MOSCOW MATHEMATICAL JOURNAL
Volume 3, Number 3, July–September 2003, Pages 1–10

APPARENT CONTOURS AND THEIR LEGENDRIAN DEFORMATIONS

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Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday

Abstract. After reviewing the notion of apparent contours of a smooth map φ from a compact manifold N to another manifold M, we recall the construction of an associated Legendrian subvariety in the space of contact elements of the goal manifold M and we study various examples. The main result is that, in some sense, non-trivial Legendrian deformations of apparent contours do not exist: In the space of contact elements of a real projective space, the set of the Legendrian submanifolds obtained in this way is closed under Legendrian isotopy.

2000 Math. Subj. Class. Primary 53C15.

Key words and phrases. Contact topology.

1. Introduction

This paper deals with variations on the theme of generating families in symplectic and contact topology. These ideas were initiated by Arnold and motivated by his conjecture about symplectic intersections of Lagrangian submanifolds. This theory, which is the result of the work of many people, is reviewed in an extensive paper by Eliashberg and Gromov [EG]. Most of the current state of the art in this theory is contained in that paper. In particular, Theorem A below is probably already implicit in [EG]. In the classical theory, one has to introduce mappings and functions defined over open manifolds, and hence to impose conditions at infinity. In theorem B below, only compact objects are involved. It is a close relative of Chekanov's theorem [Ch] (see also [Fe]). This paper is self-contained, apart from some references to classical textbooks [Ar], [AG], [AGV].

Throughout this paper, M and N denote smooth manifolds of dimensions m and n, respectively. In addition, N is assumed to be closed (i. e., compact without boundary). The real projective space of dimension k is denoted by P^k. P always denotes the real projective space of dimension m. The projectivization of a non-zero vector v (a non-zero covector p) is denoted by $[v]$ (respectively, $[p]$).

Received July 14, 2002.

A *contact element* of M is the collection of a point $q \in M$ and of an hyperplane in the tangent space of M at q. In other words, the set of all contact elements of M is PT^*M, the projectivized cotangent bundle of M. The tangent bundle of the manifold PT^*M carries a natural sub-bundle of codimension one, the *contact distribution*: Denote by π the natural projection $PT^*M \to M$ and by $D\pi(z)$ its differential at some point $z \in PT^*M$. At z, the natural hyperplane is the pre-image of z by $D\pi(z)$. This makes sense since z can be viewed as an hyperplane in $T_{\pi(z)}M$. This tautological construction is the contact analogue of the construction of the Liouville form in cotangent bundles. A submanifold l of PT^*M is called *Legendrian* if it is everywhere tangent to the contact distribution, and if it has the maximal possible dimension among such integral submanifolds, that is $m-1$. The *wave front* of l is its projection $\pi(l) \subset M$. See [Ar, Ch. 3] for more details.

Consider a smooth map $\varphi\colon N \to M$. Its differential at $x \in N$ will be denoted by $D\varphi(x)\colon T_xN \to T_{\varphi(x)}M$. By a *critical point* of φ, we mean a point $x \in N$ such that the rank of $D\varphi(x)$ is less than m (and hence, if $n < m$, any $x \in N$ is critical). $\varphi(x)$ is then a *critical value* of φ. The *critical set* is the set of all critical values of φ.

Definition. The *Legendrian reduction of N by φ*, denoted by l_φ, is the subset of PT^*M, defined by

$$l_\varphi = \{z \in PT^*M : \exists x \in N,\, \pi(z) = \varphi(x),\, \operatorname{Im} D\varphi(x) \subset z\}.$$

If $D\varphi(x)$ has rank m at x, then there is no z in l_φ corresponding to x. In other words $\pi(l_\varphi)$ is the set of critical values of φ. If y is a critical value of φ, then $l_\varphi \cap \pi^{-1}(y)$ consists in all the hyperplanes of T_yM which contain the image of $D\varphi(x)$, for all the critical points x in $\varphi^{-1}(y)$.

If N is an immersed surface in the three dimensional Euclidian space, and if φ is the projection on the Euclidian plane along some fixed direction, then $\pi(l_\varphi)$ is really the *apparent contour* of N viewed from this direction (assuming that N is somehow transparent). This justifies the title of this paper.

The range of Legendrian submanifolds which arise as Legendrian reductions is fairly large. Several example are given in Section 2. They may help to get acquainted with this definition.

In general l_φ is not smooth, but for an typical φ, it is: Denote by $J^1(N,\, M)$ the space of one-jets of mappings from N to M. Denote by Σ the stratification of $J^1(N,\, M)$ induced by the rank of the differential.

Definition. φ is called a *Morse* mapping if its one-jet extension $j^1\varphi$ is transversal to Σ. It is called *excellent* if in addition, for any two distinct critical points x_1 and x_2 with the same critical value, the images of $D\varphi(x_1)$ and $D\varphi(x_2)$ are transverse subspaces of $T_{\varphi(x_1)}M$.

By Thom's transversality theorem (see [AGV]), excellent mappings form an open dense subset in $\mathcal{C}^\infty(N,\, M)$ with the C^∞-topology.

Theorem A. *If φ is excellent, then l_φ is an* embedded Legendrian *submanifold of PT^*M.*

The proof is given in Section 3.

Denote by $\{\Psi_t\}_{t\in[0,1]}$ a *contact isotopy*, i. e., a smooth one-parameter family of contactomorphisms of PT^*P such that Ψ_0 is the identity. Fix any smooth map $\varphi\colon N \to P$ (l_φ needs not to be embedded or even immersed). The following statement says that the property of being the Legendrian reduction of a closed manifold is preserved by contact isotopies.

Theorem B. *There exists a closed manifold $\mathcal{N}$, and a one parameter family of maps $\Phi_t\colon \mathcal{N} \to P$ such that $\Psi_t(l_\varphi) = l_{\Phi_t}$.*

Nota. $\mathcal{N}$ belongs to a family of manifolds $\{\mathcal{N}^k\}_{k\in\mathbb{Z}}$ whose topology is explicitly controlled. $\mathcal{N}^0$ is N, and $\mathcal{N}^{k+1}$ is a bundle over $\mathcal{N}^k$, with fiber P^{m-1}, whose structure is completely determined by φ.

Sections 5 to 7 are devoted to the proof of this theorem.

Corollary. *If a path-connected component of the space of Legendrian embeddings in PT^*M contains an element of the form l_φ, then any other element of this component is also of this form.*

This is due to the fact that any path in a connected component of the space of Legendrian embeddings in PT^*M can be induced by some contact isotopy (the corresponding contact Hamiltonian can be constructed explicitly. See [AG, Ch. 4]).

2. Examples of Reductions of Closed Manifolds

2.1. Excellent Morse functions. Here N is any closed compact manifold, and φ a smooth function with values in the real line M. Then the definition of Morse maps coincides with the classical definition of Morse functions: φ is Morse iff the graph of its differential is transversal to the zero section of T^*N. If furthermore φ is excellent, then two critical points must have distinct critical values.

2.2. One Point. Here, N is one point, M is arbitrary and $\varphi(N) = y$ is the inclusion of this point in M. Then l_φ is the fiber $\pi^{-1}(y)$. It is an embedded Legendrian submanifold in PT^*M.

2.3. An immersed curve in the plane. Here N is the circle, and M is the projective plane. If φ is an immersion of N into the plane, then φ is Morse. l_φ is the projectivization of the conormal bundle of the image of φ. It is an immersed Legendrian submanifold. φ is excellent if all crossings in its image are transverse. Under this assumption, l_φ is embedded.

2.4. A cusped curve in the plane. Here N is again the circle, and M is still the projective plane P^2. But we assume that φ is an immersion of N everywhere except at some isolated points where φ has non-degenerate cusps (locally equivalent to $t \to (t^2, t^3)$), so that φ is not Morse. At a cusp point, the image of φ is singular, but nevertheless there is a well defined tangent. It is a typical wave front, and there is a well-defined Legendrian lift of this curve. ¿From the definition, it follows that l_φ is the union of this Legendrian lift, and of all the fibers above the cusp points. These pieces intersect tangentially in PT^*P^2.

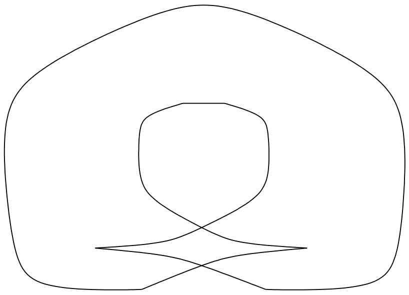

Figure 1. The apparent contour of a Klein Bottle, or of a torus

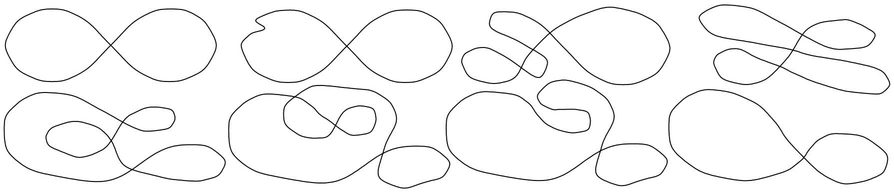

Figure 2. Slices of an immersed cylinder

2.5. Maps from a surface to a surface. In this example N and M are both compact surfaces, and φ is a stable map. According to Whitney, the set of all critical values of φ is a collection of curves in M with non-degenerate cusps and transverse self-intersections (like the curve of Fig. 1). Again this is a typical wave front (possibly the same as in Example 2.4). But in this case, l_φ consists only in the Legendrian lifts of the curves. This shows that the knowledge of the critical set of φ is not sufficient to determine l_φ.

Note that the same Legendrian submanifold might associated to several source manifolds. For example the curve of Fig. 1 is the contour of a Klein bottle and also the contour of a torus, both immersed in $\mathbb{R}^3$ and projected on the plane. To construct an immersed torus whose contour in the plane is that shown in Fig. 1, consider the homotopy of immersed circles of Fig. 2. This defines an immersed cylinder in the three-dimensional "space-time". Glue the two "figure eight" curves at the ends of this cylinder by the trivial gluing, and project the resulting immersed torus along the vertical axis. It is simpler to get a Klein bottle: replace the homotopy of Fig. 2 by a rotation by π of the figure eight.

There exists however many restrictions on the topology of the contours that are reminiscent of the topology of the source manifold (see for example [Hae]).

2.6. A stable mapping between two compact manifolds is excellent. The critical set always lifts to an embedded Legendrian submanifold. However, in many instances, the corresponding l_φ is not in general position with respect to the projection π. For example, a Whitney umbrella is not a stable wave front.

3. Proof of Theorem A

In this section, we suppose that $\varphi\colon N \to M$ is an excellent map, and we show that l_φ is an embedded Legendrian submanifold of PT^*M.

We begin by some local considerations. Assume that φ is a map defined over $\mathbb{R}^n$ with values in $\mathbb{R}^m$. Denote by $\mathcal{M}_{n,m}$ the space of real matrices of width n and height m, and by Δ^k the set of matrices whose rank is exactly k. We look at $D\varphi$ as a map from $\mathbb{R}^n$ to $\mathcal{M}_{n,m}$. Assume that $D\varphi$ is transversal to all the strata Δ^k, $k = 0, \ldots, \min(n, m)$.

Denote by $\mathcal{K} = \{(A, [p]),\, A \in \mathcal{M}_{n,m},\, p \in \mathbb{R}^{m*},\, pA = 0\} \subset \mathcal{M}_{n,m} \times P^{m-1}$. The map: $(A, p) \to pA$ is transversal over $0 \in \mathbb{R}^{n*}$ whenever $p \neq 0$, so that $\{(A, p),\, pA = 0\}$ is a smooth submanifold of $\mathcal{M}_{n,m} \times (\mathbb{R}^{m*} \setminus \{0\})$. Hence $\mathcal{K}$ is a smooth submanifold of $\mathcal{M}_{n,m} \times P^{m-1}$. Denote by $\kappa\colon \mathbb{R}^n \times P^{m-1} \to \mathcal{M}_{n,m} \times P^{m-1}$ the map defined by $\kappa(x, [p]) = (D\varphi(x), [p])$.

Lemma. *If φ is Morse, then κ is transversal to $\mathcal{K}$.*

Proof. Roughly speaking, this follows from the fact that the stratified set $\Delta^0 \cup \Delta^1 \cup \cdots \cup \Delta^{\min(n,m)-1}$ of matrices whose rank is less than $\min(m, n)$ is the apparent contour of $\mathcal{K}$ viewed along the projection $\Pi\colon \mathcal{M}_{n,m} \times P^{m-1} \to \mathcal{M}_{n,m}$. More precisely, consider some $(A, [p]) \in \mathcal{K}$. A belongs to some strata Δ^k, $k \in \{0, \ldots, \min(n, m)\}$. On can check that, if $k < \min(n, m)$, the tangent space to Δ^k at A is the intersection for all $[p] \in \Pi^{-1}(A) \cap \mathcal{K}$ of all the projections of the tangent spaces $T_{(A,[p])}\mathcal{K}$. Hence, if $D\varphi$ is transversal to Δ^k at A, then, for any $[p] \in \Pi^{-1}(A) \cap \mathcal{K}$, κ is transversal to $\mathcal{K}$ at $(A, [p])$. □

In particular, $\mathcal{K}_\varphi = \kappa^{-1}(\mathcal{K})$ is a smooth submanifold of $\mathbb{R}^n \times P^{m-1}$.

Lemma. *The restriction to $\mathcal{K}_\varphi$ of the map $\phi\colon (x, [p]) \to (\varphi(x), [p])$ is a Legendrian immersion into $PT^*\mathbb{R}^m$.*

Proof. We identify $T_{(x,p)}(\mathbb{R}^n \times P^{m-1})$ with the set of all $(X, \wp)$, where $X \in \mathbb{R}^n$, $\wp \in \mathbb{R}^m$, $\wp$ orthogonal to p. Denote by $DD\varphi$ the second differential of φ with respect to our coordinates. The tangent space to $\mathcal{K}_\varphi$ at $(x, [p]) \in \mathbb{R}^n \times P^{m-1}$ consists of those $(X, \wp)$ such that $pDD\varphi(x) \cdot X + \wp D\varphi(x) = 0$. On the other hand, a vector $(X, \wp)$ lies in the kernel of $D\phi(x, [p])$ if $\wp = 0$ and $D\varphi(x) \cdot X = 0$.

Hence, if $(X, \wp) \in Ker D\phi(x, [p]) \cap T_{(x,[p])}\mathcal{K}_\varphi$, then X is such that $D\varphi(x) \cdot X = 0$ and $pDD\varphi(x) \cdot X = 0$. This implies that $X = 0$, otherwise the $(m+n) \times m$-matrix $(pDD\varphi(x), {}^tD\varphi(x))$ would have rank less than m, which would contradict the fact the map $(x, p) \to pD\varphi(x)$ is transversal to 0. Hence, $\phi(\mathcal{K}_\varphi)$ is immersed.

In addition, $\phi(\mathcal{K}_\varphi)$ is Legendrian in $PT^*\mathbb{R}^m$ because

- $pD\varphi(x) = 0$ if $(x, [p]) \in \mathcal{K}_\varphi$, which ensures that $\phi(\mathcal{K}_\varphi)$ is tangent to the contact distribution.
- It has dimension $m - 1$ (since $\mathcal{K}$ has codimension n). □

In the non-local setting, $\mathcal{K}_\varphi$ is defined as follows: There is a natural map from $\varphi^*(T^*M)$ to T^*N. To a couple (x, p), $p \in T^*_{\varphi(x)}M$, associate the covector $\tilde{p} \in T^*_xN$ defined by $\tilde{p}(v) = p(D\varphi(v))$. $\mathcal{K}_\varphi \subset \varphi^*(PT^*M)$ is the projectivization of the preimage of the zero-section of T^*N by this map. According to the local study,

$\mathcal{K}_\varphi$ is an embedded submanifold of $\varphi^*(T^*M)$, and l_φ is an immersed Legendrian submanifold parametrized by $\mathcal{K}_\varphi$. If in addition φ is excellent, then two distinct points in $\mathcal{K}_\varphi$ must have distinct images, so that l_φ is embedded.

4. Projective Duality

The following interpretation of projective duality in terms of contact geometry was made popular by Arnold.

Denote by $P^\vee$ the *dual space* of P, i. e., the set of all projective hyperplanes of P. It is a projective space, whose dual is canonically identified with P. An element of PT^*P can be seen as a couple $(y,\, y^\vee)$, where $y^\vee$ is a projective hyperplane that goes through y. Conversely, an element of $PT^*P^\vee$ can be identified with a couple $(y^\vee,\, y)$, where y is a point in P that belongs to $y^\vee$.

Arnold's contactomorphism. The map $(y,\, y^\vee) \to (y^\vee,\, y)$ induces a contactomorphism between PT^*P and $PT^*P^\vee$.

See [Ar, Ch. 3] for details. Hence PT^*P is endowed with two natural Legendrian fibrations $\pi\colon PT^*P \to P$ and $\pi^\vee\colon PT^*P \to P^\vee$. At each point $z \in PT^*P$, the contact hyperplane at z is the direct sum of the tangent spaces of the two fibers $\pi^{-1}(\pi(z))$ and ${\pi^\vee}^{-1}(\pi^\vee(z))$.

A subset of P and a subset of $P^\vee$ which are projectively dual according to the classical point of view are just the two projections $\pi(l)$ and $\pi^\vee(l)$ of some Legendrian subvariety in $l \subset PT^*P$.

5. Small Deformation of a Fiber

In this section, we explain the central idea of this paper by considering a trivial particular case. Pick some contactomorphism Ψ of PT^*M. Assume that it satisfies the following:

Hypothesis H. *Ψ is such that, for any $y \in P$, the restriction of $\pi^\vee$ to the deformed fiber $\Psi(\pi^{-1}(y))$ is an embedding in $P^\vee$.*

Lemma. *If Ψ is close enough to the identity, then H holds.*

Proof. Consider $A_\Psi = \{(y,\, \pi^{-1}(y))\} \subset P \times PT^*P$. The restriction to A_Ψ of the map $(y,\, z) \to (y,\, \pi^\vee(z))$ is an embedding if Ψ is the identity map. It remains an embedding if Ψ is close enough to it. □

Suppose that N is one point, and that φ is the inclusion $\varphi(N) = y \in M$. Then the image by Ψ of the Legendrian submanifold l_φ, that is of the fiber $\pi^{-1}(y)$, can be identified with l_Φ, where Φ is the embedding of $\pi^{-1}(y) = P^{m-1}$ in $P^\vee$ induced by $\pi^\vee \circ \Psi$. Hence we see that, in this example, *the property of being a Legendrian reduction is preserved by the action of small contactomorphisms at the price of increasing the dimension of the source manifold*. As we shall see later, the fact that Φ does not take its values in the same space as φ can be overcome.

6. Stabilization by Enveloppes of Families of Projective Hyperplanes

This is an elaboration on the preceding example. Consider now any closed manifold N, and any smooth mapping $\varphi\colon N \to P$.

Define $\mathcal{N}$ to be $\varphi^*(PT^*P)$, the pull-back by φ of the bundle $PT^*P \to P$. Denote by Φ the natural map from $\mathcal{N} \to P^\vee$. Φ is called the *stabilization* of φ.

Proposition. $l_\varphi = l_\Phi$.

Nota. This makes sense only via the natural contactomorphism between PT^*P and $PT^*P^\vee$.

Example. *The figure-eight curve and its dual* (see Fig. 3): Suppose that $N = S^1$ and that φ is an excellent immersed curve in P^2. The fiber above x of $\mathcal{N} \to N$ is sent by Φ to $\pi^\vee(\pi^{-1}(\varphi(x)))$. It is a line in ${P^2}^\vee$, tangent at some point $y^\vee$ to the curve dual to γ ($y^\vee$ represents the line tangent to the image of φ at $\varphi(x)$).

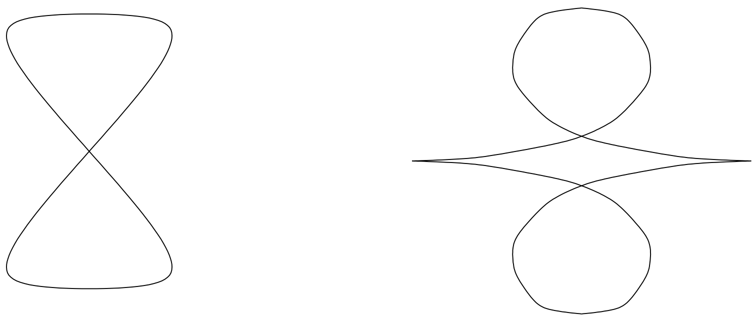

Figure 3. The figure eight and its dual curve

Hence the image of Φ is swept by a family of projective lines parametrized by N. In this example, the enveloppe of this family of lines is precisely the dual curve, since the dual curve is locally convex, which in turn follows from the fact that φ is an immersion. If one accepts the claim (which will be clarified in Section 7) that l_Φ is the Legendrian lift of the enveloppe, then this confirms the Lemma.

Example. *A non-generic curve in the plane*: If the image of φ is a curve with some isolated non-degenerate cusps (a non-generic situation), the dual curve has some inflexions, and the enveloppe of the family of lines tangent to the dual curve consists in the union of the dual curve and the lines tangent to it at the inflexion points. These lines are precisely the projection of the fibers $\pi^{-1}(y)$, for y a cusp point of φ. But these fibers belong to the Legendrian subvariety l_φ.

Proof of the proposition. This is an explicit computation. First, we define an open set U in PT^*P. Fix an hyperplane $H \subset P$, and consider some Euclidean coordinates $y_1, \dots, y_m$ on $P \setminus H = \mathbb{R}^m$. In $PT^*(\mathbb{R}^m)$, U is the set of all contact elements which are non-vertical with respect to the projection $(y_1, \dots, y_m) \to (y_1, \dots, y_{m-1})$. Such a contact element is tangent to the graph of a unique affine function $y_m = \sum_1^{m-1} p_i y_i + c$ of the variables $(y_1, \dots, y_m)$. Hence we get natural coordinates

$(y_1, \dots, y_m, p_1, \dots, p_{m-1})$ on U (the contact distribution is the kernel of the one-form $dy_m - \sum_1^{m-1} p_i\, dy_i$).

We now define some coordinates $(y_1^\vee, \dots, y_m^\vee)$ on $\pi^\vee(U)$, the set of all non vertical Euclidean hyperplanes in $P \setminus H$: These are in one-to-one correspondence with affine functions of the form $y_m = \sum_1^{m-1} y_i^\vee y_i + y_m^\vee$. Hence $(y_1^\vee, \dots, y_m^\vee)$ are coordinates on $\pi^\vee(U)$. This induces natural coordinates $(y_1^\vee, \dots, y_m^\vee, p_1^\vee, \dots, p_{m-1}^\vee)$ on the set V of all contact elements of $\pi^\vee(U)$ which are non-vertical with respect to the projection $(y_1^\vee, \dots, y_m^\vee) \to (y_1^\vee, \dots, y_{m-1}^\vee)$.

In these coordinates, Arnold's contactomorphism reduces to the classical Legendre transform between U and V: it is the map

$$(y_1, \dots, y_m, p_1, \dots, p_{m-1}) \to$$
$$\left(y_1^\vee = p_1, \dots, y_{m-1}^\vee = p_{m-1}, y_m^\vee = y_m - \sum_1^{m-1} p_i y_i, p_1^\vee = -y_1, \dots, p_{m-1}^\vee = -y_{m-1}\right).$$

We show now that $l_\varphi \cap U = l_\Phi \cap V$ modulo this contactomorphism. Denote by p the covector $(p_1, \dots, p_{m-1}, -1)$. By definition,

$$l_\varphi \cap U = \{(y, p) \colon \exists x \in \varphi^{-1}(P \setminus H),\, y = \varphi(x),\, pD\varphi(x) = 0\}.$$

Denote by $\varphi_1, \dots, \varphi_m$ the components of φ restricted to $\varphi^{-1}(P \setminus H)$. $\Phi^{-1}(\pi^\vee(U))$ is the total space of a trivial bundle with base space $\varphi^{-1}(P \setminus H)$ and fiber $\mathbb{R}^{m-1} \times \{-1\}$. Elements of $(s_1, \dots, s_{m-1}, -1)$ of $\mathbb{R}^{m-1} \times \{-1\}$ will be denoted by s. Hence, restricted to $\Phi^{-1}(\pi^\vee(U))$, Φ is the map which associates to (x, s) the hyperplane orthogonal to s that goes through $\varphi(x)$. It has the following form:

$$(x, s) \to \left(y_1^\vee = s_1, \dots, y_{m-1}^\vee = s_{m-1}, y_m^\vee = \varphi_m(x) - \sum_1^{m-1} s_i \varphi_i(x)\right). \qquad (*)$$

We abbreviate the covector $(p_1^\vee, \dots, p_{m-1}^\vee, -1)$ by $p^\vee$. By definition

$$l_\Phi \cap V = \{(y^\vee, p^\vee) \colon \exists (x, s) \in \varphi^{-1}(P \setminus H) \times \mathbb{R}^{m-1} \times \{-1\},$$
$$y^\vee = \Phi(x),\, p^\vee D\Phi(x, s) = 0\}.$$

The explicit formula $(*)$ implies that $(y_1^\vee, \dots, y_m^\vee, p_1^\vee, \dots, p_{m-1}^\vee)$ belongs to l_Φ iff there exists some (x, s) in $\varphi^{-1}(P \setminus H) \times \mathbb{R}^{m-1}$ such that:

(i) $y_1^\vee = s_1, \dots, y_{m-1}^\vee = s_{m-1}$,
(ii) $y_m^\vee = \varphi_m(x) - \sum_1^{m-1} s_i \varphi_i(x)$,
(iii) $p_1^\vee = -\varphi_1(x), \dots, p_m^\vee = -\varphi_{m-1}(x)$,
(iv) $sD\varphi(x) = 0$.

Equations (iii) and (iv) follow from $p^\vee D\Phi(x, s) = 0$. In these equations, s can be eliminated, and one gets precisely the image of $l_\varphi \cap U$ by Arnold's contactomorphism. □

Remark. The idea to realize singularities as enveloppes is not new: It was used by O. Shcherbak [Sh] for local considerations.

7. Proof of Theorem B

In this section, we first prove theorem B under the assumption that Ψ_t satisfies the Hypothesis H of Section 5 for all $t \in [0, 1]$: for any $y \in P$, $\pi^\vee$ restricted to any deformed fiber $\Psi_t(\pi^{-1}(y))$ is an embedding. The map Φ below is the stabilization defined in the preceding section.

Proposition. *There exists a family of maps* $\Phi_t \colon \mathcal{N} \to P^\vee$ *such that*

- $\Phi_0 = \Phi$,
- *for any* $t \in [0, 1]$, $\Psi_t(l_\varphi) = l_{\Phi_t}$.

Proof. The mappings Φ_t are defined as follows: Consider some $X \in \mathcal{N}$. This corresponds to some $x \in N$ and to some $z \in \pi^{-1}(\varphi(x))$. $\Phi_t(X)$ is defined to be $\pi^\vee(\Psi_t(z))$. The map Φ_0 is nothing but Φ. The image by Φ_t of the fiber over x is a smooth embedded hypersurface $H_{x,t}$ in $P^\vee$, which is a small perturbation of the hyperplane $H_x = \pi^\vee(\pi^{-1}(\varphi(x)))$.

Roughly speaking, the proposition holds because one has the following informal picture at the wave front level:

(i) $\pi^\vee(l_{\Phi_t})$ is the enveloppe of the family of hyperplanes $H_{x,t}$,
(ii) $H_{x,0}$ is "sent" to $H_{x,t}$ by Ψ_t,
(iii) a contact transformation "preserve the enveloppes".

This can be made precise as follows: consider the subset W of $PT^*(P^\vee \times N)$ which consists in all the contact elements of $P^\vee \times N$ which are *not* tangent to the fibers of the projection $P^\vee \times N \to N$.

Denote by $\mathcal{C}$ the subset of W which consist in all the contact elements of $P^\vee \times N$ which are tangent to the fibers of the projection $\eta \colon P^\vee \times N \to P^\vee$, and by R the natural map $R \colon \mathcal{C} \to PT^*P^\vee$.

Denote by π^* the natural projection $\mathcal{N} \to N$. The map $\Gamma_t \colon \mathcal{N} \to P^\vee \times N$, $\mathcal{X} \mapsto (\Phi_t(\mathcal{X}), \pi^*(\mathcal{X}))$ is an embedding of $\mathcal{N}$ into $P^\vee \times N$. The associated Legendrian submanifold $l_{\Gamma_t} \subset PT^*(P^\vee \times N)$ lies in fact in W. Since $\Phi_t = \eta \circ \Gamma_t$, $l_{\Phi_t} = R(l_{\Gamma_t})$.

Lemma. *There exists a contactomorphism* $\tilde{\Psi}_t$ *of* W *which "lifts"* Ψ_t *as follows*:

(a) $\tilde{\Psi}_t$ *preserve* $\mathcal{C}$, *and for any* $L \subset W$, $R(\tilde{\Psi}_t(L)) = \Psi_t(R(L))$,
(b) $\tilde{\Psi}_t(l_{\Gamma_0}) = L_{\Gamma_t}$.

Assuming this lemma: $\Psi_t(l_\Phi) = \Psi_t(R(l_{\Gamma_0})) = R(\tilde{\Psi}_t(l_{\Gamma_0})) = R(l_{\Gamma_t}) = l_{\Phi_t}$. This proves the proposition. $\square$

Proof of the lemma. This construction is easy to explain with the help of a contact form. Denote by $ST^*P^\vee$ the set of *cooriented contact elements* of $P^\vee$, i. e., the spherization of $T^*P^\vee$. Choose any Riemannian metric on $P^\vee$ and identify $ST^*P^\vee$ with the hypersurface of $T^*P^\vee$ made of all unit covectors. The Liouville form of $T^*P^\vee$, restricted to this hypersurface, is a contact from and induces the contact structure which lifts that of $PT^*P^\vee$. Denote by α this contact form on $ST^*P^\vee$. Denote by λ the Liouville form of T^*N. $\alpha - \lambda$ is a contact form on $ST^*P^\vee \times T^*N$. This manifold can be viewed as the set of cooriented contact elements of $P^\vee \times N$ which are not vertical with respect to the projection on N. It is a double cover of W. Denote by $\hat{\Psi}_t$ the lift of Ψ_t to $ST^*P^\vee$. We have that $\hat{\Psi}_t^*\alpha = \mu_t\alpha$, when

μ_t is a positive function on $PT^*P^\vee$. Suppose that $q = (q_1, \ldots, q_n)$ is some local coordinate system on N. Denote by (q, p) the associated canonical coordinates on T^*N. We have: $\lambda = p\,dq$. Define $\hat{\hat{\Psi}}_t(z, q, p) = (\hat{\Psi}_t(z), q, \mu_t p)$. $\hat{\hat{\Psi}}_t$ descends to a contactomorphism $\tilde{\Psi}_t$ of W.

In the preceding notations, $\mathcal{C}$ corresponds to the set $p = 0$, hence $\mathcal{C}$ is preserved by $\tilde{\Psi}_t$. On $\mathcal{C}$, $\tilde{\Psi}_t$ acts as the product of Ψ and the identity map. Claim (a) follows. $\tilde{\Psi}_t$ is constructed above such that the projections in $P^\vee \times N$ of $\tilde{\Psi}_t(l_{\Gamma_0})$ and of l_{Γ_t} are the same. The iamge of this projection is an embedding of $\mathcal{N}$ in $P^\vee \times N$ which completely determines the Legendrian submanifold which lies above. This proves claim (b). $\square$

Now that $\Psi_t(l_\varphi)$ is realized as the Legendrian reduction of $\mathcal{N}$ by a mapping with values in $P^\vee$, we can stabilize once more to realize the same Legendrian submanifold as a Legendrian reduction by a mapping with values in P. This proves Theorem B in the case of small deformations.

To remove the smallness assumption on $\{\Psi_t\}_{t\in[0,1]}$, we use a classical trick in the folklore of generating families theory: There exists $K \in \mathbb{N}$ and a subdivision $t_0 = 0 < t_1 < t_2 < \cdots < t_K = 1$ of $[0, 1]$ such that, for $k \in 1, \ldots, K-1$ and $t \in [t_k, t_{k+1}]$, $\Psi_t(l_\varphi) = \Xi_t(\Psi_{t_k}(l_\varphi))$, where Ξ_t is a contact isotopy which satisfies H.

We already know that there exists a family of mappings $\{\Phi^1_t\}_{t\in[0,t_1]}$, such that, for $t \in [0, t_1]$, $\Psi_t(l_\varphi) = l_{\Phi^1_t}$. There also exists $\{\Phi^2_t\}_{t\in[t_1,t_2]}$ such that, for $t \in [t_1, t_2]$, $\Psi_t(l_\varphi) = l_{\Phi^2_t}$. Moreover, $\Phi^2_{t_1}$ is the stabilization of $\Phi^1_{t_1}$. For $t \in [0, t_1]$, define Φ^2_t to be the stabilization of Φ^1_t. Then $t \in [0, t_2]$, $\Psi_t(l_\varphi) = l_{\Phi^2_t}$ for $t \in [0, t_2]$.

This procedure can be continued so that the whole family of Legendrian submanifolds $\{\Psi_t(l_\varphi)\}_{t\in[0,1]}$ is induced by a single family of mappings $\{\Phi_t\}_{t\in[0,1]}$. This proves Theorem B.

References

[Ar] V. I. Arnold, *Singularities of caustics and wave fronts*, Mathematics and its Applications (Soviet Series), vol. 62, Kluwer Academic Publishers Group, Dordrecht, 1990. MR **93b:**58019

[AG] V. I. Arnold and A. B. Givental, *Symplectic geometry*, Dynamical systems, IV, Encyclopaedia Math. Sci., vol. 4, Springer, Berlin, 2001, pp. 1–138. MR 1 866 631

[AGV] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. I*, Monographs in Mathematics, vol. 82, Birkhäuser Boston Inc., Boston, MA, 1985. MR **86f:**58018

[Ch] Y. V. Chekanov, *Critical points of quasifunctions, and generating families of Legendrian manifolds*, Funktsional. Anal. i Prilozhen. **30** (1996), no. 2, 56–69, 96 (Russian). MR **97f:**58016. English translation in: Funct. Anal. Appl. **30** (1996), no. 2, 118–128.

[EG] Y. Eliashberg and M. Gromov, *Lagrangian intersection theory: finite-dimensional approach*, Geometry of differential equations, Amer. Math. Soc. Transl. Ser. 2, vol. 186, Amer. Math. Soc., Providence, RI, 1998, pp. 27–118. MR **2002a:**53102

[Fe] E. Ferrand, *On a theorem of Chekanov*, Symplectic singularities and geometry of gauge fields (Warsaw, 1995), Banach Center Publ., vol. 39, Polish Acad. Sci., Warsaw, 1997, pp. 39–48. MR **98i:**53039

[Hae] A. Haefliger, *Quelques remarques sur les applications différentiables d'une surface dans le plan*, Ann. Inst. Fourier. Grenoble **10** (1960), 47–60. MR 22 #7145

[Sh] O. P. Shcherbak, *Projectively dual space curves and Legendre singularities*, Trudy Tbiliss. Univ. **232/233** (1982), 280–336 (Russian). MR **85g:**58022

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