COVERINGS IN p-ADIC ANALYTIC GEOMETRY AND LOG COVERINGS I: COSPECIALIZATION OF THE (p')-TEMPERED FUNDAMENTAL GROUP FOR A FAMILY OF CURVES

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Abstract. The tempered fundamental group of a p-adic analytic space classifies coverings that are dominated by a topological covering (for the Berkovich topology) of a finite étale covering of the space. Here we construct cospecialization homomorphisms between (p') versions of the tempered fundamental groups of the fibers of a smooth family of curves with semistable reduction. To do so, we will translate our problem in terms of cospecialization morphisms of fundamental groups of the log fibers of the log reduction and we will prove the invariance of the geometric log fundamental group of log smooth log schemes over a log point by change of log point.

Introduction

In general topology, the fundamental group of a connected locally contractible space classifies its (unramified) coverings. A. Grothendieck developed an avatar in abstract algebraic geometry: he attached to any algebraic variety a profinite fundamental group, which classifies its finite étale coverings. For a complex algebraic variety, Grothendieck's fundamental group is canonically isomorphic to the profinite completion of the topologic fundamental group of the corresponding topological space.

Here we are interested in an analog in p-adic geometry. More precisely we will study the *tempered fundamental group* of p-adic varieties defined by Y. André. The profinite completion of the tempered fundamental group of any smooth p-adic algebraic variety coincides with Grothendieck's algebraic fundamental group. It also accounts for the usual (infinite) "uniformizations" in p-adic analytic geometry such as the uniformization of Tate elliptic curves, which are historically at the very basis of p-adic rigid geometry. Such uniformizations give infinite discrete quotients of the tempered fundamental group.

Since the analytification (in the sense of V. Berkovich or of rigid geometry) of a finite étale covering of a p-adic algebraic variety is not necessarily a topological covering, André had to consider a category of coverings slightly bigger than just the category of topological coverings. He defined tempered coverings, which are (possibly infinite) étale coverings in the sense of A.J. de Jong (that is to say, which are, Berkovich-locally on the base, direct sums of finite coverings) such that, after pulling back by some finite étale covering, they become topological coverings (for the Berkovich topology). The *tempered fundamental group* is the prodiscrete group that classifies those tempered coverings. To give a more handful description, if one has a sequence of pointed finite Galois connected coverings $((S_i, s_i))_{i\in\mathbf{N}}$ such

that the corresponding pointed pro-covering of (X,x) is the universal pro-covering of (X,x), and if (S_i^∞, s_i^∞) is a universal topological covering of S_i, the tempered fundamental group of X can be seen as $\pi_1^{\mathrm{temp}}(X,x) = \varprojlim_i \mathrm{Gal}(S_i^\infty/X)$. Therefore, to understand the tempered fundamental group of a variety, one has to understand the topological behavior of its finite étale coverings.
In the case of a curve, the question becomes more concrete since there is a natural embedding of the geometric realization of the graph of its stable model into the Berkovich space of the curve which is a homotopy equivalence.
Among applications of tempered fundamental groups, let us cite in passing the theory of p-adic orbifolds and p-adic triangle groups [And03b] and a p-adic version of Grothendieck-Teichmüller theory [And03a].

In this article we will be interested in the variation of the tempered fundamental group of the fibers of a family of curves. This article will be followed by another one [Lep], in which we will consider higher dimensional families.
For a proper morphism of schemes $f: X \to Y$ with geometrically connected fibers and a specialization $\bar{y}_1 \to \bar{y}_2$ of geometric points of Y, A. Grothendieck has defined algebraic fundamental groups $\pi_1^{alg}(X_{\bar{y}_i})$ and a specialization homomorphism $\pi_1^{alg}(X_{\bar{y}_1}) \to \pi_1^{alg}(X_{\bar{y}_2})$. Grothendieck's specialization theorem tells that this homomorphism is surjective if f is separable and induces an isomorphism between the prime-to-p quotients if f is smooth (here, p denotes the characteristic of $\bar{y}_2$), cf. [Gro71, cor. X.2.4, cor. X.3.9].
In complex analytic geometry, a smooth and proper morphism is locally a trivial fibration of real differential manifolds, so that, in particular, all the fibers are homeomorphic, and thus have isomorphic (topological) fundamental groups.
The aim of this paper is to find some analog of the specialization theorem of Grothendieck in the case of the tempered fundamental group.
In this paper, we will concentrate on the case of curves.

One problem which appears at once in looking for some non archimedean analog of Grothendieck's specialization theorems is that there are in general no non trivial specializations between distinct points of a non archimedean analytic (Berkovich or rigid) space: for example a separated Berkovich space has a Hausdorff underlying topological space, so that if there is a cospecialization (for the Berkovich topology, the étale topology...) between two geometric points of a Berkovich space, the two geometric points must have the same underlying point. Thus we will assume we have a model over the ring of integers of our non-archimedean field (with good enough properties) and we will look at the specializations in the special fiber.

We want to understand how the tempered fundamental group of the geometric fibers of a smooth family varies. Let us for instance consider a family of elliptic curves. The tempered fundamental group of an elliptic curve over a complete algebraically closed non archimedean closed field is $\widehat{\mathbf{Z}}^2$ if it has good reduction, and $\widehat{\mathbf{Z}} \times \mathbf{Z}$ if it is a Tate curve. In particular, by looking at a moduli space of stable pointed elliptic curves with level structure[1], the tempered fundamental group (or

[1] to avoid stacks. However, the cospecialization homomorphisms we will construct will be local for the étale topology of the special fiber of the base. Thus, the fact that the base is a Deligne-Mumford stack is not really a problem.

any reasonable (p')-version) cannot be constant.
Moreover, if one looks at the moduli space over $\mathbf{Z}_p$, and considers a curve E_1 with bad reduction and a curve E_0 with generic reduction (hence good reduction), there cannot be a morphism $\pi_1^{\mathrm{temp}}(E_0) \to \pi_1^{\mathrm{temp}}(E_1)$ which induces Grothendieck's specialization on the profinite completion, although the reduction point corresponding to E_1 specializes to the reduction point corresponding to E_0. Therefore there cannot be any reasonable specialization theory.
On the contrary, if one has two geometric points η_1 and η_2 of the moduli space such that the reduction of η_1 specializes to the reduction of η_2, then E_{η_1} has necessarily better reduction than E_{η_2} and there is some morphism $\pi_1^{\mathrm{temp}}(E_{\eta_2}) \to \pi_1^{\mathrm{temp}}(E_{\eta_1})$ that induces an isomorphism between the profinite completions. Thus we want to look for a *cospecialization* of the tempered fundamental group.

The topological behavior of general finite étale coverings is too complicated to hope to have a simple cospecialization theory without adding a (p') condition on the coverings: for example two Mumford curves over some finite extension of $\mathbf{Q}_p$ with isomorphic geometric tempered fundamental group have the same metrized graph of stable reduction. Thus even if two Mumford curves have isomorphic stable reduction (and thus the point corresponding to their stable reduction is the same), they may not have isomorphic tempered fundamental group in general. Thus we will only study here finite coverings that are dominated by a finite Galois covering whose order is prime to p, where p is the residual characteristic (which can be 0; such a covering will be called a (p')-finite covering). Then, it becomes natural to introduce a (p')-tempered fundamental group which classifies tempered coverings that become topological coverings after pullback along some (p')-finite covering. It should be remarked that this (p')-tempered fundamental group cannot be in general recovered from the tempered fundamental group: for example, if $p=2$, $\pi_1^{\mathrm{temp}}(X)^{(p')} = \{1\}$ for an Enriques surface over $\mathbf{C}_p$ with good reduction and $\pi_1^{\mathrm{temp}}(X)^{(p')} = \mathbf{Z}/2\mathbf{Z}$ for an Enriques surface with semistable reduction whose dual simplicial complex is homotopic to a real projective plane, whereas, in both cases, $\pi_1^{\mathrm{temp}}(X) = \mathbf{Z}/2\mathbf{Z}$. However, in the case of a curve, one can recover the (p')-tempered fundamental group from the tempered fundamental group.
The (p')-tempered fundamental group of a curve was already studied by S. Mochizuki in [Moc06]. It can be described in terms of a graph of profinite groups. From this description, one easily sees that the isomorphism class of the (p')-tempered fundamental group of a p-adic curve depends only of the stratum of the Knudsen stratification of the moduli space of stable curves in which the stable reduction lies. Moreover if one has two strata x_1 and x_2 in the moduli space of stable curves such that x_1 is in the closure of x_2, one can easily construct morphisms from the graph of groups corresponding to x_1 to the graph of groups corresponding to x_2 (inducing morphisms of tempered fundamental groups which induce isomorphisms of the pro-(p') completions).

We shall study the following situation. Let O_K be a complete discrete valuation ring, K be its fraction field, k be its residue field and p be its characteristic (which can be 0). Let X be a proper semistable pointed curve over O_K smooth over K and let $U \subset X_\eta$ be the complement of the marked points. Let us describe the tempered fundamental group of $U_{\bar{\eta}}^{\mathrm{an}}$ in terms of $X_{\bar{s}}$ ([Moc06]).

Let us make sure at first that we can get such a description for the pro-(p') completion, *i.e.* the algebraic fundamental group. One cannot apply directly Grothendieck's specialization theorems (even if $U = X_\eta$) since the special fiber is not smooth but only semistable. Indeed, a pro-(p') geometric covering of the generic fiber will generally only induce a Kummer covering on the special fiber. These are naturally described in terms of *log geometry*, more precisely in terms of *Kummer-étale* (két) coverings of a log scheme. One can endow X (and thus X_s too by restriction) with a natural log structure such that the pro-(p') fundamental group of U is isomorphic to a pro-(p') log fundamental group (as defined in [Ill02]) of $X_{\bar{s}}$. One then gets a description of $\pi_1^{\mathrm{alg}}(U_{\bar{\eta}})$ by taking the projective limit under tame coverings of K, or equivalently under két coverings of s endowed with its natural log structure: there is an equivalence between finite étale coverings of $U_{\bar{\eta}}$ and "geometric két coverings" of $X_{\bar{s}}$.
A két covering of X will still be a semistable model of its generic fiber if one replaces K by some tame extension. Thus, one can describe the topology of the corresponding covering of $U_{\bar{\eta}}$ in terms of the graph of the corresponding geometric két covering of $X_{\bar{s}}$.

Let us now come back to the problem of cospecialization. Let $X \to Y$ be a semistable curve over O_K with $X \to Y$ endowed with compatible log structure (see definition 2.11).
Let $\bar{\eta}_1$ (resp. $\bar{\eta}_2$) be a (Berkovich) geometric point of $Y_0 := Y_{\mathrm{tr}}^{\mathrm{an}} \cap \mathfrak{Y}_\eta \subset Y_K^{\mathrm{an}}$, where Y_{tr} is the locus of Y where the log structure is trivial and $\mathfrak{Y}_\eta$ is the generic fiber of the formal completion of Y along its closed fiber (if Y is proper, then $Y_0 = Y_{\mathrm{tr}}^{\mathrm{an}}$).
Let $\bar{s}_1$ (resp. $\bar{s}_2$) be its log reduction in Y_s .
To use the previous description of the tempered fundamental group of $U_{\bar{\eta}_1}$ and $U_{\bar{\eta}_2}$ in terms of $X_{\bar{s}_1}$ and $X_{\bar{s}_2}$, we have to assume that $\bar{\eta}_1$ and $\bar{\eta}_2$ lie over Berkovich points with discrete valuation.

The main result of this paper is the following:

Theorem 0.1 (th. 3.8)**.** *Let K be a complete discretely valued field. Let $\mathbb{L}$ be a set of primes that does not contain the residual characteristic of K. Let $Y \to O_K$ be a morphism of log schemes. Let $Y_0 = Y_{\mathrm{tr}}^{\mathrm{an}} \cap \mathfrak{Y}_\eta \subset Y^{\mathrm{an}}$ where $\mathfrak{Y}$ is the completion of Y along its closed fiber. Let $X \to Y$ be a proper semistable curve with compatible log structure. Let $U = X_{\mathrm{tr}}$. Let η_1 and η_2 be two Berkovich points of Y_0 whose residue fields have discrete valuation, and let $\bar{\eta}_1, \bar{\eta}_2$ be geometric points above them. Let $\bar{s}_2 \to \bar{s}_1$ be a log specialization of their log reductions such that there exists a compatible specialization $\bar{\eta}_2 \to \bar{\eta}_1$, then there is a cospecialization homomorphism $\pi_1^{\mathrm{temp}}(U_{\bar{\eta}_1})^{\mathbb{L}} \to \pi_1^{\mathrm{temp}}(U_{\bar{\eta}_2})^{\mathbb{L}}$. Moreover, it is an isomorphism if $\overline{M}_{Y,\bar{s}_1} \to \overline{M}_{Y,\bar{s}_2}$ is an isomorphism.*

Let us come back to our example of the moduli space of pointed stable elliptic curves with high enough level structure M over O_K, and let C be the canonical stable elliptic curve on M. If η_1 and η_2 are two Berkovich points of M_η, they are in M_η^{tr} if and only if C_{η_1} and C_{η_2} are smooth. $C \to M$, endowed with their natural log-structures over (O_K, O_K^*), is a semistable morphism of log schemes. One thus get a cospecialization outer morphism $\pi_1^{\mathrm{temp}}(C_{\overline{\eta}_1}) \to \pi_1^{\mathrm{temp}}(C_{\overline{\eta}_2})$ for every specialization $\bar{s}_2 \to \bar{s}_1$, which is an isomorphism if $\bar{s}_1$ and $\bar{s}_2$ are in the same stratum of M_s. Since the moduli stack of pointed stable elliptic curves over $\operatorname{Spec} k$ has only two strata, one corresponding to smooth elliptic curves M_0 and one to singular curves M_1, one gets that $\pi_1^{\mathrm{temp}}(E_1)^{(p')} \simeq \pi_1^{\mathrm{temp}}(E_2)^{(p')}$ if E_1 and E_2 are two curves with good reduction or two Tate curves (the isomorphism depends on choices of cospecializations). Since M_1 is in the closure of M_0 one gets a morphism from the tempered fundamental group of a Tate curve to the tempered fundamental group of an elliptic curve with good reduction.

The first thing we need in order to construct the cospecialization homomorphism for tempered fundamental groups is a specialization morphism between the (p')-log geometric fundamental groups of $X_{\bar{s}_1}$ and $X_{\bar{s}_2}$. Such a specialization morphism will be constructed by proving that one can extend any (p')-log geometric covering of X_{s_1} to a két covering of X_U where U is some két neighborhood of s_1. If one has such a specialization morphism, by comparing it to the fundamental groups of $X_{\bar{\eta}_1}$ and $X_{\bar{\eta}_2}$ and using Grothendieck's specialization theorem, we will easily get that it must be an isomorphism. This specialization morphism is easily deduced from [Org] if s_1 is a strict point of Y (*i.e.* the log structure of s_1 is simply the one induced by Y), *i.e.* the log structure of s_1 is just the pull back of the log structure of Y, but is not straightforward when the log structure is really modified. Thus we will study the invariance of the log geometric fundamental group by change of fs base point. The main result we will prove (in any dimension) is the following :

Theorem 0.2 (th. 2.15)**.** *Let $s' \to s$ be a morphism of fs log points with isomorphic algebraically closed underlying fields. Let $X \to s$ be a saturated morphism of log schemes with X noetherian and let $X' \to s'$ be the pull back to s'. Then the map $\pi_1^{\text{log-geom}}(X'/s', \bar{x}') \to \pi_1^{\text{log-geom}}(X/s, \bar{x})$ is an isomorphism.*

It is interesting to notice that, in this situation, this is an isomorphism for the full fundamental group, and not only of the pro-(p') part. This mainly comes from the fact that the morphism of underlying schemes $\mathring{X}' \to \mathring{X}$ is an isomorphism (so that the problem only comes from the logarithmic structure and not the schematic structure). This result is proved by a local study on X for the strict étale topology. The log geometric fundamental group can be described in terms of an inductive filtered limit of categories of coverings. One then has to study an inductive filtered limit of stacks over $X_{\text{ét}}$.

Then we have to construct cospecialization topological morphisms for a semistable curve, more precisely cospecialization morphisms of the graphs of the geometric fibers. This will be done étale locally. These morphisms are not morphisms of graphs in the usual sense, since an edge can be contracted over a vertex, but still give a map between their geometric realizations, whence a map of homotopy types $U_{\bar{\eta}_1} \to U_{\bar{\eta}_2}$. This can also be done for any két covering of $X_{\bar{s}_1}$: we thus get such a map of homotopy types for every (p')-covering of $U_{\bar{\eta}_1}$. Those maps are compatible, and thus glue together to give the wanted cospecialization of tempered fundamental groups.

The paper is organized as follows.

In the first section, we will recall the basic notions about tempered fundamental groups and define the (p')-versions. We will also recall what we will need about graphs.

In the second section, we will recall the definition of log fundamental groups. We will then study specialization of log fundamental groups in paragraph 2.4.

Finally, in a third part, we will construct the cospecialization morphisms of graphs of the geometric fibers of a semistable curve. We will verify the compatibility with két morphisms to obtain the wanted cospecialization morphisms of tempered fundamental groups.

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1. Preliminaries

1.1. **Tempered fundamental groups.** Let K be a complete nonarchimedean field.
Let $\mathbb{L}$ be a set of prime numbers (for example, we will denote by (p') the set of all primes except the residual characteristic p of K). An $\mathbb{L}$-integer will be an integer which is a product of elements of $\mathbb{L}$.
If $X \subset \overline{X}$ is a K-algebraic variety, X^{an} will be the K-analytic space in the sense of Berkovich associated to X. A morphism $f: S' \to S$ of analytic spaces is said to be an *étale covering* if S is covered by open subsets U such that $f^{-1}(U) = \coprod V_j$ and $V_j \to U$ is étale finite ([dJ95]).
For example, étale $\mathbb{L}$-finite coverings (*i.e.* finite étale coverings that are dominated by a Galois covering S'' of S such that $\#\,\mathrm{Gal}(S''/S)$ is an $\mathbb{L}$-integer), also called $\mathbb{L}$-*algebraic coverings*, and coverings in the usual topological sense for the Berkovich topology, also called *topological coverings*, are étale coverings.
Then, André defines tempered coverings in [And03b, def. 2.1.1]. We generalize this definition to $\mathbb{L}$-tempered coverings as follows:

Definition 1.1. An étale covering $S' \to S$ is $\mathbb{L}$-*tempered* if it is a quotient of the composition of a topological covering $T' \to T$ and of a $\mathbb{L}$-finite étale covering $T \to S$.

This is equivalent to say that it becomes a topological covering after pullback by some $\mathbb{L}$-finite étale covering.
Let X be a K-analytic space. We denote by $\mathrm{Cov}^{\mathrm{temp}}(X)^{\mathbb{L}}$ (resp. $\mathrm{Cov}^{\mathrm{alg}}(X)^{\mathbb{L}}$, $\mathrm{Cov}^{\mathrm{top}}(X)$) the category of $\mathbb{L}$-tempered coverings (resp. $\mathbb{L}$-algebraic coverings, topological coverings) of X (with the obvious morphisms).

A geometric point of a K-analytic space X is a morphism of Berkovich spaces $\mathcal{M}(\Omega) \to X$ where Ω is an algebraically closed complete isometric extension of K. Let $\bar{x}$ be a geometric point of X. Then one has a functor

$$F_{\bar{x}}^{\mathbb{L}} : \mathrm{Cov}^{\mathrm{temp}}(X)^{\mathbb{L}} \to \mathrm{Set}$$

which maps a covering $S \to X$ to the set $S_{\bar{x}}$.
The tempered fundamental group of X pointed at $\bar{x}$ is

$$\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}} = \mathrm{Aut}\, F_{\bar{x}}^{\mathbb{L}}.$$

When X is a smooth algebraic K-variety, $\mathrm{Cov}^{\mathrm{temp}}(X^{\mathrm{an}})^{\mathbb{L}}$ and $\pi_1^{\mathrm{temp}}(X^{\mathrm{an}}, \bar{x})^{\mathbb{L}}$ will also be denoted simply by $\mathrm{Cov}^{\mathrm{temp}}(X)^{\mathbb{L}}$ and $\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}}$.
By considering the stabilizers $(\mathrm{Stab}_{F_{\bar{x}}^{\mathbb{L}}(S)}(s))_{S \in \mathrm{Cov}^{\mathrm{temp}}(X)^{\mathbb{L}}, s \in F_{\bar{x}}^{\mathbb{L}}(S)}$ as a basis of open subgroups of $\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}}$, $\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}}$ becomes a topological group. It is a prodiscrete topological group.
When X is algebraic, K of characteristic zero and has only countably many finite extensions in a fixed algebraic closure $\overline{K}$, $\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}}$ has a countable fundamental system of neighborhood of 1 and all its discrete quotient groups are finitely generated ([And03b, prop. III.2.1.7]).
When $\mathbb{L}$ is the set of all primes, we often forget it in the notations. It should be remarked that in general, for a given $\mathbb{L}$, one cannot recover $\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}}$ from $\pi_1^{\mathrm{temp}}(X, \bar{x})$.

If $\bar{x}$ and $\bar{x}'$ are two geometric points, then $F_{\bar{x}}^{\mathbb{L}}$ and $F_{\bar{x}'}^{\mathbb{L}}$ are (non canonically) isomorphic ([dJ95, th. 2.9]). Thus, as usual, the tempered fundamental group depends on the basepoint only up to inner automorphism (this topological group, considered up to conjugation, will sometimes be denoted simply $\pi_1^{\mathrm{temp}}(X)^{\mathbb{L}}$).
The full subcategory of tempered coverings S for which $F_{\bar{x}}^{\mathbb{L}}(S)$ is $\mathbb{L}$-finite is equivalent to $\mathrm{Cov}^{\mathrm{alg}}(S)^{\mathbb{L}}$, hence

$$\left(\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}}\right)^{\mathbb{L}} = \pi_1^{\mathrm{alg}}(X, \bar{x})^{\mathbb{L}}$$

(where $(\ )^{\mathbb{L}}$ denotes the pro-$\mathbb{L}$ completion).
For any morphism $X \to Y$, the pullback defines a functor $\mathrm{Cov}^{\mathrm{temp}}(Y)^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{temp}}(X)^{\mathbb{L}}$. If $\bar{x}$ is a geometric point of X with image $\bar{y}$ in Y, this gives rise to a continuous homomorphism

$$\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}} \to \pi_1^{\mathrm{temp}}(Y, \bar{y})^{\mathbb{L}}$$

(hence an outer morphism $\pi_1^{\mathrm{temp}}(X)^{\mathbb{L}} \to \pi_1^{\mathrm{temp}}(Y)^{\mathbb{L}}$).
One has the analog of the usual Galois correspondence:

Theorem 1.2 (([And03b, th. III.1.4.5])). *$F_{\bar{x}}^{\mathbb{L}}$ induces an equivalence of categories between the category of direct sums of $\mathbb{L}$-tempered coverings of X and the category $\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}}$-Set of discrete sets endowed with a continuous left action of $\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}}$.*

If S is a $\mathbb{L}$-finite Galois covering of X, its universal topological covering S^{∞} is still Galois and every connected $\mathbb{L}$-tempered covering is dominated by such a Galois $\mathbb{L}$-tempered covering.
If $((S_i, \bar{s}_i))_{i \in \mathbf{N}}$ is a cofinal projective system (with morphisms $f_{ij} : S_i \to S_j$ which maps s_i to s_j for $i \geq j$) of geometrically pointed Galois $\mathbb{L}$-finite étale coverings of $(X, \bar{x})$, let $((S_i^{\infty}, \bar{s}_i^{\infty}))_{i \in \mathbf{N}}$ be the projective system of its pointed universal topological coverings (the transition maps will be denoted by f_{ij}^{∞}). It induces a projective system $(\mathrm{Gal}(S_i^{\infty}/X))_{i \in \mathbf{N}}$ of discrete groups. For every i, $\mathrm{Gal}(S_i^{\infty}/X)$ can be identified with $F_{\bar{x}}^{\mathbb{L}}(S_i^{\infty})$: this gives us compatible morphisms $\pi_1^{\mathrm{temp}}(X, \bar{x})^{\mathbb{L}} \to \mathrm{Gal}(S_i^{\infty}/X)$. Then, thanks to [And03b, lem. III.2.1.5],

Proposition 1.3.

$$\pi_1^{\mathrm{temp}}(X,\bar{x})^{\mathbb{L}} \to \varprojlim \mathrm{Gal}(S_i^\infty/X)$$

is an isomorphism.

In a more categorical way, we have a fibered category $\mathcal{D}_{\mathrm{top}}(X) \to \mathrm{Cov}^{\mathrm{alg}}(X)$, where the fiber $\mathcal{D}_{\mathrm{top}}(X)_S$ in an algebraic covering S of X is $\mathrm{Cov}^{\mathrm{top}}(X)$.
Since algebraic coverings are of effective descent for tempered coverings, the full subcategory of tempered coverings T of X such that $T_S \to S$ is a topological covering is naturally equivalent to the category $\mathrm{DD}_{\mathrm{temp}_S}$ of descent data in the fibered category $\mathcal{D}_{\mathrm{top}}(X)$ with respect to $S \to X$.
If "$\varprojlim$" S_i is a universal procovering of (X,x), one gets a natural equivalence

$$\mathrm{Cov}^{\mathrm{temp}}(X) = \underset{\longleftarrow i}{\mathrm{Lim}}\, \mathrm{DD}_{\mathrm{temp}_{S_i}}$$

In particular one can recover the tempered fundamental group from the fibered category $\mathcal{D}_{\mathrm{top}}(X) \to \mathrm{Cov}^{\mathrm{alg}}(X)$.
If $S \to S$ is an isomorphism, the induced functor $\mathrm{DD}_{\mathrm{temp}_S} \to \mathrm{DD}_{\mathrm{temp}_S}$ is naturally isomorphic to the identity. Thus if $\alpha :$ "$\varprojlim_i$" $S_i \to$ "$\varprojlim_i$" S_i is an automorphism of the universal pro-covering, the induced functor $\underset{\longleftarrow i}{\mathrm{Lim}}\, \mathrm{DD}_{\mathrm{temp}_{S_i}} \to \underset{\longleftarrow i}{\mathrm{Lim}}\, \mathrm{DD}_{\mathrm{temp}_{S_i}}$ is naturally isomorphic to the identity. Thus the construction does not depend of the choice of the universal procovering.

To give a more stacky and functorial description, let us consider $\mathrm{Cov}^{\mathrm{alg}}(X)$ with its canonical topology. $\mathcal{D}_{\mathrm{top}}(X)$ is a separated prestack on $\mathrm{Cov}^{\mathrm{alg}}(X)$ (*i.e.* just a fibered category; it is not a prestack of groupoids).
Let $\mathcal{D}_{\mathrm{temp}}(X) \to \mathrm{Cov}^{\mathrm{alg}}(X)$ be the fibered category whose fiber over U is the category $\mathrm{Cov}^{\mathrm{temp}}(U)$ of tempered coverings of U. Then $\mathcal{D}_{\mathrm{temp}}(X)$ is a stack. The fully faithful cartesian functor of prestacks $\mathcal{D}_{\mathrm{top}}(X) \to \mathcal{D}_{\mathrm{temp}}(X)$ induces a fully faithful cartesian functor of stacks $\mathcal{D}_{\mathrm{top}}(X)^a \to \mathcal{D}_{\mathrm{temp}}(X)$ where $\mathcal{D}_{\mathrm{top}}(X)^a$ is the stack associated to $\mathcal{D}_{\mathrm{top}}(X)$. Since a tempered covering is a topological covering locally on $\mathrm{Cov}^{\mathrm{alg}}(X)$, this functor is in fact an equivalence ([Gir71, th. II.2.1.3]).
In a similar way:

Proposition 1.4. *The stack* $(\mathcal{D}_{\mathrm{top}}(X)|_{\mathrm{Cov}^{\mathrm{alg}}(X)^{\mathbb{L}}})^a$ *is the stack* $\mathcal{D}_{\mathrm{temp}}(X)^{\mathbb{L}}$ *of* $\mathbb{L}$*-tempered coverings on* $\mathrm{Cov}^{\mathrm{alg}}(X)^{\mathbb{L}}$.

1.2. **Graphs.** A *graph* $\mathbb{G}$ is given by a set of edges $\mathcal{E}$ a set of vertices $\mathcal{V}$ and for any $e \in \mathcal{E}$ a set of branches $\mathcal{B}_e$ of cardinality 2 and a map $\psi_e : \mathcal{B}_e \to \mathcal{V}$. A branch b of e can be thought of as an orientation of e (or a half-edge), and $\psi_e(b)$ is to be thought of as the ending of e when e is oriented according to b.
One can also equivalently replace the data of edges and branches of each edge by the datum of the set of all branches $\mathcal{B} = \coprod_e \mathcal{B}_e$, with an involution ι without fixed points (which corresponds heuristically to the reversing of the orientation given by the branch), and a map $\psi : \mathcal{B} \to \mathcal{V}$. $\mathcal{E}$ is then the set of orbits of branches for ι.
A *genuine morphism of graphs* $\phi : \mathbb{G} \to \mathbb{G}'$ is given by a map $\phi_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}'$, a map $\phi_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$ and for every $e \in \mathcal{E}$ a bijection $\phi_e : \mathcal{B}_e \to \mathcal{B}'_{\phi_{\mathcal{E}}(e)}$ such that the following

diagram commutes:

$$\begin{array}{ccc} \mathcal{B}_e & \longrightarrow & \mathcal{B}'_{\phi_{\mathcal{E}}(e)} \\ \downarrow & & \downarrow \\ \mathcal{V} & \longrightarrow & \mathcal{V}' \end{array}$$

Remark that $\phi_{\mathcal{E}}$ and $\phi_{\mathcal{V}}$ are not enough to define ϕ if $\mathbb{G}$ has a loop (*i.e.* an edge whose to branches abut to the same vertex): one can define an automorphism of $\mathbb{G}$ just by inverting the two branches of the loop. Thus, to know how the branches are mapped is important as soon as $\mathbb{G}$ or $\mathbb{G}'$ has loops.

The topological cospecialization for semistable curves will be given by maps of graphs which are not genuine morphisms. A *generalized morphism of graphs* $\phi : \mathbb{G} \to \mathbb{G}'$ will be given by:

- a map $\phi_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$,
- a map $\phi_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}' \coprod \mathcal{V}'$,
- for any $e \in \mathcal{E}$ such that $\phi_{\mathcal{E}}(e) \in \mathcal{E}'$, a bijection $\phi_e : \mathcal{B}_e \to \mathcal{B}'_{\phi_{\mathcal{E}}(e)}$ such that the obvious diagram commutes (it is the same diagram as in the case of genuine morphisms).

One can replace the last two data by the data of $\phi_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}' \coprod \mathcal{V}'$ such that if $\phi_{\mathcal{B}}(b) \in \mathcal{B}'$ then $\phi_{\mathcal{B}}(\iota(b)) = \iota'(\phi_{\mathcal{B}}(b))$ and if $\phi_{\mathcal{B}}(b) \in \mathcal{V}'$, $\phi_{\mathcal{B}}(\iota(b)) = \phi_{\mathcal{B}}(b)$.
In particular, a genuine morphism is a generalized morphism. There is an obvious composition of morphisms and generalized morphisms of graphs.
One thus gets a category Graph of graphs with genuine morphisms and a category GenGraph of graphs with generalized morphisms.

There is a geometric realization functor $|\ | : \mathrm{GenGraph} \to \mathrm{Top}$ which maps a graph $\mathbb{G}$ to

$$|\mathbb{G}| := \mathrm{Coker}(\coprod_{b \in \mathcal{B}} \mathrm{pt}_{1,b} \amalg \mathrm{pt}_{2,b} \rightrightarrows \coprod_{v \in \mathcal{V}} \mathrm{pt}_v \amalg \coprod_{b \in \mathcal{B}} [1/2, 1]_b),$$

where

- the upper map sends:
  - $\mathrm{pt}_{1,b}$ to $1/2$ in $[1/2, 1]_b$
  - $\mathrm{pt}_{2,b}$ to 1 in $[1/2, 1]_b$,
- the lower map sends
  - $\mathrm{pt}_{1,b}$ to $1/2$ in $[1/2, 1]_{\iota(b)}$
  - $\mathrm{pt}_{2,b}$ to $\mathrm{pt}_{\psi(b)}$.

If $\phi : \mathbb{G} \to \mathbb{G}'$ is a generalized morphism, $|\phi|$ is obtained by mapping

- pt_v to $\mathrm{pt}_{\phi_{\mathcal{V}}(v)}$,
- $[1/2, 1]_b$ to $[1/2, 1]_{\phi_{\mathcal{B}}(b)}$ if $\phi_{\mathcal{B}}(b) \in \mathcal{B}'$ (by the identity of $[1/2, 1]$),
- $[1/2, 1]_b$ to $\mathrm{pt}_{\phi_{\mathcal{B}}(b)}$ if $\phi_{\mathcal{B}}(b) \in \mathcal{V}'$.

Remark that, if $\mathbb{G}$ is just a loop, then the geometric realization of the morphism induced by inverting the two branches is not homotopic to the identity: thus $\phi_{\mathcal{E}}$ and $\phi_{\mathcal{V}}$ are not enough in general to characterize the topological behavior of ϕ.

2. Specialisation of log fundamental groups

The main result of this part will be the invariance of the log geometric fundamental group announced in theorem 0.2. We will deduce from it morphisms of specialization for the pro-(p') log geometric fundamental group of the fibers of a proper log smooth saturated morphism.

2.1. Log fundamental groups.

For a curve with bad reduction, one cannot apply Grothendieck's specialization theorem to describe the geometric fundamental group of the curve in terms of the fundamental group of its stable reduction since the family is not smooth. However, such a comparison result exists in the realm of log geometry. More precisely, if one considers a smooth and proper variety with semistable reduction, the semistable model can naturally be endowed with a log structure, and the pro-(p') fundamental group of the variety is canonically isomorphic to the pro-(p') log fundamental group of the semistable reduction. Here we recall the basic definitions and results about log fundamental groups.

First, recall some usual notations about log schemes. If X is a log scheme, the sheaf of monoid defining its log structure will usually be denoted by M_X, the underlying scheme will be denoted by $\mathring{X}$, and the open subset of $\mathring{X}$ where the log structure is trivial will be denoted X_{tr}.
If P is a monoid, one denotes by $\operatorname{Spec} P$ the set of primes of P. There is a natural map $\operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} P$.

Definition 2.1. A morphism $h : Q \to P$ of fs monoids is *Kummer* (resp. $\mathbb{L}$-Kummer) if h is injective and for every $a \in P$, there exists a positive integer (an $\mathbb{L}$-integer) n such that $a^n \in h(Q)$ (note that if $Q \to P$ is Kummer, $\operatorname{Spec} P \to \operatorname{Spec} Q$ is an homeomorphism).
A morphism $f : X \to Y$ of fs log schemes is said to be *Kummer* (resp. *exact*) if for every geometric point $\bar{x}$ of X, $\overline{M}_{Y,f(\bar{x})} \to \overline{M}_{X,\bar{x}}$ is Kummer (resp. exact).
A morphism of fs log scheme is *Kummer étale* (or két for short) if it is Kummer and log étale.

A morphism f is két if and only if étale locally it is deduced by strict base change and étale localization from a map $\operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} \mathbf{Z}[Q]$ induced by a Kummer map $Q \to P$ such that $nP \subset Q$ for some n invertible on X.
In fact if $f : Y \to X$ is két, $\bar{y}$ is a geometric point of Y, and $P \to M_X$ is an exact chart of X at $f(\bar{y})$, there is an étale neighborhood U of $\bar{x}$ and a Zariski open neighborhood $V \subset f^{-1}(U)$ of $\bar{y}$ such that $V \to U$ is isomorphic to $U \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$ with $P \to Q$ a $\mathbb{L}$-Kummer morphism where $\mathbb{L}$ is the set of primes invertible on U ([Sti02, Prop. 3.1.4]).
Két morphisms are open and quasi-finite.

The category of két fs log schemes over X (any X-morphism between two such fs log schemes is then két) where the covering families $(T_i \to T)$ of T are the families that are set-theoretical covering families (being a set-theoretical covering két family is stable under fs base change) is a site. We will denote by $X_{\text{két}}$ the corresponding topos. Any locally constant finite object of $X_{\text{két}}$ is representable.

Definition 2.2. A két fs log scheme over X that represents such a locally constant finite sheaf is called a *két covering* of X. The category of két coverings of $X_{\text{két}}$ is denoted by $\mathrm{KCov}(X)$.

A log geometric point is a log scheme s such that $\mathring{s}$ is the spectrum of a separably closed field k and M_s is saturated and multiplication by n on $\overline{M}_s$ is an isomorphism for every n prime to the characteristic of k.
A log geometric point of X is a morphism $x : s \to X$ of log schemes where s is a log geometric point. A két neighborhood U of s in X is a morphism $s \to U$ of X-log schemes where $U \to X$ is két. Then if x is a log geometric point of X, the functor F_x from $X_{\text{két}}$ to Set defined by $\mathcal{F} \mapsto \varinjlim_U \mathcal{F}(U)$ where U runs through the directed category of két neighborhoods of x is a point of the topos $X_{\text{két}}$ and any point of this topos is isomorphic to F_x for some log geometric point and the family of points (F_x) where x runs through log geometric points of X is a conservative system of points.

Definition 2.3. The inverse limit in the category of saturated log schemes of the két neighborhoods of x is called the *log strict specialization*, and is denoted by $X(x)$. If x and y are log geometric points of x, a *specialization* of log geometric points $x \to y$ is a morphism $X(x) \to X(y)$ over X.

A specialization $x \to y$ induces a canonical morphism $F_y \to F_x$ of functors.
If there is a specialization $x \to y$ of the underlying topological points, then there is some specialization $x \to y$ of log geometric points.
If X is connected, for any log geometric point x of X, F_x induces a fundamental functor $\mathrm{KCov}(X) \to \mathrm{fSet}$ of the Galois category $\mathrm{KCov}(X)$.

Definition 2.4. The *két fundamental group* $\pi_1^{\log}(X, x)$ is the profinite group of automorphisms of the fundamental functor $\mathrm{KCov}(X) \to \mathrm{fSet}$.

Strict étale surjective morphisms satisfy effective descent for két coverings ([Sti02, prop. 3.2.19]).
If $f : S' \to S$ is an exact morphism of fs log schemes such that $\mathring{f}$ is proper, surjective and of finite presentation, then f satisfies effective descent for két coverings ([Sti02, th. 3.2.25]).

Let us now state the main results to compare log fundamental groups of different log schemes (in particular specialization comparisons). According to [Ill02, th. 7.6], if X is a log regular fs log scheme, $\mathrm{KCov}(X)$ is equivalent to the category of tamely ramified coverings of X_{tr}. If $\mathbb{L}$ is a set of primes invertible on X, by taking the pro-$\mathbb{L}$ completion, one gets:

Theorem 2.5. *If X is a log regular fs log scheme and all the primes of $\mathbb{L}$ are invertible on X, then $\mathrm{KCov}(X)^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(X_{\text{tr}})^{\mathbb{L}}$ is an equivalence of categories.*

For example, if X is a regular scheme and D is a normal crossing divisor, $M_X = O_X \cap j_* O^*_{X \setminus D}$ is a log structure on X for which X is log regular and $X_{\text{tr}} = U := X \setminus D$. Thus there is an equivalence of categories $\mathrm{KCov}(X)^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(U)^{\mathbb{L}}$.

Proposition 2.6 (([Org, cor. 2.3])). *Let S be a strictly local scheme with closed point s and let X be a connected fs log scheme such that $\mathring{X}$ is proper over S. Then*

$$\mathrm{KCov}(X) \to \mathrm{KCov}(X_s)$$

is an equivalence of categories.

One can extend this result to henselian schemes:

Theorem 2.7. *Let S be a henselian scheme with closed point s, and let X be a connected fs log scheme such that $\mathring{X}$ is proper over S. Then*

$$\mathrm{KCov}(X) \to \mathrm{KCov}(X_s)$$

is an equivalence of categories.

Proof. First assume X_s to be geometrically connected. Then X is also connected and we have to prove that $\pi_1^{\log}(X_s) \to \pi_1^{\log}(X)$. Let $\overline{s}$ be a strict localization of s and let $\overline{S}$ be the strict localization of S at $\overline{s}$. Let S_i be a pointed Galois covering of S, let G_i be its Galois group and let $s_i = s \times_S S_i$. Then we have a diagram with exact lines:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\log}(X_{s_i}) & \longrightarrow & \pi_1^{\log}(X_s) & \longrightarrow & G_i & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1^{\log}(X_{S_i}) & \longrightarrow & \pi_1^{\log}(X) & \longrightarrow & G_i & \longrightarrow & 1 \end{array}$$

By taking the projective limit when S_i describes the category of pointed Galois covering of S, one gets a diagram with exact lines

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varprojlim_{S_i} \pi_1^{\log}(X_{s_i}) & \longrightarrow & \pi_1^{\log}(X_s) & \longrightarrow & \pi_1^{\mathrm{alg}}(S) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \varprojlim_{S_i} \pi_1^{\log}(X_{S_i}) & \longrightarrow & \pi_1^{\log}(X) & \longrightarrow & \pi_1^{\mathrm{alg}}(S) & \longrightarrow & 1 \end{array}$$

But $\pi_1^{\log}(X_{\overline{S}}) \to \varprojlim_{S_i} \pi_1^{\log}(X_{S_i})$ is an isomorphism (for this one does not need $X \to S$ to be proper but simply to be quasicompact and separated). To show this is equivalent to prove that $F : \varinjlim \mathrm{KCov}(X_{S_i}) \to \mathrm{KCov}(X_{\overline{S}})$ is an equivalence of category. If $T_i \to X_{S_i}$ is a két covering such that $T_i \times_{X_{S_i}} X_{S_j}$ is connected for every j, then $T_i \times X_{\overline{S}}$ is also connected. Thus $\varinjlim \mathrm{KCov}(X_{S_i}) \to \mathrm{KCov}(X_{\overline{S}})$ is fully faithful. Let $T \to X_{\overline{S}}$ be a covering. Let $\coprod_{k \in K} U_k \to X_{overlineS}$ be a surjective etale morphism such that U_k has a chart $U_k \to \mathrm{Spec}\,\mathbf{Z}[P]$, and there is an étale morphism $T_k := T \times_{X_{\overline{S}}} U_k \to U_{k,Q} := U_k \times_{\mathrm{Spec}\,\mathbf{Z}[P]} \mathrm{Spec}\,\mathbf{Z}[Q]$. Assume K to be finite and U_k to be quasicompact. Then there exists i and $U_{k,i} \to X_{S_i}$ such that $U_k = U_{k,i} \times_{X_{S,i}} X_{\overline{S}}$. If i is big enough, there is a chart $U_{k,i} \to \mathrm{Spec}\,\mathbf{Z}[P]$ extending the chart of U_k and $U_{k,Q} = U_{k,Q,i} \times_{X_{S_i}} X_{\overline{S}}$ where $U_{k,Q,i} = U_{k,i} \times_{\mathrm{Spec}\,\mathbf{Z}[P]} \mathrm{Spec}\,\mathbf{Z}[Q]$. If i is big enough, the étale morphism $T_k \to U_{k,Q}$ comes from some étale morphism $T_{k,i} \to U_{k,Q,i}$. As F is fully faithful for any X quasicompact and separated over S, for i big enough, $(T_{k,i})_k$ can be enriched into a descent datum with respect to $\coprod_k U_{k,i} \to X_{S_i}$ inducing after pullback to $X_{\overline{S}}$ the descent datum given by T. Since surjective étale morphisms satisfy effective descent for két coverings, one gets for i big enough a két covering $T_i \to X_{S_i}$ such that $T = T_i \times_{X_{S_i}} X_{\overline{S}}$. Thus

$\pi_1^{\log}(X_{\overline{S}}) \to \varprojlim_{S_i} \pi_1^{\log}(X_{S_i})$ is an isomorphism.
Similarly $\pi_1^{\log}(X_{\bar{s}}) \to \varprojlim_{S_i} \pi_1^{\log}(X_{s_i})$ is an isomorphism. Thanks to proposition 2.6, $\pi_1^{\log}(X_{\bar{s}}) \to \pi_1^{\log}(X_{\overline{S}})$ is an isomorphism. Thus $\pi_1^{\log}(X_s) \to \pi_1^{\log}(X)$ is also an isomorphism.
In the general case, let $X \to S'$ be the Stein factorization of $X \to S$. For every connected component S'_j of S', let $X_j = X \times_{S'} S'_j$. Since S'_j is henselian and $X_j \to S'_j$ has geometrically connected fibers, one gets that $\mathrm{KCov}(X_{j,s}) \to \mathrm{KCov}(X_j)$ is an equivalence of category. Since $\mathrm{KCov}(X) = \prod_j \mathrm{KCov}(X_j)$ and $\mathrm{KCov}(X_s) = \prod \mathrm{KCov}(X_{j,s})$, one gets that $\mathrm{KCov}(X) \to \mathrm{KCov}(X_s)$ is an equivalence of categories. □

Corollary 2.8. *Let O_K be a complete discretely valued ring endowed with its natural log structure and let $\mathbb{L}$ a set of prime numbers invertible in O_K. Let $X \to \operatorname{Spec} O_K$ be a proper and log smooth morphism and let $U := X_{\mathrm{tr}} \subset X_\eta$. There is a natural equivalence of categories*

$$\mathrm{KCov}(X_s)^{\mathbb{L}} \simeq \mathrm{Cov}^{\mathrm{alg}}(U)^{\mathbb{L}}.$$

In particular, if $\mathring{X} \to \operatorname{Spec} O_K$ is a semistable model of X_η, and the log structure on X is given by $M_X = O_X \cap j_* O^*_{X_\eta}$ where $j : X_\eta \to X$, then $X \to \operatorname{Spec} O_K$ is log smooth and $X_{\mathrm{tr}} = X_\eta$. We get an equivalence of categories $\mathrm{KCov}(X_s)^{\mathbb{L}} \simeq \mathrm{Cov}^{\mathrm{alg}}(X_\eta)^{\mathbb{L}}$, and thus an isomorphism

$$\pi_1^{\mathrm{alg}}(X_\eta)^{\mathbb{L}} \to \pi_1^{\log}(X_s)^{\mathbb{L}}.$$

Here we recall basic results about saturated morphisms of fs log schemes. The main reference on the subject is [Tsu97], which is unfortunately unpublished.

Definition 2.9. A morphism of fs monoids $P \to Q$ is *integral* if, for any morphism of integral monoids $P \to Q'$, the amalgamated sum $Q \oplus_P Q'$ is still integral.
A integral morphism of fs monoids $P \to Q$ is *saturated* if, for any morphism of fs monoids $P \to Q'$, the amalgamated sum $Q \oplus_P Q'$ is still fs.
A morphism $f : Y \to X$ of fs log schemes is *saturated* if for any geometric point $\bar{y}$ of Y, $\bar{M}_{X,f(\bar{y})} \to \bar{M}_{Y,\bar{y}}$ is saturated.

If $Y \to X$ is saturated and $Z \to X$ is a morphism of fs log schemes, then the underlying scheme of $Z \times_X Y$ is $\mathring{Z} \times_{\mathring{X}} \mathring{Y}$.
If $P \to Q$ is a local and integral (resp. saturated) morphism of fs monoids and P is sharp, the morphism $\operatorname{Spec} \mathbf{Z}[Q] \to \operatorname{Spec} \mathbf{Z}[P]$ is flat (resp. separable, *i.e.* flat with geometrically reduced fibers, cf. [Ogu, cor. I.4.3.16] and [IKN05, rem. 6.3.3]).
Let $f : X \to Y$ be log smooth, let $\bar{x}$ be a geometric point of X and let $\bar{y}$ be its image in Y. Etale locally on Y, there is a chart $Y \to \operatorname{Spec} P$ such that $P \to \overline{M}_{Y,\bar{y}}$ is an isomorphism. Then, according to [Kat89, th. 3.5], there is étale locally at x a fs chart $\phi : P \to Q$ of $X \to Y$ such that $Y \to \operatorname{Spec} \mathbf{Z}[Q] \times_{\mathbf{Z}[P]} X$ is étale such that ϕ is injective and the torsion part of $\mathrm{Coker}(\phi^{\mathrm{gp}})$ has order invertible on X. Up to localizing Q by the face corresponding to $\bar{x}$, one can assume that $Q \to M_{X,\bar{x}}$ is local (and thus exact according to [Ogu, def. II.2.2.8]). Thus if f is integral (resp. saturated), $P \to Q$ is a local and integral (resp. saturated) morphism of fs monoids and P is sharp. Thus f is flat (resp. separable).
If $P \to Q$ is an integral morphism of fs monoids, there exists an integer n such that

the pullback $P_n \to Q'$ of $P \to Q$ along $P \xrightarrow{n} P = P_n$ is saturated (theorem [IKN05, A.4.2]).
Moreover if $P \to Q$ factors through Q_0 such that $P \to Q_0$ is saturated and $Q_0 \to Q$ is $\mathbb{L}$-Kummer, n can be chosen to be an $\mathbb{L}$-integer.

2.2. **Log geometric fundamental groups.** Let $X \to s$ be a morphism of fs log schemes, where s is an fs log point. Let $\bar{x}$ be a log geometric point of X and let $\bar{s}$ be its image in s.
Then we define the *log geometric fundamental group* of X at $\bar{x}$ to be

$$\pi_1^{\text{log-geom}}(X/s, \bar{x}) = \operatorname{Ker}(\pi_1^{\log}(X, \bar{x}) \to \pi_1^{\log}(s, \bar{s})).$$

If $(t, \bar{t}) \to (s, \bar{s})$ is a Galois connected pointed két covering of $(s, \bar{s})$, one defines $X_t = X \times_s t$ and $\bar{x}_t = (\bar{x}, \bar{t})$.
Then one has an exact sequence:

$$1 \to \pi_1^{\log}(X_t, \bar{x}_t) \to \pi_1^{\log}(X, \bar{x}) \to \operatorname{Gal}(t/s),$$

and the right map is onto if X_t is connected (we will say that X is *log geometrically connected* if X_t is connected for any connected két covering t of s).
By taking the projective limit of the previous exact sequence when $(t, \bar{t})$ runs through the directed category of pointed Galois connected coverings of $(s, \bar{s})$, one gets a canonical isomorphism $\pi_1^{\text{log-geom}}(X/s, \bar{x}) = \varprojlim_{(t,\bar{t})} \pi_1^{\log}(X_t, \bar{x}_t)$.
Actually, if $\tilde{t} \to t$ is the reduced subscheme of t endowed with the inverse image log structure, the map $\pi_1^{\log}(X_{\tilde{t}}, \bar{x}_t) \to \pi_1^{\log}(X_t, \bar{x}_t)$ is an isomorphism, so that one can replace X_t in the previous projective limit by $X_{\tilde{t}}$.
The category $\pi_1^{\text{log-geom}}(X/s, \bar{x})$-fSet is naturally equivalent to the category

$$\operatorname{KCov}_{\text{geom}}(X/s) := \varprojlim_{\tilde{t}} \operatorname{KCov}(X_{\tilde{t}}).$$

If one has a commutative square of pointed fs log schemes:

$$\begin{tikzcd} (X', \bar{x}') \arrow[r, "\phi"] \arrow[d] & (X, \bar{x}) \arrow[d] \\ (s', \bar{s}') \arrow[r, "\psi"] & (s, \bar{s}) \end{tikzcd}$$

where s' and s are log points, one gets a commutative diagram of profinite groups:

$$\begin{tikzcd} \pi_1^{\log}(X', \bar{x}') \arrow[r] \arrow[d] & \pi_1^{\log}(X, \bar{x}) \arrow[d] \\ \pi_1^{\log}(s', \bar{s}') \arrow[r] & \pi_1^{\log}(s, \bar{s}) \end{tikzcd}$$

By taking the kernel of the vertical arrows, one gets a map $\pi_1^{\text{log-geom}}(X'/s', \bar{x}') \to \pi_1^{\text{log-geom}}(X/s, \bar{x})$, which is functorial with respect to (ϕ, ψ).
One also has, by definition of the universal pro-pointed covering of $(s', \bar{s}')$, a canonical morphism of pro-$(s', \bar{s}')$-pointed két covering

$$\varprojlim_{(t', \bar{t}')} (t', \bar{t}') \to \varprojlim_{(t,\bar{t})} \psi^*(t, \bar{t})$$

(where $(t', \bar{t}')$ runs through pointed Galois connected két coverings of $(s', \bar{s}')$ and $(t, \bar{t})$ runs through pointed Galois connected két coverings of $(s, \bar{s})$), and thus a morphism of pro-pointed fs log schemes

$$\varprojlim_{(t',\bar{t}')} (t', \bar{t}') \to \varprojlim_{(t,\bar{t})} (t, \bar{t}).$$

This induces a morphism of pro-pointed fs log schemes

$$\varprojlim (X'_{t'}, \bar{x}'_{t'}) \to \varprojlim (X_t, \bar{x}_t),$$

hence a map of profinite groups

$$\varprojlim \pi_1^{\log}(X'_{t'}, \bar{x}'_{t'}) \to \varprojlim \pi_1^{\log}(X_t, \bar{x}_t),$$

such that the following square of profinite groups is commutative:

$$\begin{tikzcd}
\varprojlim \pi_1^{\log}(X'_{t'}, \bar{x}'_{t'}) \arrow[r] \arrow[d, equal] & \varprojlim \pi_1^{\log}(X_t, \bar{x}_t) \arrow[d, equal] \\
\pi_1^{\text{log-geom}}(X'/s', \bar{x}') \arrow[r] & \pi_1^{\text{log-geom}}(X/s, \bar{x})
\end{tikzcd}$$

Let O_K be a complete discretely valued ring endowed with its natural log structure and let $\mathbb{L}$ a set of prime numbers invertible in O_K. Let $X \to \operatorname{Spec} O_K$ be a proper and log smooth morphism and let $U := X_{\text{tr}} \subset X_\eta$. There is a geometric analog to the specialization isomorphism $\pi_1^{\text{alg}}(X_\eta)^{\mathbb{L}} \to \pi_1^{\log}(X_s)^{\mathbb{L}}$ of corollary 2.8:

Theorem 2.10 ([Kis00, th. 1.4])**.** *There is a natural equivalence of categories*

$$\mathrm{KCov}_{\mathrm{geom}}(X/s)^{\mathbb{L}} \simeq \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{\eta}})^{\mathbb{L}}.$$

It can be deduced from corollary 2.8 thanks to the fact that any algebraic covering of $U_{\bar{\eta}}$ is already defined over a tamely ramified extension of K ([Kis00, prop. 1.15]).

2.3. Semistable log curves.

Definition 2.11. A morphism $X \to S$ of fs log schemes is a *semistable log curve* if étale locally on S there is a chart $S \to \operatorname{Spec} P$ such that one of the following is satisfied:

- $X \to S$ is a strict smooth curve,
- $X \to S$ factorizes through a strict étale morphism $X \to S \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$ with $Q = (P \oplus < u, v >)/u + v = p$ and $p \in P$,
- $X \to S$ factorizes through a strict étale morphism $X \to S \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[P \oplus \mathbf{N}]$.

A semistable log curve is *strictly semistable* if étale locally on S, there are such maps locally for the Zariski topology of X.

Proposition 2.12. *A morphism $X \to S$ is a semistable log curve if and only if it is a log smooth and saturated morphism purely of relative dimension 1.*

Proof. The direct sense is obvious. Let $X \to S$ be a saturated log smooth scheme of pure dimension 1. As the definition of a semistable log curve is local on the étale topology of X and S, one can assume that S has a chart $S \to \operatorname{Spec} \mathbf{Z}[P]$ and $X = S \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$ where $P \to Q$ is an injective local and saturated

morphism of monoids, P is sharp and $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ is invertible on S. In particular $T^{\mathrm{gp}} := \overline{Q}^{\mathrm{gp}}/\overline{P}^{\mathrm{gp}}$ is torsionfree. Since $P \to Q$ is saturated, $\operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} \mathbf{Z}[Q]$ is flat and $1 = \dim \operatorname{Spec} \mathbf{Z}[P] - \dim \operatorname{Spec} \mathbf{Z}[Q] = \operatorname{rk} P^{\mathrm{gp}} - \operatorname{rk} Q^{\mathrm{gp}} = \operatorname{rk} Q^{\mathrm{gp}}/P^{\mathrm{gp}} \geq \operatorname{rk} T^{\mathrm{gp}}$. Thus T^{gp} is $\{1\}$ or $\mathbf{Z}$. . For every $x \in T$, there exists a unique $\psi(x) \in \overline{Q}$ such that $f^{-1}(x) \cup Q = \psi(x) + \overline{P}$ where $f : \overline{Q}^{\mathrm{gp}} \to T^{\mathrm{gp}}$ ([Ogu, prop. I.4.3.14]). In particular, if $T^{\mathrm{gp}} = \{1\}$, then $\overline{P} \to \overline{Q}$ is bijective, thus $X \to S$ is strict and thus $X \to S$ is smooth.
Assume $T^{\mathrm{gp}} = \mathbf{Z}$. Then $\operatorname{rk} Q^{\mathrm{gp}}/P^{\mathrm{gp}} = \operatorname{rk} T^{\mathrm{gp}}$ and thus $\operatorname{rk} Q^{\mathrm{gp}} = \operatorname{rk} \overline{Q}^{\mathrm{gp}}$. Since $\overline{Q}^{\mathrm{gp}}$ is a free abelian group, one can choose a splitting $Q = \overline{Q} \oplus Q^*$. Since $Q^* \hookrightarrow Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ is finite of order invertible on S, $X \to S \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[\overline{Q}]$ is étale. Thus one can assume that Q is sharp. Let T be the image of Q in T^{gp}. Then $T = \mathbf{N}$ or $T = \mathbf{Z}$.
First assume $T = \mathbf{N}$. Then $n\psi(1) = \psi(n) + p$ with $p \in P$. Since $P \to Q$ is saturated and $p \leq n\psi(1)$, there exists $p' \in P$ such that $p \leq np'$ and $p' \leq \psi(1)$. Thus, by definition of $\psi(1)$, $p' = 0$, thus $p = 0$ and $\psi(n) = n\psi(1)$. Thus $Q = P \oplus \mathbf{N}\psi(1)$.
If $T = \mathbf{Z}$, let $u = \psi(1)$ and $v = \psi(-1)$. Since $\psi(u + v) = 0$, $p := u + v \in P$. As in the previous case, if $n \geq 0$, then $\psi(n) = n\psi(1)$ and $\psi(-n) = n\psi(-1)$. Thus $Q = P \oplus < u, v > /(u + v = p)$. $\square$

The underlying morphism of schemes $\mathring{X} \to \mathring{S}$ is a semistable curve of schemes. In particular, if $\mathring{S}$ is a geometric point, one can associate to X a graph $\mathbb{G}(X)$ in the following way: the vertices are the irreducible components of X, the edges are the nodes. If x is a node, then the henselization $X(x)$ of X at x has two irreducible components: these components are the branches of the edge corresponding to x. If z is an irreducible component of $X(x)$ and z' is the irreducible component containing the image of z in X, the branch corresponding to z abuts to the vertex corresponding to z' (this graph does not depend of the log structure).

If $X \to S$ is a proper semistable log curve and $X' \to X$ is a két covering, then for any log geometric point $\bar{s}$ of S, there is a két neighborhood U of $\bar{s}$ such that $X'_U \to U$ is saturated. Then $X'_U \to U$ is also a semistable curve.
The morphism $\mathring{X}'_{\bar{s}} \to \mathring{X}_{\bar{s}}$ induces a genuine morphism $\mathbb{G}(X'_{\bar{s}}) \to \mathbb{G}(X_{\bar{s}})$ of graphs.

If K is a complete nonarchimedean field with separably closed residue field k, O_K is its ring of integers and $X \to O_K$ is a proper semistable curve with smooth generic fiber, there is a canonical embedding $|\mathbb{G}(X_k)| \to X_\eta^{\mathrm{an}}$ which is a homotopy equivalence ([Ber90, th. 4.3.2]). It is compatible with any isometric extension of K.
Moreover, if U is any dense Zariski open subset of X_η, $|\mathbb{G}(X_k)|$ is mapped into U^{an} and $|\mathbb{G}(X_k)| \to U^{\mathrm{an}}$ is still a homotopy equivalence.
If $X \to O_K$ is a semistable log curve and $X' \to X$ is a két morphism such that X' is still a semistable curve, the following diagram is commutative:

$$\begin{array}{ccc} |\mathbb{G}(X'_k)| & \to & X'^{\mathrm{an}}_\eta \\ \downarrow & & \downarrow \\ |\mathbb{G}(X_k)| & \to & X^{\mathrm{an}}_\eta \end{array}$$

2.4. **Specialization of log fundamental groups.** Let us study specialization of log geometric fundamental groups (that is the projective limit of the log fundamental groups after taking két extensions of the base log point).
The only result we will need later on is the following:

Proposition 2.13 ((cor. 2.18))**.** *Let $X \to S$ be a proper and saturated morphism of log schemes, and let $Y \to X$ be a két covering. Let s and s' be two log points of S and assume that one has a specialization $\bar{s}' \to \bar{s}$ (where $\bar{s}$ and $\bar{s}'$ are some log geometric points over s and s'). Let p be the characteristic of s. One has a specialization morphism*

$$\pi_1^{\text{log-geom}}(Y_{s'}/s')^{(p')} \to \pi_1^{\text{log-geom}}(Y_s/s)^{(p')}.$$

Moreover this morphism factorizes through $\varprojlim_U \pi_1^{\log}(Y_U)^{(p')}$, where U runs through the két neighborhoods of $\bar{s}$.

To prove this, our main result will be the invariance of the log geometric fundamental group of an fs log scheme X saturated and of finite type over an fs log point S with separably closed field by fs base change that is an isomorphism on the underlying scheme. The assumptions implies that our base change induces an isomorphism of underlying schemes. Working étale locally on this scheme, we are reduced to the case where this scheme is strictly local, where the log geometric fundamental group can be explicitly described in terms of the morphism of monoids $\overline{M}_X \to \overline{M}_S$.
Combining this base change invariance result with strict base change invariance of the (p')-log geometric fundamental group and strict specialization of the (p')-log geometric fundamental group ([Org]), we will get that if $X \to S$ is proper log smooth saturated morphism, and s_2, s_1 are fs points of S and $\bar{s}_2 \to \bar{s}_1$ is a specialization of log geometric points of S over s_2 and s_1, then there is a specialization morphism $\pi_1^{\text{log-geom}}(X_{s_2}) \to \pi_1^{\text{log-geom}}(X_{s_1})$ (this is the only result we will really need in the following).

Lemma 2.14. *Let $s' \to s$ be a strict morphism of log points such that $\mathring{s}'$ and $\mathring{s}$ are geometric points of characteristic p. Let $X \to s$ be a morphism of fs log schemes such that $\mathring{X} \to \mathring{s}$ is of finite type.*
Then $F : \mathrm{KCov}(X)^{(p')} \to \mathrm{KCov}(X_{s'})^{(p')}$ is an equivalence of categories.

Proof. If T is a connected két covering of X, $T \mathring{\times}_s s' \to \mathring{T} \times_{\mathring{s}} \mathring{s}'$ is an isomorphism since $s' \to s$ is strict. $\mathring{T} \times_{\mathring{s}} \mathring{s}'$ is connected too, so we get that the functor F is fully faithful.
As one already knows that F is fully faithful for any X, and as strict étale surjective morphisms satisfy effective descent for két coverings, one may prove the result étale locally, and thus assume that X has a global chart $X \to \operatorname{Spec} \mathbf{Z}[P]$.
Let S' be a két covering of $X_{s'}$. Then there exists a (p')-Kummer morphism of monoids $P \to Q$ such that

$$S'_Q := S' \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q] \to X_{s',Q} := X_{s'} \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$$

is strict étale (and surjective).
But, since $\mathring{X}_{s',Q} \to \mathring{X}_Q \times_{\mathring{s}} \mathring{s}'$ is an isomorphism of schemes, $\mathrm{Cov}^{\mathrm{alg}}(\mathring{X}_{s',Q}) \to \mathrm{Cov}^{\mathrm{alg}}(\mathring{X}_Q)$ is an equivalence of categories ([Org03, cor 4.5]). Thus, there is a strict étale covering S_Q of X_Q (and thus $S_Q \to X$ is a két covering) such that S'_Q is $X_{s',Q}$-isomorphic to $S_Q \times_s s'$.
Thus F is an equivalence of categories. □

Let now $s' \to s$ be a morphism of fs log points, such that the underlying morphism of schemes $\mathring{s}' \to \mathring{s}$ is an isomorphism of geometric points, and let $X \to s$ be a

saturated morphism of fs log schemes with $\mathring{X}$ nœtherian and $\mathring{X} \to \mathring{s}$ geometrically connected. Since $X \to s$ is saturated, it is log geometrically connected.
Let $\bar{x}'$ be a log geometric point of $X' = X \times_s s'$ and let $\bar{x}$, $\bar{s}'$ and $\bar{s}$ be its image in X, s' and s respectively.
We have a commutative diagram

$$\begin{tikzcd}
\pi_1^{\log}(X', \bar{x}') \arrow[r] \arrow[d] & \pi_1^{\log}(X, \bar{x}) \arrow[d] \\
\pi_1^{\log}(s', \bar{s}) \arrow[r] & \pi_1^{\log}(s, \bar{s})
\end{tikzcd}$$

Theorem 2.15. *The map* $\pi_1^{\text{log-geom}}(X'/s', \bar{x}') \to \pi_1^{\text{log-geom}}(X/s, \bar{x})$ *is an isomorphism.*

Proof. Let $(s_i, \bar{s}_i)_{i \in I}$ be a cofinal system of pointed Galois connected két coverings of $(s, \bar{s})$. Let $\tilde{s}_i$ be the reduced subscheme of s_i endowed with the inverse image log structure. Let us write $(X_i, \bar{x}_i) = (X \times_s \tilde{s}_i, \bar{x} \times_{\bar{s}} \bar{s}_i)$.
Let $(s'_j, \bar{s}'_j)_{j \in J}$ be a cofinal system of pointed Galois connected két coverings of $(s', \bar{s}')$. Let $\tilde{s}'_j$ be the reduced subscheme of s'_j endowed with the inverse image log structure. Let us write $(X'_j, \bar{x}'_j) = (X \times_{s'} s'_j, \bar{x}' \times_{\bar{s}'} \bar{s}'_j)$.
One has to prove that

$$\varprojlim_j \pi_1^{\log}(X'_j, \bar{x}'_j) \to \varprojlim_i \pi_1^{\log}(X_i, \bar{x}_i)$$

is an isomorphism.

If Y is an fs log scheme, $\varepsilon_Y : Y_{\text{két}} \to \mathring{Y}_{\text{ét}}$ denotes the usual morphism of topoi from the két topos of Y to the étale topos of the underlying scheme of Y.
The log scheme $\tilde{s}_i$ has the same underlying scheme as s, so $\mathring{X}_i \to \mathring{X}$ is an isomorphism of schemes since $X \to s$ is assumed to be saturated. For the same reason, $\mathring{X}'_j \to \mathring{X}'$ and $\mathring{X}' \to \mathring{X}$ are also isomorphisms of schemes.
More precisely, for any i there is j_0 such that for $i \geq i'$ and $j \geq j' \geq j_0$, one has a 2-commutative diagram:

$$\begin{tikzcd}
X'_{j,\text{két}} \arrow[rr] \arrow[dr] \arrow[dd] & & X'_{j',\text{két}} \arrow[dr] \arrow[dd] & \\
& X_{i,\text{két}} \arrow[rr] \arrow[dd] & & X_{i',\text{két}} \arrow[dd] \\
\mathring{X}'_{j,\text{ét}} \arrow[rr] \arrow[dr] & & \mathring{X}'_{j',\text{ét}} \arrow[dr] & \\
& \mathring{X}_{i,\text{ét}} \arrow[rr] & & \mathring{X}_{i',\text{ét}}
\end{tikzcd}$$

where all the morphisms of schemes of the bottom square are isomorphisms.
If G is a finite group, one thus has 2-commutative diagrams of stacks over $\mathring{X}_{\text{ét}}$:

$$\begin{array}{ccc} \varepsilon_{X'_{j'}*}\operatorname{Tors}_{X'_{j',\text{két}}}(G) & \longrightarrow & \varepsilon_{X'_j*}\operatorname{Tors}_{X'_{j,\text{két}}}(G) \\ \downarrow & & \downarrow \\ \varepsilon_{X_{i'}*}\operatorname{Tors}_{X_{i',\text{két}}}(G) & \longrightarrow & \varepsilon_{X_i*}\operatorname{Tors}_{X_{i,\text{két}}}(G) \end{array}$$

More precisely, let us write $IJ = I\coprod J$ endowed with the partial order such that:

- the restriction of the partial order to I (resp. J) is the usual one,
- if $i \in I$ and $j \in J$, then $i \leq j$ if and only if there exists a (necessarily unique) morphism of pointed fs log schemes $(s'_j, \bar{s}'_j) \to (s_i, \bar{s}_i)$ that makes the square
$$\begin{array}{ccc} (s'_j, \bar{s}'_j) & \to & (s_i, \bar{s}_i) \\ \downarrow & & \downarrow \\ (s', \bar{s}') & \to & (s, \bar{s}) \end{array}$$
commutative,
- if $j \in J$ and $i \in I$, then $j \not\leq i$.

Let us also denote by IJ the corresponding category. Then one has a fibered category over $IJ^{\text{op}} \times \mathring{X}_{\text{ét}}$, whose fiber in (i, U) is $\varepsilon_{X_i*}\operatorname{Tors}_{X_{i,\text{két}}}(G)(U)$ and the fiber in (j, U) is $\varepsilon_{X'_j*}\operatorname{Tors}_{X'_{j,\text{két}}}(G)(U)$.

By taking the inductive limit over $i \in I$, which is directed, one gets, according to [Gir71, I.1.10], a fibered category over $\mathring{X}_{\text{ét}}$ whose fiber in U is $\varinjlim_i \varepsilon_{X_i*}\operatorname{Tors}_{X_{i,\text{két}}}(G)(U)$; let us denote by $\varinjlim_i \varepsilon_{X_i*}\operatorname{Tors}_{X_{i,\text{két}}}(G)$ the stack associated to this fibered category (and we do the same for X' and J).
Since $\mathring{X}$ is assumed to be noetherian (and thus $\mathring{X}_{\text{ét}}$ is a coherent topos), the functor from $\varinjlim_j \operatorname{Tors}(G, X'_{j,\text{két}})$ to the category of global sections of $\varinjlim_j \varepsilon_{X'_j*}\operatorname{Tors}_{X'_{j,\text{két}}}(G)$ is an equivalence of categories. Moreover

$$\varinjlim_j \operatorname{Tors}(G, X'_{j,\text{két}}) \simeq \varinjlim_j \operatorname{Tors}(G, \pi_1^{\log}(X'_j, \bar{x}'_j)\text{−Set}) \simeq \operatorname{Tors}(G, \varprojlim \pi_1^{\log}(X'_j, \bar{x}'_j)\text{−Set}).$$

The 2-commutative diagrams induce a morphism of stacks:

$$\varinjlim_j \varepsilon_{X'_j*}\operatorname{Tors}_{X'_{j,\text{két}}}(G) \to \varinjlim_i \varepsilon_{X_i*}\operatorname{Tors}_{X_{i,\text{két}}}(G) \tag{1}$$

We thus have to prove that (1) is an equivalence of stacks, which can be proved on the fibers (since $\mathring{X}_{\text{ét}}$ has enough points; [Gir71, cor. III.2.1.5.8]). This does not depend on the base points anymore (we thus may forget those, so that we may use the notations again to denote other points).
Let x be a point of $\mathring{X}_{\text{ét}}$, $\bar{x}'$ a point of $X'_{\text{két}}$ above x and $\bar{s}'$ with image $\bar{x}$ in $X_{\text{két}}$. Let $\bar{x}_i = (\bar{x}, \bar{s})$ and let $\bar{x}'_j = (\bar{x}', \bar{s}'_j)$. let V_x be the category of étale neighborhoods of x in $\mathring{X}$. Then one has (by coherence of the morphism of topoi ε_{X_i}, as in [Org, proof of 2.4]):

$$\begin{aligned} \varinjlim_{U \in V_x} \varinjlim_i \varepsilon_{X_i*}\operatorname{Tors}_{X_{i,\text{két}}}(G)(U) &= \varinjlim_i \varinjlim_{U \in V_x} \varepsilon_{X_i*}\operatorname{Tors}_{X_{i,\text{két}}}(G)(U) \\ &= \varinjlim_i \operatorname{Tors}(G, X(x)_{i,\text{két}}) \\ &= \operatorname{Tors}(G, \varprojlim_i \pi_1^{\log}(X(x)_{i,\text{két}}, \bar{x}_i) - \text{Set}), \end{aligned}$$

and one has a similar result for X'.
One thus only have to prove that

$$\varprojlim_j \pi_1^{\log}(X'(x)_{j,\text{két}}, \bar{x}'_j) \to \varprojlim_i \pi_1^{\log}(X(x)_{i,\text{két}}, \bar{x}_i)$$

is an isomorphism.
We are thus reduced to the case where $\mathring{X}$ is a strictly local and noetherian scheme.
But then ([Sti02, prop. 3.1.11]), for $\mathring{X}$ a strictly local and noetherian scheme,

$$\begin{aligned}
\varprojlim_i \pi_1^{\log}(X_{i,\text{két}}, \bar{x}_i) &= \varprojlim_i \operatorname{Hom}(\overline{M}^{\text{gp}}_{X_i,x_i}, \widehat{\mathbf{Z}}^{(p')}) \\
&= \operatorname{Hom}(\varinjlim_i \overline{M}^{\text{gp}}_{X_i,x_i}, \widehat{\mathbf{Z}}^{(p')}) \\
&= \operatorname{Hom}(\operatorname{Coker}(\overline{M}^{\text{gp}}_s \to \overline{M}^{\text{gp}}_{X,x}), \widehat{\mathbf{Z}}^{(p')}),
\end{aligned}$$

and one has a similar result for X'.
Since $\overline{M}_{X',x'} = \overline{M}_{X,x} \oplus_{\overline{M}_s} \overline{M}_{s'}$, one has $\overline{M}^{\text{gp}}_{X',x'} = \overline{M}^{\text{gp}}_{X,x} \oplus_{\overline{M}^{\text{gp}}_s} \overline{M}^{\text{gp}}_{s'}$. Thus, $\operatorname{Coker}(\overline{M}^{\text{gp}}_s \to \overline{M}^{\text{gp}}_{X,x}) \to \operatorname{Coker}(\overline{M}^{\text{gp}}_{s'} \to \overline{M}^{\text{gp}}_{X',x'})$ is an isomorphism.
One thus gets the wanted result. $\square$

Let us assume now that $(s', \bar{s}') \to (s, \bar{s})$ is simply assumed to be a morphism of log points, that $Y \to s$ is a saturated morphism and that $X \to Y$ is a két morphism with $\mathring{X}$ of finite type over s.

Corollary 2.16. *The map of profinite groups*

$$\pi_1^{\text{log-geom}}(X/s, \bar{x})^{(p')} \to \pi_1^{\text{log-geom}}(X'/s', \bar{x}')^{(p')}$$

is an isomorphism.

Proof. By replacing s (resp. s') by the closed reduced subscheme of a connected két covering of s (resp. s'), one can assume that $X \to s$ is saturated ($\mathring{X} \to \mathring{s}$ will still be of finite type).
If $(t, \bar{t}) \to (s, \bar{s})$ is a strict étale covering, then $\pi_1^{\text{log-geom}}(X_t/t, \bar{x}_t) \to \pi_1^{\text{log-geom}}(X/s, \bar{x})$ is an isomorphism. Thus, by writing s_0 for the separable closure of s and by taking the projective limit over pointed strict étale coverings (since $\pi_1^{\log}(X_{s_0}) = \varprojlim \pi_1^{\log}(X_t)$, where t runs through pointed strict étale coverings of s), one gets that $\pi_1^{\text{log-geom}}(X_{s_0}/s_0, \bar{x}_0) \to \pi_1^{\text{log-geom}}(X/s, \bar{x})$ is an isomorphism.
One thus may assume that $\mathring{s}$ and $\mathring{s}'$ are geometric points.
Let us consider the fs log scheme s'' whose underlying scheme is $\mathring{s}'$ and whose log structure is the inverse image of the log structure of s.
Thus, one has morphisms $s' \to s'' \to s$, where $s' \to s''$ is an isomorphism on the underlying schemes and $s'' \to s$ is strict.
But according to lemma 2.14, $\pi_1^{\log}(X_{s''})^{(p')} \to \pi_1^{\log}(X)^{(p')}$ and $\pi_1^{\log}(s'')^{(p')} \to \pi_1^{\log}(s)^{(p')}$ are isomorphisms. Thus,

$$\pi_1^{\text{log-geom}}(X_{s''}/s'')^{(p')} \to \pi_1^{\text{log-geom}}(X/s)^{(p')}$$

is an isomorphism.
By 2.15, $\pi_1^{\text{log-geom}}(X_{s'}/s')^{(p')} \to \pi_1^{\text{log-geom}}(X_{s''}/s'')^{(p')}$ is also an isomorphism. $\square$

Definition 2.17. Let S be a fs log scheme such that $\mathring{S}$ is strictly local. Let $Y \to S$ be a morphism of fs log schemes whose special fiber is log geometrically connected.

The log geometric fundamental group of Y over S is

$$\pi_1^{\text{log-geom}}(Y_S/S) := \varprojlim_T \pi_1^{\log}(Y_T),$$

where T runs through pointed két coverings of $(S, \bar{s})$.

Recall that if S is a strictly local scheme with closed point s and X is a connected fs log scheme such that $\mathring{X}$ is proper over S, then

$$\mathrm{KCov}(X) \to \mathrm{KCov}(X_s)$$

is an equivalence of categories (proposition 2.6).
Let $X \to S$ be a proper and saturated morphism of log schemes, and let $Y \to X$ be a két covering. Let s and s' be two log points of S and assume that one has a specialization $\bar{s}' \to \bar{s}$ (where $\bar{s}$ and $\bar{s}'$ are some log geometric points over s and s'). Let Z be the strictly local scheme of S at s endowed with the inverse image log structure, and let z be its closed point, endowed with the inverse image log structure. One also has an isomorphism $\pi_1^{\text{log-geom}}(Y_Z/Z) := \varprojlim_U \pi_1^{\log}(Y_U)$, where U runs through két neighborhoods of $\bar{s}$ in S.
One has the following arrows (defined up to inner homomorphisms):

$$\pi_1^{\text{log-geom}}(Y_s/s)^{(p')} \overset{\simeq}{\to} \pi_1^{\text{log-geom}}(Y_z/z)^{(p')} \overset{\simeq}{\to} \pi_1^{\text{log-geom}}(Y_Z/Z)^{(p')} \leftarrow \pi_1^{\text{log-geom}}(Y_{s'}/s')^{(p')}$$

where the first two homomorphism are isomorphisms according to corollary 2.16 and theorem 2.6.

Corollary 2.18. *One has a specialization morphism*

$$\pi_1^{\text{log-geom}}(Y_{s'}/s')^{(p')} \to \pi_1^{\text{log-geom}}(Y_s/s)^{(p')}$$

that factors through $\pi_1^{\text{log-geom}}(Y_Z/Z)^{(p')}$.

3. Cospecialization of tempered fundamental groups

3.1. **Topological cospecialization of semistable curves.** Let $f : X \to Y$ be a semistable curve, and let $\bar{y}_2 \to \bar{y}_1$ be a specialization of geometric points of Y. In this section we will define a cospecialization map of graphs $\mathbb{G}(X_{\bar{y}_1}) \to \mathbb{G}(X_{\bar{y}_2})$.

Lemma 3.1. *Assume Y is strictly local of special point y_1, and $X \to Y$ strictly semistable. Let x be a node or a generic point of X_{y_1}. Then $X(x)_{y_2}$ is either contained in the smooth locus of a geometrically irreducible component $F(x)$ of $X(x)_{y_2}$ or contains a single node $F(x)$ of $X(x)_{y_2}$, which is rational.*

Proof. By replacing Y by a closed subscheme, one can assume that y_2 is the generic point of Y and Y is integral.

(i) If x is in the smooth locus of X_{y_1}, $X \to Y$ is smooth at x, and $X(x)_{y_2}$ is geometrically connected by local 0-acyclicity of smooth morphisms.
(ii) If x is a node, one can assume that $Y = \operatorname{Spec} A$ and f factorizes through an étale morphism $X \to \operatorname{Spec} B$ with $B = A[u,v]/uv - a$ and $a(y_1) = 0$.
If $a = 0$ let $Z = X \times_{\operatorname{Spec} B} \operatorname{Spec} A$ where $g : B \to A$ is defined by $g(u) = g(v) = 0$ (this is the closed subscheme of X defined by the node; in particular

Z_{y_2} is the union of all the nodes of X_{y_2}). $Z \to Y$ is étale and thus $Z(x) \to Y$ is an isomorphism. Thus $Z(x)_{y_2}$ is just a rational point $F(x)$.
If $a \neq 0$, $a(y_2) \neq 0$ and thus X_{y_2} is smooth. Since $X \to Y$ is a semistable curve, it is separable (*i.e.* flat with separable geometric fibers). By applying [GD67, cor. 18.9.8] to $X(x) \to Y$, one gets that $X(x)_{y_2}$ is geometrically connected.

□

Let x be a point of $X_{\bar{y}_1}$. If $F(x)$ is a rational node of X_{y_2}, then it defines an edge $F_0(x)$ of $\mathbb{G}_{\bar{y}_2}$. If $F(x)$ is a geometrically irreducible component of X_{y_2}, then it defines a vertex $F_0(x)$ of $\mathbb{G}_{\bar{y}_2}$.

Lemma 3.2. *If $\phi : X' \to X$ is a finite open morphism of strictly semistable curves over Y which maps nodes to nodes on every fiber, then $\phi F' = F\phi$.*

Proof. Indeed $\phi(x)$ is in the closure of $\phi F'(x)$, thus $F\phi(x)$ is in the closure of $\phi F'(x)$. One only has to prove that if $F\phi(x)$ is a node, then $\phi F'(x)$ is also a node. Let us assume that $F\phi(x)$ is a node of X_{y_2}. Let z_1 and z_2 be the two generic points of the irreducible components of X_{y_2} whose closures contain $F\phi(x)$ (and thus also $\phi(x)$). Since ϕ is open, there exists z'_1 and z'_2 in $X(x)_{y_2}$ such that $\phi(z'_1) = z_1$ and $\phi(z'_2) = z_2$. Thus $X(x)_{y_2}$ cannot be in a single irreducible component of X_{y_2}, and thus $F'(x)$ is a node of X'_{y_2}. By assumption, $\phi F'(x)$ is a node of X_{y_2}. □

Corollary 3.3. *There is a unique generalized morphism of graphs*

$$\psi : \mathbb{G}(X_{\bar{y}_1}) \to \mathbb{G}(X_{\bar{y}_2})$$

which is

- *functorial for étale morphisms $X' \to X$,*
- *compatible with base change $Y' \to Y$,*
- *such that if $f : X \to Y$ is strictly semistable and Y is strictly local with special point $\bar{y}_1$, $\psi(x) = F_0(x)$ for any node or generic point x.*

Proof. Let $f : X \to Y$ be a strictly semistable curve, and let $\bar{y}_2 \to \bar{y}_1$ be a specialization of geometric points of Y. After replacing Y by its strict localization at $\bar{y}_1$, onne can assume that Y is strictly local and $\bar{y}_1$ is the closed point.
If e is a vertex of $\mathbb{G}(X_{\bar{y}_1})$, then $\psi(e) := F_0(x)$ where x is the node of $X_{\bar{y}_1}$ corresponding to e. If v is a vertex of $\mathbb{G}(X_{\bar{y}_1})$, then $\psi(v) := F_0(x)$ where x is the generic point of the irreducible component of $X_{\bar{y}_1}$ corresponding to e. Let b be a branch of an edge e in $\mathbb{G}(X_{y_1})$ that abuts to a vertex v. Then $F(x) \subset F(y)$, where x is the node corresponding to e and y is the generic point of the irreducible component corresponding to v. If $F(x) = F(y)$, then $\psi(b) := F_0(x) = F_0(y)$. Otherwise, $\psi(e)$ is an edge and $\psi(v)$ is a vertex, and there is a branch b' of $\psi(e)$ abutting to $\psi(v)$. Since $X_{\bar{y}_2}$ is strictly semistable), this branch is unique. Let $\psi(b) = b'$.
This morphism is clearly compatible with étale morphisms of strictly semistable curve.

If $X \to Y$ is now a general semistable curve, one choses an étale covering family $X' \to X$, and let $X'' = X' \times_X X'$.

One has a commutative diagram:

$$\begin{array}{ccccc} \mathbb{G}(X''_{\bar{y}_1}) & \rightrightarrows & \mathbb{G}(X'_{\bar{y}_1}) & \to & \mathbb{G}(X_{\bar{y}_1}) \\ \downarrow & & \downarrow & & \\ \mathbb{G}(X''_{\bar{y}_2}) & \rightrightarrows & \mathbb{G}(X'_{\bar{y}_2}) & \to & \mathbb{G}(X_{\bar{y}_2}) \end{array}$$

There is a unique generalized morphism of graphs $\psi : \mathbb{G}(X_{\bar{y}_1}) \to \mathbb{G}(X_{\bar{y}_2})$ making the diagram commutative. $\square$

The cospecialization generalized morphisms of graphs are also compatible with finite open morphisms of strictly semistable curves that maps nodes to nodes fiber-wise.

We want to know when this generalized morphism of graphs is an isomorphism.

Proposition 3.4. *If $\psi : \mathbb{G}(X_{\bar{y}_1}) \to \mathbb{G}(X_{\bar{y}_2})$ is a genuine morphism of graphs and f is proper, then ψ is an isomorphism.*

Proof. One may assume $Y = \operatorname{Spec} A$ to be strictly local and integral with special point y_1 and generic point y_2.
The assumption means that étale locally on the special fiber (and thus on X by properness), X is isomorphic to $\operatorname{Spec} A[u, v]/uv$ or is smooth.

Let $Z \subset X$ be the non smooth locus of $X \to Y$, endowed with the reduced subscheme structure. $Z \to Y$ is étale (as can be seen étale locally over X), and proper. One thus gets that F induces a bijection between nodes of $X_{\bar{y}_1}$ and $X_{\bar{y}_2}$.
Let $\widetilde{X}$ be the blowup of X along Z. When $X = \operatorname{Spec} A[u, v]/(uv)$, Z is defined by the ideal generated by u and v, and $\widetilde{X} = \operatorname{Spec} A[u] \coprod \operatorname{Spec} A[v]$.
Thus by looking étale locally over X, one sees that $\widetilde{X}$ is smooth over Y, and that $\widetilde{X}_y$ is simply the normalization of X_y.
Since we assumed $X \to Y$ to be proper, $\widetilde{X} \to Y$ is smooth and proper, thus its Stein factorization induces a bijection between the connected components of $\widetilde{X}_{\bar{y}_1}$ and $\widetilde{X}_{\bar{y}_2}$, and thus the map between the irreducible components of $X_{\bar{y}_1}$ and $X_{\bar{y}_2}$ is a bijection too. $\square$

Proposition 3.5. *Let $f : X \to Y$ be a log semistable curve and let $\bar{y}_2 \to \bar{y}_1$ be a specialization of log geometric point.*
Assume $\overline{M}_{\bar{y}_1} \to \overline{M}_{\bar{y}_2}$ is an isomorphism. Then $\psi : \mathbb{G}(X_{y_1}) \to \mathbb{G}(X_{y_2})$ is a genuine morphism of graphs.

Proof. One can assume Y to be strictly local, integral with generic point y_2: $Y = \operatorname{Spec} A$, with a chart $P \to A$.

To show that it is a genuine morphism, one only has to prove that $\psi(e)$ is an edge if e is an edge of $\mathbb{G}(X_{y_1})$. This is not changed by an étale morphism, so that one can simply assume $X = \operatorname{Spec} A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ with $Q = (P \oplus < u, v >)/(u+v = p)$ and $p \in P$, such that the image of p in $M_{\bar{y}_1}$ is not invertible. Thus the image of p in $M_{\bar{y}_2}$ is not invertible and thus $X = \operatorname{Spec} A[u, v]/(uv = 0)$, which gives the wanted result.

$\square$

3.2. **Topological cospecialization and két morphisms.**

Proposition 3.6. *Let $f : X \to Y$ be a proper log semistable curve. Let $\bar{y}_2 \to \bar{y}_1$ be a specialization of log geometric points. Let $S_{\bar{y}_1}$ be a log geometric két covering of $X_{y_1}/\bar{y}_1$. Let $S_{\bar{y}_2}$ be the corresponding log geometric covering of $X_{y_2}/\bar{y}_2$ defined by proposition 2.13. Then there is a canonical generalized morphism of graphs*

$$\mathbb{G}(S_{\bar{y}_1}) \to \mathbb{G}(S_{\bar{y}_2}).$$

Proof. According to proposition 2.13, $S_{\bar{y}_1}$ extends to a két neighborhood U of $\bar{y}_1$ in a két covering $S \to X_U$. After further két localization, one may even assume $S \to X_U$ to be a log semistable curve.
Thus one can use corollary 3.3 and get a generalized morphism of graphs $\mathbb{G}(S_{\bar{y}_1}) \to \mathbb{G}(S_{\bar{y}_2})$. It does not depend on the choice of U or S since any two extensions of S_0 to a két neighborhood of $\bar{y}_1$ are isomorphic over a smaller két neighborhood and this isomorphism is unique up to further két localization. $\square$

If $S'_{\bar{y}_1} \to S_{\bar{y}_2}$ is a morphism of log geometric két coverings, then the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{G}(S'_{\bar{y}_1}) & \to & \mathbb{G}(S'_{\bar{y}_2}) \\ \downarrow & & \downarrow \\ \mathbb{G}(S_{\bar{y}_1}) & \to & \mathbb{G}(S_{\bar{y}_2}) \end{array}$$

If $\overline{M}_{Y,y_1} \to \overline{M}_{Y,y_2}$ is an isomorphism, $\overline{M}_{U,y_1} \to \overline{M}_{U,y_2}$ is still an isomorphism, so that one can still apply proposition 3.5 to S: the morphism $\mathbb{G}(S_{\bar{y}_1}) \to \mathbb{G}(S_{\bar{y}_2})$ is a genuine morphism of graphs.

3.3. **Cospecialization of tempered fundamental groups.** Let $Y \to O_K$ be a morphism of fs log schemes.
Let Y_{tr} be the open locus of Y where the log structure is trivial ($Y_{\mathrm{tr}} \subset Y_\eta$). Let $\mathfrak{Y}$ be the completion of Y along its closed fiber. Then $\mathfrak{Y}_\eta$ is an analytic domain of Y^{an}. Let $Y_0 = \mathfrak{Y}_\eta \cap Y_{\mathrm{tr}}^{\mathrm{an}} \subset Y^{\mathrm{an}}$.
Let $\tilde{y}$ be a K'-point of Y_0 where K' is a complete extension of K. One has a canonical morphism of log schemes $\operatorname{Spec} O_{K'} \to Y$ where $\operatorname{Spec} O_{K'}$ is endowed with the log structure given by $O_{K'}\backslash\{0\} \to O_{K'}$. The *log reduction* $\tilde{s}$ of $\tilde{y}$ is the log point of Y corresponding to the special point of $\operatorname{Spec} O_{K'}$ with the inverse image of the log structure of $\operatorname{Spec} O_{K'}$. If K' has discrete valuation, then $\tilde{s}$ is a fs log point. If K' is algebraically closed, $\tilde{s}$ is a geometric log point.

Definition 3.7. The category $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$ is the category whose objects are the geometric points $\bar{y}$ of Y_0 such that $\mathcal{H}(y)$ is discretely valued (where y is the underlying point of $\bar{y}$) and $\mathrm{Hom}_{\mathrm{Pt}^{\mathrm{an}}(Y)}(\bar{y}, \bar{y}')$ is the set of két specializations in Y_k from the log reduction $\bar{s}$ of $\bar{y}$ to the log reduction $\bar{s}'$ of $\bar{y}'$ such that there exists some specialization $\bar{y} \to \bar{y}'$ of geometric points in the sense of algebraic étale topology for which the following diagram commutes:

$$\begin{array}{ccc} \bar{y} & \longrightarrow & \bar{s} \\ \downarrow & & \downarrow \\ \bar{y}' & \longrightarrow & \bar{s}' \end{array}$$

The category $\mathrm{Pt}_0^{\mathrm{an}}(Y)$ is the category obtained from $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$ by inverting the class of morphisms $\bar{y} \to \bar{y}'$ such that $\overline{M}_{Y,\bar{s}'} \to \overline{M}_{Y,\bar{s}}$ is an isomorphism.

Let $\mathrm{OutGp}_{\mathrm{top}}$ be the category of topological groups with outer morphisms.

Theorem 3.8. *Let O_K be a complete discretely valued field of residue characteristic $p \geq 0$, let $\mathbb{L}$ be a set of integers such that $p \notin L$. Let $Y \to O_K$ be a morphism of fs log schemes and $X \to Y$ be a proper log semistable curve. Let U be the open locus of X where the log structure is trivial. Then there is a functor $\pi_1^{\mathrm{temp}}(U_{(\cdot)})$: $\mathrm{Pt}_0^{\mathrm{an}}(Y)^{\mathrm{op}} \to \mathrm{OutGp}_{\mathrm{top}}$ sending $\bar{y}$ to $\pi_1^{\mathrm{temp}}(U_{\bar{y}})$.*

Proof. Let $\bar{y}_2 \to \bar{y}_1$ be a morphism $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$. One has to construct a cospecialization morphism $\pi_1^{\mathrm{temp}}(X_{\bar{y}_1}) \to \pi_1^{\mathrm{temp}}(X_{\bar{y}_2})$.
One has a cospecialization functor

$$F : \mathrm{KCov}_{\mathrm{geom}}(X_{s_1}/s_1)^{\mathbb{L}} \to \mathrm{KCov}_{\mathrm{geom}}(X_{s_2}/s_2)^{\mathbb{L}}$$

which factors through $\mathrm{KCov}_{\mathrm{geom}}(X_T/T)^{\mathbb{L}}$ where T is the strict localization at s_1.
The cospecialization functor $\mathrm{KCov}_{\mathrm{geom}}(X_{s_i}/s_i)^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{y}_i})$ is an equivalence since $y_i \in Y_{\mathrm{tr}}$ (2.10). If one choses a specialization $\bar{y}_2 \to \bar{y}_1$ above $\bar{s}_2 \to \bar{s}_1$, then one can apply [Gro71, cor. XIII.2.9] to $U_K \subset X_K \to Y_K$: one gets that the functor $\mathrm{Cov}^{\mathrm{alg}}(U_{\bar{y}_i})^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{y}_2})^{\mathbb{L}}$ is also an equivalence. Thus F is an equivalence.
Let $S_{\bar{s}_1}$ be a log geometric két covering of $X_{\bar{s}_1}$ and let $S_{\bar{s}_2}$ (resp. S_1, S_2) be the corresponding covering of $X_{\bar{s}_2}$ (resp. $U_{\bar{y}_1}$, $U_{\bar{y}_2}$).

There are maps (functorially in S):

$$|S_{\bar{s}_1}^{\mathrm{an}}| \leftarrow |\mathbb{G}(S_{\bar{s}_1})| \to |\mathbb{G}(S_{\bar{s}_2})| \to |S_{\bar{s}_2}^{\mathrm{an}}|$$

where the first and third map are the embedding of the skeleton of an anlytic curve. They are homotopy equivalences.
One thus get a morphism of homotopy types $|S_{\bar{s}_1}^{\mathrm{an}}| \to |S_{\bar{s}_2}^{\mathrm{an}}|$ functorially in S. According to propositions 3.5 and 3.4, if $\overline{M}_{Y,\bar{s}_1} \to \overline{M}_{Y,\bar{s}_2}$ is an isomorphism, $|S_{\bar{s}_1}^{\mathrm{an}}| \to |S_{\bar{s}_2}^{\mathrm{an}}|$ is an isomorphism of homotopy types.
With the notations of proposition 1.4, one thus gets a functor of fibered categories:

$$\begin{array}{ccc} \mathcal{D}_{\mathrm{top}}(U_{\bar{y}_2}) & \to & \mathcal{D}_{\mathrm{top}}(U_{\bar{y}_1}) \\ \downarrow & & \downarrow \\ \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{y}_2})^{\mathbb{L}} & \simeq & \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{y}_1})^{\mathbb{L}} \end{array}$$

Using proposition 1.4, this induces a functor of associated stacks:

$$\begin{array}{ccc} \mathcal{D}_{\mathrm{temp}}(U_{\bar{y}_2})^{\mathbb{L}} & \to & \mathcal{D}_{\mathrm{temp}}(U_{\bar{y}_1})^{\mathbb{L}} \\ \downarrow & & \downarrow \\ \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{y}_2})^{\mathbb{L}} & \simeq & \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{y}_1})^{\mathbb{L}} \end{array}$$

By taking the global sections one gets a functor:

$$\mathrm{Cov}^{\mathrm{temp}}(U_{\bar{y}_2})^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{temp}}(U_{\bar{y}_1}))^{\mathbb{L}},$$

which is an equivalence if $\overline{M}_{Y,\bar{s}_1} \to \overline{M}_{Y,\bar{s}_2}$ is an isomorphism. It induces a cospecialization outer morphism of tempered fundamental groups

$$\pi_1^{\mathrm{temp}}(U_{\bar{y}_1})^{\mathbb{L}} \to \pi_1^{\mathrm{temp}}(U_{\bar{y}_2})^{\mathbb{L}},$$

which is an isomorphism if $\overline{M}_{Y,\bar{s}_1} \to \overline{M}_{Y,\bar{s}_2}$ is an isomorphism. $\square$

Remark. Such a functor cannot exist if $p \neq 0$ and $\mathbb{L}$ is the set of all primes. Indeed, if X_1 and X_2 are two Mumford curves with isomorphic stable reduction but with different metrics on the graphs of their stable models, then their tempered fundamental groups are not isomorphic ([Lep10]). Let us consider a moduli space of stable curves with level structure, endowed with its canonical log structure, and a geometric point $\bar{s}$ in the special fiber of the moduli space such that the corresponding stable curve has totally degenerate reduction. In particular, it has at least two double points, and thus the rank of $\overline{M}_{\bar{s}}^{\mathrm{gp}}$ is at least two. Let us take two valuative fs log points s_1 and s_2 (*i.e.* $\overline{M}_{s_i} \simeq \mathbf{N}$) such that the corresponding morphisms $\overline{M}_{\bar{s}}^{\mathrm{gp}} \to \mathbf{Z}$ are not collinear. Let η_1 and η_2 be discretely valued points of the analytic geometric fiber whose log reductions are s_1 and s_2. Then the two corresponding geometric Mumford curves have different metric on the graph of their stable model, and thus have non isomorphic tempered fundamental groups. But two geometric log points over s_1 and s_2 are isomorphic with respect to specialization for két topology.

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