PASCAL AND SIERPINSKI MATRICES, AND THE THREE STRAND BRAID GROUP

EMMANUEL FERRAND

1. Introduction

In a recent paper *Pascal Matrices* [ES] published in the American Mathematical Monthly, A. Edelman and G. Strang discuss several interpretations of the LU decomposition of the symmetric Pascal matrix. In the last part of their paper, devoted to the functional interpretation of Pascal matrices, they suggest "to look for matrices of Pascal type in a group representation". In this note we follow their idea. We will consider the group presented by two generators, say a and b, and the relation $aba = bab$. This group is known in topology as the *three strand braid group.*

2. Two linear transformations of the space of polynomials

Denote by Φ the endomorphism of the space $\mathbb{C}^n[X]$ of polynomials of degree n with complex coefficients, which maps a polynomial $p(X)$ to the polynomial $p(X+1)$. Denote by Ψ the endomorphism of $\mathbb{C}^n[X]$ which maps a polynomial $p(X)$ to $(1-X)^n p(\frac{X}{1-X})$, which of course is also a polynomial.

Theorem. Φ and Ψ verify a braid-like relation : $\Phi\Psi\Phi = \Psi\Phi\Psi$.

Proof : It is a short computation.

$$\Psi \circ \Phi(p)(X) = (1-X)^n p(\frac{X}{1-X}+1)) = (1-X)^n p(\frac{1}{1-X}))$$

and

$$\Phi \circ \Psi \circ \Phi(p)(X) = (-X)^n p(\frac{1}{1-(X+1)})) = (-X)^n p(\frac{-1}{X}).$$

On the other hand :

$$\Phi \circ \Psi(p)(X) = (-X)^n p(\frac{X+1}{-X})$$

and

$$\Psi \circ \Phi \circ \Psi(p)(X) = (1-X)^n (-\frac{X}{1-X})^n p(\frac{\frac{X}{1-X}+1}{-\frac{X}{1-X}}) = (-X)^n p(\frac{-1}{X}).$$

□

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3. The Pascal triangle

In fact this theorem is not new. Its matrix version, described below, was discovered by Humphries in [Hu]. For any integer n, denote by P the $(n+1)\times(n+1)$ matrix whose entries are $P_{i,j} = \binom{j-1}{i-1}$, by D the diagonal matrix whose entries are $D_{i,j} := (-1)^{i-1}$ if $i=j$ and 0 otherwise, and by A the matrix whose entries are $A_{i,j} := 1$ if $i = n-j+1$ and 0 otherwise.

For $n=3$ $P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

P is the matrix of Φ in the canonical basis of $\mathbb{C}^n[X]$. Since Φ^{-1} sends a polynomial $p(X)$ to the polynomial $p(X-1)$, P^{-1} is the matrix whose entries are $P^{-1}_{i,j} = (-1)^{i+j}\binom{j-1}{i-1}$. In other words,

$$P^{-1} = DPD = DPD^{-1}.$$

Denote by Q the matrix of Ψ in the canonical basis. Since A is the matrix of the map which sends a polynomial $p(X)$ to the polynomial $X^n p(1/X)$, one has that

$$Q = AP^{-1}A.$$

For $n=3$, $Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$.

Q is also a version of Pascal's triangle. It is obtained from P by a "central symmetry", and a "checkerboard" change of sign. By the preceding functional interpretation, we know that P and Q are involved in a braid-like relation :

$$PQP = QPQ$$

These matrices were described in [Hu], where the above identity is proved coefficient-wise by checking the corresponding combinatorial identities.

4. Sierpinski matrices

In this section, we restrict ourselves to the case when $n+1 = 2^k$, for some integer k. Denote by $B(k)$ the $2^k \times 2^k$ matrix with integer coefficients whose entries are $B(k)_{i,j} = 1$ if $\binom{j-1}{i-1}$ is odd and 0 otherwise. Denote by $U \otimes V$ the *Kronecker product* (also called *tensor product*) of two square matrices U and V of size u and v, respectively. It is the square matrix of size uv defined block-wise by

$$P = \begin{pmatrix} U_{1,1}V & \dots & U_{1,n}V \\ \vdots & \ddots & \vdots \\ U_{n,1}V & \dots & U_{n,n}V \end{pmatrix}.$$

One can check that $(U\otimes V)(U'\otimes V') = UU' \otimes VV'$, where U and U' (resp. V and V') are square matrices of size u (resp. v). Denote by $U^{\otimes k}$ the Kronecker product of k copies of U.

The following lemma describes the classical self-similarity property of Pascal's triangle reduced modulo 2. It looks like a Sierpinski gasket, and this suggest the name *Sierpinski matrix of order k* for $B(k)$.

Lemma 2. $B(k) = B(1)^{\otimes k}$

Proof : Consider the $(2^{k-1}+l)$-th column of $B(k)$. Its coefficients are those of the reduction modulo 2 of $(1+X)^{2^{k-1}+l-1}$.

Claim. $(1+X)^{2^{k-1}+l-1}$ is equal to $(1+X)^{l-1} + X^{2^{k-1}}(1+X)^{l-1}$ modulo 2.

This claim implies that

$$B(k) = \begin{pmatrix} B(k-1) & B(k-1) \\ 0 & B(k-1) \end{pmatrix} = B(1) \otimes B(k-1)$$

and lemma 2 follows.

To prove the claim, observe that

$$(1+X)^{2^{k-1}+l-1} - (1+X)^{l-1} - X^{2^{k-1}}(1+X)^{l-1} = (1+X)^{l-1}((1+X)^{2^{k-1}} - (1+X^{2^{k-1}}))$$

and that $((1+X)^{2^{k-1}} - (1+X^{2^{k-1}}))$ has only even coefficients (this last statement can be proved by induction on k). $\square$

Denote by $C(k)$ the matrix $AB(k)^{-1}A$, where A is the anti-diagonal matrix of size $n+1 = 2^k$ defined in the preceding section. Note that $B(k)^{-1} = (B(1)^{-1})^{\otimes k}$ is easy to compute. For $k = 2$ these matrices are :

$$B(2) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C(2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Lemma 2'. $C(k) = C(1)^{\otimes k}$ $\square$

Theorem. The matrices $B(k)$ and $C(k)$ verify a braid-like relation :

$$B(k)C(k)B(k) = C(k)B(k)C(k)$$

Proof. The theorem is true for $k = 1$, by a direct calculation. By lemma 2, lemma 2' and the properties of the Kronecker product, it is true for any k. $\square$

This argument shows that any relation verified by $B(1)$ and $C(1)$ in the group $SL_2(\mathbb{Z})$ of integer matrices with determinant one will also hold for $B(k)$ and $C(k)$. The Pascal triangle modulo 2 $B(k)$ and its cousin $C(k)$ just arise in the k-fold tensor product of the standard representation of $SL_2(\mathbb{Z})$ as the image of two generators of this group.

References

[ES] Edelman, A.; Strang, G. *Pascal matrices* Amer. Math. Monthly 111 (2004), no. 3, 189–197. 15A57

[Hu] Humphries, S. P. *Some linear representations of braid groups* J. Knot Theory Ramifications 9 (2000), no. 3, 341–366.

Institut Fourier, BP 74, 38402 St Martin d'Hères Cedex, France.
E-mail address: emmanuel.ferrand@ujf-grenoble.fr