}_1$, invariant by I_0 and $I_1(\theta_1)$ is the one with center determined by $r(\theta_1)$ in the surface of centers. Call that center M_1 and M_2 the center of the $\mathbf{C}$-circle $\mathbf{C}_2$.

For each one of the three surfaces we need to define the $\mathbf{C}$-circles composing them and that will be done by describing a curve of centers of $\mathbf{C}$-circles with an appropriate function of those centers describing their *radii*.

Standard fundamental domain.
The invariant surface for $\mathbf{R}_0$ will be given by its axis $\rho(\theta)$ with $\theta_1 \leq \theta \leq \theta_2$ (see Lemma 6.3).

In the standard embedding, M_1 is on the vertical axis with height $\cos(\pi/p)$ and the $\mathbf{C}$-circle $\mathbf{C}_1$ is a horizontal $\mathbf{C}$-circle with center in M_1 of *radius* $\sqrt{\sin(\pi/p)}$. A fundamental domain in that case is obtained by defining S_1 to be the union of concentric $\mathbf{C}$-circles, analogously for S_2 and S_0 to be the union of $\mathbf{C}$-circles with centers on the vertical axis from M_2 to M_1.

Deformation of the standard fundamental domain.
Let $N_1 \in \mathbf{R}_1$ and $N_2 \in \mathbf{R}_2$ whose projections are the intersection of projections of $\mathbf{R}_1$ and $\mathbf{R}_2$ on the z-plane. For $\mathbf{R}_1$ we will take centers in the segment $[M_1, N_1]$ with appropriate *radii.* Then we will complete the surface by a union of $\mathbf{C}$-circles of centers N_1. We proceed analogously with $\mathbf{R}_2$ (see Lemma 6.4).
1. Chimney. If $\mathbf{C}_1$ and $\mathbf{C}_2$ will be near the corresponding $\mathbf{C}$-circle of the standard embedding, we will be able to choose an axis such that the corresponding $\mathbf{C}$-circles will be above the plane determined by $\mathbf{C}_1$ $(P(M_1))$ and bellow the plane determined by $\mathbf{C}_2$ $(P(M_2))$.
2. Beginning of S_1 and S_2. We then may choose concentric circles centered on M_1 and M_2 with increasing *radii* until their projections are large contain the projection of S_0. This assures that the families S_1 and S_2 do not intersect the family S_0. If the deformation is small enough we are, moreover, certain that the families S_1 and S_2 do not intersect (see Prop. 6.5).
3. Middle of S_1 and S_2. We should then move the centers of the $\mathbf{C}$-circles from M_1 to N_1 on $\mathbf{R}_1$ and from M_2 to N_2 on $\mathbf{R}_2$. Again, if the deformation is small enough this can be done in such a way that there will be no intersection between the families S_1 and S_2 (see Lemma 6.4).
4. End of S_1 and S_2. Finally, from the points N_1 and N_2 we complete the construction of S_1 and S_2 with concentric $\mathbf{C}$-circles. They are parallel.

The final construction is explicit in Proposition 6.5. Observe that, for the standard case, part 2 and 3 are empty. □

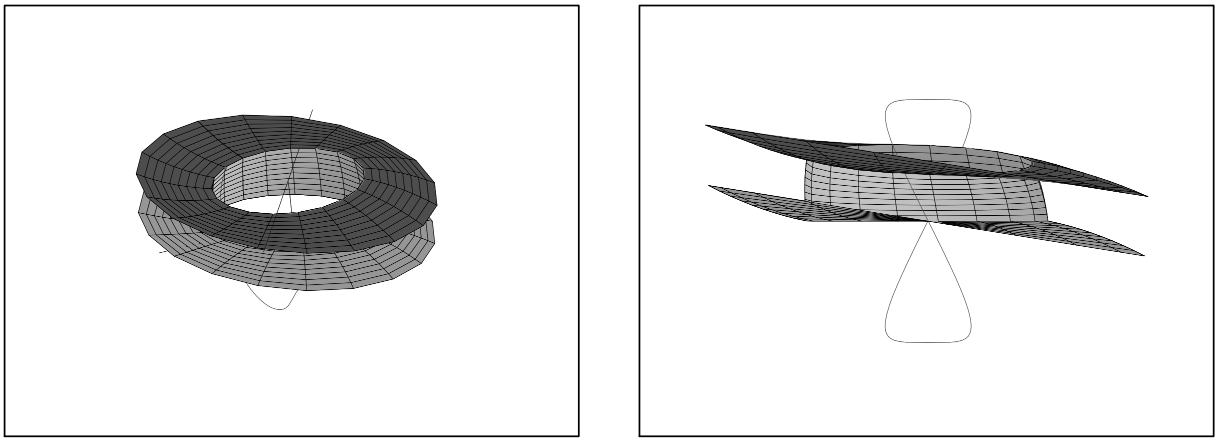


Figure 7: two views of the fundamental domain

Remark 5.3 *Figures 6 and 7 show the deformation of the triangle group corresponding to the configuration* $\mathbf{C}(\pi/2, 0, \pi/3, 2\pi/3)$. *That group contains the modular group as an index two subgroup.*

Remark 5.4 *Analogously, combining present methods and those of [FK], a triangle group of type* (p, ∞, ∞) *has Teichmüller neighborhoods of dimension two and three if* $p > 2$. *The triangle group of type* $(2, \infty, \infty)$ *has Teichmüller neighborhoods of dimension two.*

Remark 5.5 *We did not determine explicitly the complete Teichmüller space. We conjecture that it contains the points of the configuration space where the* $\mathbf{C}$*-circles* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are not linked. In particular, using the methods of this paper one can prove that the* $\mathbf{R}$*-Fuchsian standard embeddings* $\mathbf{R}(\pi/2, \pi/3)$ *are in the same component as* $\mathbf{C}(\pi/3, -\pi/3, \pi/2, -\pi/2)$.

6 Technical Lemmas

Lemma 6.1 *Let* $P(M_1)$ *be the contact plan at* $M_1 = (r_1, \theta_1, t_1)$*. Let* C_2 *be a* $\mathbf{C}$*-circle of center* $M_2 = (r_2, \theta_2, t_2)$ *and radius* R_2*. For any point* M *in* C_2 *(see notations in Prop. 3.6)*

$$|h + 4S| - 2dR_2 \leq \sqrt{1 + 4r_1^2}\, d(M, P(M_1)) \leq |h + 4S| + 2dR_2$$

Proof. For $M = (x, y, z)$, we have $d(M, P(M_1))\sqrt{1 + 4r_1^2} = |z - t_1 + 2r_1(\cos(\theta_1)y - \sin(\theta_1)x)|$. Here $x = r_2\cos(\theta_2) + R\cos(\varphi), y = r_2\sin(\theta_2) + R\sin(\varphi), z = t_2 + 2r_2R\sin(\varphi - \theta_2)$ so

$$d(M, P(M_1)) = \frac{1}{\sqrt{1 + 4r_1^2}}|t_2 - t_1 + 2r_1r_2\sin(\theta_2 - \theta_1) + 2R(r_2\sin(\varphi - \theta_2) - r_1\sin(\varphi - \theta_1))|$$

But $|r_2\sin(\varphi - \theta_2) - r_1\sin(\varphi - \theta_1)| \leq d$ so the announced result. □

Lemma 6.2 (Upper Chimney) *Let* $M_0 = (0, \theta_0, \sin(2\theta_0)$ *and* $M_2 = (r_2, \theta_2, \sin(2\theta_2))$ *with* $-\pi/4 < \theta_1 < \theta_2 < \pi/4$ *and let* $r_2 = a\sin(\theta_2 - \theta_0)$*. Then the axis* $r(\theta) = a\sin(\theta - \theta_0), \theta_0 \leq \theta \leq \theta_2$ *is a good axis when* $r_2^2 < m_2(\theta_0, \theta_2)$*. Furthermore, for* $\theta_0 \leq \theta \neq \theta' \leq \theta_2$*, the* $\mathbf{C}$*-circle* $C(\theta)$ *does not intersect the plane containing the* $\mathbf{C}$*-circle* $C(\theta')$*.*

Proof. Let $M = (r(\theta), \theta, \sin(2\theta))$, $M' = (r(\theta'), \theta', \sin(2\theta'))$ with $r(\theta) = a\sin(\theta - \theta_0)$. We have

$$\begin{aligned} 4d^2R(\theta)^2 - (h + 4S)^2 &= 4a^2\sin(\theta - \theta')^2 \cdot (a^2\sin(\theta - \theta_0)^2 + \cos(2\theta)) \\ &\quad - \left(2\sin(\theta' - \theta)(\cos(\theta' + \theta) + a^2\sin(\theta - \theta_0)\sin(\theta' - \theta_0))\right)^2 \\ &= 4\sin(\theta - \theta')^2 \\ &\quad \left(a^2(a^2\sin(\theta - \theta_0)^2 + \cos(2\theta)) - (\cos(\theta + \theta') + a^2\sin(\theta - \theta_0)\sin(\theta' - \theta_0))^2\right). \end{aligned}$$

If $a^2 < a^2\sin(\theta' - \theta_0)^2 + \cos(2\theta')$ then $a^2\cos(2\theta_0) < 1$ and

$$\begin{aligned} &a^2(a^2\sin(\theta - \theta_0)^2 + \cos(2\theta)) - (\cos(\theta + \theta') + a^2\sin(\theta - \theta_0)\sin(\theta' - \theta_0))^2 \\ \leq\ &(a^2\sin(\theta' - \theta_0)^2 + \cos(2\theta'))(a^2\sin(\theta - \theta_0)^2 + \cos(2\theta)) - (\cos(\theta + \theta') + a^2\sin(\theta - \theta_0)\sin(\theta' - \theta_0))^2 \\ =\ &(a^2\cos(2\theta_0) - 1)\sin(\theta - \theta')^2 < 0 \end{aligned}$$

For instance we must have $a^2 < \min(\cos(2\theta_0), \cos(2\theta_2)/\cos(\theta_2 - \theta_0)^2) = m_2(\theta_0, \theta_2)/\sin(\theta_2 - \theta_0)^2$.
□

We thus deduce the

Lemma 6.3 (Chimney) *Let* $M_1 = (r_1, \theta_1, \sin(2\theta_1))$ *and* $M_2 = (r_2, \theta_2, \sin(2\theta_2))$ *with* $-\pi/4 < \theta_1 < \theta_2 < \pi/4$. *If* r_1 *and* r_2 *are small enough then there exists a good axis from* M_1 *to* M_2. *Furthermore corresponding* $\mathbf{C}$*-circles are between the contact planes* $P(M_1)$ *at* M_1 *and* $P(M_2)$ *at* M_2.

Proof. As in Lemma 6.2 we build a lower chimney from $M_1 = (r_1, \theta_1, \sin(2\theta_1))$ and $M_0 = (0, \theta_0, \sin(2\theta_0))$ with $-\pi/4 < \theta_1 < \theta_0 < \pi/4$ under the condition:

$$r_1^2 \le m_1(\theta_0, \theta_1) = \min(\cos(2\theta_0), \cos(2\theta_1)/\cos(\theta_1 - \theta_0)^2) \sin(\theta_1 - \theta_0)^2.$$

Let us consider now $\theta_0 = 1/2(\theta_1 + \theta_2)$. If $r_1^2 < m_1(\theta_0, \theta_1)$ and $r_2^2 < m_2(\theta_0, \theta_2)$ the concatenation of the lower and upper chimney gives us the announced result. $\square$

Observe that in this case for any $\mathbf{C}$-circle, we have $R^2 = r^2 + \cos(2\theta) \le \max(r_1^2, r_2^2) + 1$.

Lemma 6.4 *Let* $\mathbf{R}_1 = \mathbf{R}_{r_1,\theta_1,\alpha_1}$ *and* $\mathbf{R}_2 = \mathbf{R}_{r_2,\theta_2,\alpha_2}$ *be two infinite* $\mathbf{R}$*-circles and let* C_1, C_2 *be two* $\mathbf{C}$*-circles centered at* M_1 *and* M_2 *with radii* R_1 *and* R_2. *Let* d *be the horizontal distance between* M_1 *and* M_2. *If*

$$\frac{2d}{|\sin(\alpha_1 - \alpha_2)|}\left(|r_1| + |r_2| + 2\max(R_1, R_2) + \frac{2d}{|\sin(\alpha_1 - \alpha_2)|}\right) \le |\sin(2\theta_2 - 2\theta_1)|$$

then there exist an invariant surface S_1 *of* $\mathbf{R}_1$ *and an invariant surface* S_2 *for* $\mathbf{R}_2$ *that do not intersect.*

Proof. Let $M = (r, \theta, 0)$ the intersection point of projections of $\mathbf{R}_1$ and $\mathbf{R}_2$. It is also the projection of N_1 and N_2. We have

$$r\sin(\theta - \alpha_1) = r_1 \sin(\theta_1 - \alpha_1), r\sin(\theta - \alpha_2) = r_2 \sin(\theta_2 - \alpha_2).$$

$$\begin{aligned} d(N_1, M_1) &= \left|\frac{r_2\sin(\theta_2 - \alpha_2) - r_1\sin(\theta_1 - \alpha_2)}{\sin(\alpha_1 - \alpha_2)}\right| \le \frac{d}{|\sin(\alpha_1 - \alpha_2)|}, \\ d(N_2, M_2) &= \left|\frac{r_2\sin(\theta_2 - \alpha_1) - r_1\sin(\theta_1 - \alpha_1)}{\sin(\alpha_1 - \alpha_2)}\right| \le \frac{d}{|\sin(\alpha_1 - \alpha_2)|}. \end{aligned}$$

Consider the family of $\mathbf{C}$-circles centered at $Q_1 = (r, \theta, Z(\theta)) \in [M_1, N_1] \subset \mathbf{R}_1$, of *radius* $R_1 + \lambda_1 d(Q_1, M_1) \le R$. If $\lambda_1 > 1$ then this family is not linked because projections of $\mathbf{C}$-circles are not. For any point Q_1 of this family of $\mathbf{C}$-circles we have

$$d(Q_1, P(M_1)) \le 2d(N_1, M_1)(R_1 + \lambda_1 d(Q_1, M_1)) \le 2\frac{dR}{|\sin(\alpha_1 - \alpha_2)|}.$$

Consider the family of $\mathbf{C}$-circles centered at $Q_2 = (r, \theta, Z(\theta)) \in [M_2, N_2] \subset \mathbf{R}_2$. of *radius* $R_2 + \lambda_2 d(Q_2, M_2) \le R$ with $\lambda_2 > 1$. For any point Q_2 of this family. We have (in $d(Q_2, P(M_1))$)

$$\begin{aligned} |h| - |t_2 - t_1| &\le 2\,|r_2|\,|\sin(\alpha_2 - \theta_2)|\,d(N_2, M_2) \\ |4S| &\le 2\,|r_1|\,|\sin(\alpha_1 - \theta_1)|\,d(Q_2, M_1) \end{aligned}$$

so

$$d(Q_2, P(M_1)) \geq |t_2 - t_1| - 2\frac{(|r_1| + |r_2| + R)d}{|\sin(\alpha_1 - \alpha_2)|}.$$

We choose λ_1 and λ_2 such that $\lambda_i > 1$ and

$$R = R_1 + \lambda_1 d(N_1, M_1) = R_2 + \lambda_2 d(N_2, M_2) < \max(R_1, R_2) + d/\left|\sin(\alpha_1 - \alpha_2)\right|.$$

We then have $d(Q_1, P(M_1)) < d(Q_2, P(M_2))$ for any points $(Q_1, Q_2) \in S_1 \times S_2$ □

Proposition 6.5 (Non-intersecting invariant surfaces) *Let* $\mathbf{R}_1 = \mathbf{R}_{r_1,\theta_1,\alpha_1}$ *and* $\mathbf{R}_2 = \mathbf{R}_{r_2,\theta_2,\alpha_2}$ *be two infinite* $\mathbf{R}$*-circles with* $\sin(\alpha_1 - \alpha_2) \neq 0$*. If* r_1 *and* r_2 *are small enough then there exist 3 invariant surfaces* S_0, S_1, S_2 *for corresponding inversions* I_0, I_1, I_2*.*

Proof. Let $\theta_0 = 1/2(\theta_1 + \theta_2)$. Suppose we have $r_1^2 \leq m_1(\theta_0, \theta_1)$ and $r_2^2 \leq m_2(\theta_0, \theta_2)$.

1. We first build an invariant surface S_0, using the chimney lemma. Its axis is given by the concatenation of the two axis:

$$r(\theta) = r_1\frac{\sin(\theta_0 - \theta)}{\sin(\theta_0 - \theta_1)}, \theta_1 \leq \theta \leq \theta_0, r(\theta) = r_2\frac{\sin(\theta - \theta_0)}{\sin(\theta_2 - \theta_0)}, \theta_0 \leq \theta \leq \theta_2,$$

Radii of corresponding $\mathbf{C}$-circle are bounded by $\sqrt{1 + r_1^2}$ and $\sqrt{1 + r_2^2}$ respectively. S_0 is inside the cylinder $d(M, M_1) = 1 + 2r_1 = R_1$ and the cylinder $d(M, M_2) = 1 + 2r_2 = R_2$.

2. We then build a family of concentric $\mathbf{C}$-circles at M_1 with *radius* from $\sqrt{r_1^2 + \cos(2\theta_1)}$ to R_1. This family of $\mathbf{C}$-circle is part of S_1 and is a subset of $P(M_1)$ so does not intersect S_0. In the same way we build a family of concentric $\mathbf{C}$-circles with *radius* from $\sqrt{r_2^2 + \cos(2\theta_2)}$ to R_2.

If $|r_1 r_2 \sin(\theta_1 - \theta_2)| + d(1 + 2\max(|r_1|, |r_2|)) < 1/2\left|\sin(2\theta_1) - \sin(2\theta_2)\right|$, these family do not intersect.

3. Using the Lemma 6.4 we then build two family of invariant $\mathbf{C}$-circles for I_1 and I_2 that do not intersect if

$$\frac{2d}{|\sin(\alpha_1 - \alpha_2)|}\left(|r_1| + |r_2| + 2 + 4\max(|r_1|, |r_2|) + \frac{2d}{|\sin(\alpha_1 - \alpha_2)|}\right) < \left|\sin(2\theta_1 - 2\theta_2)\right|.$$

All the $\mathbf{C}$-circles we have already built are inside the cylinder $d(M_0, M) = R$.

4. We then consider the two concentric families of $\mathbf{C}$-circle centered at N_1 and N_2 with growing *radius* from R to infinity. There are lying in the two parallel contact planes at N_1 and N_2. □

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