Chapter 1
Tempered fundamental group and graph of the stable reduction

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Abstract The tempered fundamental group of a hyperbolic curve over an algebraically closed nonarchimedean field is an invariant that does not depend only on the genus of the curve. In this paper we review some results about what can be recovered of a hyperbolic curve from its tempered fundamental group. S. Mochizuki proved that, for a curve over $\overline{\mathbb{Q}}_p$, one can recover the graph of the stable reduction of the curve. For Mumford curves, one can also recover a natural metric on this graph.

1.1 Introduction

In characteristic 0, one cannot recover much of a proper curve over an algebraically closed field from the geometric fundamental group: it only depends on the genus. In p-adic analytic geometry, the homotopy type of a curve cannot be described in terms of the genus of the curve. Here we will be interested in what one can recover of a p-adic curve from a category of geometric analytic coverings, including finite étale coverings and infinite coverings from analytic geometry. More precisely, we will be interested in tempered coverings, i.e., coverings that become topological coverings after pullback by some finite étale covering. These coverings are classified by a topological group called the *tempered fundamental group*.

This paper reviews what can be recovered of a curve from its geometric tempered fundamental group of a hyperbolic curve. Mochizuki proved in [9] that one can recover the graph of its stable reduction:

Theorem 1 ([9, Cor 3.11]) *If X_α and X_β are two hyperbolic $\overline{\mathbb{Q}}_p$-curves, every (outer) isomorphism $\phi : \pi_1^{\mathrm{temp}}(X_{\alpha,\mathbb{C}_p}) \simeq \pi_1^{\mathrm{temp}}(X_{\beta,\mathbb{C}_p})$ determines, functorially in ϕ, an isomorphism of graphs of the stable reductions $\bar{\phi} : \mathbf{G}_{X_\alpha} \simeq \mathbf{G}_{X_\beta}$.*

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More precisely, one can recover from the tempered fundamental group a (p')-version of the fundamental group (which classifies coverings that become topological after pullback by some finite Galois covering of order prime to p), and one can describe the graph of the stable reduction from this (p')-tempered fundamental group in the following way:

- vertices correspond to conjugacy class of maximal compact subgroups of the (p')-tempered fundamental group,
- edges correspond to conjugacy class of nontrivial intersection of two different maximal compact subgroups.

In [8], we were interested in recovering a natural metric of the graph of the stable reduction from the tempered fundamental group. The metric is defined so that the length of an edge is the width of the annulus which is the generic fiber of the formal completion of the node corresponding to this edge. We proved the following:

Theorem 2 ([8, Thm 4.13]) *If X_1 and X_2 are hyperbolic Mumford $\overline{\mathbb{Q}}_p$-curves (i.e., with totally degenerate stable reduction) and $\phi : \pi_1^{\mathrm{temp}}(X_{1,\mathbb{C}_p}) \simeq \pi_1^{\mathrm{temp}}(X_{2,\mathbb{C}_p})$ is an isomorphism, then the induced $\bar{\phi} : \mathbf{G}_{X_1} \to \mathbf{G}_{X_2}$ is an isomorphism of metric graphs.*

In this paper, we will explain the proof of this result. In contrast to the previous result, one cannot recover this metric from the (p')-tempered fundamental group: we will also have to study the topological behavior of wildly ramified coverings.

We will mainly focus on wildly ramified abelian torsors on a Mumford curve X and follow the study made in [11] and [10]. The abelian torsors on Mumford curves can be described in terms of currents on the graph of the stable reduction. Indeed, the pullback of a μ_{p^n}-torsor of X to the universal covering Ω of X can be obtained by pulling back the canonical μ_{p^n}-torsor on $\mathbb{G}_{\mathrm{m}}$ along some theta function $\Omega \to \mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$.

Theta functions can be described in terms of currents on the graph of the stable reduction: given a theta function $f : \Omega \to \mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$, the potential associated to the corresponding current is the function $x \mapsto |f(x)|$. This gives a surjective map

$$\mathrm{Hom}(\pi_1^{\mathrm{temp}}(X), \mu_n) \to C(\mathbf{G}_X, \mathbb{Z}/n\mathbb{Z})$$

from the set of μ_n-torsors on X to the set of currents on X with value in $\mathbb{Z}/n\mathbb{Z}$. We will show in Proposition 13 that these morphisms for X_1 and X_2 are compatible with ϕ and $\bar{\phi}$ up to a scalar. For the canonical μ_{p^h}-torsor on $\mathbb{G}_{\mathrm{m}}$, the splitting of the torsor at a Berkovich point depends on the distance of this point to the skeleton of $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$ (i.e., the line linking 0 to ∞). For a given theta function $\Omega \to \mathbb{G}_{\mathrm{m}}$, one can then get information about the splitting of the torsor in a point in terms of the distance of the point to the support of the corresponding current.

In §1 we first give a brief description of the Berkovich space of an algebraic variety and, for a curve, we recall that it is naturally homotopy equivalent to the graph of the stable reduction. Then we define the tempered fundamental group. In §2 we explain Theorem 1, and in §3 we study abelian coverings of Mumford curves. In §4 we give a proof Theorem 2.

1.2 Tempered fundamental group

1.2.1 Berkovich analytification of algebraic varieties and curves

Let K be a complete nonarchimedean field. We will mostly be interested lateron in the case where $K = \mathbb{C}_p$. The norm will be chosen so that $|p| = p^{-1}$ and the valuation so that $v(p) = 1$. All valued fields will have valuations with values in $\mathbb{R}$.

If X is an algebraic variety over K, one can associate to X a topological set X^{an} with a continuous map $\phi : X^{\mathrm{an}} \to X$ defined in the following way. A point of X^{an} is an equivalence class of morphisms $\operatorname{Spec} K' \to X$ over $\operatorname{Spec} K$ where K' is a complete valued extension of K. Two morphisms $\operatorname{Spec} K' \to X$ and $\operatorname{Spec} K'' \to X$ are equivalent if there exists a common valued extension L of K' and K'' such that

$$\begin{array}{ccc} \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} K'' \\ \downarrow & & \downarrow \\ \operatorname{Spec} K' & \longrightarrow & X \end{array}$$

commutes. In fact, for any point $x \in X^{\mathrm{an}}$, there is a unique smallest such complete valued field defining x denoted by $\mathscr{H}(x)$ and called the completed residue field of x. Forgetting the valuation, one gets points $\operatorname{Spec}(K) \to X$ from the same equivalence class of points: this defines a point of X, hence the map $X^{\mathrm{an}} \to X$. If $U = \operatorname{Spec} A$ is an affine open subset of X, every $x \in \phi^{-1}(U)$ defines a seminorm $|\ |_x$ on A. The topology on $\phi^{-1}(U)$ is defined to be the coarsest such that $x \mapsto |f|_x$ is continuous for every $f \in A$.

The space X^{an} is locally compact, and even compact if X is proper. In fact X^{an} is more than just a topological space: it can be enriched into a K-analytic space, in the sense of and as defined by Berkovich in [2].

Let us assume for simplicity that X is irreducible and reduced. One can describe the sheaf O of analytic functions on X^{an} as follows (but recall that an analytic space in the sense of Berkovich is not just given by a locally ringed space, thus more data should be given to get a well defined analytic space). If U is an open subset of X^{an}, then $O(U)$ is the ring of functions $f : U \to \coprod_x \mathscr{H}(x)$ such that $f(x) \in \mathscr{H}(x)$ and f is locally a uniform limit of rational functions: for every $x \in U$, there is an open neighborhood V of x and a sequence (g_n) of rational functions on X with no poles in U such that $\sup_{x \in V} |f(x) - g_n(x)| \to 0$. The sheaf $\mathscr{M}$ of meromorphic functions on X is the sheaf associated to the presheaf mapping an open subset U of X^{an} to the total ring of fractions of $O(U)$.

For a hyperbolic curve X over an algebraically closed complete nonarchimedean field K, the homotopy type of X^{an} can be described in terms of the stable model $\mathscr{X}$ of X, see [6, def. 1.1] for the definition of stable curves in the non-proper case. Indeed, consider the *graph* $\mathbf{G}_X$ *of the stable reduction* of X: the vertices correspond to the irreducible component of the stable reduction and the edges correspond to the nodes of the stable reduction. There is a canonical embedding $\mathbf{G}_X \hookrightarrow X^{\mathrm{an}}$ which admits a canonical strong deformation retraction Φ. In particular, it is a homotopy equivalence. The image of $\mathbf{G}_X$ is called the *skeleton* of X.

Let e be an edge corresponding to a double point of the special fiber $\mathscr{X}_s$ of the stable model of X. Locally for the étale topology, $\mathscr{X}$ is isomorphic to $\operatorname{Spec} O_K[x,y]/(xy-a)$ with $a \in K$ and $|a|<1$. If $\mathring{e}$ is an open edge of $\mathbf{G}_X$, its preimage $\Phi^{-1}(\mathring{e})$ is isomorphic to the open annulus $\{z \mid |a|<|z|<1\}$. One can define a metric on $\mathbf{G}_X$ by setting the *length* of e to be

$$\lg(e) = v(a).$$

A metric graph $\mathbf{G}_{\mathscr{X}'}$ can also be defined for any semistable model $\mathscr{X}'$ of X, and there is also a natural embedding $\mathbf{G}_{\mathscr{X}'} \hookrightarrow X^{an}$. Those metrics are compatible under blow-up.

Let us assume $K = \mathbb{C}_p$. Points of $\mathbb{A}^{1,an}$ are of four different types and are described in the following way:

- A closed ball $B = B(a,r) \subset \mathbb{C}_p$ of center a and radius r defines a point $b = b_{a,r}$ of $\mathbb{A}^{1,\mathrm{an}}$ by
$$|f|_b = \sup_{x\in B} |f(x)|.$$
The point $b_{a,r}$ is said to be of type 1 if $r=0$, of type 2 if $r \in p^{\mathbb{Q}}$ and of type 3 otherwise. The pairs (a,r) and (a',r') define the same point if and only if $B(a,r) = B(a',r')$, i.e., $r=r'$ and $|a-a'| \le r$.
- A decreasing family of balls $E = (B_i)$ with empty intersection defines a point by
$$|f|_E = \inf |f|_{b_i}.$$
Such a point is said to be of type 4.

The analytic projective line $\mathbb{P}^{1,\mathrm{an}}$ is obtained from $\mathbb{A}^{1,\mathrm{an}}$ by adding a point at infinity. There is a natural metric on the set of points of type 2 and 3 of $\mathbb{P}^{1,\mathrm{an}}$ defined by the following formula:

$$d(b_{a,r}, b_{a',r'}) = \begin{cases} \log_p(|a-a'|/r) + \log_p(|a-a'|/r') & \text{if } |a-a'| \ge \max(r,r') \\ |\log_p(r'/r)| & \text{if } |a-a'| \le \max(r,r') \end{cases}$$

This metric is compatible with the metrics of the graphs of the semistable reductions. It is invariant under automorphisms of $\mathbb{P}^{1,\mathrm{an}}$. The metric topology defined on the set of points of type 2 and 3 is much finer than the Berkovich topology.

If $x \neq y \in \mathbb{P}^{1,\mathrm{an}}$, there is a unique smallest connected subset of $\mathbb{P}^{1,\mathrm{an}}$ that contains x and y. It is homeomorphic to a closed interval and is denoted $[x,y]$. We also set $]x,y[= [x,y] \setminus \{x,y\}$, it consists of points of type 2 and 3. The topology induced by the restriction of the metric d to $]x,y[$ is the topology induced by the topology of $\mathbb{P}^{1,\mathrm{an}}$. For example, if $x,y \in \mathbb{A}^1(\mathbb{C}_p)$,

$$]x,y[= \{b_{x,r}\}_{0<r\le|x-y|} \cup \{b_{y,r}\}_{0<r\le|x-y|}.$$

1.2.2 Definition of the tempered fundamental group

Let K be a complete nonarchimedean field, and let X be a connected smooth algebraic variety over K.

The usual definition of tempered fundamental groups, as given in [1, def. 2.1.1], uses a notion of étale topology on Berkovich spaces defined in [3]. However, we will start with a description of the tempered fundamental group that only uses the topology of the analytifications of the finite étale coverings of X. This description will be enough for our purposes.

Let $x : \operatorname{Spec} K' \to X$ be a geometric point of X and assume that K' is provided with a complete valuation extending the valuation of K so that x also defines a point of X^{an}.

Let $(Y,y) \to (X,x)$ be a pointed Galois finite étale covering of (X,x). Then y also defines a point of the Berkovich space Y^{an}. Let $\phi : (Y^\infty, y^\infty) \to (Y^{\mathrm{an}}, y)$ be the pointed universal covering of (Y^{an}, y). Let us consider the following group:

$$H_Y := \{(g,h) \in \mathrm{Gal}(Y/X) \times \mathrm{Aut}_{X^{\mathrm{an}}}(Y^\infty) | \phi h = g^{\mathrm{an}} \phi\}.$$

Heuristically, H_Y can be thought of as the Galois group of Y^∞ over X. There is a natural homomorphism $\pi_1^{\mathrm{top}}(Y^{\mathrm{an}}, y) \to H_Y$ that maps $h \in \mathrm{Gal}(Y^\infty/Y)$ to (id_Y, h) and a natural homomorphism $H_Y \to \mathrm{Gal}(Y/X)$ mapping (g,h) to g. One thus get an exact sequence

$$1 \to \pi_1^{\mathrm{top}}(Y^{\mathrm{an}}, y) \to H_Y \to \mathrm{Gal}(Y/X) \to 1.$$

Tthe surjectivity of the morphism on the right comes from the extension property of universal topological coverings.

By the strong deformation retraction recalled above, the group $\pi_1^{\mathrm{top}}(Y^{\mathrm{an}})$ is isomorphic to $\pi_1^{\mathrm{top}}(\mathbf{G}_Y)$ and the extension of $\mathrm{Gal}(Y/X)$ by $\pi_1^{\mathrm{top}}(\mathbf{G}_Y)$ can also be directly described in terms of the action of $\mathrm{Gal}(Y/X)$ on $\mathbf{G}_Y$.

For a morphism $\psi : (Y_1, y_1) \to (Y_2, y_2)$ of pointed Galois finite étale coverings, let $\psi^\infty : (Y_1^\infty, y_1^\infty) \to (Y_2^\infty, y_2^\infty)$ be the morphism of pointed topological spaces extending ψ. One defines a morphism $H_{Y_1} \to H_{Y_2}$ by mapping (g,h) to (g',h') such that $h'\psi = \psi h$ and $g'\psi^\infty = \psi^\infty g$.

The *tempered fundamental group* of X, pointed at x, is the topological group

$$\pi_1^{\mathrm{temp}}(X,x) = \varprojlim_{(Y,y)\in\mathscr{C}_0} H_Y,$$

where $\mathscr{C}_0$ is the filtered category of pointed Galois finite étale coverings. Thanks to a result of J. de Jong, the group $\pi_1^{\mathrm{temp}}(X,x) \to H_Y$ is surjective for any (Y,y) and $\pi_1^{\mathrm{temp}}(X,x)$ does not depend on x up to inner automorphism.

The tempered fundamental group is functorial: if $(Y,y) \to (X,x)$ is a morphism of geometrically pointed smooth varieties, one gets a morphism of topological groups $\pi_1^{\mathrm{temp}}(X,x) \to \pi_1^{\mathrm{temp}}(Y,y)$. If we forget base points, one gets a functor π_1^{temp} from smooth K-varieties to topological groups with outer morphisms.

We will be mainly interested in curves over $\overline{\mathbb{Q}}_p$ in this paper. If X is a curve over $\overline{\mathbb{Q}}_p$, then X^{an} will be an abbreviation for $X^{\mathrm{an}}_{\mathbb{C}_p}$ and $\pi_1^{\mathrm{temp}}(X)$ will be an abbreviation for $\pi_1^{\mathrm{temp}}(X_{\mathbb{C}_p})$.

As stated before, the tempered fundamental group classifies a category of analytic coverings. A morphism of K-analytic spaces $f : S \to X^{\mathrm{an}}$ is said to be an *étale covering* if X^{an} is covered by open subsets U such that $f^{-1}(U) = \bigsqcup V_j$ and $V_j \to U$ is finite étale, see [5]. For example, finite étale coverings, also called *algebraic coverings*, and coverings in the usual topological sense for the Berkovich topology, also called *topological coverings*, are tempered coverings. Then, André defines tempered coverings as follows:

Definition 3 ([1, def. 2.1.1]) *An étale covering $S \to X^{\mathrm{an}}$ is* tempered *if it is a quotient of the composition of a topological covering $T' \to T$ with a finite étale covering $T \to X$.*

Here are two properties of the category of tempered coverings.

Proposition 4 *If X is a proper curve, then the category of tempered coverings of X is equivalent to the category of locally constant sheaves for the Berkovich étale topology on X^{an}.*

Proposition 5 *There is an equivalence between the category of sets endowed with an action of $\pi_1^{\mathrm{temp}}(X,x)$ that goes through a discrete quotient and the category of tempered coverings of X^{an}.*

1.3 Mochizuki's results on the pro-(p') tempered group of a curve

Mochizuki proves in [9] the folllowing theorem.

Theorem 6 ([9, cor. 3.11]) *If X_α and X_β are two hyperbolic $\overline{\mathbb{Q}}_p$-curves, every (outer) isomorphism $\gamma : \pi_1^{\mathrm{temp}}(X_{\alpha,\mathbb{C}_p}) \simeq \pi_1^{\mathrm{temp}}(X_{\beta,\mathbb{C}_p})$ determines, functorially in γ, an isomorphism of graphs $\bar{\gamma} : \mathbf{G}_{X_\alpha} \simeq \mathbf{G}_{X_\beta}$.*

Let us explain this result. In fact, the graph of the stable reduction of the curve can even be recovered from a prime-to-p version of the fundamental group. Let

$$\pi_1^{\mathrm{temp}}(X,x)^{(p')} = \varprojlim_{(Y,y)\in\mathscr{C}} H_Y$$

where $\mathscr{C}$ is the category of pointed Galois finite étale coverings (Y,y) of (X,x) such that the order of $\mathrm{Gal}(Y/X)$ is prime to p. Any morphism $\pi_1^{\mathrm{temp}}(X_1) \to \pi_1^{\mathrm{temp}}(X_2)$ induces a morphism

$$\pi_1^{\mathrm{temp}}(X_1)^{(p')} \to \pi_1^{\mathrm{temp}}(X_2)^{(p')}.$$

Indeed, $(H_Y)_{p\wedge\sharp\mathrm{Gal}(Y/X)=1}$ is cofinal among discrete quotients of $\pi_1^{\mathrm{temp}}(X)$ that are extensions of a finite prime-to-p group by a torsionfree group. Hence, if $\mathscr{C}$ is the class of discrete groups that have a normal torsionfree subgroup of finite prime-to-p index, then $\pi_1^{\mathrm{temp}}(X)^{(p')}$ is the pro-$\mathscr{C}$ completion of $\pi_1^{\mathrm{temp}}(X)$.

A finite prime-to-p covering $Y \to X$ extends as a Kummer covering $\mathscr{Y} \to \mathscr{X}$, where $\mathscr{Y}$ and $\mathscr{X}$ are the stable models of Y and X. The map $\mathscr{Y} \to \mathscr{X}$ induces a morphism of graphs $\mathbf{G}_Y \to \mathbf{G}_X$ and a commutative diagram

$$\begin{array}{ccc} \mathbf{G}_Y & \hookrightarrow & Y^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathbf{G}_X & \hookrightarrow & X^{\mathrm{an}}. \end{array}$$

Let z be a vertex (resp. an edge) of $\mathbf{G}_X$. Let us consider a compatible family $(z_Y^\infty)_{Y\in\mathscr{C}}$ where z_Y^∞ is a vertex (resp. an edge) of $\mathbf{G}_Y^\infty$ over z. Then $\pi_1^{\mathrm{temp}}(X,x)^{(p')}$ acts on $\mathbf{G}_Y^\infty$ for every Y. Let D_z be the subgroup of $\pi_1^{\mathrm{temp}}(X,x)^{(p')}$ that stabilizes z_Y for every Y. Changing the family $(z_Y^\infty)_Y$ would replace D_z by a conjugate subgroup, so that D_z only depends on z up to conjugacy. The group D_z is called the *decomposition subgroup* of z. It is a profinite subgroup of $\pi_1^{\mathrm{temp}}(X)^{(p')}$, and in fact it can be identified with the decomposition group of z in $\pi_1^{\mathrm{alg}}(X)^{(p')}$, which is the prime-to-$p$ completion of $\pi_1^{\mathrm{temp}}(X)^{(p')}$.

If e is an edge that ends at the vertex v, then D_e is a subgroup of D_v. This gives a natural structure of a graph of profinite groups on $\mathbf{G}_X$ The important facts for Theorem 6 are that (1) every compact subgroup of $\pi_1^{\mathrm{temp}}(X)^{(p')}$ is contained in some decomposition group of some vertex of $\mathbf{G}_x$, and that (2) if the intersection of two different decomposition subgroups of a vertex is non trivial, then this intersection is the decomposition subgroup of a unique edge, and that (3) the intersection of three different decomposition subgroups of a vertex is trivial. Hence $\mathbf{G}_X$ can be recovered from $\pi_1^{\mathrm{temp}}(X)^{(p')}$ together with the structure of graph of profinite groups on it in the following way.

- The vertices of $\mathbf{G}_X$ correspond to conjugacy classes of maximal compact subgroups of $\pi_1^{\mathrm{temp}}(X)^{(p')}$. Such a maximal compact subgroup is called a *vertical subgroup* of $\pi_1^{\mathrm{temp}}(X)^{(p')}$.
- The edges of $\mathbf{G}_X$ correspond to conjugacy classes of nontrivial intersections of two different maximal compact subgroups. Such an intersection is called an *edge-like* subgroup of $\pi_1^{\mathrm{temp}}(X)^{(p')}$.

A connected finite étale covering $f : Y \to X$ induces a morphism of stable models $\mathscr{Y} \to \mathscr{X}$. One can recover $\mathbf{G}_X$ from $\pi_1^{\mathrm{temp}}(X)$ and $\mathbf{G}_Y$ from $\pi_1^{\mathrm{temp}}(Y)$. We are now interested in the combinatorial data of the morphism $\mathscr{Y}_s \to \mathscr{X}_s$ which can be recovered from the embedding $\iota : \pi_1^{\mathrm{temp}}(Y) \hookrightarrow \pi_1^{\mathrm{temp}}(X)$. If H is a vertical subgroup of $\pi_1^{\mathrm{temp}}(Y)^{(p')}$ corresponding to an irreducible component y of $\mathscr{Y}_s$, then $\iota^{(p')}(H)$ is

- either a finite index subgroup of a unique vertical subgroup H' of $\pi_1^{\mathrm{temp}}(X)^{(p')}$ if y maps onto an irreducible component x of $\mathscr{X}_s$, and then H' is the vertical subgroup corresponding to x,
- or a commutative group and hence not a finite index subgroup of any vertical subgroup.

Thus, for a given irreducible component x of $\mathscr{X}_s$, one can recover from ι the set of irreducible components of $\mathscr{Y}_s$ that map onto x. Translated into Berkovich spaces, one gets:

Proposition 7 *One can recover from ι the preimage of any vertex of the skeleton of X^{an}. In particular, one can know if the covering is split at this vertex.*

1.4 Abelian coverings of Mumford curves

1.4.1 Definition of Mumford curves

A proper curve X over $\overline{\mathbb{Q}}_p$ is a *Mumford curve* if the following equivalent properties are satisfied:

- all normalized irreducible components of its stable reduction are isomorphic to $\mathbb{P}^1$,
- X^{an} is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$,
- its Jacobian variety J has multiplicative reduction,
- the universal topological covering of J^{an} is a torus $\widetilde{J}$.

The universal topological covering Ω of X^{an} for a Mumford curve X is an open subset of $\mathbb{P}^{1,\mathrm{an}}$. More precisely there is a Shottky subgroup Γ of $\mathrm{PGL}_2(\mathbb{C}_p)$, i.e., a free finitely generated discrete subgroup of $\mathrm{PGL}_2(\mathbb{C}_p)$, such that $\Omega = \mathbb{P}^{1,\mathrm{an}}\backslash\mathscr{L}$ where $\mathscr{L}$ is the closure of the set of $\mathbb{C}_p$-points stabilized by some nontrivial element of Γ. The points of $\mathscr{L}$ are of type 1, i.e., are $\mathbb{C}_p$-points. Then X is p-adic analytically uniformized as

$$X^{\mathrm{an}} = \Omega/\Gamma$$

and $\Gamma = \pi_1^{\mathrm{top}}(X)$.

Let $\mathbf{G}_X$ be the graph of the stable reduction of X and $\mathbf{T}_X$ be its universal topological covering. The graph $\mathbf{T}_X$ embeds in Ω and can be described as the smallest subset of Ω such that $\mathbf{T}_X \cup \mathscr{L}$ is connected, i.e., $\mathbf{T} = \cup_{(x,y)\in\mathscr{L}^2}]x,y[$.

1.4.2 Abelian torsors and invertible functions on Ω

Let X be a Mumford curve of genus $g \geq 2$ over $\overline{\mathbb{Q}}_p$, let $\Omega \subset \mathbb{P}^1$ be the universal topological covering of X^{an}, and $\Gamma = \mathrm{Gal}(\Omega/X)$, so that $X^{\mathrm{an}} = \Omega/\Gamma$. All the cohomology groups will be cohomology groups for étale cohomology in the sense of algebraic geometry or in the sense of Berkovich. One can replace étale cohomology of X^{an} by étale cohomology of X thanks to [4, thm. 3.1]. Kummer theory gives us the following diagram with exact lower row (see [3, prop. 4.1.7] for the Kummer exact sequence in Berkovich étale topology):

$$\begin{array}{ccccccc} & & & & H^1(X,\mu_n) & \longrightarrow & H^1(X,O^*) \\ & & & & \downarrow & & \downarrow \\ 1 & \longrightarrow & O(\Omega)^*/(O(\Omega)^*)^n & \longrightarrow & H^1(\Omega,\mu_n) & \longrightarrow & H^1(\Omega,O^*) \end{array}$$

The map $H^1(X,O^*) \to H^1(\Omega,O^*)$ is zero, and thus $H^1(X,\mu_n) \to H^1(\Omega,\mu_n)$ goes through $O(\Omega)^*/(O(\Omega)^*)^n$. Let us explain why $H^1(X,O^*) \to H^1(\Omega,O^*)$ is zero. There is a commutative diagram with exact lines:

$$\begin{array}{ccccc} \mathbb{C}_p(X)^* & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & H^1(X,O^*) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{M}(\Omega)^* & \longrightarrow & \mathrm{Div}(\Omega) & \longrightarrow & H^1(\Omega,O^*) \end{array}$$

where $\mathscr{M}(\Omega)^*$ is the group of nonzero meromorphic functions on Ω and $\mathrm{Div}(\Omega)$ is the group of divisors on $\Omega(\mathbb{C}_P)$ with discrete support, i.e.,

$$\mathrm{Div}(\Omega) = \{f : \Omega(\mathbb{C}_p) \to \mathbb{Z} \mid \text{ the support } \{x \in \Omega(\mathbb{C}_p) \mid f(x) \neq 0\} \text{ is discrete } \}.$$

Since $\mathrm{Div}(X) \to H^1(X,O^*)$ is surjective, the next proposition tells us that the map $H^1(X,O^*) \to H^1(\Omega,O^*)$ is zero.

Proposition 8 ([11, prop. 1.3]) $\mathscr{M}(\Omega)^* \to \mathrm{Div}(\Omega)$ *is surjective.*

Proof. Assume $\infty \in \Omega$. Let $D = \sum_{i\in I} n_i x_i \in \mathrm{Div}(\Omega)$. Let us choose $y_i \in \mathscr{L}$ such that $|y_i - x_i| = \min_{z\in\mathscr{L}} |z - x_i|$. Then the infinite product $\prod_{i\in I}(\frac{z-x_i}{z-y_i})^{n_i}$ is uniformly convergent on every compact of Ω and thus defines a meromorphic function f such that $\mathrm{div}(f) = D$. □

Combining the above, we find morphisms for every $n \in \mathbb{N}$

$$H^1(X,\mu_n) \to O(\Omega)^*/(O(\Omega)^*)^n \hookrightarrow H^1(\Omega,\mu_n).$$

Moreover, $H^1(X,\mu_n)$ is mapped into the set of Γ-equivariant elements of $H^1(\Omega,\mu_n)$. There is a spectral sequence

$$H^p(\Gamma, H^q(\Omega,\mu_n)) \Rightarrow H^n(X^{\text{an}},\mu_n) = H^n(X,\mu_n)$$

associated to the Galois étale covering $\Omega \to X^{\text{an}}$. It gives a five-term exact sequence

$$0 \to H^1(\Gamma,\mu_n) \to H^1(X,\mu_n) \to H^1(\Omega,\mu_n)^\Gamma \to H^2(\Gamma,\mu_n).$$

Since Γ is free, $H^2(\Gamma,\mu_n)=0$, and thus $H^1(X,\mu_n)\to H^1(\Omega,\mu_n)^\Gamma$ is surjective, which in turn implies that

$$(O(\Omega)^*/(O(\Omega)^*)^n)^\Gamma \to H^1(\Omega,\mu_n)^\Gamma$$

is an isomorphism. The kernel of $H^1(X,\mu_n)\to H^1(\Omega,\mu_n)$ is denoted by $H^1_{\mathrm{top}}(X,\mu_n)$ and consists of μ_n-torsors that are already locally constant for the topology of X^{an}.

1.4.3 Invertible functions on Ω and currents on $\mathbf{T}$

We recall the combinatorial description of $O(\Omega)^*$ in terms of currents of the graph is given in [11].

If $\mathbf{G}_0$ is a locally finite graph and A is an abelian group, a *current* $\mathscr{C}$ on $\mathbf{G}_0$ with coefficients in A is a function $\mathscr{C}:\{\text{oriented edges of } \mathbf{G}_0\}\to A$ such that

- $\mathscr{C}(e)=-\mathscr{C}(e')$ if e and e' are the same edge but with reversed orientation,
- if v is a vertex of $\mathbf{G}_0$, $\sum_{e \text{ ending at } v}\mathscr{C}(e)=0$.

The group of currents on $\mathbf{G}_0$ with coefficients in A will be denoted $C(\mathbf{G}_0,A)$. We will simply write $C(\mathbf{G}_0)$ for $C(\mathbb{G}_0,\mathbb{Z})$.

Proposition 9 ([11, prop. 1.1]) *There is an exact sequence*

$$1\to\mathbb{C}_p^*\to O(\Omega)^*\to C(\mathbf{T})\to 0.$$

Proof (sketch). The morphism $O(\Omega)^*\to C(\mathbf{T})$ assigns to an $f\in O(\Omega^*)$ a current $\mathscr{C}_f$ on $\mathbf{T}$ as follows.

For any oriented open edge $\mathring{e}$, the preimage $\Phi^{-1}(\mathring{e})$ is isomorphic to an open annulus $\{z\in\mathbb{P}^1|1<|z|<r\}$ where the beginning of the edge tends to 1 and the end tends to r. Let $i:\{z\in\mathbb{P}^1|1<|z|<r\}\to\Phi^{-1}(\mathring{e})$ be such an isomorphism. Then fi can be written in a unique way as $z\mapsto z^m g(z)$ with $m\in\mathbb{Z}$ and $|g|$ is constant. Then we set $\mathscr{C}_f(e)=m$, which does not depend on the choice of i.

For $x,y\in\mathscr{L}$, let $f_{x,y}:\mathbb{P}^1\to\mathbb{P}^1$ be an automorphism of $\mathbb{P}^1$ that maps x to 0 and y to ∞. It restricts to a map $\Omega\to\mathbb{G}_{\mathrm{m}}$, i.e., an element $f_{x,y}\in O(\Omega)^*$ that depends on the choice of the homography only up to multiplication by a scalar. Let us choose $x_0\in\Omega(\mathbb{C}_p)$ and fix $f_{x,y}$ by imposing $f_{x,y}(x_0)=1$. The corresponding current is denoted by $c_{]x,y[}$. We find $c_{]x,y[}(e)=\pm1$ for every edge that belongs to $]x,y[$ and $c_{]x,y[}(e)=0$ for any other edge.

Every current $c\in C(\mathbf{T})$ can be written as a locally finite sum $c=\sum_{i\in I}n_i c_{]x_i,y_i[}$. Then $f=\prod f_{x_i,y_i}$ is a locally uniformly convergent product and defines a preimage of c in $O(\Omega)^*$. The exactness in the middle comes from the fact that every bounded analytic function on Ω is constant. □

The Kummer exact sequence of Ω gives a map

$$C(\mathbf{T},\mathbb{Z}/n\mathbb{Z})=C(\mathbf{T})/nC(\mathbf{T})\simeq O(\Omega)^*/(O(\Omega)^*)^n\to H^1(\Omega,\mu_n),$$

that combined with $(O(\Omega)^*/(O(\Omega)^*)^n)^\Gamma = C(\mathbf{T},\mathbb{Z}/n\mathbb{Z})^\Gamma = C(\mathbf{G},\mathbb{Z}/n\mathbb{Z})$ yields an exact sequence

$$0 \to H^1_{\mathrm{top}}(X,\mu_n) \to H^1(X,\mu_n) \to C(\mathbf{G},\mathbb{Z}/n\mathbb{Z}) \to 0. \tag{1.1}$$

1.4.4 Splitting of abelian torsors

We now assume $n = p^h$ for some positive integer h. Let c be a current on $\mathbf{T}$ with coefficients in $\mathbb{Z}/p^h\mathbb{Z}$. The subtree $\cup_{e|c(e)\neq 0}$ of $\mathbf{T}$ is called the support of c and is denoted by $\mathrm{supp}\, c$. It can also be viewed as a subset of Ω. We show that the corresponding μ_{p^h}-torsor of Ω is split at some point if this point is far enough from the support of c.

Let us begin with the simplest case, the torsor corresponding to the current $c_{]x,y[}$.

Lemma 10 *Let $z \in \Omega$ be of type 2 or 3, and let h be a positive integer. The μ_{p^h}-torsor corresponding to the current $c_{]x,y[}$ is split over z if and only if*

$$d(z,]x,y[) > h + 1/(p-1)$$

where d is the metric defined in §1.2.1.

Proof. Up to changing the embedding $\Omega \to \mathbb{P}^1$, one can assume $x = 0$ and $y = \infty$. Then the μ_{p^h}-torsor is just $f : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, with $f(t) = t^{p^h}$.

We first assume that $h = 1$. Let $b_{a,r}$ be a preimage of z. Then $f(b_{a,r}) = b_{a^p,r'} = z$ with $r' = \sup_{|x-a|\leq r} |f(x) - a^p|$. If $a = 0$, then $r' = p^h$. Otherwise we compute

$$f(y+a) - a^p = \sum_{k=1}^{p} \binom{p}{k} a^{p-k} y^k$$

and $r' = \sup_{k=1\dots p} |\binom{p}{k}| \cdot |a|^{p-k} r^k$ with $|\binom{p}{k}| = p^{-1}$ if $1 \leq k \leq p-1$ and $|\binom{p}{p}| = 1$. Thus

$$r' = \begin{cases} r^p & \text{if } \ r \leq |a| p^{-\frac{1}{p-1}} \\ |a|^{p-1} r & \text{if } \ r \geq |a| p^{-\frac{1}{p-1}} \end{cases}$$

Let ζ be a generator of μ_p. The torsor is split over z if and only if the orbit of $b_{a,r}$ under the action of μ_p is not reduced to one point, if and only if $b_{\zeta a,r} \neq b_{a,r}$, if and only if $|\zeta a - a| > r$, i.e., $a \neq 0$ and $|\zeta - 1| > r/|a|$. But $|\zeta - 1| = p^{-\frac{1}{p-1}}$. If $a = 0$, $z \in]0,\infty[$ and the torsor is not split. Otherwise, one can assume $r' < |a^p|$. The torsor is split over $z = b_{a^p,r'}$ if and only if $r'/|a^p| < p^{-\frac{p}{p-1}}$. But

$$d(b_{a^p,r'},]0,\infty[) = \inf_{r''} d(b_{a^p,r'}, b_{0,r''}) = \log_p |a^p|/r'.$$

The result follows for any h by induction. □

Every current c on $\mathbf{T}$ can be decomposed as a locally finite sum $c = \sum_{i\in I} n_i c_{]x_i,y_i[}$. For any point z of type 2 or 3, the set

$$I_z = \{i \mid d(z,]x_i, y_i[) \leq h + 1/(p-1)\}$$

is finite, and locally around z, the μ_{p^h} torsor $Y_c \to \Omega$ defined by c is isomorphic to the μ_{p^h}-torsor defined by $c' = \sum_{i\in I_z} n_i c_{]x_i,y_i[}$. One can moreover choose the decomposition $c = \sum_{i\in I} n_i c_{]x_i,y_i[}$ such that $\operatorname{supp} c = \cup]x_i, y_i[$. This proves the folowing proposition.

Proposition 11 *If z is a point of type 2 or 3 of Ω such that $d(z, \operatorname{supp} c) > h + \frac{1}{p-1}$, then the μ_{p^h}-torsor of Ω defined by c is split over z.*

1.5 Metric graph and tempered fundamental group

We consider now two Mumford curves X_1 and X_2 over $\overline{\mathbb{Q}}_p$ of genus $g \geq 2$, and an isomorphism

$$\phi : \pi_1^{\text{temp}}(X_1) \xrightarrow{\sim} \pi_1^{\text{temp}}(X_2),$$

together with the induced isomorphism of graphs

$$\bar{\phi} : \mathbf{G}_1 \xrightarrow{\sim} \mathbf{G}_2,$$

hence an isomorphism $\bar{\phi} : \mathbf{T}(\Omega_1) \xrightarrow{\sim} \mathbf{T}(\Omega_2)$.

Theorem 12 *The isomorphism $\bar{\phi} : \mathbf{G}_1 \to \mathbf{G}_2$ is an isomorphism of metric graphs.*

We will sketch the proof of this result. The metric d_i on $\mathbf{T}_i$ obtained by pullback of the metric on $\mathbf{G}_i$ is equal to the one induced by the natural metric of $\mathbb{P}^{1,\text{an}}$. Consider the diagram

$$\begin{array}{ccc} H^1(X_2, \mu_n) & \longrightarrow & H^1(X_1, \mu_n) \\ \downarrow & & \downarrow \\ C(\mathbf{G}_2, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & C(\mathbf{G}_1, \mathbb{Z}/n\mathbb{Z}) \end{array} \qquad (1.2)$$

where the vertical arrows are given by equation (1.1), the upper arrow is $\operatorname{Hom}(\phi, \mu_n)$ after identifying $H^1(X_i, \mu_n) = \operatorname{Hom}(\pi_1^{\text{temp}}(X_i), \mu_n)$. The lower arrow is $C(\bar{\phi}, \mathbb{Z}/n\mathbb{Z})$.

Proposition 13 *The diagram (1.2) above commutes up to multiplication by a scalar $\lambda \in (\mathbb{Z}/n\mathbb{Z})^*$.*

Proof. For $i = 1, 2$, a μ_n-torsor on X_i is in $H^1_{\text{top}}(X_i, \mu_n)$ if and only if it is dominated by a Galois tempered covering with torsion free Galois group. Thus the isomorphism $H^1(X_2, \mu_n) \to H^1(X_1, \mu_n)$ is compatible with a unique isomorphism

$$\tilde{\phi} : C(\mathbf{G}_2, \mathbb{Z}/n\mathbb{Z}) \to C(\mathbf{G}_1, \mathbb{Z}/n\mathbb{Z}).$$

We have to show that there exists $\lambda \in (\mathbb{Z}/n\mathbb{Z})^*$ such that for every $c \in C(\mathbb{G}_2, \mathbb{Z}/n\mathbb{Z})$ and every edge e of $\mathbf{G}_1$, we have $\tilde{\phi}(c)(e) = \lambda c(\bar{\phi}(e))$.

Let e_1 be an edge of $\mathbf{G}_1$ such that $\mathbf{G}_1 \backslash e_1$ is connected, and let $e_2 = \bar{\phi}(e_1)$. By contracting $\mathbb{G}_i \backslash e_i$ to a point, one gets a map $\mathbf{G}_i \to \mathbf{S}^1$ to the circle $\mathbf{S}^1$. There is a unique connected Galois covering $\mathbf{S}^1 \to \mathbf{S}^1$ of order n and Galois group $G = \mathbb{Z}/n\mathbb{Z}$, the pullback of which to $\mathbf{G}_i$ we denote by $\psi_i : \mathbf{G}_i^{(n)} \to \mathbf{G}_i$. Let $X_i^{(n)}$ be the corresponding topological covering of X_i. We will use the following lemma, where for sake of clarity we have omitted the indices $i = 1, 2$.

Lemma 14 *For a current $c \in C(\mathbf{G}, \mathbb{Z}/n\mathbb{Z})$ we have $c(e) = 0$ if and only if there exists $c' \in C(\mathbf{G}^{(n)}, \mathbb{Z}/n\mathbb{Z})$ such that $\psi^* c = \sum_{g \in G} g^* c'$.*

Proof. Assume there is such a current c'. Then, the fact that c' is a current implies that $c'(e')$ is the same for every preimage e' of e. Thus if e' is such a preimage of e, then

$$c(e) = \psi^* c(e') = \sum_{e'' \in \psi^{-1}(e)} c'(e'') = n \cdot c'(e') = 0.$$

If $c(e) = 0$, then c induces a current on $\mathbf{G} \backslash e$. Let A be a connected component of $\psi^{-1}(\mathbf{G} \backslash e)$. Since ψ is a trivial covering above $\mathbf{G} \backslash e$, the component A is isomorphic to $\mathbf{G} \backslash e$ and c thus induces a current on A. One extends this current by 0 on $\mathbf{G}^{(n)} \backslash A$ to get a current c' on $\mathbf{G}^{(n)}$ for which $\psi^* c = \sum_{g \in \Gamma} g^* c'$. $\square$

Let us come back to the proof of Proposition 13. The map ϕ induces a commutative diagram:

$$\begin{array}{ccc}
\pi_1^{\text{temp}}(X_1^{(n)}) & \xrightarrow{\phi^{(n)}} & \pi_1^{\text{temp}}(X_2^{(n)}) \\
\big\downarrow & & \big\downarrow \\
\pi_1^{\text{temp}}(X_1) & \xrightarrow{\phi} & \pi_1^{\text{temp}}(X_2).
\end{array}$$

It induces a commutative diagram compatible with the actions of G

$$\begin{array}{ccc}
C(\mathbf{G}_2, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\tilde{\phi}} & C(\mathbf{G}_1, \mathbb{Z}/n\mathbb{Z}) \\
\big\downarrow \psi_2^* & & \big\downarrow \psi_1^* \\
C(\mathbf{G}_2^{(n)}, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\tilde{\phi}^{(n)}} & C(\mathbf{G}_1^{(n)}, \mathbb{Z}/n\mathbb{Z}).
\end{array}$$

Lemma 14 shows that $\tilde{\phi}(c)(e_1) = 0$ if and only if $\psi_1^* \tilde{\phi}(c)$ is a norm in $C(\mathbf{G}_1^{(n)}, \mathbb{Z}/n\mathbb{Z})$ for the G-action. This holds if and only if $\psi_2^*(c)$ is a norm in $C(\mathbf{G}_2^{(n)}, \mathbb{Z}/n\mathbb{Z})$ for the G-action, or again by Lemma 14 if and only if $c(e_2) = 0$.

An edge e of a graph $\mathbf{G}$ is said to be *unconnecting* if $\pi_0(\mathbf{G} \backslash \{e\}) \to \pi_0(\mathbf{G})$ is injective. If the evaluation map $\mathrm{ev}_e : C(\mathbf{G}_1, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ is nonzero, then e is unconnecting and ev_e is surjective.

Let e be an unconnecting edge of $\mathbf{G}_1$. Since $\tilde{\phi}$ maps $\mathrm{Ker}(\mathrm{ev}_{\bar{\phi}(e)})$ to $\mathrm{Ker}(\mathrm{ev}_e)$ one gets an isomorphism

$$\mathbb{Z}/n\mathbb{Z} = C(\mathbf{G}_2, \mathbb{Z}/n\mathbb{Z})/\,\mathrm{Ker}(\mathrm{ev}_{\bar{\phi}(e)}) \to C(\mathbf{G}_1, \mathbb{Z}/n\mathbb{Z})/\,\mathrm{Ker}(\mathrm{ev}_e) = \mathbb{Z}/n\mathbb{Z}$$

induced by $\tilde{\phi}$, that is multiplication by a unique $\lambda_e \in (\mathbb{Z}/n\mathbb{Z})^*$. This means that for every $c \in C(\mathbf{G}_1, \mathbb{Z}/n\mathbb{Z})$, we have $\tilde{\phi}(c)(e) = \lambda_e c(\bar{\phi}(e))$. One has now to prove that λ_e does not depend of e.

Let $\pi_2 : \mathbf{G}_2' \to \mathbf{G}_2$ be a finite topological covering, let $X_2' \to X_2$ be the corresponding finite topological covering of X_2, let $\pi_1 : \mathbf{G}_1' = \bar{\phi}^* \mathbf{G}_2' \to \mathbf{G}_1$ and let X_1' be the corresponding finite topological covering of X_1. Let $\phi' : \pi_1^{\mathrm{temp}}(X_1) \to \pi_1^{\mathrm{temp}}(X_2)$ be the induced isomorphism. Let e' be a preimage of e which is also unconnecting. The scalar $\lambda_{e'} \in \mathbb{Z}/n\mathbb{Z}$ induced by ϕ' turns out to be equal to λ_e because for every current $c \in C(\mathbf{G}_2, \mathbb{Z}/n\mathbb{Z})$ we have

$$\lambda_e \cdot \tilde{\phi}'(\pi_2^* c)(e') = \lambda_e \lambda_{e'} \cdot \pi_2^* c(\bar{\phi}'(e')) = \lambda_e \lambda_{e'} \cdot c(\bar{\phi}(e))$$
$$= \lambda_{e'} \cdot \tilde{\phi}(c)(e) = \lambda_{e'} \cdot \pi_1^* \tilde{\phi}(c)(e') = \lambda_{e'} \cdot \tilde{\phi}'(\pi_2^* c)(e').$$

If e_a and e_b are two unconnecting edges of $\mathbf{G}_1$, there exists a finite topological covering $\mathbf{G}_1' \to \mathbf{G}_1$, a preimage e_a' (resp. e_b') of e_a (resp. e_b) and a cycle of $\mathbf{G}_1'$ that goes through e_a' and e_b'. Let c be the $\mathbb{Z}/n\mathbb{Z}$-current on $\mathbf{G}_2'$ which follows this cycle and is zero everywhere else. Since $\tilde{\phi}'(c)$ must also be a current, one gets that λ must be constant on the cycle and thus $\lambda_{e_a} = \lambda_{e_a'} = \lambda_{e_b'} = \lambda_{e_b}$. Therefore the scalar $\lambda := \lambda_e$ does not depend of e and, for every $c \in C(\mathbf{G}_2, \mathbb{Z}/n\mathbb{Z})$ and every edge e of $\mathbf{G}_1$, we have $\tilde{\phi}(c)(e) = \lambda c(\bar{\phi}(e))$. □

We continue the proof of Theorem 12. Let L_1 be an oriented loop in $\mathbf{G}_1$, and set $L_2 = \bar{\phi}(L_1)$. Let $\widetilde{L}_1$ be an oriented path lifting L_1 in $\mathbf{T}_1$ and set $\widetilde{L}_2 = \bar{\phi}(\widetilde{L}_1)$. Then $\widetilde{L}_i =]x_i, y_i[$ for some points $x_i, y_i \in \Omega_i$. Let z_1 be a vertex of $\mathbf{T}_1$ and set $z_2 = \bar{\phi}(z_1)$. The stabilizer $H \subset \Gamma_1$ of $\widetilde{L}_1$ is the image of $\pi_1(L_1) \to \pi_1(\mathbf{G}_1) = \Gamma_1$.

We fix an integer $h' > 0$. Let Γ' be a finite index subgroup of Γ_1 such that, for every $g \in \Gamma' \backslash H$,

$$\min\{d_1(\widetilde{L}_1, g\widetilde{L}_1), d_2(\widetilde{L}_2, \bar{\phi}(g\widetilde{L}_1))\} > h'.$$

The current $c_1 = \sum_{g \in \Gamma'/(H \cap \Gamma')} g^* c_{]x_1, y_1[}$ is Γ'-equivariant, and we set $c_2 = (\bar{\phi}^{-1})^* c_1$. We consider the finite topological covering $X_1' = \Omega_1/\Gamma' \to X_1$, and set $X_2' = \phi^* X_1' = \Omega_2/\phi(\Gamma')$.

Let Y_1 be a μ_{p^h}-torsor of X_1', whose pullback S_1 to Ω_1 is induced by c_1. Let $Y_2 = (\phi^{-1})^* Y_1$ and let S_2 be its pullback to Ω_2. According to Proposition 13, S_2 is the μ_{p^h}-torsor induced by λc_2 for some $\lambda \in \mathbb{Z}/p^h\mathbb{Z}$.

For h' chosen big enough, locally around z_1 (resp. z_2), S_1 (resp. S_2) is isomorphic to the μ_{p^h}-torsor induced by $c_{]x_1, y_1[}$ (resp. $\lambda c_{]x_2, y_2[}$), hence is split at z_1 (resp. z_2) if and only if $d_1(]x_1, y_1[, z_1) > h + \frac{1}{p-1}$ (resp. $d_2(]x_2, y_2[, z_2) > h + \frac{1}{p-1}$). According to Proposition 7, those two conditions must be equivalent for any z_1 and h. In particular,

$$\left| \, d_1(]x_1, y_1[, z_1) - d_2(]x_2, y_2[, z_2) \, \right| \leq 2. \tag{1.3}$$

Let L'_1 be a loop in $\mathbf{G}_1$ and $\widetilde{L}'_1$ be a lifting of the universal covering of L'_1 to $\mathbf{T}_1$ such that $\widetilde{L}'_1 \neq \widetilde{L}_1$. This is possible since X_1 is hyperbolic and thus $\widetilde{L}_1 \neq \mathbf{T}_1$. Let $\lg_1(L'_1)$ be the length of this loop and $\lg_2(\bar{\phi}(L'_1))$ be the length of $\bar{\phi}(L'_1)$.

Let $(z_1^n)_{n\in\mathbb{Z}}$ be the family of preimages in $\widetilde{L}'_1$ of a vertex of L'_1, numbered compatibly with an orientation of $\widetilde{L}'_1$. Let z_2^n be the image of z_1^n in $\mathbf{T}_2$. Then there exists constants c_1 and c_2 such that, for $n \gg 0$, $d_1(]x_1,y_1[,z_1^n) = n\lg_1(L'_1)+c_1$ and $d_2(]x_2,y_2[,z_2^n) = n\lg_2(\bar{\phi}(L'_1))+c_2$. Thus, for $n \gg 0$,

$$n|\lg_1(L'_1)-\lg_2(\bar{\phi}(L'_1))| = |d_1(]x_1,y_1[,z_1)-d_2(]x_2,y_2[,z_2)+c_1-c_2| \leq 2+|c_1|+|c_2|.$$

Since the sequence $(n|\lg_1(L'_1)-\lg_2(\bar{\phi}(L'_1))|)_{n\in\mathbb{N}}$ is bounded, $\lg_1(L'_1) = \lg_2(\bar{\phi}(L'_1))$. Theorem 12 is thus a consequence of the following purely combinatorial statement applied to $\mathbf{G} = \mathbf{G}_1$ and $f = \lg_1 - \lg_2 \circ \bar{\phi}$:

Proposition 15 ([8, prop. A.1]) *Let $\mathbf{G}$ be finite graph such that the valency of every vertex is at least 3. Let $f : \{\text{edges of } \mathbf{G}\} \to \mathbb{R}$ be any function. Let us denote also by f the induced function on the set of edges of a topological covering of $\mathbf{G}$. Let us set $f(C) = \sum_{x\in\{\text{edges of } C\}} f(x)$ for C a loop of a covering of $\mathbf{G}$.*

If $f(C) = 0$ for every loop C of every covering of $\mathbf{G}$, then $f = 0$.

Remark 1. Theorem 12 is also true for open Mumford curves, see [7, cor. 3.4.7].

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