A Volume function for Spherical CR tetrahedra

Elisha Falbel
Institut de Mathématiques
Université Pierre et Marie Curie
4, place Jussieu
F-75252 Paris
e-mail: `falbelmath.jussieu.fr`

Abstract

We define a volume function on configurations of four points in the sphere S^3 which is invariant under the action of $PU(2,1)$, the automorphism group of the CR structure defined on S^3 by its embedding in $\mathbb{C}^2$. We show that the volume function, constructed using appropriate combinations of the dilogathm function of Bloch-Wigner, satisfies a five term relation in a more general context which includes at the same time CR and real hyperbolic geometry.

1 Introduction

The volume of ideal tetrahedra in real three dimensional hyperbolic space is a fundamental geometric invariant. One can interpret it as a function defined on ordered quadruples of distinct points in the Riemann sphere $\mathbb{C}P^1$. It is invariant by the diagonal action of $PSL(2,\mathbb{C})$ and it satisfies a 5 term relation. It defines an element of $H^3_{cont}(PSL(2,\mathbb{C}),\mathbb{R})$ (the continuous group cohomology of $PSL(2,\mathbb{C})$).

The goal of this paper is to define a volume function defined on ordered quadruples of points in a more general context which includes at the same time real hyperbolic geometry and CR geometry. In particular it is defined for ordered quadruples of points in the sphere $S^3 \subset \mathbb{C}^2$. It is invariant under the diagonal action of $PU(2,1)$ on the configuration of points and satisfies a 5 term relation. Although $H^3_{cont}(PU(2,1),\mathbb{R}) = 0$ and therefore, in that case, this volume is a coboundary it is sufficiently interesting to be singled out.

Recall that ordered triples of (pairwise distinct) points in $\mathbb{C}$ are classified, up to similarity, by a complex parameter $z \in \mathbb{C} \setminus \{0,1\}$. A natural compactification of this space, being obtained collapsing pairs of points, is identified to $\mathbb{C}P^1$.

More precisely, to each vertex of an ordered triple $p_1, p_2, p_3 \in \mathbb{C}$ we associate a coordinate: $z_1 = \frac{p_3-p_1}{p_2-p_1}, z_2 = \frac{p_1-p_2}{p_3-p_2}, z_3 = \frac{p_2-p_3}{p_1-p_3}$. They satisfy

$$z_2 = \frac{1}{1-z_1}, \quad z_3 = 1 - \frac{1}{z_1}.$$

One can chose one of the coordinates (say $z_1 \in \mathbb{C} \setminus \{0,1\}$) to parametrize the configuration up to similarity.

The main idea of this paper is that each point in a configuration of four points might see the other three points as forming an Euclidean triangle. The mean value of the Bloch-Wigner dilogarithm of the invariants of those four triangles satisfies a five term relation if natural compatibilities are imposed between the triangles (up to similarity) associated to a configuration of five points.

For a compact manifold the total volume is a function on an algebraic variety of "geometric structures" given by invariants z_{ij} (satisfying certain compatibility conditions) associated to simplices of a triangulation. The volume at a hyperbolic structure coincides with the hyperbolic volume and the volume at closed spherical CR structure is always null. I thank Riccardo Benedetti, Herbert Gangl, Juliette Genzmer, Julien Marché, John Parker, Luc Pirio, Qingxue Wang and Pierre Will for the discussions leading to that paper.

2 Tetrahedra with cross-ratio structures

Consider a set of four elements $\Delta = \{p_1, p_2, p_3, p_4\}$. We call p_i, $1 \leq i \leq 4$ the vertices of Δ. Let $O\Delta$ be the set of all orderings of Δ. We will denote an element of $O\Delta$ by $[p_i, p_j, p_k, p_l]$ (where $\{i, j, k, l\} = \{1, 2, 3, 4\}$) and call it a simplex although we only deal with configurations of four points. Given Δ, there are 24 simplices divided in two classes $O\Delta^+$ (containing $[p_1, p_2, p_3, p_4]$) and Δ^- (containing $[p_1, p_2, p_4, p_3]$) of 12 elements each. Each class is an orbit of the even permutation group acting on $O\Delta$.

The following definition assigns similarity invariants to each vertex of a configuration of four points. That is, to each vertex, it is assigned a triangle in $\mathbb{C}$, up to similarity, defined by coordinates as in the introduction.

Definition 2.1 *A cross-ratio structure on a set of four points $\Delta = \{p_1, p_2, p_3, p_4\}$ is a function defined on the ordered quadruples*

$$\mathbf{X} : O\Delta \to \mathbb{C} \setminus \{0, 1\}$$

satisfying, if (i, j, k, l) is any permutation of $(1, 2, 3, 4)$, the relations

1.
$$\mathbf{X}(p_i, p_j, p_k, p_l) = \frac{1}{\mathbf{X}(p_i, p_j, p_l, p_k)}.$$

2. (similarity relations)
$$\mathbf{X}(p_i, p_j, p_k, p_l) = \frac{1}{1 - \mathbf{X}(p_i, p_l, p_j, p_k)}.$$

Remarks

1. To visualize the definition we refer to Figure 1. For each $[p_i, p_j, p_k, p_l] \in O\Delta^+$ we define

$$z_{ij} = \mathbf{X}(p_i, p_j, p_k, p_l).$$

   We interpret z_{ij} as a cross-ratio associated to the edge $[ij]$ at the vertex i. Cross-ratios of elements of $O\Delta^-$ are obtained taking inverses by the first symmetry.

2. A cross-ratio structure defined on Δ is a point in the variety in $(\mathbb{C} \setminus \{0, 1\})^{12}$ with coordinates z_{ij}, $1 \leq i \neq j \leq 4$ defined by the usual similarity constraints: if (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$ then

$$z_{ik} = \frac{1}{1 - z_{ij}}.$$

3. The similarity relations can be used to reduce the number of variables to four, one for each vertex. One can use, for instance, $(z_{12}, z_{21}, z_{34}, z_{43}) \in (\mathbb{C} \setminus \{0, 1\})^4$.

In the following we will denote by a sequence of numbers $ijkl$ the corresponding invariant $\mathbf{X}(u_i, u_j, u_k, u_l)$. Given a simplex $[u_1, u_2, u_3, u_4]$, the twelve coordinates of a cross-ratio structure introduced above can be listed as follows

$$\begin{pmatrix} z_{12} \\ z_{13} \\ z_{14} \\ z_{21} \\ z_{24} \\ z_{23} \\ z_{34} \\ z_{31} \\ z_{32} \\ z_{43} \\ z_{42} \\ z_{41} \end{pmatrix} = \begin{pmatrix} 1234 \\ 1342 \\ 1423 \\ 2143 \\ 2431 \\ 2314 \\ 3412 \\ 3124 \\ 3241 \\ 4321 \\ 4213 \\ 4132 \end{pmatrix}$$

By the similarity relations, the cross-ratio structure is defined by $z_{12}, z_{21}, z_{34}, z_{43}$ so it is convenient to use the following notation.

Definition 2.2

$$[[u_1, u_2, u_3, u_4]] = \begin{pmatrix} z_{12} \\ z_{21} \\ z_{34} \\ z_{43} \end{pmatrix} = \begin{pmatrix} 1234 \\ 2143 \\ 3412 \\ 4321 \end{pmatrix}$$

2.1 Hyperbolic configurations

This section is not used in the next sections. It identifies a subset of cross-ratio structures closely related to real hyperbolic ideal tetrahedra. I thank J. Genzmer (see [Ge] for more details) for correcting an earlier version of it.

Proposition 2.3 *The complex algebraic variety in* $(\mathbb{C}^* \setminus \{1\})^{12}$ *with coordinates* z_{ij}, $1 \le i \ne j \le 4$, *defined by, for* (i, j, k, l) *an even permutation of* $(1, 2, 3, 4)$, *the usual similarity constraints*

$$z_{ik} = \frac{1}{1 - z_{ij}}$$

and the three complex equations

$$z_{ij} z_{ji} = z_{kl} z_{lk} \tag{1}$$

has two irreducible components:

- *One branch is parametrised by* $(z_{12}, z_{21}) \in (\mathbb{C}^* \setminus \{1\})^2$*:*

$$z_{34} = -z_{12} \frac{1 - z_{21}}{1 - z_{12}} \quad z_{43} = -z_{21} \frac{1 - z_{12}}{1 - z_{21}}.$$

- *The other branch is parametrised by* $\mathbb{C}^* \setminus \{1\}$*:*

$$z_{12} = z_{21} = z_{34} = z_{43}.$$

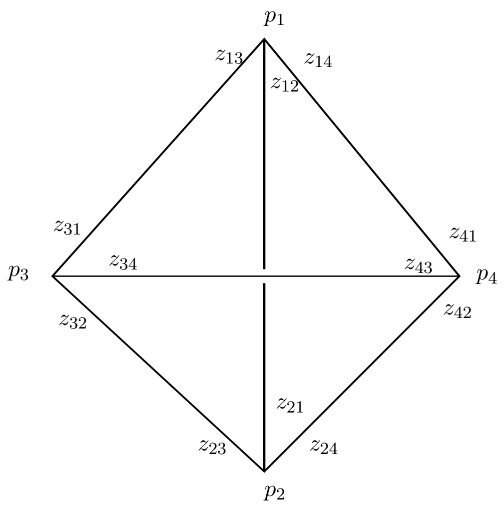

Figure 1: *Parameters for a cross-ratio structure*

Proof. By the previous remarks the cross-ratio variety defined by the similarity constraints is parametrized by $(z_{12}, z_{21}, z_{34}, z_{43}) \in (\mathbb{C} \setminus \{0,1\})^4$. It suffices now to solve equations 1 in these coordinates. The equations are

$$z_{12}z_{21} = z_{34}z_{43}$$

$$(1-z_{12})(1-z_{34}) = (1-z_{21})(1-z_{43})$$

$$(1-z_{12})(1-z_{43})z_{21}z_{32} = (1-z_{21})(1-z_{34})z_{12}z_{43}$$

and dividing the last by the second and the first, we obtain

$$(1-z_{12})^2 z_{34}^2 = (1-z_{21})^2 z_{12}^2$$

so

$$z_{34} = \pm z_{12}\frac{1-z_{21}}{1-z_{12}}.$$

This equation and the first one give then

$$z_{43} = \pm z_{21}\frac{1-z_{12}}{1-z_{21}}.$$

Substituting the solution $z_{34} = z_{12}\frac{1-z_{21}}{1-z_{12}}$, $z_{43} = z_{21}\frac{1-z_{12}}{1-z_{21}}$ back in the second equation we obtain that $z_{12} = z_{21} = z_{34} = z_{43}$. On the other hand the solution $z_{34} = -z_{12}\frac{1-z_{21}}{1-z_{12}}$, $z_{43} = -z_{21}\frac{1-z_{12}}{1-z_{21}}$ satisfies the second equation without constraints. $\square$

Note that the branch

$$z_{34} = -z_{12}\frac{1-z_{21}}{1-z_{12}} \quad z_{43} = -z_{21}\frac{1-z_{12}}{1-z_{21}}$$

has the property that the coeficients

$$z_1 = z_{12}z_{21},\ z_2 = z_{31}z_{13},\ z_3 = z_{14}z_{41}$$

satisfy the similarity conditions

$$z_2 = \frac{1}{1-z_1} \ \text{ and } z_3 = \frac{1}{1-z_2}.$$

There exists, for each hyperbolic configuration z_1, z_2, z_3, a $\mathbb{C} \setminus \{0,1\}$-parameter lift in the complex variety $(\mathbb{C} \setminus \{0,1\})^{12}$. In particular, the configurations $z_1^{1/2} = z_{12} = z_{21} = -z_{34} = -z_{43}$ are in 2-1 correspondence to ideal hyperbolic configurations.

On the other hand the branch given by $z_{12} = z_{21} = z_{34} = z_{43}$ satisfy the condition

$$z_1z_2z_3 = 1$$

which should be opposed to $z_1z_2z_3 = -1$ in the hyperbolic case. Part of these configurations can be interpreted geometrically. Namely, the real points ($z_{12} \in \mathbb{R}^* \setminus \{1\}$) parametrise configurations of points, up to the action of $PU(2,1)$, in $S^3 \in \mathbb{C}^2$, contained in an $\mathbb{R}$-circle (cf. Lemma 3 in [W1] and section 6).

Remark: Define the map $(\mathbb{C} \setminus \{0,1\})^{12} \to (\mathbb{C} \setminus \{0\})^6$ by taking $a_{ij} = z_{ij}z_{ji}$. Its image is of complex codimension 2 and outside the hyperbolic configurations the map is injective onto its image. On the other hand, the fiber above an ideal hyperbolic configuration is $\mathbb{C} \setminus \{0,1\}$ as computed in the proposition above.

3 Triangulations

Let T be an ideal triangulation of a 3-manifold. By this we mean a simplicial complex whose underlying topological space is a manifold if the vertices are deleted. Let $\mathbf{X}(p_i, p_j, p_k, p_l)$ be a function defined on the simplices.

We impose the following compatibility conditions:

1. Edge compatibility: If $[p_i, p_j, p_{m_0}, p_{m_1}], [p_i, p_j, p_{m_1}, p_{m_2}], \cdots, [p_i, p_j, p_{m_n}, p_{m_0}]$ are simplices having the edge $[p_i, p_j]$ in common then

$$\mathbf{X}(p_i, p_j, p_{m_0}, p_{m_1}) \cdots \mathbf{X}(p_i, p_j, p_{m_n}, p_{m_0}) = 1$$

2. Face compatibility: If $[p_i, p_j, p_k, p_l]$ and $[p_{i'}, p_j, p_k, p_l]$ are two simplices with a common face $[p_j, p_k, p_l]$ then

$$\mathbf{X}(p_j, p_i, p_k, p_l)\mathbf{X}(p_k, p_i, p_l, p_j)\mathbf{X}(p_l, p_i, p_j, p_k) = \mathbf{X}(p_j, p_{i'}, p_k, p_l)\mathbf{X}(p_k, p_{i'}, p_l, p_j)\mathbf{X}(p_l, p_{i'}, p_j, p_k)$$

Remarks

1. A more constrained definition of cross-ratio structure associated to a triangulation is to give a function $\mathbf{X}(p_i, p_j, p_k, p_l)$ defined on all configurations of four distinct vertices in the complex, not necessarily in the same simplex. The cross-ratios are then defined on the product of four copies of the 0-skeleton. The compatibility conditions are more difficult to verify in that case but are equivalent to the previous definition in the case of hyperbolic and CR geometry because the cross-ratios define actual vertices in an appropriate model ($\mathbb{C}P^1$ and S^3 respectively) and conversely any four points in the model define a simplex with cross-ratios defined by these points. One can define an even more constrained condition, which is clearly satisfied for both geometries, by imposing that we can add one vertex to the 0-skeleton of the complex keeping the compatibility relations.

   In the following section we impose constraints on a triangulation in order that, once compatibility conditions are verified for a triangulation, they are valid for other triangulations obtained through Pachner moves.

2. The definition here should be compared with cross-ratio coordinates as in [F]. The relation is that the coordinates, say $\mathbf{Y}$, defined in [F] are given by

$$\mathbf{Y}(p_i, p_j, p_k, p_l) = \mathbf{X}(p_i, p_j, p_k, p_l)\mathbf{X}(p_j, p_i, p_l, p_k).$$

   The definition of a $\mathbf{T}$-structure is essentially the same but the computations seem to be more natural. As it will be shown bellow, the relation can be interpreted as a blow-up of cross-ratio coordinates along ideal real hyperbolic tetrahedra.

4 Configurations of five points

The goal of this section is to obtain all relations between cross-ratios obtained from choosing four among five points. These relations will be used to prove a five term relation satisfied by the dilogarithm in the next section.

In order to simplify certain formulae, we sometimes denote $\mathbf{X}(p_i, p_j, p_k, p_l)$ by $(ijkl)$ or simply by $ijkl$. So, we are looking for all relations between all quadruples $(ijkl)$ with pairwise distinct i, j, k, l chosen among $1, 2, 3, 4, 5$. We describe the relations by explicitly writing the following formal sum with a minimal set of independent variables

$$[[u_1, u_2, u_3, u_4]] - [[u_1, u_2, u_3, u_5]] + [[u_1, u_2, u_4, u_5]] - [[u_1, u_3, u_4, u_5]] + [[u_2, u_3, u_4, u_5]]$$
$$= \begin{pmatrix} 1234 \\ 2143 \\ 3412 \\ 4321 \end{pmatrix} - \begin{pmatrix} 1235 \\ 2153 \\ 3512 \\ 5321 \end{pmatrix} + \begin{pmatrix} 1245 \\ 2154 \\ 4512 \\ 5421 \end{pmatrix} - \begin{pmatrix} 1345 \\ 3154 \\ 4513 \\ 5431 \end{pmatrix} + \begin{pmatrix} 2345 \\ 3254 \\ 4523 \\ 5432 \end{pmatrix}$$

Let $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$.

We impose the following relations:

1. The edge compatibility conditions

$$(ijkl) = (ijkm)(ijml).$$

2. The face compatibility conditions

$$(ijkl)(ljik)(kjli) = (imkl)(lmik)(kmli).$$

The edge equations correspond to (oriented) pairs (ij) of points. That makes 20 equations not all of them independent. The number of face equations correspond to (non-oriented) pairs $\{j,m\}$ of points (opposite to each face). That makes 10 equations not all of them independent.

In principle there are 4 independent invariants for each configuration of four points. Given five points, there are then $4\times 5 = 20$ invariants arranged in five columns.

Using the compatibility conditions we may establish relations between the columns. In Proposition 4.1 we first relate the cross-ratio invariants using only the edge compatibilities. In fact, the invariants of two sets of configurations of four points determine those of the other three as is shown in Proposition 4.2 by adding the face compatibilities. The proofs are straightforward computations with the compatibility relations.

Proposition 4.1 *If the only relations between the cross-ratios are the edge compatibilities, the space of cross-ratios of a configuration of five points is of dimension 10. A possible set of coordinates is*

$$\{x_1, x_2, x_3, y_1, y_2, y_3, z_3, z_4, w_3, w_4\}:$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \frac{z_3(1-w_3)}{w_3(1-z_3)} \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \frac{1-w_4}{1-z_4} \end{pmatrix} + \begin{pmatrix} \frac{y_1}{x_1} \\ \frac{y_2}{x_2} \\ \frac{w_3}{z_3} \\ \frac{w_4}{z_4} \end{pmatrix} - \begin{pmatrix} \frac{1-y_1}{1-x_1} \\ \frac{(1-y_3)}{(1-x_3)} \\ w_3 \\ w_4 \end{pmatrix} + \begin{pmatrix} \frac{x_2(1-y_2)}{y_2(1-x_2)} \\ \frac{x_3(1-y_3)}{y_3(1-x_3)} \\ z_3 \\ z_4 \end{pmatrix}$$

Proof. The proof follows writing all edge compatibility relations. For instance, the 4th element in the first column is (4321). We have

$$(4321) = (4325)(4351) = \frac{1}{1-\frac{1}{z_3}}(1-\frac{1}{w_3}) = \frac{z_3(1-w_3)}{w_3(1-z_3)}.$$

Also, the 4th element in the second column is (5321). We have

$$(5321) = (5324)(5341) = \frac{1}{1-z_4}(1-w_4).$$

The other terms are obtained similarly. □

Proposition 4.2 *If the only relations between the cross-ratios are edge and face compatibilities, the space of cross-ratios of a configuration of five points is of dimension 7. A possible set of coordinates is*

$$\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}:$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} \frac{y_1}{x_1} \\ \frac{y_2}{x_2} \\ \frac{w_3}{z_3} \\ \frac{w_4}{z_4} \end{pmatrix} - \begin{pmatrix} \frac{1-y_1}{1-x_1} \\ \frac{(1-y_3)}{(1-x_3)} \\ w_3 \\ w_4 \end{pmatrix} + \begin{pmatrix} \frac{x_2(1-y_2)}{y_2(1-x_2)} \\ \frac{x_3(1-y_3)}{y_3(1-x_3)} \\ z_3 \\ z_4 \end{pmatrix}$$

with

$$y_3 = \frac{y_1x_3(y_2-1)(x_1-1)}{x_1(y_1-1)(x_2-1)}.$$

$$z_3 = \frac{x_4(-y_1x_1y_2 - x_1x_2 + y_1x_1x_2 + x_1 + y_1y_2 - y_1)}{y_1(x_4-1)(x_2-y_2)(x_1-1)}$$

$$w_3 = \frac{-y_1x_1y_2 - x_1x_2 + y_1x_1x_2 + x_1 + y_1y_2 - y_1}{(x_4-1)(x_1-y_1)(x_2-1)}$$

$$z_4 = \frac{x_1(y_4-1)(y_1-1)(x_2-y_2)}{y_4(-y_1x_1y_2 - x_1x_2 + y_1x_1x_2 + x_1 + y_1y_2 - y_1)}.$$

$$w_4 = \frac{(y_4-1)(y_2-1)(x_1-y_1)}{-y_1x_1y_2 - x_1x_2 + y_1x_1x_2 + x_1 + y_1y_2 - y_1}.$$

PROOF. It remains to use the face compatibilities:

1. From

$$(1524)(4512)(2541) = (1324)(4312)(2341)$$

we get

$$\frac{(x_1-y_1)x_2}{y_1(z_3x_4 - x_4 - z_3)(-y_2+x_2)} = -\frac{x_2(-1+x_1)}{(x_2-1)x_4}$$

Therefore

$$z_3 = \frac{x_4(-y_1x_1y_2 - x_1x_2 + y_1x_1x_2 + x_1 + y_1y_2 - y_1)}{y_1(x_4-1)(x_2-y_2)(x_1-1)}.$$

Substituting the above value for z_3 in $x_4 = \frac{z_3(1-w_3)}{w_3(1-z_3)}$ we obtain

$$w_3 = \frac{-y_1x_1y_2 - x_1x_2 + y_1x_1x_2 + x_1 + y_1y_2 - y_1}{(x_4-1)(x_1-y_1)(x_2-1)}$$

2. Analogously, from

$$(1452)(2415)(5421) = (1352)(2315)(5321)$$

we get

$$z_4 = \frac{x_1(y_4-1)(y_1-1)(x_2-y_2)}{y_4(-y_1x_1y_2 - x_1x_2 + y_1x_1x_2 + x_1 + y_1y_2 - y_1)}.$$

Again, substituting the above value for z_4 in $y_4 = \frac{1-w_4}{1-z_4}$ we obtain

$$w_4 = \frac{(y_4-1)(y_2-1)(x_1-y_1)}{-y_1x_1y_2 - x_1x_2 + y_1x_1x_2 + x_1 + y_1y_2 - y_1}.$$

3. The equation

$$(1523)(3512)(2531) = (1423)(3412)(2431)$$

gives

$$y_3 = \frac{y_1(y_2-1)(x_1-1)x_3}{(y_1-1)(x_2-1)x_1}.$$

4. A computation shows that the other equations don't give any new relations.

□

5 Dilogarithm and volume

In this section we define a volume of a cross-ratio structure on a simplex. For preliminaries on the dilogarithm we refer the reader to [Z]. Consider the function (Bloch-Wigner) (see [B], section 3 of [Z] or formula 19 in [O])

$$D(z) = log|z|arg(1-z) - \operatorname{Im} \int_0^z \frac{log(1-t)}{t} dt$$

which is well defined and analytic on $\mathbb{C} \setminus \{0,1\}$ and extends to a continuous function on $\mathbb{C}P^1$ by defining $D(0) = D(1) = D(\infty) = 0$. It satisfies the 5-term relation (see formula 34 in [O])

$$D(x) - D(y) + D(\frac{y}{x}) - D(\frac{1-y}{1-x}) + D(\frac{1-y^{-1}}{1-x^{-1}}) = 0.$$

There are many equivalent forms of the five term relations. Each one is obtained from the other by a change of coordinates. For instance the five term relation in [Z], formula 4 is

$$D(u) + D(v) + D(\frac{1-u}{1-uv}) + D(1-uv) + D(\frac{1-v}{1-uv}) = 0$$

which can be obtained from the previous by writing $v = 1/y$.

Recall that the Bloch-Wigner function can be interpreted as a volume function on the space of ideal hyperbolic tetrahedra (see section 4 in [Z]). Indeed, $\mathbb{C} \setminus \{0,1\}$ parametrises configurations of four distinct points in $\mathbb{C}P^1$ which is identified to the boundary of real hyperbolic space, $H^3_{\mathbb{R}}$. The convex hull (inside $H^3_{\mathbb{R}}$) of four points in $\mathbb{C}P^1$ with cross-ratio z defines an ideal simplex , up to translations by $PSL(2,\mathbb{C})$, whose volume is $D(z)$.

We will define next a function defined on cross-ratio structures. Associated to a cross-ratio structure are the invariants z_{ij}. Recall that four invariants, one at each vertex, determine the whole set of invariants, so the we might chose $z_1 = z_{12}$, $z_2 = z_{21}$, $z_3 = z_{34}$, $z_4 = z_{43}$. It is reasonable to expect that the following definition will be an analog of the volume of an ideal hyperbolic simplex. But the true reason behind it will be the fact that it satisfies a 5 term relation.

Definition 5.1 *The volume of a cross-ratio structure* $\mathbf{z} = (z_1, z_2, z_3, z_4) \in (\mathbb{C} \setminus \{0,1\})^4$ *on a simplex* $[p_1, p_2, p_3, p_4]$ *is*

$$\mathcal{D}(\mathbf{z}) = D(z_1) + D(z_2) + D(z_3) + D(z_4).$$

Using formula (see [Z], formula 2)

$$D(z) = \frac{1}{2}\left(D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-z^{-1}}{1-\bar{z}^{-1}}\right) + D\left(\frac{1-\bar{z}}{1-z}\right)\right)$$

we obtain (cf. Lemma 2 for hyperbolic geometry in [M])

$$\mathcal{D}(\mathbf{z}) = \frac{1}{2}\sum_{ij} D(e^{2i\theta_{ij}}) = \sum_{ij} \Lambda(\theta_{ij}),$$

where $\theta_{ij} = \arg\, z_{ij}$ and $\Lambda(\theta)$ is Lobachevsky function as defined by [Co, M].

Let T be an ideal triangulation with a cross-ratio structure as above satisfying edge and face compatibilities. We define a function on simplices of the triangulation by using the volume function of a generic tetrahedron. Observe that to each simplex $[p_1, p_2, p_3, p_4]$ we associate four complex coordinates $\mathbf{z} = (x_1, x_2, x_3, x_4) \in (\mathbb{C} \setminus \{0, 1\})^4$. If the cross-ratio structure is fixed we will also write

$$\mathcal{D}([p_1, p_2, p_3, p_4]) = \mathcal{D}(\mathbf{z}) = D(x_1) + D(x_2) + D(x_3) + D(x_4).$$

Theorem 5.2 *The function $\mathcal{D}$ satisfies the 5-term relation:*

$$\mathcal{D}([p_1, p_2, p_3, p_4]) - \mathcal{D}([p_1, p_2, p_3, p_5]) + \mathcal{D}([p_1, p_2, p_3, p_4]) - \mathcal{D}([p_1, p_3, p_4, p_5]) + \mathcal{D}([p_2, p_3, p_4, p_5]) = 0.$$

PROOF. Recall that

$$D(x) - D(y) + D(\frac{y}{x}) - D(\frac{1-y}{1-x}) + D(\frac{1-y^{-1}}{1-x^{-1}}) = 0.$$

It gives rise to the following relations:

$$D(x_1) - D(y_1) + D(\frac{y_1}{x_1}) - D(\frac{1-y_1}{1-x_1}) + D(\frac{1-y_1^{-1}}{1-x_1^{-1}}) = 0,$$

$$D(x_2) - D(y_2) + D(\frac{y_2}{x_2}) - D(\frac{1-y_2}{1-x_2}) + D(\frac{1-y_2^{-1}}{1-x_2^{-1}}) = 0,$$

$$D(x_3) - D(y_3) + D(\frac{y_3}{x_3}) - D(\frac{1-y_3}{1-x_4}) + D(\frac{1-y_3^{-1}}{1-x_3^{-1}}) = 0,$$

$$D(x_4) - D(y_4) + D(\frac{y_4}{x_4}) - D(\frac{1-y_4}{1-x_4}) + D(\frac{1-y_4^{-1}}{1-x_4^{-1}}) = 0.$$

We also have

$$D(\frac{1-w_3^{-1}}{1-z_3^{-1}}) - D(\frac{1-w_3}{1-z_3}) + D(\frac{w_3}{z_3}) - D(w_3) + D(z_3) = 0$$

and

$$D(\frac{1-w_4^{-1}}{1-z_4^{-1}}) - D(\frac{1-w_4}{1-z_4}) + D(\frac{w_4}{z_4}) - D(w_4) + D(z_4) = 0.$$

Comparing these relations to the sum of the lines in the formal sum of Proposition 4 we obtain

$$\mathcal{D}([p_1, p_2, p_3, p_4]) - \mathcal{D}([p_1, p_2, p_3, p_5]) + \mathcal{D}([p_1, p_2, p_3, p_4]) - \mathcal{D}([p_1, p_3, p_4, p_5]) + \mathcal{D}([p_2, p_3, p_4, p_5]) =$$

$$-D(\frac{1-y_1^{-1}}{1-x_1^{-1}}) + D(\frac{1-y_2}{1-x_2}) - D(\frac{y_3}{x_3}) + D(\frac{1-w_3}{1-z_3}) - D(\frac{1-w_4^{-1}}{1-z_4^{-1}})$$

Now, substituting the values of y_3, z_3, w_3, z_4, w_4 in terms of $x_1, x_2, x_3, x_4, y_1, y_2, y_4$ we obtain

$$-D(\frac{1-y_1^{-1}}{1-x_1^{-1}}) + D(\frac{1-y_2}{1-x_2}) - D(\frac{(-1+x_1)(-y_2+x_2)y_1}{(x_1-y_1)(x_2-1)}) + D(\frac{x_1(-1+y_1)(-y_2+x_2)}{(x_1-y_1)(y_2-1)}).$$

Call $a = \frac{1-y_1^{-1}}{1-x_1^{-1}}$ and $b = \frac{1-y_2}{1-x_2}$. Then, a simple calculation gives that the above expression is

$$-(D(a) - D(b) + D(\frac{b}{a}) - D(\frac{1-b}{1-a}) + D(\frac{1-b^{-1}}{1-a^{-1}})) = 0.$$

□

Consider a triangulation of a three manifold with a cross-ratio structure. The volume is obtained by adding over all simplices:

$$Vol = \sum_i \epsilon_i \mathcal{D}(T_i)$$

where the sum is taken over all 3-simplices with a factor $\epsilon = \pm 1$ which is $+1$ if the orientation of the simplex is the same as of the space and -1 if the orientation of the simplex is the opposite. By the five term relation, it does not depend on the triangulation. The volume defines an element in $H^3(M, \mathbb{R})$. See [T, NZ] for the case of real hyperbolic geometry.

Remark: For the hyperbolic configurations, that is, when

$$z_{34} = -z_{12}\frac{1-z_{21}}{1-z_{12}} \quad z_{43} = -z_{21}\frac{1-z_{12}}{1-z_{21}},$$

a simple use of the five term relation shows that

$$D(z_{12}z_{21}) = D(z_{12}) + D(z_{21}) + D(z_{34}) + D(z_{43}).$$

This shows that the function defined above coincides with the usual volume function for ideal hyperbolic tetrahedra.

6 CR geometry (see [BS, G, J])

CR geometry is modeled on the *Heisenberg group* $\mathfrak{N}$, the set of pairs $(z,t) \in \mathbb{C} \times \mathbb{R}$ with the product

$$(z,t)\cdot(z',t') = (z+z', t+t' + 2\mathrm{Im}\, z\overline{z}').$$

The one point compactification of the Heisenberg group, $\overline{\mathfrak{N}}$, of $\mathfrak{N}$ can be interpreted as S^3 which , in turn, can be identified to the boundary of Complex Hyperbolic space.

We consider the group $U(2,1)$ preserving the Hermitian form $\langle z, w\rangle = w^* J z$ defined by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the following subspaces in $\mathbb{C}^3$:

$$V_0 = \left\{ z \in \mathbb{C}^3 - \{0\} \quad : \quad \langle z, z\rangle = 0 \ \right\},$$

$$V_- = \left\{ z \in \mathbb{C}^3 \quad : \quad \langle z, z\rangle < 0 \ \right\}.$$

Let $\mathbb{P} : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}P^2$ be the canonical projection. Then $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_-)$ is the complex hyperbolic space and $S^3 = \mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0)$ can be identified to $\overline{\mathfrak{N}}$.

The group of biholomorphic transformations of $\mathbf{H}^2_{\mathbb{C}}$ is then $PU(2,1)$, the projectivization of $U(2,1)$. It acts on S^3 by CR transformations. We define $\mathbb{C}$-circles as boundaries of complex lines in $\mathbf{H}^2_{\mathbb{C}}$. Analogously, $\mathbb{R}$-circles are boundaries of totally real totally geodesic two dimensional submanifolds in $\mathbf{H}^2_{\mathbb{C}}$. Using the identification $S^3 = \mathfrak{N} \cup \{\infty\}$ one can define alternatively a $\mathbb{C}$-circle as any circle in S^3 which is obtained from the vertical line $\{(0,t)\} \cup \{\infty\}$ in the compactified Heisenberg space by translation by an element of $PU(2,1)$. Analogously, $\mathbb{R}$-circles are all obtained by translations of the horizontal line $\{(x,0)\} \cup \{\infty\}$, $x \in \mathbb{R}$.

A point $p=(z,t)$ in the Heisenberg group and the point ∞ are lifted to the following points in $\mathbb{C}^{2,1}$:

$$\hat{p}=\begin{bmatrix}\frac{-|z|^2+it}{2}\\ z\\ 1\end{bmatrix} \quad \text{and} \quad \hat{\infty}=\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Definition 6.1 *Given any three ordered points p_1, p_2, p_3 in $\partial\mathbf{H}^2_{\mathbb{C}}$ we define Cartan's angular invariant $\mathbb{A}$ as*

$$\mathbb{A}(p_1,p_2,p_3)=\arg(-\langle\hat{p}_1,\hat{p}_2\rangle\langle\hat{p}_2,\hat{p}_3\rangle\langle\hat{p}_3,\hat{p}_1\rangle).$$

The Cartan's angular invariants classifies ordered triples of points in S^3:

Proposition 6.2 ([C], see also[G]) *There exists an element of $PU(2,1)$ which translates an ordered triple of points in S^3 to another if and only if their corresponding Cartan's invariants are equal.*

The CR cross ratio is given by the Koranyi-Reimann invariant introduced in [KR] (see [KR] and [G] for its properties):

Definition 6.3 *The CR cross-ratio associated to four distinct points in S^3 is*

$$KR(p_1,p_2,p_3,p_4)=\frac{\langle p_4,p_2\rangle\langle p_3,p_1\rangle}{\langle p_3,p_2\rangle\langle p_4,p_1\rangle}.$$

Here, we choose lifts for the points p_i which we denote by the same letter. The invariant does not depend on the choice of lifts. The product of the three cross ratios gives the Cartan invariant (see [KR, G])

$$KR(p_1,p_2,p_3,p_4)KR(p_1,p_4,p_2,p_3)KR(p_1,p_3,p_4,p_2)=e^{2i\mathbb{A}(p_2,p_3,p_4)}$$

6.1 Configurations of four points

We refer to Figure 1 to describe the parameters of a tetrahedron (see also [F]). Consider a generic configuration of four (ordered) points in S^3 (any three of them not contained in a $\mathbb{C}$-circle). Fix one of them say p_1 and consider the projective space of complex lines passing through it. Then p_2,p_3,p_4 determine three points t_2,t_3,t_4 on $\mathbb{C}P^1$. The fourth point corresponds to the complex line passing through p_1 and tangent to S^3, call it t_1. The cross-ratio of those four points in $\mathbb{C}P^1$ is $z_{12}=\mathbf{X}(t_1,t_2,t_3,t_4)$ (here, $\mathbf{X}$ is the usual cross-ratio of four points in $\mathbb{C}P^1$). We define analogously the other invariants. If we take $p_1=\infty$, the complex lines passing through p_1 intersect $\mathfrak{N}$ in vertical lines which are then determined by a coordinate in $\mathbb{C}$. Up to Heisenberg translations, we can assume that $p_2=(0,0)$ and $p_3=(1,s_3)$ and $p_2=(z_{12},s_4)$, $s_3,s_4\in\mathbb{R}$. The corresponding points in $\mathbb{C}P^1$ will be $\infty,0,1,z_{12}$. Therefore one "sees" at the vertex p_1 the Euclidean triangle determined by $0,1,z_{12}\in\mathbb{C}$.

We associate to each vertex $i\in[ij]$ inside an edge the invariant $(ijkl)$ where the order k and l is fixed by the right hand with the thumb pointed from j to i. A shortcut notation for the invariants is therefore

$$z_{ij}=(ijkl),$$

the indices kl being determined by the choice ij. It satisfies the relation $(ijlk)=(ijkl)^{-1}$.

They satisfy the following relations:

$$z_{ij}z_{ji}=\overline{z_{kl}z_{lk}}$$

Remarks:

1. An explicit formula for the invariants is given in [W2];

$$(p_1, p_2, p_3, p_4) = \frac{\langle \hat{p}_4, c_{12}\rangle\langle \hat{p}_3, \hat{p}_1\rangle}{\langle \hat{p}_3, c_{12}\rangle\langle \hat{p}_4, \hat{p}_1\rangle},$$

   where $\hat{p}_i$ are lifts of p_i and $c_{12} \in \mathbb{C}^{2,1}$ is a vector orthogonal to the complex plane defined by $\hat{p}_1$ and $\hat{p}_2$.

2. The formulae relating the cross-ratio invariants and those defined above are

$$(p_1, p_2, p_3, p_4) = \frac{KR(p_1, p_2, p_3, p_4)KR(p_1, p_3, p_4, p_2)KR(p_2, p_3, p_1, p_4) + 1}{1 + KR(p_1, p_4, p_2, p_3)(KR(p_4, p_2, p_1, p_3) - 1)}$$

   and conversely

$$KR(p_1, p_2, p_3, p_4) = (p_1, p_2, p_3, p_4)(p_2, p_1, p_4, p_3).$$

3. For other descriptions of configurations of four points in S^3 and their applications we refer to [W1, W2, PP, PP1, FP].

Proposition 6.4 *(cf. [F]) Configurations of four distinct points in S^3 such that any three points are not contained in a $\mathbb{C}$-circle or an $\mathbb{R}$-circle are parametrised by the real algebraic variety in $(\mathbb{C}^* \setminus \{1\})^{12} \setminus (\mathbb{R}^* \setminus \{1\})^{12}$ with coordinates z_{ij}, $1 \leq i \neq j \leq 4$, defined by, for (i, j, k, l) an even permutation of $(1, 2, 3, 4)$, the usual similarity constraints*

$$z_{ik} = \frac{1}{1 - z_{ij}}$$

and the three complex equations

$$z_{ij} z_{ji} = \overline{z_{kl} z_{lk}} \tag{2}$$

Remarks:

1. The real solutions are contained in two different branches (cf. Proposition 2.1) . One parametrises configurations with four points contained in an $\mathbb{R}$-circle. The other branch corresponds to degenerate hyperbolic ideal tetrahedra with four points contained in the boundary of a totally geodesic plane in real hyperbolic space. I thank J. Genzmer for correcting an earlier version which appeared in [F]. For more details see [Ge].

2. In fact, eliminating two variables at each vertex, one can write the six real equations directly in $(\mathbb{C} \setminus \{0, 1\})^4$ with variables $z_{12}, z_{21}, z_{34}, z_{43}$. The equations in these variables are:

$$z_{12} z_{21} = \overline{z_{34} z_{43}}$$

$$\frac{1}{1 - z_{12}} \frac{1}{1 - z_{34}} = \frac{1}{1 - \bar{z}_{21}} \frac{1}{1 - \bar{z}_{43}}$$

$$(1 - \frac{1}{z_{12}})(1 - \frac{1}{z_{43}}) = (1 - \frac{1}{\bar{z}_{34}})(1 - \frac{1}{\bar{z}_{21}})$$

3. Recall that (see [KR, G])

$$KR(p_1,p_2,p_3,p_4)KR(p_1,p_4,p_2,p_3)KR(p_1,p_3,p_4,p_2) = e^{2i\mathbb{A}(p_2,p_3,p_4)}$$

gives Cartan's invariant in terms of cross ratios. One can write then

$$e^{2i\mathbb{A}(p_2,p_3,p_4)} = z_{12}z_{21}z_{14}z_{41}z_{13}z_{31} = -z_{21}z_{41}z_{31}.$$

A common face of two tetrahedra has opposite orientations and as $\mathbb{A}(p_3,p_2,p_4) = -\mathbb{A}(p_2,p_3,p_4)$, the face gluing conditions between tetrahedra with invariants z_{ij} and $w_{i'j'}$ are given by expressions of the form

$$z_{il}z_{jl}z_{kl}w_{i'l'}w_{j'l'}w_{k'l'} = 1.$$

where l and l' correspond to points oposed to the common face. This explains the face compatibility conditions in the CR case.

4. Writing

$$z_{ij} = r_{ij}e^{i\theta_{ij}}$$

we observe that the angles θ_{ij} determine the parameters z_{ij}. The equations defining the possible values of θ_{ij} are:

(a) For each vertex i:

$$\sum_j \theta_{ij} = \pm\pi.$$

(b) Two sets of three CR conditions (there are only four independent equations, two from each set):

$$\theta_{ij} + \theta_{ji} + \theta_{kl} + \theta_{lk} = 0 \ (2\pi)$$

$$r_{ij}r_{ji} = r_{kl}r_{lk}.$$

Using the relations at each vertex of the form $r_{12} = \frac{\sin\theta_{13}}{\sin\theta_{14}}, r_{21} = \frac{\sin\theta_{24}}{\sin\theta_{23}}, \cdots$, we may write the last conditions in terms of angles as

$$\frac{\sin\theta_{13}}{\sin\theta_{14}}\frac{\sin\theta_{24}}{\sin\theta_{23}} = \frac{\sin\theta_{31}}{\sin\theta_{32}}\frac{\sin\theta_{42}}{\sin\theta_{41}}$$

$$\frac{\sin\theta_{14}}{\sin\theta_{12}}\frac{\sin\theta_{32}}{\sin\theta_{34}} = \frac{\sin\theta_{23}}{\sin\theta_{21}}\frac{\sin\theta_{41}}{\sin\theta_{43}}$$

$$\frac{\sin\theta_{12}}{\sin\theta_{13}}\frac{\sin\theta_{43}}{\sin\theta_{42}} = \frac{\sin\theta_{21}}{\sin\theta_{24}}\frac{\sin\theta_{34}}{\sin\theta_{31}}.$$

The last equation is clearly obtained from the first two. There are 12 variables θ_{ij}, 4 equations at each vertex and 4 equations corresponding to the CR conditions. That makes a total of 4 independent parameters.

6.2 The CR volume as a coboundary

As it was pointed out to me by Qingxue Wang the fact that $H^3_{cont}(PU(2,1),\mathbb{R}) = 0$ implies that $\mathcal{D}$ (which can be seen as a measurable 3-cocycle in $PU(2,1)$) is a coboundary and, therefore, the volume function of a CR structure of any closed three manifold is null.

For background on continous cohomology we refer to Lecture 3 in [B1] (a more comprehensive introduction is [Gu]). The vanishing of the continuous cohomology group follows from Van Est theorem that $H^3_{cont}(PU(2,1),\mathbb{R}) = H^3(\mathfrak{g},\mathfrak{u},\mathbb{R})$, where $\mathfrak{g}$ is the Lie algebra of $PU(2,1)$ and $\mathfrak{u}$ the Lie algebra of the maximal compact subgroup $U(2)$. Indeed, write a Cartan decomposition $\mathfrak{g} = \mathfrak{u}+\mathfrak{p}$ and let $\mathfrak{g}_{\mathfrak{u}} = \mathfrak{u}+i\mathfrak{p}$ be another compact form in the complexified Lie algebra. We obtain (cf. Lecture 3 in [B1] or chapter III, 7 in [Gu]) that $H^3(\mathfrak{g},\mathfrak{u},\mathbb{R}) = H^3(\mathfrak{g}_{\mathfrak{u}},\mathfrak{u},\mathbb{R})$ and $H^3(\mathfrak{g}_{\mathfrak{u}},\mathfrak{u},\mathbb{R}) = H^3(\mathbb{C}P^2,\mathbb{R}) = 0$ ($\mathbb{C}P^2$ being the compact symmetric space associated to non-compact symmetric space $H^2_{\mathbb{C}}$).

Continuous cohomology can also be computed using measurable cochains (see [B1]). Fix a point $\infty \in S^3$ and consider the measurable cochain defined outside a set of zero measure in $PU(2,1)^4$, equipped with a Haar measure, by

$$\mathcal{D}(g_1,g_2,g_3,g_4) = \mathcal{D}([g_1\infty, g_2\infty, g_3\infty, g_4\infty]).$$

The set of measure zero, where the 2-cochain is not defined, is the set of quadruples such that the points $g_1\infty, g_2\infty, g_3\infty, g_4\infty$ are either not pairwise distinct or degenerate (three of them belong to a $\mathbb{C}$-circle). Theorem 5.2 is the statement that the measurable cochain is a measurable cocycle.

In the following we will determine $\mathcal{D}$ as a coboundary. Define the measurable 2-cochain in $PU(2,1)$ by

$$c_2(g_1,g_2,g_3) = \frac{1}{2}D(-e^{2i\mathbb{A}(g_1\infty,g_2\infty,g_3\infty)}).$$

Remark: We can also define $\mathcal{D}$ and c_2 as cochains for the simplicial complex defined by a triangulation of a three manifold with a cross-ratio structure. In that case, if $[p_1,p_2,p_3,p_4]$ is a simplex, $\mathcal{D}(p_1,p_2,p_3,p_4) = \mathcal{D}(\mathbf{z})$ as in Definition 5.1 and $c_2(p_2,p_3,p_4) = D(z_{21}z_{31}z_{41})/2$. But in the general case, the volume is not a coboundary. The following proposition is, therefore, special to the CR case.

Proposition 6.5

$$\partial c_2 = \mathcal{D}$$

Proof. Using the definition of a coboundary, we have to prove that

$$D(z_{21}z_{41}z_{31}) + D(z_{12}z_{32}z_{42}) + D(z_{13}z_{23}z_{43}) + D(z_{14}z_{24}z_{34}) = 2(D(z_{12}) + D(z_{21}) + D(z_{34}) + D(z_{43})).$$

$\mathcal{D}$ is a continuous 3-chain defined on generic tetrahedra which extends to degenerate tetrahedra of the form of Figure 2. In that case $\mathcal{D} = D(z)$. Suppose $\mathcal{D}$ is a coboundary. We may suppose that there exists a function $F : U(1) \to \mathbb{R}$ such that

$$F(z_{21}z_{41}z_{31}) + F(z_{12}z_{32}z_{42}) + F(z_{13}z_{23}z_{43}) + F(z_{14}z_{24}z_{34}) = 2(D(z_{12}) + D(z_{21}) + D(z_{34}) + D(z_{43})).$$

That function might be extended to degenerate tetrahedra and in that case we compute, taking limits using the CR constraints,

$$z_{21}z_{41}z_{31} = \frac{1-\bar{z}}{1-z}$$

$$z_{12}z_{32}z_{42} = \frac{1-\frac{1}{z}}{1-\frac{1}{\bar{z}}}$$

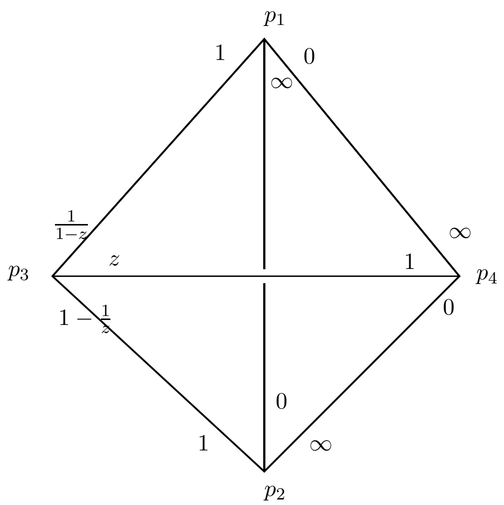

Figure 2: *Parameters for a degenerate tetrahedron*

$$z_{14}z_{24}z_{34} = \frac{z}{\bar{z}}$$

$$z_{13}z_{23}z_{43} = 1.$$

Therefore, fixing $F(1) = 0$, we have

$$F\left(\frac{z}{\bar{z}}\right) + F\left(\frac{1-\bar{z}}{1-z}\right) + F\left(\frac{1-\frac{1}{z}}{1-\frac{1}{\bar{z}}}\right) = 2D(z).$$

Using angle variables we obtain

$$F(e^{2i\theta_1}) + F(e^{2i\theta_2}) + F(e^{2i\theta_3}) = D(e^{2i\theta_1}) + D(e^{2i\theta_2}) + D(e^{2i\theta_3})$$

where $\theta_1 + \theta_2 + \theta_3 = 0\ (\pi)$. Taking derivatives with respect to θ_1 and supposing θ_2 independent we get

$$(F-D)'(e^{2i\theta_1}) - (F-D)'(e^{2i\theta_3}) = 0.$$

As θ_2 is independent we conclude that $(F-D)'$ is constant. In fact, it is null as the functions are periodic. This implies that $F = D$ up to an additive constant which must be zero as $F(1) = D(1) = 0$.

□

References

[B] S. Bloch ; Applications of the dilogarithm function in algebraic K-theory and algebraic geometry, in: Proc. of the International Symp. on Alg. Geom., Tokyo, 1978.

[B1] S. Bloch ; Higher regulators, algebraic K-theory, and zeta functions of elliptic curves. CRM Monograph Series, 11. American Mathematical Society, Providence, RI, 2000.

[BS] D. Burns, S. Shnider ; Spherical Hypersurfaces in Complex Manifolds. Invent. Math. 33 (1976), 223-246.

[C] E. Cartan ; Sur le groupe de la géométrie hypersphérique, Comm. Math. Helv. 4 (1932), 158-171.

[Co] H. S. M. Coxeter ; The functions of Schläfli and Lobatchevsky, Q. J. Math. 6 (1935), 13-29.

[F] E. Falbel ; A spherical CR structure on the complement of the figure eight knot with discrete holonomy. Journal of Differential Geometry 79 (2008) 69-110.

[F1] E. Falbel ; Geometric structures associated to triangulations as fixed point sets of involutions. Topology and its Applications 154 (2007), no. 6, 1041-1052.

[FP] E. Falbel, I. D. Platis ; The $PU(2,1)$ configuration space of four points in S^3 and the cross-ratio variety. Math. Ann. 340 (2008), no. 4, 935–962.

[Ge] J. Genzmer ; On spherical CR structures on 3-manifolds. Thesis, Paris VI (2009) to appear.

[Gu] A. Guichardet ; Cohomologie des groupes topologiques et des algèbres de Lie. Textes Mathématiques 2. CEDIC, Paris, 1980.

[G] W. M. Goldman ; Complex Hyperbolic Geometry. Oxford Mathematical Monographs. Oxford University Press (1999).

[J] H. Jacobowitz ; An Introduction to CR Structures. Mathematical Surveys and Monographs **32**, American Math. Soc. (1990).

[KR] A. Korányi, H. M. Reimann ; The complex cross ratio on the Heisenberg group. Enseign. Math. (2) 33 (1987), no. 3-4, 291–300.

[M] J. Milnor ; Hyperbolic geometry: the first 150 years., Bull. Am. Math. Soc. 6, (1982), 9-24.

[NZ] W. Neumann, D. Zagier ; Volumes of hyperbolic three-manifolds. Topology 24 (1985), no. 3, 307–332.

[O] J. Osterlé ; Polylogarithmes. Séminaire Bourbaki, exp. 762, 1992-1993.

[PP] J. R. Parker, I. D. Platis ; Complex hyperbolic Fenchel-Nielsen coordinates. Topology 47 (2008), no. 2, 101–135.

[PP1] J. R. Parker, I. D. Platis ; Global, geometrical coordinates on Falbel's cross-ratio variety Canadian Mathematical Bulletin 52 (2009) 285-294.

[T] W. Thurston ; The geometry and topology of 3-manifolds. Lecture notes 1979.

[W] P. Will ; The punctured torus and Lagrangian triangle groups in $PU(2,1)$. J. Reine Angew. Math. 602 (2007), 95–121.

[W1] P. Will ; Traces, Cross-ratios and 2-generator Subgroups of PU(2,1). To appear in Can. J. Math.

[W2] P. Will ; Bending Fuchsian representations of fundamental groups of cusped surfaces in PU(2,1). Preprint 2008.

[Z] D. Zagier ; The dilogarithm function. Frontiers in number theory, physics and geometry. II, 3-65, Springer, Berlin, 2007.