MORSE THEORY AND GLOBAL COEXISTENCE OF SINGULARITIES ON WAVE FRONTS

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Abstract. We produce examples of Legendrian isotopy classes in $ST^*\mathbb{R}^n$ (the space of co-oriented contact elements of $\mathbb{R}^n$, endowed with its standard contact structure) such that any element of those classes has a singular (non immersed) *wave front*, i.e the projection to the basis of the fibration $ST^*\mathbb{R}^n \to \mathbb{R}^n$, when restricted to such a Legendrian embedding, has local singularities. Furthermore our examples are such that the underlying Legendrian homotopy classes contain some Legendrian embeddings whose wave front are immersed. These examples are motivated by a question of Arnold about the removability of singularities of wave fronts by means of Legendrian isotopy. The two key points, which are independent interest are :

1. A duality argument which allows to translate the question into a problem about Legendrian submanifolds of the one-jets space of the sphere, and hence to use the technology of *generating families.*
2. Several independent constructions of families of functions on compact manifolds such that the "critical sets" of these families are connected. For example we prove that, given two compact, connected manifolds M and N of dimension bigger than zero, there exists

$$f\ : N\times M\to\mathbb{R}, (x,y)\to f(x,y),\quad x\in N,\quad y\in M$$

such that the equation $\frac{\partial f}{\partial y}(x,y)=0$ is regular and defines a smooth and *connected* submanifold of $M\times N$.

1. Introduction.

The results presented in this paper were motivated by a question of Arnold [Ar0, Ar1] about the removability of singularities of wave fronts by means of Legendrian isotopy. This question was solved (simultaneously and independently) by Misha Entov [En1] and in our note [FP1]. In the present paper, we explicit and generalize some constructions which were only sketched in [FP1] [1]. The methods of [En1] are close to ours (generating families, Morse theory) but nevertheless different. We use a version of projective duality and exploit the relationship between the singularities of a wave front and those of its dual (Section 2 below). We present four natural constructions of families of functions with a connected "critical set" (propositions

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[1]The present paper is an evolution of the preprint [FP2], which already contained the full proofs of the results sketched in the note [FP1]

A, B, C and C' below). Manifolds with this property are a key ingredient to produce examples that answer the question of Arnold.

A co-oriented contact element at some point q of a manifold M is a co-oriented hyperplane in T_qM. The space ST^*M of all co-oriented contact elements of M is the fiberwise spherization of the cotangent bundle T^*M. It has a natural contact structure (i.e. an everywhere maximally non-integrable hyperplane field), defined by the following construction. Denote by $\pi : ST^*M \to M$ the natural projection. A vector X, tangent to ST^*M at some contact element κ, belongs to the contact distribution if $\pi_* X \in TM$ lies in κ. A submanifold which is everywhere tangent to some contact structure is called *isotropic*. Its dimension cannot exceed $n-1$, where $2n-1$ is the dimension of the ambient contact manifold. An isotropic submanifold of maximal dimension is called *Legendrian*. For instance, the fibers of π are Legendrian spheres. The projection $\pi(L)$ of a Legendrian $L \subset ST^*M$ is called the wave front (or simply *the front*) of L. In general, $\pi_{|L}$ is not an immersion. When M is a surface, a generic front is a curve on M whose singularities are semi-cubic cusps and transverse self-intersections.

Definition. Two compact Legendrian submanifolds are said to be *Legendrian isotopic* (resp. *Legendrian homotopic*) if they can be joined by a smooth one parameter family of Legendrian embeddings (resp. immersions).

Let M be a manifold whose universal cover is $\mathbb{R}^n$.

Theorem. There exists an embedded Legendrian submanifold $L \subset ST^*M$ such that

- any L' Legendrian isotopic to L has a non-immersed front : $\pi_{|L'}$ is *not* an immersion.
- there exists L'', Legendrian homotopic ot L, whose front is an immersed (and even embedded) hyper-surface in M.

In the case when M is the plane $\mathbb{R}^2$, this theorem answers a question formulated by Arnold in [Ar0] and in [Ar1], section 11 : Is it possible to deform a singular planar wave front into a smooth immersed curve by means of a Legendrian isotopy of the underlying Legendrian knot ? This theorem implies that the answer is negative, even when some obvious obstructions like the Maslov class are trivial. Remark that it is enough to prove the theorem in the case $M = \mathbb{R}^n$.

A crucial step in the proof (section 3.4 below) can be interpreted as a particular case of the second author's "Sturm-like" theory of the Maslov class (see [Pu1]), but our proof is exposed here in a self-contained way.

In the case when $n = 2$, this theorem contrasts with the following result, proved in [Fe1]: Consider a front on a surface for which there is no homotopical obstruction to cusps cancellation. Then one can achieve cusps cancellation by a Legendrian isotopy *after stabilization* : assume that one can remove the singularities of the front of some Legendrian knot L by a deformation through Legendrian immersions. Then, after "connected sum"[2] with sufficiently many copies of the same standard Legendrian knot, on can cancel the singularities of the front of L by a deformation of L through Legendrian embeddings.

See [AG] for all the basic results of contact topology used throughout this paper without proof.

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2. A preliminary construction.

The following construction is a contact-geometric version of the theory of support functions of convex bodies in the Euclidean space.

2.1. **One-jets spaces.** Given some compact manifold N, denote by $J^1(N, \mathbb{R}) = T^*N \times \mathbb{R}$ the space of one-jets of functions on the N. It has a natural contact structure, induced by the contact form $du - pdq$ where pdq stands for the canonical one-form (the Liouville form) of T^*N, and u denotes a coordinate along the $\mathbb{R}$ factor.

Denote by σ the projection $J^1(S^{n-1}, \mathbb{R}) \to J^0(S^{n-1}, \mathbb{R}) = S^{n-1} \times \mathbb{R}$. This projection has Legendrian fibers with respect to the above contact structure. We describe below a classical contactomorphism between $ST^*(\mathbb{R}^n)$ and $J^1(S^{n-1}, \mathbb{R})$, endowed with their natural contact structures.

2.2. Choose an origin in $\mathbb{R}^n$. Identify $ST^*\mathbb{R}^n$ with $\mathbb{R}^n \times S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n = T_0^*\mathbb{R}^n$. Denote by (Q, θ) a point in $\mathbb{R}^n \times S^{n-1}$, with $Q \in \mathbb{R}^n$ and θ a unit vector, and by $\langle \cdot \, , \, \cdot \rangle$ the standard scalar product of $\mathbb{R}^n$. The one-form $\langle \theta, dQ \rangle$ induces the contact structure on $ST^*\mathbb{R}^n$. Using the standard metric of the Euclidean sphere to identify its tangent and cotangent bundles, a point in $J^1(S^{n-1}, \mathbb{R})$ can be identified with (θ, p, u), where p is a (co)vector (co)tangent to the sphere at θ, and u is a real number. In these coordinates, the standard contact form of $J^1(S^{n-1}, \mathbb{R})$ is $du - pd\theta$.

Proposition. Denote by H the mapping defined by the following formula:

$$ST^*\mathbb{R}^n \to J^1(S^{n-1}, \mathbb{R})$$

$$(Q, \theta) \to (\theta, p = Q_{\theta^\perp}, u = \langle \theta, Q \rangle)$$

where $Q_{\theta^\perp}$ denotes the projection of the vector Q to the hyperplane tangent to the sphere at θ. Then, H conjugates the two standard contact forms :

$$H^*(du - pd\theta) = \langle \theta, dQ \rangle$$

[2] Legendrian connected sum in $ST^*\mathbb{R}^2$ has to be defined properly.

Proof. It is a direct computation. □

Using these notations, this mapping H has an inverse given by the formula:

$$J^1(S^{n-1}, \mathbb{R}) \to ST^*\mathbb{R}^n$$

$$(\theta, p, u) \to (Q = u \cdot \theta + p, \theta)$$

From now on, *we identify these two contact manifolds by means of this contactomorphism.* Our contact manifold is now equipped with two Legendrian fibrations, π and σ. At any point of the contact manifold, the tangent spaces to the two fibers are transversal in the contact hyperplane, and hence generate the contact hyperplane.

Terminology. Given a Legendrian submanifold $L \subset ST^*(\mathbb{R}^n) \equiv J^1(S^{n-1}, \mathbb{R})$, we call $\sigma(L)$ its 0*-jet front* and $\pi(L)$ its *Euclidean front.* These two fronts are said to be *dual* to each other.

3. The Maslov class.

Consider a Legendrian submanifold $L \subset ST^*\mathbb{R}^n \equiv J^1(S^{n-1}, \mathbb{R})$. Denote by C_σ (resp. C_π) the set of points in the neighborhood of which $\sigma_{|L}$ (resp. $\pi_{|L}$) is not an immersion. We want to show that there exists some L with the following properties: i.) it is impossible to deform L through Legendrian embeddings such that the subset C_π of this deformed submanifold becomes empty, ii.) it is possible to achieve this by a deformation through Legendrian immersions.

For L in general position with respect to the projections π and σ, C_π and C_σ are co-dimension-one singular subsets of L (see [AG]). We will call *generic* the Legendrian submanifolds having this property.

The smooth part of C_σ has a natural co orientation which we recall below in 3.3. Intersection of C_σ with loops in L induces a well defined class in $H^1(L, \mathbb{Z})$, the Maslov class. This co-homology class is an invariant of Legendrian immersions.

3.1. **The strategy of the proof of the main theorem.** Given a generic Legendrian submanifold $L \subset ST^*\mathbb{R}^n$, we will define a function ind_{C_σ} defined on paths $\gamma : [0,1] \to L$ with endpoints outside of C_σ. The construction will be such that, for any path $\gamma : [0,1] \to L \setminus C_\pi$ with endpoints outside C_σ, $|ind_{C_\sigma}(\gamma)|$ is not bigger than $n-1$ (See the proposition in section 3.4 below).

A second step in the proof is to show that there exists some Legendrian isotopy classes in $J^1(S^{n-1}, \mathbb{R})$ such that for any generic L belonging to them, there exists a path $\gamma : [0,1] \to L$ with endpoints outside of C_σ, such that $ind_{C_\sigma}(\gamma)$ is not smaller than n. Several independent constructions of such Legendrian isotopy classes are given in section 4.

Together, these two inequalities imply that C_π is non-empty and, hence, that there exists Legendrian isotopy classes all of those generic representatives have a non-immersed Euclidean front. But immersions are stable, so non-generic representatives also have a non-immersed Euclidean front.

Remark. Loosely speaking, the first step described above has the following cohomological interpretation : Using intersection duality, 1-cochains can be associated to the singular sets C_π and C_σ. These two 1-cochains happen to be cocycles. Furthermore they both induce the same class in $H^1(L, \mathbb{Z})$. Their difference is the boundary of an explicit 0-cochain whose variation is less than n. However, the formal definition of these cochains is cumbersome, and can be avoided for our purposes, as shown below.

3.2. **Generating families.** Let N and V be manifolds without boundary, not necessarily closed. A *generating family* is some function $F : N \times V \to \mathbb{R}$ to which we associate the subset $L_F \subset J^1(N, \mathbb{R})$ defined by the following parameterization:

$$L_F = \{(u,q,p) \mid \exists v \in V \text{ such that } u = F(q,v)\ ,\ p = \frac{\partial F}{\partial q}(q,v)\ ,\ \frac{\partial F}{\partial v}(q,v) = 0\}$$

where (q,p) abusively denotes an element of T^*N. If $\frac{\partial F}{\partial v}$ is transversal over 0, then L_F is an immersed (generically embedded) Legendrian submanifold. Locally, any Legendrian submanifold $L \subset J^1(N, \mathbb{R})$ is induced by such a generating family: For any point $P \in L$, there exists an *auxiliary manifold* V and a function $F : N \times V \to \mathbb{R}$ such that L coincides with L_F near P.

3.3. **The natural coorientation of C_σ.** It is given by the following construction. Consider a path $\gamma : [0,1] \to L$ with endpoints outside of C_σ , and which is completely contained in some open ball where L is identical to some L_F. The piece of C_σ which is contained in this ball corresponds to the (θ, v) such that the matrix $\frac{\partial^2 F}{\partial v^2}(\theta, v)$ is singular. The intersection between C_σ and γ is defined by the formula

$$ind_{C_\sigma}(\gamma) = ind_v(F, \gamma(1)) - ind_v(F, \gamma(0)),$$

where $ind_v(F,x)$ is the index of the quadratic form $\frac{\partial^2 F}{\partial v^2}(\theta,v)$ at the point (θ,v) which parameterizes $x \in L$. This difference is independent of the choice of the generating family F (see [AG]).

Remark. If the Legendrian submanifold L is globally defined by a single generating family[3], the formula above makes sense not only for the "small" paths as above, but also for any path γ with endpoints outside of C_σ.

Assume now that L is generic, so that C_σ is of codimension one. A path $\gamma : [0,1] \to L$ with endpoints outside of C_σ can be homotoped relatively to its endpoints to a path having only a finite number of intersections with C_σ. The integer $ind_{C_\sigma}(\gamma)$ is defined to be the sum of the local contributions of each of the intersection points, computed as above. In this way $ind_{C_\sigma}(\gamma)$ is (well) defined for all paths $\gamma : [0,1] \to L$ with endpoints outside of C_σ, provided that L is generic.

3.4. **Estimation from above of $|ind_{C_\sigma}(\gamma)|$ for paths γ in $L \setminus C_\pi$.** Denote by W a connected component of the open set $L \setminus C_\pi$.

Proposition. Assume that a path γ takes its values in W. Then $|ind_{C_\sigma}(\gamma)| \leq n-1$.

Proof. Denote by $F : S^{n-1} \times W \to \mathbb{R}$ the smooth function defined by $F(\theta,w) = \langle \pi(w), \theta \rangle$. Denote by $\mathcal{L}$ the Legendrian submanifold of $J^1(S^{n-1},\mathbb{R}) \equiv ST^*\mathbb{R}^n$ induced by F as in 3.2.

Claim. W is a connected component of $\mathcal{L}$.

The proposition follows from this claim: The "auxiliary manifold" of the generating family F is the open manifold W, whose dimension $n-1$. Hence the index of the quadratic form $\frac{\partial^2 F}{\partial w^2}(\theta,w)$ belongs to $\{0,\dots,n-1\}$. Since $ind_{C_\sigma}(\gamma)$ is by definition the difference of two such indices, the inequality follows.

To prove the claim, observe that the restriction of π to W is locally an immersion. The condition $\frac{\partial F}{\partial w}(\theta,w) = 0$ selects those $(\theta,w) \in S^{n-1} \times W$ such that θ, when interpreted as unit vector in $\mathbb{R}^n$, is orthogonal to the hyperplane $D\pi_w(T_wW) \subset T_{\pi(w)}\mathbb{R}^n$. On the other hand, a Legendrian lift of $\pi(W)$ (W itself is such a lift) is nothing but a continuous choice of one of these two vectors at each w. Hence $W \subset \mathcal{L}$. $\square$

Suppose that a generic Legendrian submanifold is such that one can find a path γ with end-points outside of C_σ such that $|ind_{C_\sigma}(\gamma)| \geq n$. Then the above proposition imply that C_π must be non-empty. In the next section, several constructions of Legendrian submanifolds satisfying this hypothesis are proposed.

[3]it will be the case of all the Legendrian submanifolds considered below

4. Constructions of Legendrian submanifolds

The following theorem is essentially due to Chekanov. As formulated below, the formal proof can be found in [Pu2], and is implicit in several other papers where proofs of Chekanov's theorem are given, such as [Cha], [Fe2], [EG]. Consider a generating family $F : S^{n-1} \times V \to \mathbb{R}$, where V is a *closed* manifold, and denote by $L_F \subset J^1(S^{n-1}, \mathbb{R})$ the Legendrian submanifold induced by F as in 3.2.

Theorem. If $L_t \subset J^1(S^{n-1}, \mathbb{R}), t \in [0,1]$ is a smooth one parameter family of Legendrian embeddings such that $L_0 = L_F$, then there exists k and a one parameter family of functions $F_t : S^{n-1} \times V \times \mathbb{R}^{2k} \to \mathbb{R}, t \in [0,1]$ such that $L_t = L_{F_t}$. Furthermore one can assume that F_t is equal to a fixed quadratic form of index k of the vector variable outside a compact set. $\square$

The Morse inequalities for these functions which are standard at infinity reads as follows:

Lemma. Let $H : V \times \mathbb{R}^{2k} \to \mathbb{R}$ be a Morse function which is equal to a quadratic form of index k outside a compact set. The number of critical points of index $k+i$ is not less than the dimension of $H^i(V, \mathbb{Z}/2\mathbb{Z})$. $\square$

This lemma is proved in [CZ] in the case when V is the torus T^n. The same argument works for any compact manifold V.

Proposition. Assume that L_F is connected. Then on any generic L Legendrian isotopic to L_F there exists a path γ such that $ind_{C_\sigma} = dim(V)$.

Proof. By the theorem above, there exists a one-parameter family of functions $F_t : S^{n-1} \times V \times \mathbb{R}^{2k} \to \mathbb{R}, t \in [0,1]$ which induces a path $L_t, t \in [0,1]$ of embedded Legendrian submanifolds such that $L_0 = L_F$ and $L_1 = L$. For a generic $\theta \in S^{n-1}$, the Morse function $F_1(\theta, \ .\ )$ has a critical point a_0 of index k and a critical point $a_{\dim(V)}$ of index $k + \dim(V)$. These two critical points correspond to two points A_0 and $A_{\dim(V)}$ on $L \setminus C_\sigma$. Since L is connected, one can join these two points by a path γ, which is such that $|ind_{C_\sigma}(\gamma)| = dim(V)$. $\square$

Remark. The definition 3.2 of generating families implies that the Legendrian submanifold associated to a generating family $F : S^{n-1} \times V \to \mathbb{R}$ is parametrized by the *critical set* of this family, that is the subset $\frac{\partial f}{\partial y}(x, y) = 0$ of $S^{n-1} \times V$, which is a smooth $n-1$-dimensional submanifold according to the transversality assumption.

To prove the main theorem, families of functions on the sphere with a connected critical set have to be constructed. This is the goal of the next propositions:

Proposition A. There exists a closed manifold V of dimension n and a generic function $F : S^{n-1} \times V \to \mathbb{R}$ such that L_F is a *connected* embedded Legendrian submanifold of $J^1(S^{n-1}, \mathbb{R})$ which is *Legendrian homotopic* to a Legendrian submanifold whose Euclidean front is *embedded.*

Proposition A, proved in 4.2, implies the theorem. We present in 4.3 a different approach which also provides another proof of the theorem in the case $n = 2$:

Proposition B. For any $k > 0$, there exists a function $F : S^1 \times S^k \to \mathbb{R}$ such that $L_F \subset J^1(S^1, \mathbb{R})$ is connected.

In dimension 2 the Maslov class is the only homotopic obstruction to cusp cancellation (see, for example, [Fe1]). It follows from its definition that the Maslov class is trivial for a Legendrian submanifold which is globally parameterized by a generating family. Hence one can deform the Legendrian submanifold whose existence is asserted in proposition B through Legendrian immersions to the effect that the resulting Euclidean front is an immersed curve.

Proposition C. For any $k > 1$ and any connected manifold N such that $\dim N > 1$, there exists a function $F : N \times \mathbb{R}P^k \to \mathbb{R}$ such that L_F is a connected Legendrian submanifold of $J^1(N, \mathbb{R})$.

Proposition C'. For any compact connected manifold W and any connected manifold N such that $\dim N > 0$, there exists a function $F : N \times W \to \mathbb{R}$ such that L_F is a connected Legendrian submanifold of $J^1(N, \mathbb{R})$.

Proposition C is a particular case of proposition C', but it has a very natural independent proof, explained in 4.4. Proposition C' is proved in 4.5.

Using another construction, Entov has given other examples of fronts in $J^0(S^2, \mathbb{R})$ with "a lot of" singularities (see [En2], proposition 10.1).

4.1. **A remark about propositions C and C'.** In the case when $N = S^{n-1}$, propositions C and C' give us some L_F for which both the 0-jets front and the Euclidean front have "a lot of" singularities which are not removable by a deformation through Legendrian embeddings. On the other hand, we cannot ensure that these L_F can be deformed through Legendrian immersions to some L' with an immersed Euclidean front.

We can however ensure that our examples are *Legendre-cobordant* to a Legendrian submanifold whose 0-jets front and Euclidean front are both immersed (see [AG] for basic details about Legendrian cobordism). In particular the obstructions to the elimination of singularities found by Vassiliev [Va] vanish for the L_F whose existence is asserted by propositions C and C'.

Proposition. Any Legendrian submanifold of $J^1(S^{n-1}, \mathbb{R})$ which is globally induced by a generating family $F : S^{n-1} \times V \to \mathbb{R}$ is Legendre-cobordant to a Legendrian submanifold L whose 0-jets and Euclidean fronts are both immersed.

Proof. Consider a "constant" generating family $F_1 : S^{n-1} \times V \to \mathbb{R}$ of the form $F_1(q, v) = f_1(v)$, where f_1 is a Morse function without multiple critical values on V.

The 0-jets front of L_{F_1} consists of a number of "parallel" copies of S^{n-1} in $S^{n-1} \times \mathbb{R}$. The corresponding Euclidean front consists in the same number of concentric round spheres provided that f_1 has no vanishing critical value. Consider some function $\widetilde{F} : S^{n-1} \times [0,1] \times V \to \mathbb{R}$ such that $\widetilde{F}(\theta, 0, v) = F(\theta, v)$ and $\widetilde{F}(\theta, 1, v) = F_1(\theta, v)$. One can assume that $\widetilde{F}$ fulfills the transversality condition, so that it induces some embedded Legendrian submanifold $L_{\widetilde{F}} \subset J^1(S^{n-1} \times [0,1], \mathbb{R})$ which realizes a Legendrian cobordism between L_F and L_{F_1}. $\square$

4.2. **Proof of the proposition A.**

4.2.1. *Apparent contours.* Let V and M be two $n-$dimensional manifolds. Suppose that V is immersed in $M \times \mathbb{R}$. We will abusively identify V with its immersed image in $M \times \mathbb{R}$. Denote by φ the standard projection $\varphi : M \times \mathbb{R} \to M$, and by $\mathcal{C} \subset ST^*(M \times \mathbb{R})$ the set of cooriented contact elements of $M \times \mathbb{R}$ which are tangent to the fiber of φ. Denote by $L_V \subset ST^*(M \times \mathbb{R})$ the set of cooriented contact elements of $M \times \mathbb{R}$ which are tangent to V. Observe that V is covered twice by L_V. It follows from [AG], p. 121, that if L_V is transversal to $\mathcal{C}$, then the image of $L_V \cap \mathcal{C}$ by the natural projection $\mathcal{C} \to ST^*M$ is an immersed (generically embedded) Legendrian submanifold $L \subset ST^*M$. Its front $\pi(L)$ is called the *apparent contour* of V. This terminology comes from the fact that, when $n = 2$ and M is the plane, this front is the usual apparent contour of the immersed surface V in $\mathbb{R}^3$. Note that $\pi(L)$ is not a generic front: $\pi_{|L} : L \to \pi(L)$ is 2 to 1. Hence L is connected iff the apparent contour of V is connected and *not* coorientable in $\mathbb{R}^n$.

4.2.2. *Generating families for apparent contours in the Euclidean space.* We keep the same notations, but we specialize to the case when M is the Euclidean space $\mathbb{R}^n$.

Lemma. Suppose that the above transversality assumption holds. The function $F : S^{n-1} \times V$ defined by

$$F : S^{n-1} \times V \to \mathbb{R}$$

$$F(\theta, v) = \langle \varphi(v), \theta \rangle$$

is a generating family for the Legendrian submanifold $L \subset J^1(S^{n-1}, \mathbb{R}) \equiv ST^*\mathbb{R}^n$.

Proof. The condition $\frac{\partial F}{\partial v}(\theta, v) = 0$ implies that we are considering those (θ, v) such that $\langle \varphi_*(X), \theta \rangle = 0$ for all $X \in T_vV$. This implies that $\varphi_{|V}$ is not a submersion at v, because the image of its differential is included in an hyperplane. In other words, the tangent hyperplane to V at v contains the line tangent to the fiber of φ at v, and $\varphi(v)$ lies in the apparent contour of V. $\square$

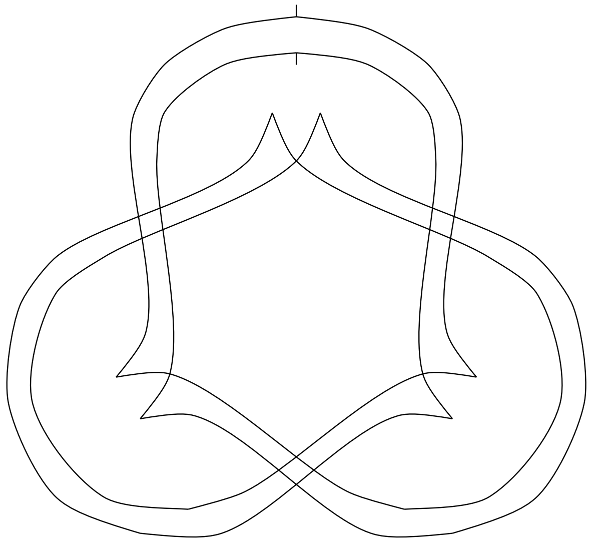

FIGURE 1. A cooriented planar front from which at least two of the cusps cannot be eliminated by means of a Legendrian isotopy.

4.2.3. *Proof of proposition A in the case $n = 2$.* In the planar case, we take advantage of the fact that the number of cusps in a generic apparent contour of a surface has the parity of the Euler characteristic of this surface (see [Mi] for references about this classical subject).

Lemma. There exists an immersion of $\mathbb{R}P^2$ in $\mathbb{R}^2 \times \mathbb{R}$ whose (non-coorientable) apparent contour is connected. Furthermore the corresponding Legendrian submanifold in $ST^*\mathbb{R}^2$ is Legendrian homotopic to a Legendrian submanifold whose Euclidean front is embedded.

Proof. The apparent contour of the Boy immersion [Bo] with respect to the projection along its axis of symmetry is connected and has three cusps. The transversality assumption is fulfilled. The corresponding Legendrian submanifold is the (connected) double cover of this front. The Euclidean front of a slight perturbation of this Legendrian submanifold is shown in fig. 1. Fig 2 details the projection to the plane of a Legendrian homotopy between this Legendrian submanifold and the Legendrian lift of a cooriented circle. In the beginning of this homotopy, the fronts (with an odd number of cusps) are kept non-generic. □

By this approach, to any generic immersion of a surface in $\mathbb{R}^2 \times \mathbb{R}$ whose apparent contour in $\mathbb{R}^2$ is a connected front with an odd number of cusps, one can associate a Legendre isotopy class all of whose representatives have a non-immersed front. Many such immersions exist for any surface with an odd Euler characteristic. Using the Boy immersion, we obtained a planar front with 6 cusps, such that at least two of them are "non-removable". The immersion of $\mathbb{R}P^2$ described in [Mi], whose

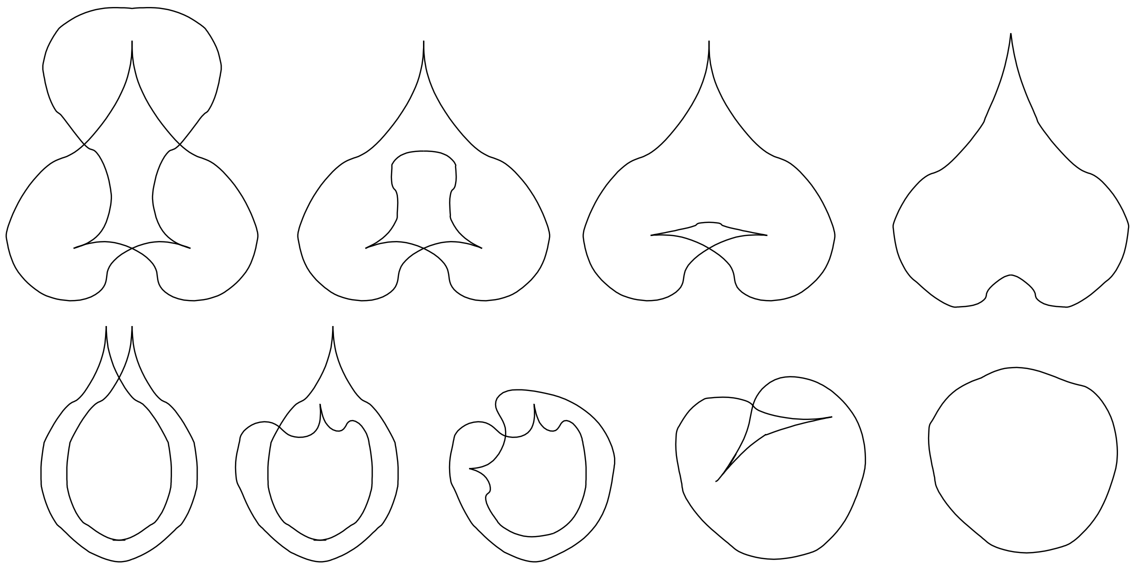

FIGURE 2. From the contour of the Boy surface to a circle.

contour consists in a fairly complicated connected curve with one single cusp, would provide us with an example with only 2 cusps, which are non-removable.

This proves proposition A in the case $n = 2$. The general case follows from this particular case via the following construction.

4.2.4. *Rotating our* 2*-dimensional example.* Consider again an immersion V in $M \times \mathbb{R}$ as in 4.2.1, an arbitrary manifold N, and the trivial product immersion of $V \times N$ in $M \times N \times \mathbb{R}$. If the transversality condition holds for the immersion of V, then it also holds for the product immersion of $V \times N$. The apparent contour of $V \times N$ in $\mathbb{R}^n \times N$ is the product by N of the apparent contour of V in M.

Hence, for any $n > 2$ there exists some immersions of $\mathbb{R}P^2 \times T^{n-2}$ in $\mathbb{R}^2 \times T^{n-2} \times \mathbb{R}$ satisfying the transversality condition, whose apparent contours in $\mathbb{R}^2 \times T^{n-2}$ are connected (T^{n-2} denotes the $(n-2)$-dimensional torus). But $\mathbb{R}^2 \times T^{n-2}$ can be realized as a small tubular neighborhood of a standard codimension 2 torus $T^{n-2} \subset \mathbb{R}^n$. In this way, we obtain immersions of $\mathbb{R}P^2 \times T^{n-2}$ in $\mathbb{R}^n \times \mathbb{R}$ whose apparent contours in $\mathbb{R}^n$ are connected, or, according to 4.2.2, connected Legendrian submanifolds of $J^1(S^{n-1}, \mathbb{R}) \equiv ST^*\mathbb{R}^n$ which are induced by a generating family with auxiliary space $\mathbb{R}P^2 \times T^{n-2}$.

By the same product trick, one can construct in a trivial way from a Legendrian homotopy which eliminates the cusps of the apparent contour of the Boy immersion in the planar case a Legendrian homotopy which eliminates C_π for this higher dimensional example. This proves proposition A, and hence, the main theorem.

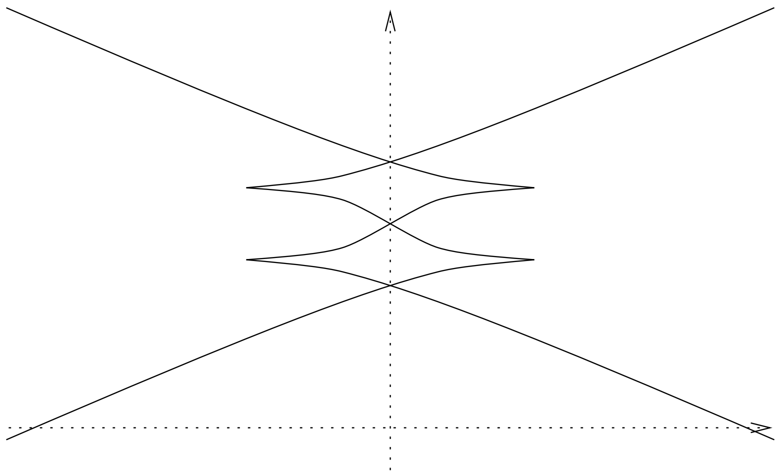

FIGURE 3. The 0-jets front of the Legendrian submanifold of $J^1(\mathbb{R},\mathbb{R})$ induced by G.

4.3. **Proof of the proposition B.** Consider a diagonal matrix $A = diag(r_1, \dots, r_k)$ such that $r_i \neq r_j$ if $i \neq j$, a vector $Y = (y_1, \dots, y_k)$ such that $y_i \neq 0 \ \ \forall i \in \{1, \dots, k\}$, and the function

$$G : \mathbb{R} \times S^{k-1} \to \mathbb{R}$$
$$G(t, X) = \frac{1}{2} \cdot \langle X, AX \rangle + t \cdot \langle Y, X \rangle$$

where the sphere S^{k-1} is seen as the unit sphere of $\mathbb{R}^k$.

Lemma. The function $t \to \#\{dG(t, .) = 0\}$, which associates to the parameter t the number of critical points of the function $G(t, .)$ on the sphere, is decreasing on $[0, \infty)$.

Proof. A critical point of $G(t, .)$ is a unit vector $X \in S^{k-1}$ such that there exists a real number r which satisfies $AX + t \cdot Y = r \cdot X$. That is, $\forall i \in \{1, \dots, k\}$, $r_i x_i + t \cdot y_i = r \cdot x_i$ under the constraint $\sum_{i=1}^{k} x_i^2 = 1$.
Hence critical points of $G(t, .)$ are in one to one correspondence with the solutions of $g(r) = \frac{1}{t^2}$, where the function g is defined by

$$g(r) = \sum_{i=1}^{k} \frac{y_i^2}{(r_i - r)^2}.$$

The fact that g is locally strictly convex ($g'' > 0$) implies the lemma.

For any $i \in [0, \dots, k-1]$, the function $G(0, .)$ has 2 critical points of index i, which are both send to the same critical value. Since critical points die in pairs of consecutive indices, the 0-jets front of the Legendrian submanifold induced by G in $J^1(\mathbb{R},\mathbb{R})$ looks like fig. 3 (in which we have set $k = 2$).

Consider a smooth one parameter family of unit vectors $Y_t,\ t \in (0,1)$ such that $Y_t = Y$ for t close to 0, and $Y_t = -Y$ for t close to 1. The function $H : (0,1) \times S^{k-1} \to \mathbb{R}$ defined by $H(t,X) = \langle X, Y_t \rangle$, viewed as generating family, induces a Legendrian submanifold of $J^1((0,1),\mathbb{R})$ whose 0-jets front is made of two horizontal lines at levels ± 1. In terms of functions over the sphere, the family $H(t,.)$ connects the two ends of the family $G(t,.)$. Hence, by gluing G and H, one obtains a generating family $F : S^1 \times S^{k-1} \to \mathbb{R}$ which induces a connected Legendrian submanifold in $J^1(S^1,\mathbb{R})$. This proves proposition B.

4.4. **Proof of proposition C.** Denote by $S^2\mathbb{R}^{k+1} = \mathbb{R}^{\frac{(k+1)\cdot(k+2)}{2}}$ the space of all quadratic forms on $\mathbb{R}^{k+1}$. Consider a smooth mapping $f : N \to S^2\mathbb{R}^{k+1}$. One can associate canonically to f a function $F : N \times \mathbb{R}P^k \to \mathbb{R}$, since a quadratic form restricted to the unit sphere is symmetric and descends to a function defined on $\mathbb{R}P^k$. The critical points (resp. critical values) of $F(x,.)$ are the eigendirections (resp. eigenvalues) of $f(x)$.

Remark that if f is a constant map with value a quadratic form all of whose eigenvalues are distinct, then the 0-jets front of the corresponding $L_F \subset J^1(N,\mathbb{R})$ is made of $k+1$ "parallel layers" (each layer being a copy of N) in $N \times \mathbb{R}$. We indicate below how to modify locally such an f to the effect that all these layers are connected.

Claim. It is enough to describe a connection model for the case $k = 1$.

Proof. Consider some $f : N \to S^2\mathbb{R}^{k+1}$ of the form $f = f_1 \oplus f_2$, with $f_i : N \to S^2\mathbb{R}^{k_i}$ ($k_1 + k_2 = k+1$). Denote by $F : N \times \mathbb{R}P^k \to \mathbb{R}$ (resp. $F_i : N \times \mathbb{R}P^{k_i - 1} \to \mathbb{R},\ \ i \in \{1,2\}$) the corresponding functions. Assuming that, for all $q \in N$, the eigenvalues of $f_1(q)$ are strictly bigger than those of $f_2(q)$, we have that L_F is the disjoint union of L_{F_1} and L_{F_2}. Hence, in order to connect two layers, it is enough to consider the case $k = 1$ provided that the resulting connecting model between the two layers produces a connected front contained within a small neighborhood of the region of $J^0(N,\mathbb{R})$ delimited by these two layers. □

An element of $S^2\mathbb{R}^2$ can be described by three coordinates (a,b,c) as follows:

$$\begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix}$$

Such a quadratic form has two equal eigenvalues iff $b = c = 0$.

Lemma. If some $f : N \to S^2\mathbb{R}^2$ is transversal to the axis $\{b = c = 0\}$, then the corresponding generating function $F : N \times \mathbb{R}P^2 \to \mathbb{R}$ is such that L_F is an embedded Legendrian submanifold of $J^1(N,\mathbb{R})$.

Proof. The transversality assumption implies that $D = f^{-1}(\{b = c = 0\})$ is a smooth submanifold of N. If q not in D, then L_F is embedded above a neighborhood of q, because in that case $F(q,.)$ is a Morse function. On the other hand, one can describe

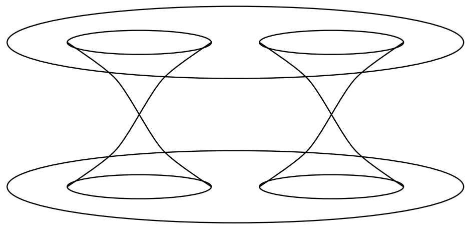

FIGURE 4. The connecting model.

a neighborhood in N of some $q \in D$ by some local coordinates (x, y, z), $(x, y) \in \mathbb{R}^2$, $z \in D$, such that the mapping f can be written

$$f : (x, y, z) \to \begin{pmatrix} a(z) + x & y \\ y & a(z) - x \end{pmatrix},$$

where a is some unknown function of z. The corresponding F is given by the formula

$$F(x, y, z, \theta) = a(z) + x \cdot cos(2 \cdot \theta) + y \cdot sin(2 \cdot \theta),$$

where the point $[cos(\theta) : sin(\theta)]$ of $\mathbb{R}P^1$ is represented by θ. From this explicit formula, a computation shows that L_F is an embedded Legendrian submanifold. This proves the lemma. $\square$

With this lemma, we can proceed to the description of the connecting model: Fix a small open ball $B \subset N$, and a smooth function $\rho : N \to [0, 1]$ such that $B = \rho^{-1}([0, 1[)$ and that $\rho^{-1}(0)$ is non-empty. Fix two constants $\lambda \in \mathbb{R}$ and $\mu \in]0, +\infty[$. Define a smooth $f : N \to S^2\mathbb{R}^2$ by the formula:

$$f(q) = \begin{pmatrix} \lambda & \rho \cdot \mu \\ \rho \cdot \mu & \lambda \end{pmatrix}$$

By construction $f^{-1}(\{b = c = 0\}) = \rho^{-1}(0)$ is non-empty, but this intersection is non-transversal. After a small perturbation inside $\rho^{-1}([0, \epsilon[)$, for some small ϵ, one can assume that $f(N)$ has a non-empty transversal intersection with $\{b = c = 0\}$ along some smooth submanifold $D \subset B$. Furthermore, if the perturbation is small enough, one can assume that the eigenvalues of $f(q)$ lie in $[\lambda - \mu, \lambda + \mu]$ for all $q \in N$. The corresponding L_F is connected: consider some point $q_0 \in D$. Denote by $J^1_{q_0}(N, \mathbb{R})$ the fiber $\sigma^{-1}(\{q_0\} \times \mathbb{R})$ of the one-jets space above q_0 (that is, $J^1_{q_0}(N, \mathbb{R}) = T^*_{q_0}N \times \mathbb{R})$. Observe that since $q_0 \in D$, then $L_F \cap J^1_{q_0}(N, \mathbb{R})$ is diffeomorphic to $\mathbb{R}P^1$, and hence, connected. Consider some point $A \in L_F$ and its projection q on N. Any path γ from q_0 to q which avoids the codimension 2 subset D (except at q_0) can be lifted unambiguously to a path in L_F joining $L_F \cap J^1_{q_0}(N, \mathbb{R})$ to A. This shows that L_F is connected, and this finishes the proof of proposition C.

Remark. The 0-jets front of this L_F is highly non-generic. In the case $k = 1$ the connection looks as in fig. 4 near D.

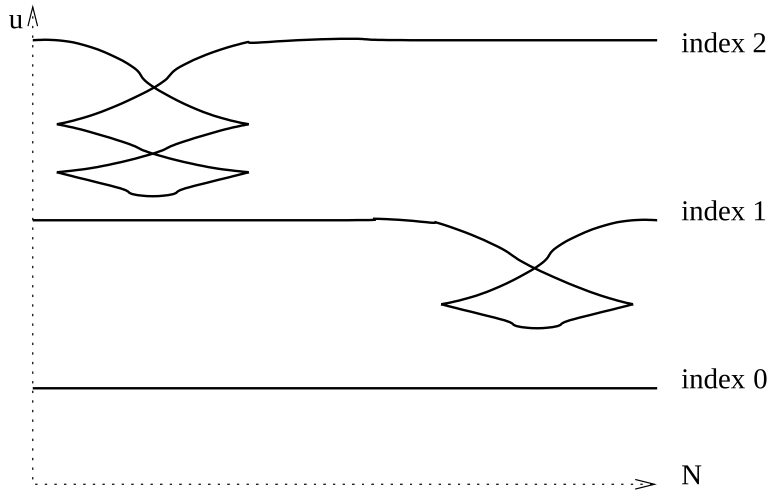

FIGURE 5. Creation of local minima.

4.5. **Proof of proposition C'.** N and W are smooth connected compact manifolds, $\dim N \geq 1$. Let $f : W \to \mathbb{R}$ be a Morse function without multiple critical values, having exactly one minimum $f(w_0)$ at some $w_0 \in W$. Denote by $w_1, \ldots, w_k$ its other critical points, and by $u_k = f(w_k)$ its critical values. Assume further that f is *ordered*, that is that critical points of bigger indices have bigger critical values : $ind(x_l) > ind(x_m) \equiv u_l > u_m$ (such a function exists). Consider the function $G : N \times W \to \mathbb{R}$, defined by $G(q, w) = f(w)$. It is a generating family of a Legendrian manifold $L_G \subset J^1(M)$ which is diffeomorphic to the disjoint union of $k+1$-copies of N : $L_G = L_0 \cup \cdots \cup L_k$. Its components $L_0, \ldots, L_k$ can be numbered according to the following rule : The value of the coordinate u is equal to the constant u_i on L_i. Fix $U_1, \ldots, U_k$ a collection of disjoint open balls in N. Fix a system of open disjoint balls in W, $V_1, \ldots, V_k$, each ball V_i being centered at the point w_i. Denote by $\mathcal{U}$ the union $U_1 \times V_1 \cup \cdots \cup U_N \times V_N$.

Lemma For any $\epsilon > 0$, there exists a smooth compactly supported function h : $\mathcal{U} \times [0, 1] \to \mathbb{R}$ such that:

- $h(., 0) \equiv 0$,
- its C^0-norm $|h|$ is not more than $\varepsilon > 0$,
- For every $t \in [0, 1]$, the function $G_t \ : (q, w) \to G(q, w) + h(q, w, t)$ is a generating family of an embedded Legendrian submanifold $L_{G_t} \subset J^1(N, \mathbb{R})$.
- For every $i = 1, \ldots, k$ there exists a point $q_i \in U_i$ such that the function $f_i(w) = G_1(q_i, w, 1)$ is a Morse function without multiple critical values and with exactly one local minimum in V_i.

The proof of this lemma, which rely on standard methods of singularity theory, is left to the reader. □

Lemma Let $g : W \to \mathbb{R}$ be a Morse function such that there exists two critical points x_1, x_2 of index 0, a critical point a of index 1 and a generic gradient-like vector field

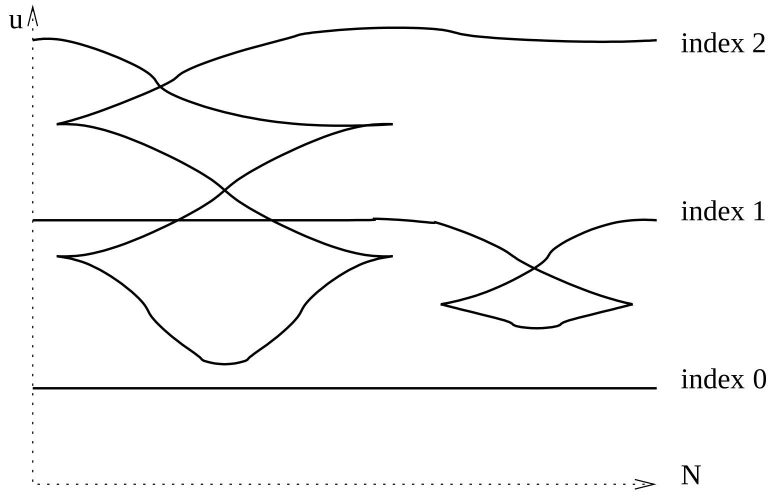

FIGURE 6. Lowering of the local minima.

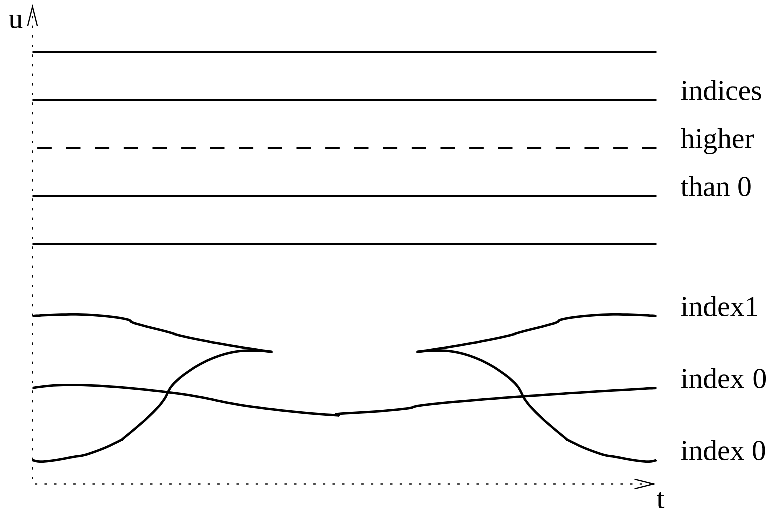

FIGURE 7. Death-birth of a pair of critical points.

for g with two trajectories connecting a with points x_1 and x_2 (these trajectories are necessarily unique since a is of index 1). Then there exists a deformation g_t : $W \times [0,1] \to \mathbb{R}$ of the function $g = g_0 = g_1$ such that the front of the Legendrian submanifold of $J^1([0,1],\mathbb{R})$ induced by the family $g_t,\ t \in [0,1]$ (this front is just the Cerf diagram of this family) has exactly two cusps, the first one corresponding to the cancellation of points a and x_1, the second to the rebirth of these two critical points (see fig. 6). $\square$

Assume first that N is 1−dimensional. After applying the first lemma, one gets a generating family G_1 which induces a front that looks like in fig. 5.

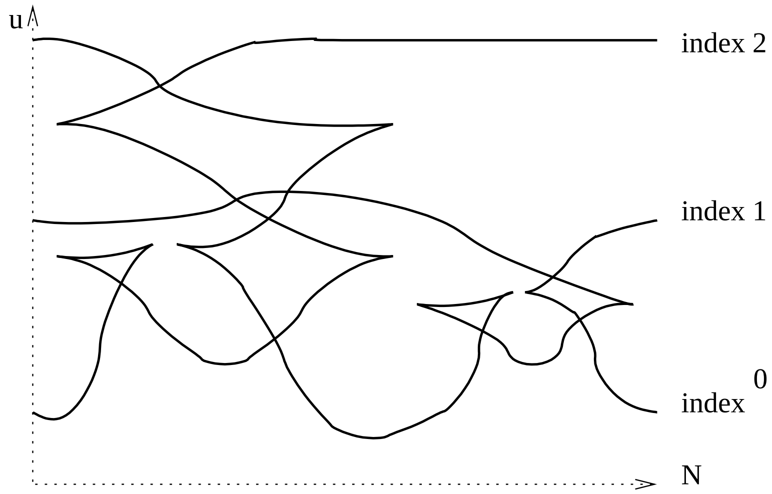

FIGURE 8. Connections of the layers.

Because the function f we started from has ordered critical values, one can deform G_1 to the effect that the local minima created after the preceding step are lowered just above the level of the global minima u_0. The corresponding front then looks like in fig. 6.

Then we are in position to apply the second lemma with each of these lowered minima, the index one critical point just above, and the global minimum. See fig. 7. The resulting Legendrian submanifold is connected.

One can adapt these arguments to the case when N is of dimension bigger than 1, using some rotation tricks similar to the one used in the proof of proposition A.

Remarks. The connected Legendrian submanifolds constructed here have only the most simple singularities (folds) with respect to the projection $J^1(N, \mathbb{R}) -> N \times \mathbb{R}$. Furthermore they are diffeomorphic to the connected sum of $k+1$ copies of N.

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