Coverings in p-adic analytic geometry and log coverings
II: Cospecialization of the \((p')\)-tempered fundamental group in higher dimensions

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Abstract
This paper constructs cospecialization homomorphisms between the \((p')\) versions of the tempered fundamental group of the fibers of a smooth morphism with polystable reduction (the tempered fundamental group is a sort of analog of the topological fundamental group of complex algebraic varieties in the p-adic world). We studied the question for families of curves in another paper. To construct them, we will start by describing the pro-\((p')\) tempered fundamental group of a smooth and proper variety with polystable reduction in terms of the reduction endowed with its log structure, thus defining tempered fundamental groups for log polystable varieties.

Introduction

This paper is a sequel to [9]. In that article we studied the behavior of the tempered fundamental groups of the fibers of a \(p\)-adic family of curves. More precisely we proved the following:

Theorem 0.1 ([9, th. 0.1]). Let \(K\) be a complete discretely valued field. Let \(L\) be a set of primes that does not contain the residual characteristic of \(K\). Let \(Y \to O_K\) be a morphism of log schemes. Let \(Y_0 = Y_{tr} \cap \mathcal{Y}_\eta \subset Y_{an}\) where \(\mathcal{Y}\) is the completion of \(Y\) along its closed fiber. Let \(X \to Y\) be a proper semistable curve with compatible log structure. Let \(U = X_{tr}\). Let \(\eta_1\) and \(\eta_2\) be two Berkovich points of \(Y_0\) whose residue fields have discrete valuation, and let \(\bar{\eta}_1, \bar{\eta}_2\) be geometric points above them. Let \(\bar{s}_2 \to \bar{s}_1\) be a log specialization of their log reductions such that there exists a compatible specialization \(\bar{\eta}_2 \to \bar{\eta}_1\). Then, there is a cospecialization homomorphism \(\pi_1^{temp}(U_{\bar{\eta}_1})^L \to \pi_1^{temp}(U_{\bar{\eta}_2})^L\). Moreover, it is an isomorphism if \(\overline{M}_{Y, \bar{s}_1} \to \overline{M}_{Y, \bar{s}_2}\) is an isomorphism.

The aim of this paper is to generalize this result in higher dimension. However, in this paper, we will only consider the case of vertical semistable morphisms \(X \to Y\) (which means mainly that \(U_{\bar{\eta}_i} = X_{\bar{\eta}_i}\)).

Recall that, if \(L\) is a set of primes, the \(L\)-tempered fundamental group is the prodiscrete group that classifies the \(L\)-tempered coverings, which are étale coverings in the sense of A.J. de Jong (that is to say that locally on the Berkovich topology, it is a direct sum of finite étale coverings) such that, after pulling back by some \(L\)-finite étale covering, they become topological coverings (for the Berkovich topology).

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In this article, we shall study the following situation. Let $K$ be a discretely valued field, $O_K$ be its valuation ring, $k$ be its residue field and $p$ its characteristics (which can be 0). Let $X \to Y$ be a proper pluristable (for example semistable) morphism of schemes over $O_K$ with geometrically connected fibers.

Let $\mathbb{L}$ be a set of prime that does not contain $p$. If $\eta_1$ is a (Berkovich) point of the generic fiber of $Y$, we first want to describe the geometric $\mathbb{L}$-tempered fundamental group of $X_{\eta_1}$ in terms of $X_{s_1}$ where $s_1$ is the reduction of $\eta_1$. To be sure that this reduction exists we have to assume $\eta_1$ is in the tube $\mathfrak{Y}_\eta$ of the special fiber of $Y$. Let us make sure at first that we can get such a description for the pro-$\mathbb{L}$ completion, i.e. the algebraic fundamental group. One cannot apply directly Grothendieck’s specialization theorems since the special fiber is not smooth but only pluristable. Indeed, a pro-$\mathbb{L}$ geometric covering of the generic fiber will in generally only induce a Kummer covering on the special fiber. These are more naturally described in terms of log geometry and of the log fundamental group.

The log fundamental group classifies Kummer log étale coverings (or, equivalently finite log étale coverings) : étale locally, these coverings are pullbacks of a morphism $\text{Spec} \mathbb{Z}[Q] \to \text{Spec} \mathbb{Z}[P]$ of a morphism of monoids $P \to Q$ where $Q$ is the saturation of $P$ in an extension of $P^\text{gp}$ of finite index invertible on the log scheme. For a proper and log smooth log scheme over a complete discrete valuation ring, there is, as in the proper and smooth case for Grothendieck’s fundamental group, a specialization morphism from the pro-$\mathbb{L}$ log fundamental group of the generic fiber (which is isomorphic to the pro-$\mathbb{L}$ algebraic fundamental group of the maximal open subset of the generic fiber where the log structure is trivial) to the pro-$\mathbb{L}$ log fundamental group of the closed fiber. We will have to assume the field $\mathcal{H}(\eta_1)$ to be with discrete valuation in order to get log schemes with good finiteness properties (more precisely to be fs). Then, one can endow $X_{s_1}$ with a natural log structure. The pro-$\mathbb{L}$ fundamental group of $X_{\eta_1}$ is isomorphic to the pro-$\mathbb{L}$ log fundamental group of $X_{s_1}$. To try to describe the $\mathbb{L}$-tempered fundamental group, one has to describe the topological behavior of any $\mathbb{L}$-algebraic covering of $X_{\eta_1}$. Berkovich, in [3], constructed a combinatorial object (more precisely a polysimplicial set) depending only on $X_{s_1}$, such that the Berkovich generic fiber $X_{\eta_1}$ is naturally homotopically equivalent to the geometric realization of this combinatorial object, thus generalizing the case of curves with semistable reduction, where the homotopy type of the generic fiber can be naturally described in terms of the graph of this semistable reduction. We will extend such a description to our log coverings: for every log covering $Y \to X_{\mathcal{O}_{\mathcal{H}(\eta_1)}}$ we will construct a combinatorial object $C(Y)$, depending only on $Y_{s_1}$, such that its geometric realization $|C(Y)|$ is naturally homotopically equivalent to the Berkovich generic fiber $Y_{\eta_1}$. This will enable us to define a $\mathbb{L}$-tempered fundamental group of our log reduction, which is isomorphic to the tempered fundamental group of the generic fiber: for any Galois két covering $f : Y \to X_{s_1}$, there is an action of $\text{Gal}(Y/X_{s_1})$ on $C(Y)$. Such an action defines an extension $G_Y$ of $\text{Gal}(Y/X_{s_1})$ by $\pi_1^\text{top}(|C(Y)|)$: $G_Y = \{(g_1, g_2) \in \text{Aut}(|C(Y)|^\infty) \times \text{Gal}(Y/X_{s_1}) | \pi g_1 = g_2 \}$, where $\pi : |C(Y)|^\infty \to |C(Y)|$ is the universal topological covering of $|C(Y)|$. The $\mathbb{L}$-tempered fundamental group of $X_{s_1}$ is the projective limits of these extensions $G_Y$, where $Y$ runs through pointed két Galois coverings of $X$ of $\mathbb{L}$ order.

In particular, one gets:

**Theorem 0.2** (see th. 3.2). The $\mathbb{L}$-tempered fundamental group of $X_{\eta_1}$ only depends on the log reduction $X_{s_1}$.

Once we have a definition for the log geometric tempered fundamental group $\pi_1^\text{temp-geom}(X_{s_1})$ of the log fibers in the special locus of $Y$, one can reformulate our cospecialization problem only in terms of this special locus.

We will prove the following:

**Theorem 0.3** (th. 4.8). Let $\eta_1$ and $\eta_2$ be two Berkovich points with discrete valuation fields of
Let \( Y_0 = \mathcal{Y}^{\text{an}}_\text{st} \cap \mathcal{Y}_\eta \). Let \( \bar{\eta}_1, \bar{\eta}_2 \) be geometric points above them. Let \( \bar{s}_2 \rightarrow \bar{s}_1 \) be a specialization of their log reductions such that there exists a compatible specialization \( \bar{\eta}_2 \rightarrow \bar{\eta}_1 \). Then there is a cospecialization homomorphism \( \pi_1^{\text{temp-geom}}(X_{\bar{\eta}_1})^L \rightarrow \pi_1^{\text{temp-geom}}(X_{\bar{\eta}_2})^L \).

Moreover, one can give a criterion for this cospecialization homomorphism to be an isomorphism. To do this, we will have to make an assumption on the combinatorial behavior of the geometric fibers of \( X \rightarrow Y \). More precisely, the poly simplicial set associated with those geometric fibers will be assumed to be interiorly free (this is for example the case if \( X \rightarrow Y \) is strictly polystable or if \( X \rightarrow Y \) is of relative dimension 1, which explains why such a condition did not appear in [9]). If the morphism of monoids \( \overline{\mathcal{M}}_{Y, s_1} \rightarrow \overline{\mathcal{M}}_{Y, s_2} \) is an isomorphism, then the cospecialization homomorphism \( \pi_1^{\text{temp-geom}}(X_{\bar{\eta}_1})^L \rightarrow \pi_1^{\text{temp-geom}}(X_{\bar{\eta}_2})^L \) is an isomorphism.

The first thing we need to construct the cospecialization homomorphism for tempered fundamental groups is a specialization morphism for the \( \mathbb{L} \)-log geometric fundamental groups of \( X_{s_1} \) and \( X_{s_2} \). More precisely, we would like to extend any \( \mathbb{L} \)-log geometric covering of \( X_{s_1} \) to a \( \mathbb{L} \)-log geometric covering of \( X_{s_2} \). By restricting this extension to \( X_{s_2} \), one obtains a functor from \( \mathbb{L} \)-log coverings of \( X_{s_1} \) to \( \mathbb{L} \)-log coverings of \( X_{s_2} \), this functor induces the wanted specialization morphism of \( \mathbb{L} \)-log geometric fundamental groups. If one has such a specialization morphism, by comparing it to the fundamental groups of \( X_{\eta_1} \) and \( X_{\eta_2} \) and using Grothendieck’s specialization theorem, we will easily get that it must be an isomorphism. These specialization morphisms has already been constructed in [9, prop. 2.10].

Then we have to study the combinatorial behavior of a \( \mathbb{L} \)-log covering with respect to cospecialization. By \( \mathbb{L} \)-log localization, one can assume that \( Y \) is strictly local with special point \( \bar{s}_1 \). Up to \( \mathbb{L} \)-log localization of \( Y \), any \( \mathbb{L} \)-log covering \( U_{s_1} \) of \( Y_{s_1} \) extends to a \( \mathbb{L} \)-log covering \( U \) of \( Y \) and \( U \rightarrow Y \) is saturated. For a stratum \( u \) of \( U_{s_1} \), there is among the strata of \( U_{s_2} \) whose closure contains \( u \) a stratum \( u' \) with smallest closure (i.e., a biggest stratum for specialization): it defines a map \( \text{Str}(U_{s_1}) \rightarrow \text{Str}(U_{s_2}) \).

The fact that \( U \rightarrow Y \) is saturated implies that the closure of the strata of \( U \) are flat over their image in \( Y \) and have geometrically reduced fibers. Thanks to [6, cor. 18.9.8]), this implies that \( u' \) is geometrically connected, whence a cospecialization map \( \text{Str}(U_{s_1}) \rightarrow \text{Str}(U_{s_2}) \). This cospecialization map can be extended into a morphism of polysimplicial sets. One gets by pullback a specialization functor between the category of topological coverings of the polysimplicial sets \( U_{s_2} \) and \( U_{s_1} \). Since the cospecialization morphisms of polysimplicial sets commute with \( \mathbb{L} \)-log coverings, the specialization functor can be seen as a functor of fibered categories over the category of \( \mathbb{L} \)-log coverings of \( X_{s_1} \) (or equivalently of \( \mathbb{L} \)-finite \( \mathbb{L} \)-log fundamental groups). But the fibered category of \( \mathbb{L} \)-finite \( \mathbb{L} \)-log coverings over the category of \( \mathbb{L} \)-finite \( \mathbb{L} \)-log coverings of \( X_{\eta_1} \) is naturally equivalent to the stack associated to the fibered categories of topological coverings over the category of \( \mathbb{L} \)-finite \( \mathbb{L} \)-log coverings of \( X_{\eta_1} \). Thus the topological specialization functor gives us the wanted tempered specialization functor.

Let us now discuss the organization of the paper.

The first section of this paper will be devoted to recall the main tools we will need later. We will recall the definition of the tempered fundamental group and its basic properties. We will also consider an \( \mathbb{L} \)-version of the tempered fundamental group, where \( \mathbb{L} \) is a set of prime numbers (\( \mathbb{L} \)-tempered fundamental groups were already introduced in [10] in the case of curves). We will then recall the basics of log geometry, especially the theory of \( \mathbb{L} \)-log coverings and \( \mathbb{L} \)-log fundamental groups. We will end this part by recalling the topological structure of the generic fiber (considered as a Berkovich space) of a pluristable formal scheme, as studied in [3] and in [4].

In §2, we define the tempered fundamental group of a connected pluristable log scheme \( X \) over
a log point. To do this, we define a functor $C$ from the Kummer étale site of our pluristable log scheme $X$ to the category of polysimplicial sets (which extends the definition of the polysimplicial set associated to a pluristable scheme defined by Berkovich in [3]). We also defines a log geometric version by taking the projective limit under connected étale extensions of the base log point.

In §3, for a connected, proper, generically smooth and pluristable scheme $X$ over a complete discretely valued ring $O_K$ (thus endowed with a canonical log structure), we construct a specialization morphism between the $\mathbb{L}$-tempered fundamental group of the generic fiber, considered as a Berkovich space, and the $\mathbb{L}$-tempered fundamental group of the special fiber endowed with the inverse image log structure, which is an isomorphism if the residual characteristic of $K$ is not in $\mathbb{L}$. This specialization morphism is induced by the specialization morphism from the algebraic fundamental group of the generic fiber to the log fundamental group of the special fiber, and by the fact that the geometric realization of the polysimplicial set $|C(Y)|$ of a két covering of the special fiber of $X$ is canonically homotopically equivalent to the Berkovich space $Y_\eta^\text{an}$ of the corresponding étale covering of the generic fiber. This homotopy equivalence is obtained by extending the strong deformation retraction $X^\eta$ to a strong deformation retraction of $Y_\eta^\text{an}$ onto a subset canonically homeomorphic to $|C(Y)|$.

In §4, we construct cospecialization morphisms between the polysimplicial sets of the geometric fibers of a polystable fibration. To do so, we first prove that, up to étale localization of $Y$ at $s_1$, for any stratum $x$ of $X_{s_1}$, the set of strata of $X_{s_2}$ whose closure contains $x$ has a biggest element (for the order induced by existence of specialization), and this biggest stratum is geometrically irreducible. This will induce cospecialization morphisms on the set of strata of the geometric fibers of $X \to Y$. Up to két localization, the same result is also true for két coverings of $Y$. This cospecialization maps of set of strata in fact come from maps of polysimplicial sets. If we identify the categories of $\mathbb{L}$-két coverings of $X_{s_1}$ and $X_{s_2}$ thanks to specialization of két coverings, one gets, for $U$ in this category, a map $|C(U_{s_1})| \to |C(U_{s_2})|$ functorially in $U$ (and in particular, when $U$ is Galois, compatibly with the action of the Galois group of $U$). We get from this cospecialization morphisms between the $\mathbb{L}$-geometric tempered fundamental groups of the fibers of our strictly polystable log fibration. Thanks to the isomorphisms between the $\mathbb{L}$-geometric tempered fundamental group of the fiber over a discretely valued Berkovich point of the generic part of our base log scheme and the $\mathbb{L}$-geometric tempered fundamental group of the fiber over the reduction log point, we will get theorem 0.3.

1. Reminder on the skeleton of a Berkovich space with pluristable reduction

1.1 Polystable morphisms

Let $K$ be a complete nonarchimedean field and let $O_K$ be its ring of integers. If $X$ is a locally finitely presented formal scheme over $O_K$, $X_\eta$ will denote the generic fiber of $X$ in the sense of Berkovich ([2, section 1]).

Recall the definition of a polystable morphism of formal schemes:

**Definition 1.1** ([3, def. 1.2], [4, section 4.1]). Let $\phi : \mathfrak{Y} \to X$ be a locally finitely presented morphism of formal schemes over $O_K$.

1. $\phi$ is said to be **strictly polystable** if, for every point $y \in \mathfrak{Y}$, there exists an open affine neighborhood $\mathfrak{X} = \text{Spf}(A)$ of $x := \phi(y)$ and an open neighborhood $\mathfrak{Y}' \subset \phi^{-1}(\mathfrak{X}')$ of $y$ such that the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$ factors through an étale morphism $\mathfrak{Y}' \to \text{Spf}(B_0) \times X_0 \cdots \times X_0 \text{Spf}(B_p)$ where each $B_i$ is of the form $A\{T_0, \ldots, T_n\}/(T_0 \cdots T_n - a_i)$ with $a \in A$ and $n \geq 0$. It is said
to be \textit{nondegenerate} if one can choose \(X', Y'\) and \((B_i, a_i)\) such that \(\{x \in (\text{Spf}(A))_0 | a_i(x) = 0\}\) is nowhere dense.

(ii) \(\phi\) is said to be \textit{polystable} if there exists a surjective \(\text{étale}\) morphism \(\mathcal{Y}' \to \mathcal{Y}\) such that \(\mathcal{Y}' \to \mathcal{X}\) is strictly polystable. It is said to be \textit{nondegenerate} if one can choose \(\mathcal{Y}'\) such that \(\mathcal{Y}' \to \mathcal{X}\) is nondegenerate.

Then a \textit{(nondegenerate) polystable fibration} of length \(l\) over \(\mathcal{S}\) is a sequence of \textit{(nondegenerate) polystable fibres} \(\mathcal{X} = (X_1 \to \cdots \to X_l \to \mathcal{S})\).

Then \(K\text{-\textit{Pstf}}^l\) \((\text{resp. } K\text{-\textit{Pstf}}^{\text{sm}}, K\text{-\textit{Pstf}}^{\text{tr}})\) will denote the category of polystable fibrations of length \(l\) over \(O_K\), where a morphism \(\mathcal{X} \to \mathcal{Y}\) is a collection of \(\text{étale}\) \((\text{resp. smooth, resp. trivially polystable})\) morphisms \(X_i \to Y_i\) which satisfies the natural commutation assumptions. \(\mathcal{Pstf}^l\) \((\text{resp. } \mathcal{Pstf}^{\text{sm}}, \mathcal{Pstf}^{\text{tr}})\) will denote the category of couples \((\mathcal{X}, K_1)\) where \(K_1\) is a complete non archimedean field and \(\mathcal{X}\) is a polystable fibration over \(O_{K_1}\), and a morphism \((\mathcal{X}, K_1) \to (\mathcal{Y}, K_2)\) is a couple \((\phi, \psi)\) where \(\phi\) is an isometric extension \(K_2 \to K_1\) and \(\psi\) is a morphism \(\mathcal{X} \to \mathcal{Y} \otimes O_{K_2} O_{K_1}\) in \(K_1\text{-}\mathcal{Pstf}^l\) \((\text{resp. } K_1\text{-}\mathcal{Pstf}^{\text{sm}}, K_1\text{-}\mathcal{Pstf}^{\text{tr}})\).

Let \(k\) be a field.

Let \(X\) be a \(k\)-scheme locally of finite type.

The normal locus \(\text{Norm}^\text{\text{red}}(X)\) is a dense open subset of \(X\). Let us define inductively \(X^{(0)} = X^\text{\text{red}}, X^{(i + 1)} = X^{(i)} \setminus \text{Norm}(X^{(i)})\). The irreducible components of \(X^{(i)} \setminus X^{(i + 1)}\) are called the strata of \(X\) (of rank \(i\)). This gives a partition of \(X\). The set of the generic points of the strata of \(X\) is denoted by \(\text{Str}(X)\) (This set is in natural bijection with the set of strata of \(X\)). There is a natural partial order on \(\text{Str}(X)\) defined by \(x \leq y\) if and only if \(y \in \{x\}\).

Berkovich defines another filtration \(X = X^{(0)} \supset X^{(1)} \supset \cdots\) such that \(X^{(i + 1)}\) is the closed subset of points contained in at least two irreducible components of \(X^{(i)}\). \(X\) is said to be \textit{quasinormal} if all of the irreducible components of each \(X^{(i)}\), endowed with the reduced subscheme structure, is normal (this property is local for the Zariski topology and remains true after \(\text{étale}\) morphisms). If \(X\) is quasinormal, then \(X^{(i)} = X^{(i)}\). \(X\) is quasinormal if and only if the closure of every stratum is normal. A strictly plurinodal scheme over a field is quasinormal \([3, \text{prop. 2.1}]\).

There is a natural partial order on \(\text{Str}(X)\) defined by \(x \leq y\) if and only if \(y \in \{x\}\).

We say that a strictly plurinodal scheme \(X\) over a field \(K\) is \textit{elementary} if \(\text{Str}(X)\) has a biggest element; we say that it is \textit{geometrically elementary} if it is elementary and all the strata are geometrically irreducible. Finally, a strictly pluristable morphism \(Y \to X\) is \textit{geometrically elementary} if all the fibers are geometrically elementary.

### 1.2 Polysimplicial sets

Berkovich defines \textit{polysimplicial sets} in \([3, \text{section 3}]\) as follows.

For an integer \(n\), let \([n]\) denote \(\{0, 1, \ldots, n\}\).

For a tuple \(\mathbf{n} = (n_0, \ldots, n_p)\) with either \(p = n_0 = 0\) or \(n_i \geq 1\) for all \(i\), let \([\mathbf{n}]\) denote the set \([n_0] \times \cdots \times [n_p]\) and \(w(\mathbf{n})\) denote the number \(p\).

Berkovich defines a category \(\Lambda\) whose objects are \([\mathbf{n}]\) and morphisms are maps \([\mathbf{m}] \to [\mathbf{n}]\) associated with triples \((J, f, \alpha)\), where:

- \(J\) is a subset of \([w(\mathbf{m})]\) asserted to be empty if \([\mathbf{m}] = [0]\),
- \(f\) is an injective map \(J \to [w(\mathbf{n})]\),
- \(\alpha\) is a collection \(\{\alpha_l\}_{0 \leq l \leq p}\), where \(\alpha_l\) is an injective map \([m_{f^{-1}(l)}] \to [n_l]\) if \(l \in \text{Im}(f)\), and \(\alpha_l\) is a map \([0] \to [n_l]\) otherwise.
The map $\gamma: [m] \to [n]$ associated with $(J, f, \alpha)$ takes $j = (j_0, \cdots, j_w(m)) \in [m]$ to $i = (i_0, \cdots, i_w(n))$ with $i_l = \alpha_l(j_{f^{-1}(l)})$ for $l \in \text{Im}(f)$, and $i_l = \alpha_l(0)$ otherwise. A polysimplicial set is a functor $\Lambda^{op} \to \text{Set}$. Polysimplicial sets form a category denoted by $\Lambda^\circ \text{Set}$.

One considers $\Lambda$ as a full subcategory of $\Lambda^\circ \text{Set}$ by the Yoneda functor. If $C$ is a polysimplicial set $\Lambda/C$ is the category whose objects are morphisms $[n] \to C$ in $\Lambda^\circ \text{Set}$ and morphisms from $[n] \to C$ to $[m] \to C$ are morphisms $[n] \to [m]$ that make the triangle commute. Objects of $\Lambda/C$ are called polysimplices of $C$, and if $x: [n] \to C$ is a polysimplex, $n$ will be denoted by $n_x$.

A polysimplex $x$ of a polysimplicial set $C$ is said to be degenerate if there is a non isomorphic surjective map $f$ of $\Lambda$ such that $x$ is the image by $f$ of a polysimplex of $C$. Let $C^\circ_n$ be the subset of non degenerate polysimplices of $C_n$.

Thanks to an analog of Eilenberg-Zilber lemma for polysimplicial sets ([3, lem. 3.2]), a morphism $C' \to C$ is bijective if and only if it maps non degenerate polysimplices to nondegenerate polysimplices and $(C')^\circ_n \to C^\circ_n$ is bijective for any $n$.

There is a functor $O: \Lambda^\circ \text{Set} \to \text{Poset}$ where $O(C)$ is the partially ordered set associated to $\text{Ob}(\Lambda/C)$ endowed with the preorder where $x \leq y$ if there is a morphism $x \to y$ in $\Lambda/C$. If one sees $O(C)$ as a category, there is an obvious functor $\Lambda/C \to O(C)$. As a set, $O(C)$ coincides with the set of equivalence classes of nondegenerate polysimplices.

A polysimplicial set $C$ is said interiorly free if $\text{Aut}(n)$ acts freely on $C^\circ_n$. If $C_1 \to C_2$ is a morphism of polysimplicial sets mapping nondegenerate polysimplices to nondegenerate polysimplices such that $O(C_1) \to O(C_2)$ is an isomorphism and $C_2$ is interiorly free, then $C_1 \to C_2$ is an isomorphism.

Berkovich also defines a strictly polysimplicial category $\Lambda$ whose objects are those of $\Lambda$, but with only injective morphisms between them. The functor $\Lambda \to \Lambda \to \Lambda^\circ \text{Set}$ extends to a functor $\Lambda \to \Lambda^\circ \text{Set}$ which commutes with direct limits (the objects of $\Lambda^\circ \text{Set}$ will be called strictly polysimplicial sets).

Berkovich then considers a functor $\Sigma: \Lambda \to \mathcal{K}e$ to the category of Kelley spaces, i.e. topological spaces $X$ such that a subset of $X$ is closed whenever its intersection with any compact subset of $X$ is closed. This functor takes $[n]$ to $\Sigma_n = \{(u_{il})_{0 \leq i \leq p, 0 \leq l \leq n_i} \in [0, 1]^{|n|} | \sum u_{il} = 1\}$, and takes a map $\gamma$ associated to $(J, f, \alpha)$ to $\Sigma(\gamma)$ that maps $u = (u_{jk})$ to $u' = (u'_{il})$ defined as follows: if $|m| \neq 0$ and $i \notin \text{Im}(f)$ or $|m| = |0|$ then $u'_{il} = 1$ for $l = \alpha_i(0)$ and $u'_{il} = 0$ otherwise; if $|m| \neq |0|$ and $i \in \text{Im}(f)$, then $u'_{il} = u_{f^{-1}(i), \alpha_i^{-1}(l)}$ for $l \in \text{Im}(\alpha_i)$ and $u'_{il} = 0$ otherwise.

This induces a functor, the geometric realization, $| |: \Lambda^\circ \text{Set} \to \mathcal{K}e$ (by extending $\Sigma$ in such a way that it commutes with direct limits). If $O(C)$ is finite (resp. locally finite), then $|C|$ is compact (resp. locally compact).

There is also a bifunctor $\Box: \Lambda^\circ \text{Set} \times \Lambda^\circ \text{Set} \to \Lambda^\circ \text{Set}$ which commutes with direct limits and defined by $[(n_0, \cdots, n_p)] \Box [(n'_0, \cdots, n'_p)] = [(n_0, \cdots, n_p, n'_0, \cdots, n'_p)]$. Thus $|C \Box C'| = |C| \times |C'|$ where the product on the right is the product of Kelley spaces (which is the same as the product of topological spaces whenever $C$ and $C'$ are locally finite).

### 1.3 Polysimplicial set of a polystable fibration

If $X$ is strictly polystable over $k$ and $x \in \text{Str}(X)$, $\text{Irr}(X, x)$ will denote the metric space of irreducible components of $X$ passing through $x$ where $d(X_1, X_2) = \text{codim}_x(X_1 \cap X_2)$. On a tuple $[n]$, one can consider the metric $d$ defined by $d((n_0, \cdots, n_p), (n'_0, \cdots, n'_p)) = \{|i \in [0, p]| n_i \neq n'_i\}$. Then there is a unique tuple $[n]$ such that $\text{Irr}(X, x)$ is bijectively isometric to $[n]$. If $[m] \to [n]$ is isometric, there exists a unique $y \in \text{Str}(X)$ with $y \leq x$ and a unique isometric bijection $[m] \to \text{Irr}(X, y)$ such that

$$
\begin{array}{ccc}
[n] & \to & \text{Irr}(X, x) \\
\uparrow & & \uparrow \\
[m] & \to & \text{Irr}(X, y)
\end{array}
$$

6
commutes.
The functor which to \([n]\) associates the set of couples \((x, \mu)\) where \(x \in \text{Str}(X)\) and \(\mu\) is an isometric bijection \([n] \to \text{Irr}(X, x)\) defines a strict polysimplicial set \(C(X)\) (and thus a polysimplicial set \(C(X)\)).

There is a functorial isomorphism of partially ordered sets \(O(C(X)) \simeq \text{Str}(X)\).

**Proposition 1.2 ([3, prop. 3.14]).** One has a functor \(C : \mathcal{P}st^{sm} \to \Lambda^\circ \text{Set}\), such that \(C(X)\) is as previously defined if \(X\) is strictly polystable and, for every étale surjective morphism \(X' \to X\):

\[
C(X) = \text{Coker}(C(X' \times_X X') \Rightarrow C(X')).
\]

This functor extends to a functor \(C\) for strictly polystable fibrations over \(K\) of length \(l\).

Let us assume we already constructed \(C\) for strictly polystable fibrations of length \(l - 1\) such that \(O(C(X)) = \text{Str}(X_{l-1})\). Let \(X : X_l \to X_{l-1} \to \cdots \to \text{Spec} k\) be a strictly polystable fibration, and let \(X_{l-1} : X_{l-1} \to \cdots \to \text{Spec} k\). Then for every \(x' \leq x \in \text{Str}(X_{l-1})\), one has:

**Lemma 1.3 ([3, cor.6.2]).** There is a canonical cospecialization morphism \(C(X_{l,x}) \to C(X_{l,x'})\) and if \(x'' \leq x' \leq x\), the morphism \(C(X_{l,x'}) \to C(X_{l,x''})\) coincides with the composition \(C(X_{l,x'}) \to C(X_{l,x''})\).

This gives a functor \(\text{Str}(X_{l-1})^{\text{op}} \to \Lambda^\circ \text{Set}\) that maps an object \(x\) in \(\text{Str}(X_{l-1})^{\text{op}}\) to \(C(X_{l,x})\) and an arrow \(x' \to x\) to the cospecialization morphism \(C(X_{l,x'}) \to C(X_{l,x})\) given by lemma 1.3. If one composes this functor with \((\Lambda^\circ / (C(X_{l-1})))^{\text{op}} \to O(C(X_{l-1})^{\text{op}} = \text{Str}(X_{l-1})^{\text{op}}\), one gets a functor

\[
D : (\Lambda^\circ / (C(X_{l-1})))^{\text{op}} \to \Lambda^\circ \text{Set}.
\]

Berkovich then defines a polysimplicial set (where we set \(C = C(X_{l-1})\)):

\[
C(X) = C \sqcap D := \text{Coker}( \prod_{N_1(\Lambda / C)} [n_y] \sqcap D_x \Rightarrow \prod_{N_0(\Lambda / C)} [n_x] \sqcap D_x),
\]

where, for \(f : y \to x \in N_1(\Lambda / C)\), the upper arrow sends \([n_y] \sqcap D_x\) to \([n_x] \sqcap D_x\) by the morphism \([f] \sqcap \text{id}_{D_x}\) and the lower arrow sends \([n_y] \sqcap D_x\) to \([n_y] \sqcap D_y\) by the morphism \(\text{id}_{[n_y]} \sqcap D_f\). This construction extends to (non necessarily strictly) polystable fibrations:

**Proposition 1.4 ([3, prop 6.9]).** There is a functor \(C : \mathcal{P}st_{l}^{\text{ps}} \to \Lambda \text{Set}\) such that:

(i) for every étale surjective morphism of polystable fibrations \(X' \to X\):

\[
C(X) = \text{Coker}(C(X' \times_X X') \Rightarrow C(X')).
\]

(ii) \(O(C(X)) \simeq \text{Str}(X)\).

### 1.4 Skeleton of a Berkovich space

Berkovich attaches to a polystable fibration \(\mathfrak{X} = (\mathfrak{X}_l \to \mathfrak{X}_{l-1} \to \cdots \to \text{Spf}(O_K))\) a subset of the generic fiber \(\mathfrak{X}_{l,0}\) of \(\mathfrak{X}_l\), the **skeleton** \(S(\mathfrak{X})\) of \(\mathfrak{X}\), which is canonically homeomorphic to \(|C(X)|\) (see [3, th. 8.2]). In fact, when \(\mathfrak{X}\) is non degenerate—for example generically smooth (we will only use the results of Berkovich to such polystable fibrations)—the skeleton of \(\mathfrak{X}\) depends only on \(\mathfrak{X}_l\) according to [4, prop. 4.3.1(ii)]; such a formal scheme that fits into a polystable fibration will be called **pluristable**, and we will note \(S(\mathfrak{X}_l)\) this skeleton.

In this case [4, prop. 4.3.1.(ii)] gives a description of \(S(\mathfrak{X}_l)\), which is independant of the retraction. For any \(x, y \in \mathfrak{X}_{l,0}\), we write \(x \preceq y\) if for every étale morphism \(\mathfrak{X}' \to \mathfrak{X}_l\) and any \(x'\) over \(x\), there exists \(y'\) over \(y\) such that for any \(f \in O(\mathfrak{X}_l)\), \(|f(x')| \leq |f(y')|\) (\(\preceq\) is a partial order on \(\mathfrak{X}_{l,0}\)). Then \(S(\mathfrak{X}_l)\) is just the set of maximal points of \(\mathfrak{X}_{l,0}\) for \(\preceq\).
Moreover there is a strong deformation retraction of \(X_{l,\eta}\) to \(S(X)\) and this construction is compatible with étale morphisms; more precisely, one has the following theorem:

**Theorem 1.5** ([3, th. 8.1]). One can construct, for every polystable fibration \(X = (X_1 \rightarrow \cdots \rightarrow X_1 \rightarrow \operatorname{Spf}(O_K))\), a proper strong deformation retraction \(\Psi : X_{l,\eta} \times [0,1] \rightarrow X_{l,\eta}\) of \(X_{l,\eta}\) onto the skeleton \(S(X)\) of \(X\) such that:

(i) \(S(X) = \bigcup_{x \in S(X_{l,\eta})} S(X_{l,x})\) (set-theoretic disjoint union), where \(X_{l-1} = (X_{l-1} \rightarrow \cdots \rightarrow \operatorname{Spf}(O_K))\);

(ii) if \(\phi : Y \rightarrow X\) is a morphism of fibrations in \(\mathcal{Psf}_{f_1}^\text{ét}\), one has \(\phi_{l,\eta}(y_t) = \phi_{l,\eta}(y)_t\) for every \(y \in Y_{l,\eta}\).

Let us describe more precisely how the retraction is defined.

If \(X = \operatorname{Spf} O_K/P/(p_i - z_i)\) where \(P\) is isomorphic to \(\bigoplus_{0 \leq i \leq p} N_i^{n+1}\), \(p_i = (1, \ldots, 1) \in N_i^{n+1}\) and \(z_i \in O_K\), let \(\mathcal{G}_m\) be the formal multiplicative group \(\operatorname{Spf} O_K\{T, \frac{1}{T}\}\) over \(O_K\), let us denote for any \(n\) by \(\mathcal{G}_m^{(n)}\) the kernel of the multiplication \(\mathcal{G}_m^{n+1} \rightarrow \mathcal{G}_m\) and let \(\mathcal{G}\) be the formal completion at the identity of \(\prod_i \mathcal{G}_m^{(n_i)}\) (it is a formal group). Then \(\mathcal{G}\) acts on \(X\). The group \(G = \mathcal{G}_{\eta}\) acts then on \(X_{\eta}\).

G has canonical subgroups \(G_t\) for \(t \in [0,1]\) defined by the inequalities \(|T_{ij} - 1| < t\) where \(T_{ij}\) are the coordinates in \(G\). \(G_t\) has a maximal point \(g_t\) Similarly, for any complete extension \(K'/K\), \(G_t \otimes_{K} K'\) has a maximal point \(g_{t,K'}\). If \(x \in X\), one defines \(x_t := g_t \ast x\) to be the image of \(g_t,\eta(x)\) by the map \(G_t \otimes_{K} K' \rightarrow (G_t \times X)_x \subset G_t \times X \rightarrow X\).

If \(X\) is étale over \(\operatorname{Spf} O_K/P/(p_i - z_i)\), the action of \(\mathcal{G}\) extends in a unique way to an action on \(X\), and \(x_t\) is still defined by \(g_t \ast x\). For any \(X\) polystable over \(O_K\), one has thus defined the strong deformation locally for the quasi-étale topology of \(X_{\eta}^n\), and Berkovich verifies that it indeed descends to a strong deformation on \(X\).

For a polystable fibration \(X \rightarrow X_{l-1} \rightarrow \cdots \rightarrow \operatorname{Spf} O_K\), we first assume that \(X \rightarrow X_{l-1}\) is of the kind \(\operatorname{Spf} B \rightarrow \operatorname{Spf} A\) with \(B = A/P/(p_i - a_i)\) (this will be called a standard polystable morphism), one first retracts fiber by fiber on \(S(X/X_{l-1})\), which are strictly polystable. The image obtained can be identified with \(S = \{(x, r_0, \ldots, r_p) \in X_{l-1,\eta}, r_0 \ldots r_p = |a_i(x)|\}\), one then has a homotopy \(\Psi : S \times [0,1] \rightarrow S\) by \(\Psi(x, r_0, \ldots, r_p, t) = (x_t, \psi_{r_0}(r_0, |a_0(x_t)|), \ldots, \psi_{r_p}(r_p, |a_p(x_t)|))\), where \(\psi_{r}\) is some strong deformation of \([0,1]^{n+1}\) to \((1, \ldots, 1) \in [0,1]^{n+1}\) defined by Berkovich (we will just need that \(\psi_{r}(r_i, t)_{\lambda} = \psi_{r_{\lambda}}(r_{\lambda} \ast t)\) for any \(\lambda \in \mathbb{R}^+\) and any \(k \in [0, n]\), and \(x_t\) is defined by the strong deformation of \(X_{l-1,\eta}\).

If \(X \rightarrow X' \rightarrow X_{l-1}\) is a geometrically elementary composition of an étale morphism and a standard polystable morphism, \(S(X/X_{l-1}) \rightarrow S(X'/X_{l-1})\) is an isomorphism, so that we deform \(X'\) fiber by fiber onto \(S(X/X_{l-1})\), then we just do the same retraction as for \(S(X'/X_{l-1})\). For an arbitrary polystable fibration \(X \rightarrow \cdots \rightarrow O_K\), this defines the retraction locally for the quasi-étale topology of \(X_{\eta}\), and Berkovich verifies that it descends to a deformation retraction on \(X\).

Berkovich deduces from (1.5.(ii)) the following corollary:

**Corollary 1.6** ([3, cor. 8.5]). Let \(K'\) be a finite Galois extension of \(K\) and let \(X\) be a polystable fibration over \(O_K\) with a normal generic fiber \(X_{l,\eta}\). Suppose we are given an action of a finite group \(G\) on \(X\) over \(O_K\) and a Zariski open dense subset \(U\) of \(X_{l,\eta}\) which is stable under the action of \(G\). Then there is a strong deformation retraction of the Berkovich space \(G\setminus U\) to a closed subset homeomorphic to \(G\setminus C(X)\).

More precisely, in this corollary, the closed subset in question is the image of \(S(X)\) (which is \(G\)-equivariant and contained in \(U\)) by \(U \rightarrow G\setminus U\).
Theorem 1.5 also implies that the skeleton is functorial with respect to pluristable morphisms:

**Proposition 1.7** [4, prop. 4.3.2.(i)]. If $\phi : X \to Y$ is a pluristable morphism between nondegenerate pluristable formal schemes over $O_K$, $\phi_\eta(S(X)) \subset S(Y)$.

In fact, more precisely, from the construction of $S$, $S(X) = \bigcup_{y \in S(Y)} S(X_y)$.

### 2. Tempered fundamental group of a polystable log scheme

In this section we define a tempered fundamental group for a polystable fibration over a field, endowed with some compatible log structure (we will call this a polystable log fibration). To define our tempered fundamental group, we will need a notion of “topological covering” of a két covering $Z$ of our polystable log fibration $X \to \cdots \to k$. To do this we will define for any $Z$ a polysimplicial set $C(Z)$ over the polysimplicial set $C(X)$, functorially in $Z$. Thus if $Z$ is a finite Galois covering of $X$ with Galois group $G$, there is an action of $G$ on $C(Z)$ which defines an extension of groups:

$$1 \to \pi_1^{\text{top}}(\{C(Z)\}) \to \Pi_Z \to G \to 1.$$  

Our tempered fundamental group will be the projective limits of $\Pi_Z$ when $Z$ runs through pointed Galois coverings of $X$.

#### 2.1 Polystable log schemes

All monoids are assumed to be commutative. We will use multiplicative notations. If $X$ is an fs log scheme, we will denote by $\check{X}$ the underlying scheme, by $M_X$ the étale sheaf of monoids on $\check{X}$ defining the log structure, and by $X_U$ the open subset of $X$ where the log structure is trivial.

A strict étale morphism of $\text{fs log scheme } Y \to X$ is a strict morphism of log schemes such that $\check{Y} \to \check{X}$ is étale. If we talk about étale topology on $X$, it will mean strict étale topology on $X$ (or equivalently étale topology on $\check{X}$), and not log étale topology.

Let $S$ be a $\text{fs log scheme}$.  

**Definition 2.1.** A morphism $\phi : Y \to X$ of $\text{fs log schemes}$ will be said:

- **standard nodal** if $X$ has an fs chart $X \to \text{Spec } P$ and $Y$ is isomorphic to $X \times_{\text{Spec } Z[P]} Z[Q]$ with $Q = (P \oplus uN \oplus vN)/(u \cdot v = a)$ with $a \in P$.

- a **strictly plurinodal morphism of log schemes** if for every point $y \in Y$, there exists a Zariski open neighborhood $X'$ of $\phi(y)$ and a Zariski open neighborhood $Y'$ of $y$ in $Y \times_X X'$ such that $Y' \to X'$ is a composition of strict étale morphisms and standard nodal morphisms.

- a **plurinodal morphism of log schemes** if, locally for the étale topology of $X$ and $Y$, it is strictly plurinodal.

- a **strictly polystable morphism of log schemes** if for every point $y \in Y$, there exists an affine Zariski open neighborhood $X' = \text{Spec } A$ of $\phi(y)$, an fs chart $P \to A$ of the log structure of $X'$ and a Zariski open neighborhood $Y'$ of $y$ in $Y \times_X X'$ such that $Y' \to X$ factors through a strict étale morphism $Y' \to X' \times_{\text{Spec } Z[P]} Z[Q]$ where $Q = (P \oplus \bigoplus_{i=0}^{P} T_{i_0, \cdots, T_{i_n} >})/(T_{i_0} \cdots T_{i_n} = a_i)$ with $a_i \in P$.

- a **polystable morphism of log schemes** if, locally for the étale topology of $Y$ and $X$, it is a strict polystable morphism of log schemes.

A **polystable log fibration** (resp. strictly polystable log fibration) $X$ over $S$ of length $l$ is a sequence of polystable (resp. strictly polystable) morphism of log schemes $X_l \to \cdots \to X_1 \to X_0 = S$.

A morphism of polystable log fibrations of length $l \phi_i : Y \to X$ is given by morphisms $f_i : Y_i \to X_i$.
of fs log schemes for every $i$ such that the obvious diagram commutes.

A morphism $f$ of polystable fibrations will be said két (resp. strict étale) if $f_i$ is két (resp. strict étale) for all $i$.

A polystable (resp. strictly polystable) morphism of log schemes is plurinodal (resp. strictly plurinodal).

A plurinodal morphism is log smooth and saturated.

**Remark.** If $\phi : X \to Y$ is a stricly polystable morphism of log schemes, then for any $y \in Y$, for any Zariski open neighborhood $X' \ni \phi(y)$ and any chart $X' \to \text{Spec } P$, there is a Zariski open neighborhood $X'' \subset X'$ of $y$ and a Zariski open neighborhood $Y' \ni y$ in $Y$ such that $Y' \to X$ factors through a strict étale morphism $Y' \to X' \times_{Z[P]} Z[Q]$ where $Q = (P \oplus \bigoplus_{i=0}^{p} < T_{i0}, \cdots, T_{i_{m_i}} >)/(T_{i0} \cdots T_{i_{m_i}} = a_i)$ with $a_i \in P$.

**Lemma 2.2.** Let $\phi : Y \to X$ be a plurinodal (resp. strictly plurinodal, resp. polystable, resp. strictly pluristable) morphism of schemes, such that $X$ has a log regular log structure $M_X$ and $\phi$ is smooth over $X_{ir}$. Then $(Y, O_Y \cap j_* O_{Y_{X_{ir}}}) \to (X, M_X)$ is a plurinodal (resp. strictly plurinodal, resp. polystable, resp. strictly pluristable) morphism of log schemes.

**Proof.** Let us prove it for the case of a stricly polystable morphism.

One can assume that $X = \text{Spec } A$ has a chart $\psi : P \to A$ and that $Y = B_0 \times_X \cdots \times_X B_p$ with $B_i = \text{Spec } A[T_{i0}, \cdots, T_{i_{m_i}}]/T_{i0} \cdots T_{i_{m_i}} - a_i$ with $a_i \in A$. Since $\phi$ is smooth over $X_{ir}$, $a_i$ is invertible over $X_{ir}$, thus after multiplying $a_i$ by an element of $A^*$ (we can do that by also multiplying $T_{i0}$ by this element), we may assume that $a_i = \psi(b_i)$ for some $b_i \in P$. Thus $Y = X \times_{Z[P]} Z[Q]$ where $Q = (P \oplus \bigoplus_{i=0}^{p} < T_{i0}, \cdots, T_{i_{m_i}} >/T_{i0} \cdots T_{i_{m_i}} = b_i)$ with $b_i \in P$. If we endow $Y$ with the log structure $M_Y$ associated with $Q$, $Y \to X$ becomes a strict polystable morphism of log schemes. In particular $Y$ is log regular ([7, th. 8.2]). Since, the set of points of $Y$ where $M_Y$ is trivial is $Y_{X_{ir}}$, $M_Y = O_Y \cap j_* O_{Y_{X_{ir}}}$ according to [11, prop. 2.6].

**2.2 Strata of log schemes**

For a polystable (log) fibration $X : X \to \cdots \to \text{Spec } k$, Berkovich defines a polysimplicial set $C(X)$. In this part we want to generalize this construction to any két log scheme $Z$ over $X$. To do this we will study the stratification of an fs log scheme defined by $\text{rk}(z) = \text{rk}(M_x^{gr})$, which corresponds to Berkovich stratification for plurinodal schemes, and we will show that étale locally a két morphism $X \to Y$ induces an isomorphism between the posets of the strata of $X$ and $Y$. This will enables us to define the polysimplicial set of $Z$ étale locally. We will then descend it so that it satisfies the same descent property as in proposition 1.4.

Let $Z$ be an fs log scheme, one gets a stratification on $Z$ by saying that a point $z$ of $Z$ is of rank $r$ if $\text{rk}^{gr}(z) = \text{rk}(M_x^{gr}/O_x^*) = r$ (where $\bar{z}$ is some geometric point over $z$ and where $\text{rk}$ is the rank of an abelian group of finite type).

The subset of points of $Z$ such that the rank is $\leq r$ is an open subset of $Z$ ([12, cor. II.2.3.5]). We thus get a good stratification.

The strata of rank $r$ of $Z$ are then the connected components of the subset of points $z$ of rank $r$.

This is a partition of $Z$, and a strata of rank $r$ is open in the closed subset of points $x$ of rank $\geq r$.

It is endowed with the reduced subscheme structure of $Z$.

The set of strata is partially ordered by $x \leq y$ if and only if $y \subset \bar{x}$. One denotes by $\text{Str}_x(Z)$ the poset of strata below $x$. More generally, if $z$ is a point of $Z$, we denote by $\text{Str}_z(Z)$ the set of strata $y$ of $Z$ such that $z \in \bar{y}$ (Str$_z(Z)$ is simply Str$_x(Z)$ where $x$ is the stratum of $z$ containing $x$). If $\bar{z}$ is
a geometric point of $Z$, let $\text{Str}_Z^{\text{geom}}(Z) = \lim_{\to} (U, \bar{u}) \rightarrow \text{Str}_{(U, \bar{u})}^L(U)$ where $(U, \bar{u})$ goes through étale neighborhoods of $\bar{z}$; it can be identified with $\text{Str}(Z(\bar{z}))$ where $Z(\bar{z})$ is the strict localization of $Z$ at $\bar{z}$.

If $f : Z' \rightarrow Z$ is a két morphism, then $\text{rk}^{\log}(x) = \text{rk}^{\log}(f(x))$, so the strata of $Z'$ are the connected components of the preimages of the strata of $Z$.

If $f : P \rightarrow O_Z$ is a chart of $Z$, it induces a continuous map $f^* : Z \rightarrow \text{Spec } P$ that maps a point $z$ to the prime $\mathfrak{p}_z = P \setminus f^{-1}(O_{\ast, Z, z})$ of $P$. Let $F_z = P \setminus \mathfrak{p}_z$ be the corresponding face. Then $\mathcal{M}_{Z,z} = P/F_z$. One deduces from it that the strata of $Z$ are exactly the connected components of the preimages by $f^*$ of points in $\text{Spec } P$. In particular one gets a map $\text{Str}(Z) \rightarrow \text{Spec } P$. If $z$ is a point of $Z$, the map $Z(z) \rightarrow \text{Spec } P$ factorizes through a map $Z(z) \rightarrow \text{Spec } M_{Z,z}$, which does not depend on the choice of the chart. One gets a map $\text{Str}_z(Z) \rightarrow \text{Spec } M_{Z,z}$. For a general log scheme $Z$, if $\bar{z}$ is a geometric point of $Z$, one gets a map $\text{Str}_z^{\text{geom}}(Z) \rightarrow \text{Spec } M_{Z,\bar{z}}$.

Let us look at the structure of the strata of $\text{Spec } k[P]$ endowed with the log structure for which $f : P \rightarrow k[P]$ is a chart. Let $f^* : \text{Spec } k[P] \rightarrow \text{Spec } P$ and let $\mathfrak{p}$ be a prime of $P$ and let $F = P \setminus \mathfrak{p}$ be the corresponding face of $P$. Then $f^{*,-1}(\mathfrak{p})$ is a closed subset of $\text{Spec } k[P]$ which, endowed with its structure of reduced closed subscheme, is $\text{Spec } k[P]/(\mathfrak{p})$ where $(\mathfrak{p}) = \bigoplus_{p_i \in \mathfrak{p}} k.p_i \subset k[P]$ ($\mathfrak{p}$ is a prime ideal of $k[P]$). Moreover, the obvious morphism of rings $k[F] \rightarrow k[P]/(\mathfrak{p})$ is an isomorphism, inducing thus an isomorphism of schemes $f^{*,-1}(\mathfrak{p}) = \text{Spec } k[P]/(\mathfrak{p}) \cong \text{Spec } k[F]$. However the log structure on $\text{Spec } k[F]$ for which $F$ is a chart is not correspond in general with the log structure on $\text{Spec } k[P]/(\mathfrak{p})$ for which $P$ is a chart. The open immersion $f^{*,-1}(\mathfrak{p}) \subset f^{*,-1}(\mathfrak{p})$ corresponds then to the open immersion $\text{Spec } k[F^{\text{gp}}] \rightarrow \text{Spec } k[F]$. In particular, since $\text{Spec } k[F^{\text{gp}}]$ is connected, there is a unique stratum of $\text{Spec } k[P]$ above $\mathfrak{p}$ and thus $\text{Str}(\text{Spec } k[P]) \rightarrow \text{Spec } P$ is bijective.

Let $Z$ be a plurinodal log scheme over some log point $(k, M_k)$ of characteristic $p$ and of rank $r_0$ and let $z$ be a point of $Z$. One has $\text{rk}^{\log}(z) = r_0 + \text{rk}(z)$ where $\text{rk}(z)$ is the codimension of the strata containing $z$ in $Z$ for the Berkovich stratification of plurinodal schemes. Thus the strata are the same for this stratification and the stratification of Berkovich. The strata of $Z$ are normal. We will often denote abusively in the same way a stratum and its generic point.

Recall that $Z$ is said to be quasinormal if the closure of any stratum endowed with its reduced scheme structure is normal.

**Lemma 2.3.** Let $f : Z \rightarrow S = \text{Spec } k$ be a log smooth morphism. Let $\bar{z}$ be a geometric point. Let $f_* : \text{Spec } M_{Z,\bar{z}} \rightarrow \text{Spec } M_{S,\bar{s}}$. Then $\phi_{Z,\bar{z}} : \text{Str}_Z^{\text{geom}}(Z) \rightarrow \text{Spec } M_{Z,\bar{z}}$ is injective and its image is $f_*^{-1}(M_{S,\bar{s}})$. Moreover $Z(\bar{z})$ is quasinormal. In particular, every stratum of $Z$ is normal.

**Proof.** Since the unique stratum of $S$ is mapped to $M_{S,\bar{s}}$ by the map $\text{Str}_S^{\text{geom}}(S) \rightarrow \text{Spec } M_{S,\bar{s}}$, one has $\text{Im } \phi_{Z,\bar{z}} \subset f_*^{-1}(M_{S,\bar{s}})$.

The lemma can be proved étale locally: one can assume that $S$ has a chart $S \rightarrow \text{Spec } k[P]$ where $P$ is sharp, and that $Z = X \times_{\text{Spec } k[P]} \text{Spec } k[Q]$ where $\psi : P \rightarrow Q$ is injective and the torsion part of $\text{Coker } \psi^{\text{gp}}$ are finite. Let $q' \in f_*^{-1}(M_{S,\bar{s}})$ and let $\mathfrak{q}$ be its image in $\text{Spec } Q$. The image of $\mathfrak{q}$ in $\text{Spec } P$ is the image $\mathfrak{p}$ of $M_{S,\bar{s}}$. Let $F = Q \setminus \mathfrak{q}$ and $F_0 = P \setminus \mathfrak{p}$. The morphism $S \rightarrow \text{Spec } k[P]$ factors through $\text{Spec } k[F^{\text{gp}}]$. Let $\phi : Z \rightarrow \text{Spec } Q$ and let $Z_F$ be the closed subset $\psi^{-1}(\{\mathfrak{q}\})$ of $Z$ ($\bar{z}$ lies in $Z_F$). Then $Z_F$ is the support of the closed subscheme $Z \times_{\text{Spec } k[Q]} \text{Spec } k[Q]/(\mathfrak{q})$, which we also denote by $Z_F$.

Then,

$Z_F = Z \times_{\text{Spec } k[Q]} \text{Spec } k[F] = S \times_{\text{Spec } k[P]} \text{Spec } k[F] = S \times_{\text{Spec } k[F_0]} \text{Spec } k[F] = S \times_{\text{Spec } k[F^{\text{gp}}]} \text{Spec } k[F^{\text{gp}}]$.

Let $T_0$ be the saturation of $F^{\text{gp}}$ in $F^{\text{gp}}$ and let $T_1$ be a subgroup of $F^{\text{gp}}$ such that $F^{\text{gp}} = T_0 \oplus T_1$. The morphism $S \times_{\text{Spec } k[F^{\text{gp}}]} \text{Spec } k[T_0] \rightarrow S$ is étale, so up to replacing $k$ by a finite extension, one
can assume $F_0^{\mathrm{gp}} = T_0$. Then $Z_F = S \times_{\text{Spec} k[T_0]} \text{Spec} k[FF_0^{\mathrm{gp}}] = \text{Spec} k[FF_0^{\mathrm{gp}} \cap T_1]$. But $FF_0^{\mathrm{gp}} \cap T_1$ is a saturated monoid, hence $Z_F$ is normal. Thus $Z_F(\bar{z})$ is irreducible. Moreover, if $F' \subseteq F$, then $Z_{F'} \subseteq Z_F$: the generic point of $Z_F$ lies above $q$. One thus obtains that there is a unique stratum of $Z(\bar{z})$ lying above $q$.

**Lemma 2.4.** Let $Z$ be a Zariski log scheme, let $Z \to \text{Spec} k$ be a log smooth morphism and let $Z' \to Z$ be a két morphism, then $\text{Str}_{x'}(Z') \to \text{Str}_x(Z)$ is an isomorphism of posets.

**Proof.** There is a commutative diagram:

$$
\begin{array}{ccc}
\text{Str}_{x'}^{\text{geom}}(Z') & \longrightarrow & \text{Str}_x^{\text{geom}}(Z) \longrightarrow \text{Spec} M_{Z,x} \\
\downarrow & & \downarrow \\
\text{Str}_{x'}(Z') & \longrightarrow & \text{Str}_x(Z) \longrightarrow \text{Spec} M_{Z,z} 
\end{array}
$$

Since $\text{Str}_x^{\text{geom}}(Z) \to \text{Spec} M_{Z,z}$ is injective, $\text{Str}_x^{\text{geom}}(Z) \to \text{Str}_x(Z)$ must be bijective. The morphism $\text{Str}_{x'}^{\text{geom}}(Z') \to \text{Str}_x^{\text{geom}}(Z)$ is bijective thanks to lemma 2.3 because $\text{Spec} M_{Z',x'} \to \text{Spec} M_{Z,z}$ is bijective since $M_{Z,z} = M_{Z',x'}$ is Kummer. Hence $\text{Str}_{x'}(Z') \to \text{Str}_x(Z)$ must also be bijective.

If $z_1'$ and $z_2'$ are elements of $\text{Str}_{x'}(Z')$, then $\text{Str}_{x'}(Z_1') \to \text{Str}_x(Z_1)$ is also bijective, so that $z_2' \in \text{Str}_{x'}(Z_1')$ if and only if $z_2 \in \text{Str}_x(Z_1)$, i.e. $z_2' \leq z_1'$ if and only if $z_2 \leq z_1$.

In particular, one can apply lemma 2.4 if $Z$ is strictly plurinodal.

### 2.3 Polysimplicial set of a két log scheme over a polystable log scheme

Let $C \to C'$ be a morphism of polysimplicial sets. Let $\alpha : S \to O(C)$ (resp. $\alpha' : S' \to O(C')$) be a morphism of posets such that $S_{\leq x} \geq O(C)_{\leq \alpha(x)}$ (resp. $S'_{\leq x} \geq O(C')_{\leq \alpha'(x)}$ for any $x$). Then $\alpha$ defines a functor $O(C)^{\text{op}} \to \text{Set}$ by sending $c$ to $\alpha^{-1}(c)$ and if $c \leq c'$, then the map $\alpha^{-1}(c') \to \alpha^{-1}(c)$ sends $x' \in \alpha^{-1}(c')$ to the unique preimage of $c$ by the map $S_{\leq c'} \to O(C)_{\leq c}$. One gets a functor $F : (\Lambda/C)^{\text{op}} \to O(C)^{\text{op}} \to \text{Set}$ (resp. $F' : (\Lambda/C')^{\text{op}} \to O(C')^{\text{op}} \to \text{Set}$), which defines a polysimplicial set $D = C \times F$ (resp. $D' = C' \times F'$):

$$D = \text{Coker}(\prod_{x \in F(x)} F(x)_{\leq n_x}) = \prod_{x \in F(x)} [n_x].$$

If we consider $F$ as a functor $(\Lambda/C)^{\text{op}} \to \Lambda^\circ \text{Set}$, then $D$ is nothing else than $C \Box F$ (but this is a very simple case of $\Box$-product where all the fibers are discrete). To give a slightly more explicit description of $D$, $D_n = \prod_{x \in C_n} F(x)$ and if $f : m \to n$ is a morphism of $\Lambda$ and $z \in F(x)$ with $x \in C_n$, $f^*(z) = F(f) \in F(f^*(x))$ where $f$ is the morphism $f^*(x) \to x$ in $\Lambda/C$. Since $F$ maps surjective morphisms to isomorphisms, a poly-simplex $z \in F(x)$ of $D$ is nondegenerate if and only if $x$ is nondegenerate. One gets that $O(D) = S$ and that $D$ is interiorly free if $C$ is.

Then any morphism of posets $f : S \to S'$ such that

$$
\begin{array}{ccc}
S & \longrightarrow & S' \\
O(C) & \longrightarrow & O(C') 
\end{array}
$$

is commutative induces a unique morphism of polysimplicial sets $F : D \to D'$ over $C \to C'$ such that $O(f) = f$.

Let us consider now a strictly polystable log fibration $X : X \to X_{l-1} \to \cdots \to s$ where $s$ is an fs log point. If $f : Z \to X$ is két, the map of posets $\text{Str}(f) : \text{Str}(Z) \to \text{Str}(X) = O(C(X))$ is such that $\text{Str}(Z)_{\leq z} \simeq \text{Str}(X)_{\leq f(z)}$ for any $z \in \text{Str}(Z)$ according to lemma 2.4. Thus one gets a functor $D_Z = (\Lambda/C(X))^{\text{op}} \to \text{Set}$ and a polysimplicial set $C_X(Z) = C(X) \Box D_Z$ (we will often write
C(Z) instead of $C_X(Z)$. This polysimplicial set is still interiorly free and $O(C(Z))$ is functorially isomorphic to $\text{Str}(Z)$.

**Lemma 2.5.** If $X \to X'$ is a két morphism of strictly polystable log fibrations, then there is a canonical isomorphism $C_{X'}(X_l) \simeq C(X)$ such that $\text{Str}(X_l) = O(C_{X'}(X_l)) \to \text{Str}(X_l) = O(C(X))$ is the identity of $\text{Str}(X_l)$.

**Proof.** Assume we already construct the isomorphism $C_{X_{l-1}}(X_{l-1}) \simeq C(X_{l-1})$. Then, $C_{X'}(X_l) = D_1 \Box C(X_{l-1})$ and $C(X) = D_2 \Box C(X_{l-1})$ where if $x$ is the generic point of a stratum of $X_{l-1}$, $D_1(x) = C_{X_{l-1}}(X_{l,x})$ and $D_2(x) = C(X_{l,x})$. By induction on $l$, the problem is thus reduced to the case where $l = 1$ and $X \to X'$ is a két morphism of strictly polystable objects over Spec $k$.

We have $C_{X'}(X) = D_X \times C(X')$ where $D_X$ maps $x' \in \text{Str}(X')$ to the set of strata of $X$ above $x'$. Then $C_{X'}(X)$ is associated to the strictly polysimplicial set $C' = D_X \times C(X')$. Then $C'_n = \{(x,x',\mu), x \in \text{Str}(X), x' = f(x), \mu : n \simeq \text{Irr}(X',x')\} = \{(x,\mu), x \in \text{Str}(X), \mu : n \simeq \text{Irr}(X,x)\}$ because $\text{Irr}(X,x) \to \text{Irr}(X',x')$ is an isomorphism. Thus $C'_n \simeq C_n$ (and the bijection is compatible with maps of $\Lambda$), which gives the wanted isomorphism.

Let us consider a commutative diagram

$$
\begin{array}{ccc}
Z & \to & Z' \\
\downarrow & & \downarrow \\
X & \to & X'
\end{array}
$$

where $X \to X'$ is a két morphism of strictly polystable log fibrations. Then

$$C_{X}(Z) = D_{Z/X} \times C(X) \simeq D_{Z/X} \times C_{X'}(X) = D_{Z/X} \times (D_{X/X'} \times C(X')) = D_{Z/X'} \times C(X') = C_{X'}(Z)$$

where $D_{Z/X}(x) = \text{Str}(Z \to X)^{-1}(x)$, $D_{X/X'}(x') = \text{Str}(X \to X')^{-1}(x')$ and $D_{Z/X'}(x') = \text{Str}(Z \to X')^{-1}(x')$. There is a morphism of functors $D_{Z/X} \to D_{Z/X'}$ which induces a morphism of polysimplicial sets

$$C_X(Z) = D_{Z/X'} \times C(X') \to D_{Z'/X'} \times C(X') = C_{X'}(Z').$$

This morphism is an isomorphism if and only if $\text{Str}(Z) \to \text{Str}(Z')$ is bijective.

Let $Z' \to Z$ be a két covering, let $Z'' = Z' \times_Z Z'$ and let $x$ be a stratum of $X$, then $D_{Z}(x) = \text{Coker}(D_{Z''}(x) \implies D_{Z'}(x))$. We deduce from it that

$$C(Z'') = \text{Coker}(C(Z') \implies C(Z)).$$

One may also define $C_X(Z)$ for $X$ a general polystable fibration. Let $X' \to X$ be an étale covering where $X'$ is strictly polystable, let $X'' = X' \times_X X'$ and let $Z'$ and $Z''$ the pullbacks of $Z$ to $X'$ and $X''$. then one defines $C_X(Z) = \text{Coker}(C_{X''}(Z'') \implies C_X(Z'))$ (it does not depend of the choice of $X'$).

If $Z' \to Z$ is a surjective két morphism over $X$ and $Z'' = Z' \times_Z Z'$, $\text{Str}(Z) = \text{Coker}(\text{Str}(Z'') \implies \text{Str}(Z))$.

One thus gets (két($X$) denotes the category of két log schemes over $X$):

**Proposition 2.6.** Let $X$ be a polystable log fibration, one has a functor $C_X : \text{két}(X) \to (\Lambda)^\circ$ Set such that:

- if $Z' \to Z$ is a két covering of két($X$),

$$C(Z) = \text{Coker}(C(Z' \times_Z Z') \implies C(Z')).$$
- $O(C(Z))$ is functorially isomorphic to $\text{Str}(Z)$.

**Remark.** If one has a két morphism $Y \to X$ of polystable fibrations of length $l$, the polysimplicial complex $C(Y)$ we have just defined by considering $Y_i$ as két over $X_i$ is canonically isomorphic to the polysimplicial complex of the polystable fibration $C(Y)$ defined by Berkovich.

We say that a fs log scheme $Z$ over a log point $s$ is log geometrically irreducible if the underlying scheme of $Z \times_s s'$ is irreducible for any morphism $s' \to s$ of log points. If $\tilde{Z}/\tilde{s}$ is geometrically irreducible and $Z \to s$ is saturated, then $Z/s$ is log geometrically irreducible since the underlying scheme of $Z \times_s s'$ is $\tilde{Z} \times_{\tilde{s}} s'$.

If $Z$ is quasicompact, then there is a connected két covering $s' \to s$ such that all the strata of $Z_{s'}$ are geometrically irreducible and $Z_{s'} \to s'$ is saturated. Then all the strata of $Z_{s'}$ are log geometrically irreducible. In particular, for any morphism of fs log points $s'' \to s'$, $C(Z_{s''}) \to C(Z_{s'})$ is an isomorphism. The polysimplicial complex $C(Z_{s'})$ for such an $s'$ is denoted by $C_{\text{geom}}(Z/s).

Let $\bar{z}$ be a geometric point of $Z$. Let $U$ be an étale neighborhood of $\bar{z}$ such that $\text{Str}_{\bar{z}}(Z) \to \text{Str}(U)$ is an isomorphism. One defines $C(Z)_{\bar{z}} := C(U)$ (it does not depend on the choice of $U$). If $Z \to X$ is két, $(C(Z)_{\bar{z}} \to C(X)_{\bar{x}})$ is an isomorphism of polysimplicial sets.

**Lemma 2.7.** The space $|C(Z)_{\bar{z}}|$ is contractible.

**Proof.** Let $\Phi_n : |[n]| \times [0, 1] \to |[n]|$ be defined by $\Phi_n((u,t), t) = (1-t)u + \frac{t}{n_i}$. This is a deformation retraction to a point. These deformation retractions are compatible with surjective maps $m \to n$.

One can assume that $X \xrightarrow{\psi} X_{l-1} \to \cdots \to s$ is a strictly polystable fibration of length $l$ and that $Z = X$. Let $\bar{x}'$ be the image of $\bar{x} := \bar{z}$ in $X_{l-1}$. One can also assume that $\text{Str}_{\bar{x}}(X) \to \text{Str}(X)$ and $\text{Str}_{\bar{x}}(X_{l-1}) \to \text{Str}(X_{l-1})$ are bijections. By induction on $l$, one can assume that $|C(X_{l-1})|$ is contractible.

If $y$ is a stratum of $X_{l-1}$, $X_{y'}$ has a biggest stratum $y$ and $C(X_{y'}) \simeq [n_y]$. Then

$$|C(X)| = \text{Coker} \left( \prod_{y \in \text{X}_{l-1}} |[n_{y'}]| \times |[n_{y}]| \right) \Rightarrow \prod |[n_{y'}]| \times |[n_y]|,$$

where $a$ maps $|[n_{y'}]| \times |[n_{y}]|$ to $|[n_{y'}]| \times |[n_{y}]|$ by $\text{id} \times f_{0}$ where $f_{0}$ is the cospecialization map $C(X_{y'}) \to C(X_{y'})$ given by lemma 1.3 and $b$ maps $|[n_{y'}]| \times |[n_{y}]|$ to $|[n_{y'}]| \times |[n_{y}]|$ $f^* \times \text{id}$.

One defines a deformation retraction $\Phi$ of $\prod_{y \in \text{X}_{l-1}} |[n_{y'}]| \times |[n_y]|$ by $\Phi(u, v, t) = (u, \Phi_{n_y}(v, t))$. Moreover, if $(z_1, z_2) \in |[n_{y_1}]| \times |[n_{y_2}]|$, $\Phi(a(z_1, z_2), t) = (z_1, \Phi_{n_{y_1}}(f_{0}(z_2), t)) = (z_1, f_{0}(\Phi_{n_{y_2}}(z_2, t))) = a(z_1, \Phi_{n_{y_2}}(z_2, t))$ because the map $n_{y_2} \to n_{y_1}$ inducing $f_{0}$ is surjective, and $\Phi(b(z_1, z_2), t) = (f^*z_1, \Phi_{n_{y_2}}(z_2, t)) = b(z_1, \Phi_{n_{y_2}}(z_2, t))$.

Thus $\Phi$ induces a deformation retraction of $C(X)$, also denoted by $\Phi$ by abuse of notation. This retraction is compatible with $\psi : |C(X)| \to |C(X_{l-1})|$ in the sense that $\psi(\Phi(z, t)) = \psi(z)$ for every $t \in [0, 1]$. Let $S$ be the image of this retraction. Let $u \in |C(X_{l-1})|$ and let $y'$ be the stratum of $X_{l-1}$ corresponding to the cell of $|C(X_{l-1})|$ containing $u$. then $\psi^{-1}(u)$ is canonically homeomorphic to $|[n_y]|$ (cf. [3, cor. 6.6]), and the deformation retraction of $\psi^{-1}(u)$ induced by $\Phi$ is just $\Phi_{n_y}$. Thus $S \cap \psi^{-1}(u)$ is reduced to a point: the map $S \to |C(X_{l-1})|$ is bijective. Since $\text{Str}(X)$ is finite, $|C(X)|$ is compact and $S$ is also compact since it is the image of $|C(X)|$ by a continuous map. The map $S \to C(X_{l-1})$ is thus an homeomorphism, and $C(X_{l-1})$ is contractible by induction. Thus $C(X)$ is contractible. □
2.4 Tempered fundamental group of a polystable log fibration

Here we define the tempered fundamental group of a log fibration $\mathcal{X}$ over an fs log point. If $T$ is a két covering of $X$, the topological coverings of $|C(T)|$ will play the role of the topological coverings of $T$.

Let us start by a categorical definition of tempered fundamental groups that we will use later in our log geometric situation.
Consider a fibered category $D \rightarrow C$ such that:

- $\mathcal{C}$ is a Galois category,
- for every connected object $U$ of $\mathcal{C}$, $D_U$ is a category equivalent to $\Pi_U$-$\text{Set}$ for some discrete group $\Pi_U$,
- if $U$ and $V$ are two objects of $\mathcal{C}$, the functor $D_U \Pi V \rightarrow D_U \times D_V$ is an equivalence,
- if $f : U \rightarrow V$ is a morphism in $\mathcal{C}$, $f^* : D_V \rightarrow D_U$ is exact.

Then, one can define a fibered category $D' \rightarrow \mathcal{C}$ such that the fiber in $U$ is the category of descent data of $D \rightarrow \mathcal{C}$ with respect to the morphism $U \rightarrow e$ (where $e$ is the final element of $\mathcal{C}$).
Let $U$ be a connected Galois object of $\mathcal{C}$ and let $G$ be the Galois group of $U/e$. Then $D'_U$ can be described in the following way:

- its objects are couples $(S_U, (\psi_g)_{g \in G})$, where $S_U$ is an object of $D_U$ and $\psi_g : S_U \rightarrow g^*S_U$ is an isomorphism in $D_U$ such that for any $g, g' \in G$, $(g^*\psi_g') \circ \psi_g = \psi_{g'g}$ (after identifying $(g^*g)^*$ and $g^*g^*$ by the canonical isomorphism to lighten the notations).
- a morphism $(S_U, (\psi_g)) \rightarrow (S'_U, (\psi'_g))$ is a morphism $\phi : S_U \rightarrow S'_U$ in $D_U$ such that for any $g \in G$, $\psi'_g \phi = (g^*\phi) \psi_g$.

There is a natural functor $F_0 : D'_U \rightarrow D_U$, which maps $(S_U, (\psi_g))$ to $S_U$. Let $F_U$ be a fundamental functor $D_U \rightarrow \text{Set}$, such that $\text{Aut} F_U = \Pi_U$.
Let $F = F_U F_0$, and $\Pi'_U = \text{Aut} F$.

Proposition 2.8. (i) The natural functor $\mathcal{F} : D'_U \rightarrow \Pi'_U$-$\text{Set}$ is an equivalence.
(ii) There is a natural exact sequence

$$1 \rightarrow \Pi_U \rightarrow \Pi'_U \rightarrow G \rightarrow 1.$$ 

Proof. First notice that $D'_U$ is a boolean topos and that $F$ is faithful and exact.
A pointed object of $D'_U$ is by definition a pair $(S, s)$ with $S$ an object of $S$, and $s \in F(S)$. Let us show that, to prove (i), it is enough to show that there exists a pointed object $(T^\infty, t^\infty)$ of $D'_U$ such that for every pointed object $(S, s)$ of $D'_U$, the map $\text{Hom}(T^\infty, S) = F(S)$ that maps $f$ to $F(f)(t^\infty)$ is bijective ($i.e.$ $T^\infty$ represents the functor $F$).

The group $\text{Aut}(T^\infty)$ acts on $\text{Hom}(T^\infty, S) = F(S)$ by action on the left compatibly for every $S$; one gets a morphism $a : \text{Aut}(T^\infty) \rightarrow \text{Aut}(F)$, which is bijective by Yoneda’s lemma.
If $S_0 \subset F(S)$ is stable by $\text{Aut}$, then the subobject $S_0$ of $S$ defined as the unions of the images of morphisms $\phi : T^\infty \rightarrow S$ such that $F(\phi)(t) \in S_0$ satisfies $F(S_0) = S_0$. Thus if $S, S'$ are objects of $D'_U$,

$$\text{Hom}(S, S') = \{S_0 \hookrightarrow S \times S'| S_0 \hookrightarrow S \}$$

$$= \{S_0 \subset F(S) \times F(S') \text{ stable by the action of } \text{Aut} F | S_0 \hookrightarrow F(S) \}$$

$$= \text{Hom}_{\Pi'_U}(F(S), F(S')).$$

Thus $\mathcal{F}$ is fully faithful. Let $S$ be a $\Pi'_U$-set. There exists an epimorphism $S' \rightarrow S$ such that $\Pi'_U$ acts freely on $S'$ and on $S'' := S' \times_S S'$. Thus there exists $S''$ and $S'$ such that $\mathcal{F}(S') = S'$ and
Let us construct $T^\infty$. If $S$ is an object of $\mathcal{D}_U$ let $	ilde{S} = \bigsqcup_{g \in G} g^* S$, et

$$\psi_h : \tilde{S} = \bigsqcup_{g \in G} g^* S = \bigsqcup_{gh \in G} (gh)^* S \xrightarrow{\sim} \bigsqcup_{g \in G} h^* g^* S = h^* (\bigsqcup_{g \in G} g^* S) = h^* \tilde{S}.$$  

This defines an object $\tilde{S}$ of $\mathcal{D}_U$. Then, for any object $S'$ of $\mathcal{D}_U$, there is a natural map

$$\hom_{\mathcal{D}_U}(\tilde{S}, T) \xrightarrow{\sim} \hom_{\mathcal{D}_U}(S, F_0(T))$$

that maps $\psi$ to the restriction of $F_0(\psi)$ to the subobject $S$ of $F_0(\tilde{S})$. The restriction of $F_0(\psi)$ to $g^* S \subset F_0(\tilde{S})$ is $\psi_g^{-1} g^* \alpha(\psi)$. Hence $F_0(\psi)$ only depends on $\alpha(\psi)$, which shows the injectivity of $\alpha$ since $F$ is faithful. Conversely, if $\beta \in \hom_{\mathcal{D}_U}(S, F_0(T))$, one defines $\beta_0 : F_0(\tilde{S}) = \bigsqcup_{g} g^* S \to F_0(T)$ by gluing the composite morphisms $g^* S \xrightarrow{g_0} g^* F_0(T) \xrightarrow{\psi_0^{-1}} F_0(T)$. The following diagram is commutative:

$$\begin{array}{ccc}
F_0(\tilde{S}) = \bigsqcup_{g} g^* S & \xrightarrow{\psi_h} & \bigsqcup_{g} g^* F_0(T) \\
\phantom{= \bigsqcup_{g} g^* S} \downarrow \psi_h & & \downarrow \psi_h \\
h^* F_0(\tilde{S}) = \bigsqcup_{g} h^* g^* S & \xrightarrow{\psi_h^{-1}} & \bigsqcup_{g} h^* g^* F_0(T)
\end{array}$$

and thus $\beta_0$ defines a morphism $\psi \in \hom_{\mathcal{D}_U}(\tilde{S}, T)$ such that $\alpha(\psi) = \beta$. Thus $\alpha$ is bijective.

If $(S^\infty, s^\infty)$ is a universal pointed object of $\mathcal{D}_U$, then, for every $T$,

$$\hom(\tilde{S^\infty}, T) \xrightarrow{\sim} \hom(S^\infty, F_0(T)) \xrightarrow{\sim} F(T).$$

Thus $(\tilde{S^\infty}, s^\infty)$ is a universal pointed object of $\mathcal{D}_U$.

The functor $F_0$ induces a morphism $\Pi_U \to \Pi'_U$. There is also a natural exact functor $F_1 : H\text{-}\mathcal{P} \to \mathcal{D}_U$ which maps a $H$-set $Y$ to $(Y = \bigsqcup_{g \in G} \{y\}, (\psi_h))$ where $Y$ is a constant object of $\mathcal{D}_U$ and $\psi_h$ maps $y$ to $h \cdot y$. $F_1$ is canonically isomorphic to the forgetful functor $H\text{-}\mathcal{P} \to \mathcal{S}$, the functor $F_1$ thus induces a morphism $\Pi'_U \to H$. Since $\Pi_U = F_U(S^\infty)$ and $\Pi'_U = F(S^\infty)$, one only has to see that the following exact sequence of pointed sets is exact:

$$1 \to F_U(S^\infty) \to F(\tilde{S^\infty}) = \bigsqcup_{g} F_U(g^* S^\infty) \xrightarrow{\psi} G \xrightarrow{1}$$

where the map $\bigsqcup_{g} F_U(g^* S^\infty) \to G$ maps $F_U(g^* S^\infty)$ to $g$. \hfill \Box

If $(U_i, u_i)_{i \in I}$ is a cofinal projective system of pointed Galois objects (and let $P$ be the corresponding object of pro-$\mathcal{C}$), one may define $\mathcal{B}^{\text{temp}}(\mathcal{D}/\mathcal{C}, P)$ to be the category $\lim \mathcal{D}_{U_i}$. An isomorphism of pro-objects $P \to P'$ induces an equivalence $\mathcal{B}^{\text{temp}}(\mathcal{D}/\mathcal{C}, P) \to \mathcal{B}^{\text{temp}}(\mathcal{D}/\mathcal{C}, P')$, so that $\mathcal{B}^{\text{temp}}(\mathcal{D}/\mathcal{C}, P)$ does not depend up to equivalence on the choice of $(U_i)$. Moreover, if $h \in G_i = \text{Gal}(U_i/e)$ the endo-functor $h^* : \mathcal{D}_{U_i} \to \mathcal{D}_{U_i}$ maps $S = (S_{U_i}, \psi_g)$ to $h^* S = (h^* S_{U_i}, \psi_h \psi_g^{-1})$. Then $\psi_g : S_{U_i} \to h^* S_{U_i}$ defines an isomorphism $S \to h^* S$ functorially in $S$. Thus $h^* : \mathcal{D}_{U_i} \to \mathcal{D}_{U_i}$ is canonically isomorphic to the identity of $\mathcal{D}_{U_i}$. Thus every automorphism of the pro-object $P$ induces an endofunctor of $\mathcal{B}^{\text{temp}}(\mathcal{D}/\mathcal{C}, P)$ which is canonically isomorphic to the identity (functorially on $\text{Aut} P$).

Let $(F_i)_{i \in I}$ be a family of fundamental functors $F_i : \mathcal{D}_{U_i} \to \mathcal{S}$ and assume one has a family $(\alpha_f)_{f : U_i \to U_j}$, indexed on the set of morphisms in $I$, of isomorphisms of functors $F_i f^* \to F_j$ such that
for any $U_i \xrightarrow{f} U_j \xrightarrow{g} U_k$, $\alpha_g(\alpha_f \cdot g^*) = \alpha_{gf}$ (after identifying $(gf)^*$ and $f^*g^*$ to lighten the notations). Such a family exists if $I$ is just $\mathbb{N}$. Then, this induces a projective system $(\Pi_{U_i}^I)_{i \in I}$ (unique up to isomorphism independantly of $(\alpha_f)$ if $I=\mathbb{N}$ and the functors $\mathcal{D}_{U_i}^I \rightarrow \mathcal{D}'_{U_i}$ are fully faithful), so that one can define

$$\pi_1^{\text{temp}}(\mathcal{D}/\mathcal{C}, (F_i)) = \varprojlim \Pi_{U_i}^I$$

Assume one has a 2-commutative diagram with fibered vertical arrows:

\[
\begin{array}{ccc}
\mathcal{D}_1 & \rightarrow & \mathcal{D}_2 \\
\downarrow & & \downarrow \\
\mathcal{C}_1 & \xrightarrow{f} & \mathcal{C}_2
\end{array}
\]

such that $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is exact, and $\mathcal{D}_{1,U} \rightarrow \mathcal{D}_{2,f(U)}$ is exact for every object $U$ of $\mathcal{C}_1$.

One then gets a functor $\mathcal{B}^{\text{temp}}(\mathcal{D}_1/\mathcal{C}_1) \rightarrow \mathcal{B}^{\text{temp}}(\mathcal{D}_2/\mathcal{C}_2)$.

For example, let $X$ be a $K$-manifold, $\mathcal{C}$ be the category of finite étale covering of $X$ and $\mathcal{D} \rightarrow \mathcal{C}$ be the fibered category such that $\mathcal{D}_U$ is the category of topological coverings of $U$. Then, since finite étale coverings are morphisms of effective descent for tempered coverings, $\mathcal{D}_U'$ can be identified functorially with the full subcategory of $\text{Cov}^{\text{temp}}(X)$ of tempered coverings $S$ such that $S_U$ is a topological covering of $U$. If $(U_i, u_i)$ is a cofinal system of pointed Galois cover of $(X, x)$, then $\mathcal{B}^{\text{temp}}(\mathcal{C}/\mathcal{D})$ becomes canonically equivalent with $\text{Cov}^{\text{temp}}(X)$.

Let us apply our categorical definition of tempered fundamental groups to our log geometrical case.

Let $X : X \rightarrow X_{l-1} \rightarrow \cdots \rightarrow \text{Spec}(k)$ be a polystable log fibration, and assume that $X$ is connected. Then one has a functor $C_{\text{top}} : \text{KCov}(X) \rightarrow \mathcal{K}e$ obtained by composing the functor $C$ of proposition 2.6 with the geometric realization functor.

One can thus define a fibered category $\mathcal{D}_{\text{top}} \rightarrow \text{KCov}(X)$ such that the fiber of a két covering of $Y$ of $X$ is the category of topological coverings of $C_{\text{top}}(Y)$ (which is equivalent to $\pi_1^{\text{top}}(C_{\text{top}}(Y))$-$\text{Set}$).

One defines a fibered category $\mathcal{D}^{\text{temp}} \rightarrow \text{KCov}(X)$ such that the fiber of a két covering $f : Y \rightarrow X$ is the category of descent data of $\mathcal{D}_{\text{top}} \rightarrow \text{KCov}(X)$ with respect to $Y \rightarrow X$ (this corresponds heuristically to the “tempered” coverings of $X$ that become topological after pullback by $Y \rightarrow X$).

Let $x$ be a log geometric point of $X$ and let $(Y, y)$ be a log geometrically pointed connected Galois két covering of $(X, x)$. Let $\tilde{y} := |C(Y)_y| \rightarrow |C(Y)|$. The space $\tilde{y}$ is contractible according to lemma 2.7. Then one has a fundamental functor $F_y : D_{\text{top}}Y \rightarrow \text{Cov}^{\text{top}}(\tilde{y}) = \text{Set}$ that corresponds to the base point $\tilde{y}$ ($F_y(S)$ is the set of connected components of $S \times |C(Y)| \tilde{y}$). Moreover, for any morphism $f : (Y', y') \rightarrow (Y, y)$, the two functors $F_{y'} f^*$ and $F_y$ are canonically isomorphic.

Then one can consider the functor $F_{(Y,y)} : \text{DD}_{\text{temp}} \rightarrow \text{Set}$ which associates to a descent datum $T$ the set $F_y(T_Y)$. The induced functor $\text{DD}_{\text{temp}} \rightarrow \text{Aut}(F_{(Y,y)})$-$\text{Set}$ is an equivalence of categories. One has an exact sequence:

$$1 \rightarrow \pi_1^{\text{top}}(|C(Y)|, \tilde{y}) \rightarrow \text{Aut}(F_{(Y,y)}) \rightarrow \text{Gal}(Y/X) \rightarrow 1.$$ 

Then one defines

$$\pi_1^{\text{temp}}(X, x)^L = \varprojlim \text{Aut}(F_{(Y,y)}),$$

where the projective limit is taken over the directed category $\mathbb{L}$-$\text{GalKcov}(X, x)$ of pointed connected Galois $\mathbb{L}$-finite két coverings of $(X, x)$.

If $x_2 \rightarrow x_1$ is a specialization of log geometric points of $X$, it induces a natural equivalence between the category of pointed coverings of $(X, x_2)$ and the category of pointed coverings of $(X, x_1)$.
(we thus identify the two categories). If \( Y \) is a pointed covering \((Y, y_1)\) of \((X, x_1)\), the corresponding pointed covering of \((X, x_2)\) is \((Y, y_2)\) where \( y_2 \) is the unique log geometric point above \( x_2 \) such that there is a specialization \( y_2 \to y_1 \) (and this specialization is unique). There is a commutative diagram

\[
\begin{array}{ccc}
\tilde{y}_1 & \longrightarrow & \tilde{y}_2 \\
\downarrow & & \downarrow \\
\mid C(Y) \mid & & 
\end{array}
\]

This induces a canonical isomorphism \( F_{y_1} \simeq F_{y_2} \), functorial in \( Y \), so that one gets a canonical isomorphism \( \pi^\text{temp}_1(X, x_1)^L \to \pi^\text{temp}_1(X, x_2)^L \). If \( X \) is connected and \( x_1, x_2 \) are two log geometric points of \( X \), there exists a sequence of specializations and cospecializations joining \( x_1 \) to \( x_2 \), so that \( \pi^\text{temp}_1(X, x_1)^L \) and \( \pi^\text{temp}_1(X, x_2)^L \) are isomorphic.

One has an equivalence of categories between

\[
B^\text{temp-geom}_{(X, x)} = \operatorname{Lim} DD_{\text{top-geom}Y} / \mathbb{L}- \text{GalK Cov}(X, x)
\]

and the category \( \pi^\text{temp}_1(X, x)^L \). Set of sets with an action of \( \pi^\text{temp}_1(X, x)^L \) that goes through a discrete quotient of \( \pi^\text{temp}_1(X, x)^L \).

Assume now that \( X \) is log geometrically connected, i.e., that \( X_{k'} \) is connected for any k't extension \( k' \) of \( k \). Let \( \bar{k} \) be a log geometric point on \( k \), let \( \bar{x} = (\bar{x}_{k'}) \) be a compatible system of log geometric points of \( X_{k'} \) where \( k' \) runs through k't extensions of \( (k, \bar{k}) \) (for every \( k' \), the set of geometric points above \( \bar{x}_{k} \) is a non empty finite set and thus the set of compatible systems of log geometric points is a non empty profinite set). Then, one defines \( \pi^\text{temp-geom}_1(X, \bar{x})^L = \lim_{k'} \pi^\text{temp}_1(X_{k'}, \bar{x}_{k'})^L \), where \( k' \) runs through k't extensions of \( k \) in a log geometric point \( \bar{k} \). Let \( \text{K Cov}_{\text{geom}}(X) = \operatorname{Lim} \text{K Cov}(X_{k'}) \) where \( k' \) runs through k't extensions of \( k \). It is the category of log geometric coverings of \( X \).

If \( Y \to X \) is a log geometric covering, defined over \( k' \), \( \text{C geom}(Y_{k'}) \) does not depend of \( k' \), so that one gets a functor \( \text{K Cov}_{\text{geom}}(X) \to \mathcal{K}e \) which maps \( Y \) to \(| \text{C geom}(Y) | \). If \( \bar{x} \) is a compatible system of points, for any pointed log geometric covering \((Y, \bar{y})\) of \((X, \bar{x})\), \( \bar{y} \) defines a fundamental functor \( F_{\bar{y}} \) of \( \mathcal{D}_{\text{top-geom}} \) which are canonically isomorphic for any morphism \((Y', \bar{y}') \to (Y, \bar{y})\). One thus get a fibered category \( \mathcal{D}_{\text{top-geom}} \to \text{K Cov}_{\text{geom}}(X) \), whose fiber in \( Y \) is the category of topological coverings of \(| \text{C geom}(Y) | \). Then

\[
\pi^\text{temp-geom}_1(X, \bar{x})^L := \pi^\text{temp}_1(\mathcal{D}_{\text{top-geom}} / \text{K Cov}_{\text{geom}}(X), (F_{\bar{y}}))^L.
\]

3. **Comparison result for the pro-(p') tempered fundamental group**

If \( X : X \to \cdots \to \text{Spec}(O_K) \) is a proper polystable log fibration, we want to compare the tempered fundamental group of the generic fiber \( X_\eta \) with the tempered fundamental group of the special fiber endowed with its natural log structure. The specialization theory of the log fundamental group already gives us a functor from k't coverings of the special fiber and algebraic coverings of the generic fiber. To extend this to tempered fundamental groups, one has to compare, for any k't covering \( T_{s} \) of the special fiber, the topological space \( \text{C}(T_{s}) \) with the Berkovich space of the corresponding covering \( T_{\eta} \) of the generic fiber. Thus we will define, as in [3], a strong deformation retraction of \( T_{\eta}^\text{an} \) to a subset canonically homeomorphic to \( | \text{C}(T_{s}) | \). We will construct this retraction étale locally, where \( T \) has a Galois covering \( V \) by some polystable log fibration over a finite tamely ramified extension.
of $O_K$. Then the retraction of the tube of $T_s$ is obtained by descending the retraction of the tube of $V'_t$, defined in [3]. We will then verify that the retraction does not depend on the choice of $V'$ so that we can descend the retraction we defined étale locally.

### 3.1 Skeleton of a két log scheme over a pluristable log scheme

If $X \to \text{Spec } O_K$ is a morphism of finite type, we denote by $\mathcal{X}$ the completion of $X$ along the closed fiber $X_s$. The generic fiber, in the sense of Berkovich, of a locally topologically finitely generated formal scheme $\mathcal{X}$ over $\text{Spf } O_K$ will be denoted by $\mathcal{X}_e$.

Let $\mathcal{X} : X \to \cdots \to \text{Spec}(O_K)$ be a polystable log fibration over $\text{Spec}(O_K)$.

**Proposition 3.1.** For every két morphism $T \to X$, let $\mathcal{X}_T$ be the generic fiber, in the sense of Berkovich, of the formal completion of $T$ along its special fiber. Then, there is a functorial map $|C(T)_s| \to \mathcal{X}_T$, which identifies $|C(T_s)|$ with a subset $S(T)$ of $\mathcal{X}_T$ on which $\mathcal{X}_T$ retracts by strong deformation.

**Remark:** $\mathcal{X}_T$ is naturally an analytic subdomain of $T^m_n$. Moreover if $T$ is proper over $O_K$ (for example if $X$ is proper, and $T$ is a finite két covering), then $\mathcal{X}_T \to T^m_n$ is an isomorphism.

**Proof.** Let $f : T \to X$ be a két morphism. Let $x \in T$. Let $U : U_1 \to \cdots \to U_0$ be a polystable fibration étale over $\mathcal{X}$ such that $(U_i, x_i)$ is an étale neighborhood of $f(x)$, such that, for every $i$, $U_i$ has an exact chart $P_i \to A_i$ and compatible morphisms $P_i \to P_{i+1}$ such that the induced morphism $U_{i+1} \to U_i \times_{\text{Spec } Z[P_i]} \text{Spec } Z[P_{i+1}]$ is étale. One has an étale neighborhood $i : (V, x') \to (T, x)$ of $x$, a $(p')$-Kummer morphism $P_i \to Q$ such that $V \to X$ factors through an étale morphism $V \to U_i \times_{\text{Spec } Z[P_i]} \text{Spec } Z[Q]$. By definition of a $(p')$-Kummer morphism, there exists $n$ prime to $p$ such that $P_i \to \frac{1}{n} P_i$ factors through $P_i \to Q$. Thus $V$ has a két Galois covering $\eta$ that comes from a polystable fibration $U'' = V' \to U''_{i-1} \to \cdots \to \text{Spec } O_K$, where $U''_i = U_i \times_{\text{Spec } Z[P_i]} \text{Spec } Z[\frac{1}{n} P_i]$ for $i \leq l$ and $V' = V \times_{\text{Spec } Z[Q]} Z[\frac{1}{n} P]$ (so that there is a strict étale morphism $V' \to U''_{i-1}$) over $O_K$, for some finite tamely ramified extension $K' = K[\pi^{1/n}]$ of $K$. Let us call $G = (\frac{1}{n} P^{\text{top}} \times Q^{\text{top}})$ the Galois group of this két covering.

The deformation retraction of $\mathcal{X}'_T$, defined in theorem 1.5 is $G$-equivariant, so that it defines a deformation retraction of $\mathcal{X}_T$. Let $S(\ )$ denote the image of the retraction of $(\ )_T$. Then $S(\mathcal{X}'_T) = G \setminus S(\mathcal{X}'_T) = G \setminus C(V'_T) = |G \setminus C(V'_T)| = |C(V_s)|$ (corollary 1.6).

Let us show that the previously defined retraction of $\mathcal{X}_T$ does not depend on $n$. Let us start by the case of a polystable morphism.

Let

$$\psi : Z_1 = \text{Spec } A[P]/(p_i - \lambda_i) \to Z_2 = \text{Spec } A[P]/(p_i - \lambda_i^s)$$

where $P = N^r = \oplus_{(i,j) \in A} N_{e_{ij}}$ and $p_i = \sum_{s} e_{ij}$ induced by the multiplication by $s$ on $P$, where $s$ is an integer prime to $p$ and $\lambda \in A$.

Let $G$ be the generic fiber of the formal completion of $G_m^{(r)}$ at the identity; it acts on $Z_1$ and $Z_2$. One has $\psi(g \cdot x) = g^s \cdot \psi(x)$.

Let $T_{ij}$ be the coordinates of $G$. Then $|T'_{ij} - 1| = |T_{ij} - 1|$ if $|T_{ij} - 1| < 1$. Thus, for $t < 1$, $(\ )^s : G \to G$ induces an isomorphism $(\ )^s : G_t \to G_t$, and $g_t^s = g_t$.

Thus, if $t < 1$ (and also for $t = 1$ by continuity),

$$\psi(x_t) = \psi(g_t \ast x) = g_t^s \ast \psi(x) = g_t \ast \psi(x) = \psi(x)_t.$$
More precisely, suppose we have the diagram:

\[ B = B'[Y_{ij}]/(Y_0 \cdots Y_m - b_i) \quad \xleftarrow{\phi} \quad B' \]

\[ A = A'[X_{ij}]/(X_0 \cdots X_m - a_i) \quad \xleftarrow{\phi'} \quad A' \]

where \( \phi(X_{ij}) = Y_{ij}^{*} \) and thus \( \phi'(a_i) = b_i^{*} \), and \( \tilde{\phi}' := \text{Spf} \phi' : \text{Spf} B' \to \text{Spf} A' \) is a két morphism of polystable log fibrations and assume by induction that we already know that \( \tilde{\phi}(x_i) = \tilde{\phi}(x) \).

Let \( \mathfrak{x} \) (resp. \( \mathfrak{x}', \mathfrak{y}, \mathfrak{y}' \)) denote \( \text{Spf} A \) (resp. \( \text{Spf} A', \text{Spf} B, \text{Spf} B' \)).

The first part of the retraction of \( \mathfrak{x}_{\eta}^n \) and \( \mathfrak{y}_{\eta}^n \) (consisting of the retraction fiber by fiber) commutes with \( \tilde{\phi} := \text{Spf} \phi \) according to the previous case. We thus just have to study the second part of the retraction.

\( \tilde{\phi} \) induces a map:

\[
S_A = \{(x, r_{ij}) \in (\mathfrak{x}')^n \times [0, 1]^n | r_0 \cdots r_m = |a_i(x)| \} \subset \mathfrak{x}_{\eta}^n
\]

\[
\downarrow
\]

\[
S_B = \{(y, r_{ij}) \in (\mathfrak{y}')^n \times [0, 1]^n | r_0 \cdots r_m = |b_i(y)| \} \subset \mathfrak{y}_{\eta}^n
\]

which maps \((x, r_{ij})\) to \((\tilde{\phi}'(x), r_{ij}^{1/s})\) (remark that \(|a_i(x)| = |b_i(\tilde{\phi}'(x))|^{s}\)).

Then, if \((x, r_{ij}) \in S_A\) (we will write \(y := \tilde{\phi}'(x)\); by induction assumption, \(\tilde{\phi}'(x_i) = y_i\))

\[
\tilde{\phi}'((x, r_{ij})_t) = \tilde{\phi}'((x_t, \psi_{n_i}(r_{ij}, |a_i(x_t)|^{1/s}))
\]

\[
= (y_t, \psi_{n_i}(r_{ij}, |a_i(x_t)|^{1/s}))
\]

\[
= (y_t, \psi_{n_i}(r_{ij}^{1/s}, |a_i(x_t)|^{1/s}))
\]

\[
= (y_t, \psi_{n_i}(r_{ij}^{1/s}, |b_i(y_t)|))
\]

\[
= (y_t, r_{ij}^{1/s})_t
\]

\[
= \tilde{\phi}(x, r_{ij})_t
\]

Thus we get that the retraction of \(\mathfrak{y}_{\eta}\) does not depend on \(n\).

Let \( W \to T \) be another neighborhood of \(x\) satisfying the same properties as \(V\), and \(W'\) defined in the same way. One may assume by the previous remark that we chose the same \(n\). Let \(W'' = V' \times_T W'\).

We have a commutative diagram

\[
\begin{array}{ccc}
W'' & \xrightarrow{p'} & W' \\
\downarrow{p} & & \downarrow{i'} \\
V' & \xrightarrow{i} & T
\end{array}
\]

where \(W'' = V' \times_T W'\). Let us show that \(p : W'' \to V'\) is étale (symmetrically, \(p'\) is étale too).

Since \(p\) is két, it is enough to prove that \(p\) is strict, i.e. that for any geometric point \(z \in W''\), \(\overline{M}_{V', p(z)} \to \overline{M}_{W'', z}\) is an isomorphism. Let \(v = p(z), w = p'(z), \tau = i(v) = i'(w)\) and \(\xi = f(\tau) \in X\).

Then \(\overline{M}_{X, \xi} = P_l/F\) where \(F\) is a face of \(P_l\). Then \(\overline{M}_{V', w} = \frac{1}{n} P_l/F_n = \frac{1}{n} \overline{M}_{X, \xi}\) where \(F_n\) is the saturation of \(F\) in \(\frac{1}{n} P\). Symmetrically, one also has \(\overline{M}_{W'', w} = \frac{1}{n} \overline{M}_{X, \xi}\). Thus,

\[
\overline{M}_{W'', z} = \overline{M}_{V', w} \oplus \overline{M}_{T', w} \overline{M}_{W', w} \\
= \overline{M}_{V', w} \oplus \overline{M}_{T', w} \overline{M}_{W', w} \\
= \frac{1}{n} \overline{M}_{X, \xi} \oplus \frac{1}{n} \overline{M}_{X, \xi}/\overline{M}_{T', w} \\
= \frac{1}{n} \overline{M}_{X, \xi} \\
= \frac{1}{n} \overline{M}_{X, \xi}
\]
where the sums are sums in the category of fs monoids. Thus \( p \) is strict, and therefore étale.

Let thus \( v \in \mathcal{V}_\eta \) and \( w \in \mathcal{W}_\eta \) with same image \( \tau \) in \( \mathfrak{X}_\eta \). Let \( z \in \mathcal{W}_\eta \) be above \( v \) and \( w \). Then, for every \( t \in [0, 1] \), \( v_t = p(z_t) \) and \( w_t = p'(z_t) \) according to theorem 1.5(ii). Thus \( i(v_t) = ip(z_t) = i'p'(z_t) = i'(y'_t) \). Thus, the retractions of the different \( \mathcal{W}_\eta \) are compatible and define a map \( \mathfrak{X}_\eta \times [0, 1] \rightarrow \mathfrak{X}_\eta \).

This map is continuous since, \( \coprod \mathfrak{X}_i \) is a covering of \( \mathfrak{X} \). \( \coprod \mathfrak{X}_i \rightarrow \mathfrak{X}_\eta \) is quasi-étale and surjective and thus a topological factor map (as in the proof of theorem 1.5 of Berkovich; cf. \cite[lem. 5.11]{3}). Moreover, if \( \phi : T_1 \rightarrow T_2 \) is a két morphism of két log schemes over \( X \), \( \phi(x_t) = \phi(x) \). As in theorem 1.5(vi), it is also compatible with isometric extensions of \( K \).

Let \( \tilde{V} = \bigcup_i V_i \) be a covering of \( T \) such that every \( V_i \) satisfies the same property as \( V \). Since \( f : \tilde{\mathfrak{X}}_\eta \rightarrow \tilde{\mathfrak{T}}_\eta \) is a topological factor map, \( S(\tilde{\mathfrak{X}}_\eta) = f^{-1}(S(\tilde{T}_\eta)) \rightarrow S(\tilde{\mathfrak{T}}_\eta) \) is also a topological factor map. Thus one gets an isomorphism, functorial in \( T \),

\[
S(\mathfrak{T}) = \text{Coker}(S(\tilde{\mathfrak{X}}_\eta) \times_{S(\mathfrak{X})} S(\tilde{\mathfrak{X}}_\eta) \rightarrow S(\tilde{\mathfrak{T}}_\eta)) = \text{Coker}(|\mathcal{C}(V)| \times |\mathcal{C}(T)|/\mathcal{C}(V)) = |\mathcal{C}(V)|/\mathcal{C}(T).
\]

\[\Box\]

### 3.2 Comparison theorem

Let \( K \) be a complete discrete valuation field. Let \( \mathfrak{X} : X \rightarrow \cdots \rightarrow \text{Spec} \mathcal{O}_K \) be a proper polystable log fibration.

Let us now compare the tempered fundamental group of the generic fiber, as a \( K \)-manifold, and the tempered fundamental group of its special fiber as defined in §2.4.

A geometric point \( \bar{x} \) of \( X^\text{an}_\eta \) is given by a algebraically closed complete nonarchimedean extension \( \Omega \) of \( K \) and a \( K \)-morphism \( \bar{x} : \text{Spec} \Omega \rightarrow X \). Since \( X \rightarrow \text{Spec} \mathcal{O}_K \) is proper, \( \bar{x} \) extends uniquely to a morphism \( \text{Spec} \mathcal{O}_\Omega \rightarrow X \). If one endows \( \text{Spec} \mathcal{O}_\Omega \) of the log structure induced by \( \mathcal{O}_\Omega \setminus \{0\} \), one can extend \( \text{Spec} \mathcal{O}_\Omega \rightarrow X \) in a morphism of log schemes. By looking at the closed fiber, one gets a morphism of log schemes \( \bar{x} : \text{Spec} \mathcal{k}_\Omega \rightarrow X_s \), where \( \text{Spec} \mathcal{k}_\Omega \) has the log structure induced by \( \mathcal{O}_\Omega \setminus \{0\} \) (it is a log geometric point). The log geometric point \( \bar{x} \) is called the log reduction of \( \bar{x} \).

**Theorem 3.2.** Let \( \bar{x} \) be a geometric point of \( X^\text{an}_\eta \), and let \( \bar{x} \) be its log reduction. One has a morphism \( \pi_1^\text{temp}(X^\text{an}_\eta, \bar{x})^L \rightarrow \pi_1^\text{temp}(X^s, \bar{x})^L \) which is an isomorphism if \( p \notin \mathbb{L} \).

These morphisms are compatible with finite extensions of \( K \).

**Proof.** One has two functors \( \mathbb{L} \text{-K Cov}(X) \rightarrow \mathbb{L} \text{-Cov}^\text{alg}(X^\eta) \), which is an equivalence of categories if \( p \notin \mathbb{L} \), and \( \mathbb{L} \text{-K Cov}(X^\eta) \rightarrow \mathbb{L} \text{-K Cov}(X^s) \) which is an equivalence of categories (theorem [9, th. 2.4]). One has a fibered category \( \mathcal{D}_{\text{top}}^\text{an}(X^\eta) \) over \( \mathbb{L} \text{-K Cov}(X^\eta) \) whose fiber at a \( \mathbb{L} \)-finite két covering \( T \) of \( X^\eta \) is the category of topological coverings of \( T^\text{an} \). Let us call \( \mathcal{D}_{\text{top}}^\text{an}(X^\eta) \) the pullback of \( \mathcal{D}_{\text{top}}^\text{an}(X^\eta)/\mathbb{L} \text{-K Cov}(X^\eta) \) to \( \mathbb{L} \text{-K Cov}(X) \): the fiber at a \( \mathbb{L} \)-finite két covering \( T \) of \( X \) is the category of topological coverings of \( T^\text{an} \). One has also another fibered category \( \mathcal{D}_{\text{top}}^\text{sp}(X) \) over \( \mathbb{L} \text{-K Cov}(X) \) obtained by pulling back the fibered category \( \mathcal{D}_{\text{top}}^\text{an}(X^\eta) \rightarrow \mathbb{L} \text{-K Cov}(X^\eta) \) defined in part 2.4 along \( \mathbb{L} \text{-K Cov}(X) \rightarrow \mathbb{L} \text{-K Cov}(X^\eta) \); the fiber at a \( \mathbb{L} \)-finite két covering \( T \) of \( X \) is the category of topological coverings of \( |\mathcal{C}(T^s)| \). Proposition 3.1 induces an equivalence of fibered categories \( \mathcal{D}_{\text{top}}^\text{an}(X^\eta) \rightarrow \mathcal{D}_{\text{top}}^\text{sp}(X) \), and thus an isomorphism \( \pi_1^\text{temp}(\mathcal{D}_{\text{top}}^\text{an}(X^\eta)/\mathbb{L} \text{-K Cov}(X^\eta)) \simeq \pi_1^\text{temp}(\mathcal{D}_{\text{top}}^\text{sp}(X)/\mathbb{L} \text{-K Cov}(X^\eta)) \).
The 2-commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}^\text{an}_{\text{top}}(X) & \longrightarrow & \mathcal{D}^\text{an}_{\text{top}}(X_\eta) \\
\downarrow & & \downarrow \\
\mathbb{L}_- \text{Kov}(X) & \longrightarrow & \mathbb{L}_- \text{Cov}^\text{alg}(X_\eta)
\end{array}
\]

induces a morphism

\[
\pi_1^{\text{temp}}(X^\text{an}_\eta) = \pi_1^{\text{temp}}(\mathcal{D}^\text{an}_{\text{top}}(X_\eta)/\mathbb{L}_- \text{Cov}^\text{alg}(X_\eta)) \to \pi_1^{\text{temp}}(\mathcal{D}^\text{an}_{\text{top}}(X)/\mathbb{L}_- \text{Kov}(X))
\]

which is an isomorphism if \( p \notin \mathbb{L} \). Similarly,

\[
\begin{array}{ccc}
\mathcal{D}^{\text{op}}_{\text{top}}(X) & \longrightarrow & \mathcal{D}^{\text{op}}_{\text{top}}(X_s) \\
\downarrow & & \downarrow \\
\mathbb{L}_- \text{Kov}(X) & \longrightarrow & \mathbb{L}_- \text{Cov}^\text{alg}(X_s)
\end{array}
\]

induces an isomorphism

\[
\pi_1^{\text{temp}}(X_s)^{\mathbb{L}} \to \pi_1^{\text{temp}}(\mathcal{D}^{\text{op}}_{\text{top}}(X)/\mathbb{L}_- \text{Kov}(X))
\]

since \( \mathbb{L}_- \text{Kov}(X) \to \mathbb{L}_- \text{Kov}(X_s) \) is an equivalence of categories. \( \square \)

3.3 Geometric comparison theorem

We will assume in this section that \( p \notin \mathbb{L} \).

**Theorem 3.3.** There is a natural isomorphism

\[
\pi_1^{\text{temp-geom}}(X_s)^{\mathbb{L}} \simeq \pi_1^{\text{temp}}(X_\eta)^{\mathbb{L}}.
\]

**Proof.** One knows, according to [1, prop 5.1.1], that

\[
\pi_1^{\text{temp}}(X_\eta) \simeq \lim_{\rightarrow} \pi_1^{\text{temp}}(X_{K_i}),
\]

where \( K_i \) runs through the finite extensions of \( K \) in \( \overline{K} \).

This induces an analog result for the \( \mathbb{L} \)-version.

However, we would like to know, in the case where \( p \notin \mathbb{L} \), if one can only take the projective limit over tamely ramified extensions of \( K \) (i.e. to \( \acute{\text{e}} \)t extensions of \( O_k \)). Then the isomorphism we want would simply be obtained from theorem 3.2 by taking the projective limit over \( \acute{\text{e}} \)t extensions of \( O_k \).

We have to show that if \( T' \) is a \( \mathbb{L} \)-finite \( \acute{\text{e}} \)t geometric covering of \( X \) (which is defined over a finite tamely ramified extension of \( K \) according to [8, prop. 1.15]: one can thus assume that \( T' \) is defined over \( K \)), the universal topological covering \( \bar{T}'_\eta \) of \( T'_\eta \) is defined over some tamely ramified extension of \( K \).

By changing \( \text{Spec} O_K \) by some \( \acute{\text{e}} \)t covering (which amounts to changing \( K \) by some tamely ramified extension) one may assume that \( T' \to \text{Spec} O_K \) is saturated.

One already knows that \( \bar{T}'_\eta \) is defined over some finite extension \( K_2 \) of \( K \) ([1, lem 5.1.3]). Let \( K_1 \) be the maximal unramified extension of \( K \) in \( K_2 \). As \( T' \to O_K \) is saturated, the underlying scheme of \( T'_{O_{K_2}} \) is obtained by the base change of schemes \( \text{Spec} O_{K_2} \to \text{Spec} O_{K_1} \) of the underlying scheme of \( T'_{O_{K_1}} \). By looking at the special fiber, as \( K_1 = K_2 \) (as schemes), the morphism \( T'_{K_2} \to T'_{K_1} \) induces an isomorphism between the underlying schemes, thus a bijection between their strata, and thus an isomorphism \(|C(T'_{K_2})| \to |C(T'_{K_1})|\) and \(|S(T'_{K_2})| \to |S(T'_{K_1})|\).
Thus $\overline{T}_{\eta}'$ is defined over $K_1$. 

This isomorphism is $\text{Gal}(\overline{K}, K)$-equivariant (since the isomorphism for each Galois extension $K_i$ of $K$ is $\text{Gal}(K_i/K)$-equivariant).

4. Cospecialization of pro-$(p')$ tempered fundamental group

Let $X \rightarrow Y$ be a proper polystable log fibration, such that $Y$ is log smooth and proper over $O_K$ (the properness of $Y \rightarrow O_K$ is only assumed so that every point of $Y_\eta$ has a reduction in $Y_\eta$, but the cospecialization morphisms we will construct only depend on $Y$ locally). In this section we will construct the cospecialization morphisms for the $(p')$-tempered fundamental group of the geometric fibers of $X_\eta \rightarrow Y_\eta$. Thanks to theorem 3.3 we will be reduced to construct cospecialization morphisms for the $(p')$-tempered fundamental group of the log geometric fibers of $X_\eta \rightarrow Y_\eta$. Let thus $\tilde{s}_2 \rightarrow \tilde{s}_1$ be a specialization of log geometric points of $Y$, where $\tilde{s}_1$ and $\tilde{s}_2$ are the reductions of geometric points $\tilde{\eta}_1, \tilde{\eta}_2$ of $Y_\eta$.

We constructed in [9, th. 0.2] an equivalence of geometric $(p')$-két coverings of $X_{\eta_1}$ and $X_{\eta_2}$. Now we must compare, for any such két covering $Z_{s_1}$ corresponding to $Z_{s_2}$ (which extends over the preimage $X_\eta$ of some két neighborhood $U$ of $s_1$ in $Y$), their polysimplicial sets as defined in proposition 2.6. First assume that $s_2$ is the generic point of its stratum. We will construct the cospecialization morphism of polysimplicial sets étale locally, so that we can assume $X$ to be strictly polystable (the properness will not be used for this). This cospecialization morphism of polysimplicial set will be constructed in the following way. Let $z$ be a geometric stratum of $Z_{s_1}$. After some két localization of the base so that $Z_U$ becomes saturated. Then the set of strata $z_2$ of $Z_{s_2}$ such that $z$ is in the closure of $z_2$ has a unique minimal element (as in lemma 1.3), which we call $z'$. Then, thanks to the fact that $Z_U \rightarrow U$ is saturated, the closure of $z'$ in the strict localization of the generic point of $z$ is separable onto its image. According to [6, cor. 18.9.8], $z'$ is geometrically connected, thus defining a geometric stratum of $Z_{s_2}$. One thus obtains a map from the set of geometric strata of $Z_{s_1}$ to the set of geometric strata of $Z_{s_2}$; this map induces a morphism of polysimplicial sets. In the case where polysimplicial sets of the geometric fibers of $Y \rightarrow X$ are interiorly free, the cospecialisation morphism of polysimplicial sets is an isomorphism if $s_1$ and $s_2$ are in the same stratum. We will end this article by glueing our specialization isomorphism of $(p')$-log tempered fundamental group with our cospecialization morphisms of polysimplicial sets in a cospecialization morphism of tempered fundamental groups.

4.1 Cospecialization of polysimplicial sets

In this section, we construct a cospecialization map of polysimplicial set for a composition of a két morphism and of a log polystable fibration.

Lemma 4.1. If $\phi : P \rightarrow Q$ is an integral (resp. saturated) morphism of fs monoids and $F'$ is a face of $Q$, let $F = \phi^{-1}(F')$. Then $F \rightarrow F'$ is also integral (resp. saturated).

Proof. To prove that $F \rightarrow F'$ is integral, thanks to [12, prop. 1.4.3.11], one only has to prove that if $f_1', f_2' \in F'$ and $f_1, f_2 \in F$ are such that $f_1' \phi(f_1) = f_2' \phi(f_2)$, there are $g' \in F'$ and $g_1, g_2 \in F$ such that $f_1' = g' \phi(g_1)$ and $f_2' = g' \phi(g_2)$.

But there exists $g' \in Q$ and $g_1, g_2 \in P$ that satisfies those properties since $P \rightarrow Q$ is integral. But, since $F'$ is a face of $Q$, $g', \phi(g_1), \phi(g_2)$ must be in $F'$, and thus $g_1$ and $g_2$ are in $F$. 

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Thanks to a criterion of T. Tsuji ([13, prop. 4.1]), an integral morphism of fs monoids $f : P_0 \to Q_0$ is saturated if and only if for any $a \in P_0$, $b \in Q_0$ and any prime number $p$ such that $f(a)[b^p]$, there exists $c \in P_0$ such that $a|c^p$ and $f(c)[b]$. Let $a \in F, b \in F'$ and $p$ be a prime such that $f(a)[b^p]$. Then since $\phi : P \to Q$ is saturated, there exists $c \in P$ such that $a|c^p$ and $f(c)[b]$. But $f(c)[b]$ implies that $f(c) \in F'$, whence $c \in F$.

**Proposition 4.2.** Let $f : X \to Y$ be a saturated log smooth morphism of fs log schemes. Assume $\hat{Y}$ is strictly henselian of special point $\hat{y}_1$ and let $y_2 \in Y$. Let $x \in X_{\hat{y}_1}$. The set $A := \{ Z \in \text{Str}(X_{y_2}) | x \in Z \}$ has a biggest element $Z_0$. Moreover, $Z_0$ is geometrically connected.

**Proof.** Up to replacing $\hat{Y}$ by a closed subscheme, one can assume that $\hat{Y}$ is integral and $y_2$ is the generic point of $Y$. One can assume that $f$ has a chart:

$$
\begin{array}{ccc}
X' & \longrightarrow & \text{Spec } Z[Q] \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \text{Spec } Z[P]
\end{array}
$$

where $P$ is sharp, $\phi : P \to Q$ is an injective saturated morphism of fs monoids, $X' \to Y_0 = Y \times_{\text{Spec } Z[P]} \text{Spec } Z[Q]$ is étale, $X' \to Y$ factorizes through $f$ and $g : X' \to X$ is étale. One also assumes that $X'$ has a unique point $x'$ above $x$. If $A' := \{ Z' \in \text{Str}(X'_{y_2}) | x' \in Z' \}$ has a biggest element $Z_0'$, $g(Z_0')$ is a biggest element of $A$. Moreover, if $Z_0'$ is geometrically connected, $g(Z_0')$ is also geometrically connected. One can thus assume $X' = X$.

Let $F_2' = P \setminus p_2$ be the kernel of $P \to M_{Y,y_2}$; since $y_2$ is a generic point of $Y$, $Y \to \text{Spec } Z[P]$ factorizes through $Y \to \text{Spec } Z[P]/(p_2) \simeq Z[F_2']$. Let $F_1 = Q \setminus q_1$ be the kernel of $Q \to M_{X,x}$. Let $F = < F_1, \phi(F_2') >$ be the face of $Q$ generated by $F_2$ and $\phi(F_1)$, and let $q_2 = Q \setminus F$. Then $q_2$ is the biggest element of $\text{Spec } Q$ above $q_1$ contained in $p_2$. Let $X_0 := X \times_{\text{Spec } Z[Q]} \text{Spec } Z[Q]/(q_2)$: it is a closed subscheme of $X$. Set-theoretically it is the union of the strata of $X$ whose image in $Q$ contains $q_2$ (it contains $x$ since $q_2 \subseteq q_1$).

Let us show that $X_0 \to Y$ is separable (i.e. flat with geometrically reduced fibers). Since $X_0 \to Y_F = Y \times_{\text{Spec } Z[P]} \text{Spec } Z[Q]/(q_2) \simeq Y \times_{\text{Spec } Z[F_2]} \text{Spec } Z[F]$ is étale, it is enough to show that $\text{Spec } Z[F_2'] \to \text{Spec } Z[F]$ is separable. But $F \to F_2'$ is saturated thanks to lemma 4.1; this implies that $\text{Spec } Z[F_2'] \to \text{Spec } Z[F]$ is separable. According to [6, cor. 18.9.8], for every $y \in Y$, $X_0(x)_y$ is geometrically connected (where $X_0(x)$ denotes the localization of $X_0$ at $x$). Set-theoretically $X_0(x)_y$ is the subset of $X_{y_2}$ consisting of points $z$ which specialize to $x$ and such that the kernel $F_z$ of $M_{X,z} \to \text{Spec } Q$ is contained in $F$. For every point of $z = X_0(x)_y$, $F_z$ is contained in $F$, contains $F_1$ because $x$ is a specialization of $z$ and contains $\phi^{-1}(F_z) = F_2'$ because the face corresponding to $y_2$ is $F_2'$; thus $F_z = F$. Thus $X_0(x)_y$ is contained in a single stratum $Z_0$ of $X_{y_2}$ ($Z_0$ is an element of $A$). Since the generic point $z_0$ of $Z_0$ is in $X_0(x)_y$, $Z_0$ must also be geometrically connected.

Let $Z \neq Z_0$ be in $A$ a maximal element and let $z$ be its generic point. Let $q_Z$ be the corresponding face of $Q$ then $q_Z \subseteq q_1$ and $\phi^{-1}(q_Z) = p_2$. Thus $q_Z \subset q_2$. Let $X_{q_Z} = X \times_{\text{Spec } Z[q]} \text{Spec } Z[Q]/(q_Z)$ (this is union of the strata of $X'$ whose image in $Q$ contains $q_Z$). As previously, $X_{q_Z}(x)_y$ is geometrically connected and contains $z$ as a generic point. It also contains $z_0$. Since $Z$ is open in $(X_{q_Z})_{y_2}$, and $Z \cap X_{q_Z}(x)_y \subseteq X_{q_Z}(x)_y$, $z$ must specialize in $X_{q_Z}(x')_{y_2}$ to an element $z'$ that is not in $Z$. The stratum containing $z'$ is in $A$ and is bigger than $Z$. Thus $A$ has no maximal element other than $Z_0$. Since $A$ is locally finite, $Z_0$ must be the biggest element of $A$.

**□**

If $f : X \to Y$ is a saturated log smooth morphism of fs log schemes and $\bar{y}_1 \to \bar{y}_1$ is a specialization
of geometric points, one can apply proposition 4.2 to the pullback of $f$ to the strict henselization of $\bar{y}_1$: one gets a nondecreasing map $\text{Str}(X_{\bar{y}_1}) \to \text{Str}(X_{\bar{y}_2})$.

If $Z \to X$ is két and $X \to Y$ is a saturated log smooth morphism of fs log schemes and $\bar{y}_2 \to \bar{y}_1$ is a két specialization of log geometric points, there exists a két neighborhood $U$ of $\bar{y}_1$ such that $X_U := X \times_Y U \to U$ is saturated. One thus gets a cospecialization map

$$\text{Str}(Z_{\bar{y}_1}) \to \text{Str}(Z_{\bar{y}_2}).$$

**Proposition 4.3.** If $X \to Y$ is proper and $\bar{M}_{Y,\bar{y}_1} \to \bar{M}_{Y,\bar{y}_2}$ is an isomorphism, then the cospecialization map $\text{Str}(X_{\bar{y}_1}) \to \text{Str}(X_{\bar{y}_2})$ is bijective.

**Proof.** Assume $\bar{Y} = \text{Spec } A$ is strictly local with special point $\bar{y}_1$, integral with generic point $\bar{y}_2$ and $X \to Y$ is saturated. By pulling back along the normalization of $\bar{Y}$, one can also assume that $A$ is normal.

Let $Z$ be a stratum of $\bar{X}_{\bar{y}_2}$ and let $z$ be its generic point. Let $\bar{Z}$ be the normalization of the closure $Z$ of $Z$ (endowed with the pullback log structure). Let $v : V \to X$ be an étale morphism such that $V \to Y$ has a global chart:

$$\begin{array}{c}
V \\
\downarrow \\
Y \\
\downarrow \\
\text{Spec } Z[P]
\end{array}$$

such that $V \to Y_Q = \text{Spec } Z[Q] \times_{\text{Spec } Z[P]} Y$ is étale and $P \to Q$ is injective and saturated.

Let $p \in \text{Spec } P$ be the image of $\bar{y}_1$ by the map $Y \to \text{Spec } P$ and let $F = P \setminus p$. The morphism $Y \to \text{Spec } Z[F]$ factors through $Y \to \text{Spec } Z[F]$, where $\text{Spec } Z[F]$ is the closure of the stratum of $\text{Spec } Z[P]$ corresponding to $p$. Since $\bar{M}_{Y,\bar{y}_1} \to \bar{M}_{Y,\bar{y}_2}$ is in an isomorphism, it even factorizes through $Y \to \text{Spec } Z[F^{\text{gp}}]$, where $\text{Spec } Z[F^{\text{gp}}]$ is the stratum of $\text{Spec } Z[P]$ corresponding to $p$. If $Z_1$ is a stratum of $V_{\bar{y}_1}$ above a face $F_0$ of $Q$, the corresponding stratum $Z_2$ of $V_{\bar{y}_2}$ by the cospecialization map is also above $F_0$; the map $\bar{M}_{V,\bar{z}_1} \to \bar{M}_{V,\bar{z}_2}$ is an isomorphism. Let $(z_i)_{i \in I}$ be the family of preimages of $z$ in $V$. Let $q_i \in \text{Spec } Q$ be the image of $z_i$ by the map $V \to \text{Spec } Q$. Let $F_i = \text{Spec } Q \setminus q_i$. According to lemma 4.1, $F \to F_i$ is a saturated morphism of fs monoids. Then $\{z_i\}$ is an irreducible component of $V_{F_i} = V \times_{\text{Spec } Z[Q]} \text{Spec } Z[F_i]$, which is étale above $Y_Q \times_{\text{Spec } Z[Q]} \text{Spec } Z[F_i] = \text{Spec } A \otimes_{Z[Q]} Z[F_i] = \text{Spec } A[F^{-1}F_i \cap T]$, where $T$ is a direct summand of $F^{\text{gp}}$ in $Q^{\text{gp}}$. $F^{-1}F_i \cap T$ is saturated: according to [12, prop. I.3.3.1], $\text{Spec } A[F^{-1}F_i \cap T]$ is normal. Hence $\{z_i\}$ is a connected component of $V_{F_i}$ and is normal. Thus $\bar{Z} \times_X V = \coprod \{z_i\}$. Since the geometric fibers of $\text{Spec } A[F^{-1}F_i \cap T] \to \text{Spec } A$ are normal for any choice of $V$, the geometric fibers of $\bar{Z} \to Y$ are also normal.

The morphism $\bar{Z} \to Y$ is proper. Let $\bar{Z} \to W \to Y$ be its Stein factorization. Since $\bar{Z} \to Y$ is separable, according to [5, prop X.1.2], $W \to Y$ is an étale covering. Since $Y$ is strictly henselian $W$ is a direct sum of copies of $Y$. Since $\bar{Z}_{\bar{y}_2}$ is connected, $W = Y$. Thus all the fibers of $\bar{Z} \to Y$ are geometrically connected. Since they are normal, they are also geometrically irreducible. Since $\bar{Z} \to Z$ is surjective, $\bar{Z}_{\bar{y}_1}$ is also irreducible. Let $z_1$ be the generic point of $\bar{Z}_{\bar{y}_1}$. Then for any specialization of geometric points $\bar{z} \to z_1$, $\bar{M}_{X,\bar{z}} \to M_{X,z}$ is an isomorphism (this can be checked on $\bar{Z} \times_X V$ if $V$ is a neighborhood of $\bar{z}_1$). The strata $Z_1$ of $X_{\bar{y}_1}$ containing $z_1$ should cospecialize to a stratum $Z'$ bigger than $Z$, but then $\bar{M}_{X,\bar{z}'_1} \to \bar{M}_{X,\bar{z}_1}$ is also an isomorphism and thus $Z' = Z$. This shows the surjectivity of $\text{Str}(X_{\bar{y}_1}) \to \text{Str}(X_{\bar{y}_2})$.

If $Z'_1 \in \text{Str}(X_{\bar{y}_1})$ cospecializes to $Z$, then $Z'_1 \subset \bar{Z}_{\bar{y}_1}$ and thus $Z'_1$ is bigger than $Z_1$ but then the morphism $\bar{M}_{X,\bar{z}'_1} \to \bar{M}_{X,\bar{z}_1}$ is also an isomorphism, and thus $Z'_1 = Z_1$, which shows the injectivity of the cospecialization map. 

We now want to define cospecialization maps of polysimplicial complexes. Let us begin with an
Lemma 4.4. Let $X \to Y$ be a strictly polystable morphism of log schemes and let $\bar{y}_2 \to \bar{y}_1$ be a specialization of geometric points of $Y$. Let $x_1$ be a stratum of $X_{\bar{y}_1}$ and let $x_2$ be its image in $\text{Str}(X_{\bar{y}_2})$ by the cospecialization map. Then, given an isometric bijection $\mu : [n] \to \text{Irr}(X_{\bar{y}_1}, x_1)$, there exists a unique couple $(I, \mu')$ consisting of a subset $I \subset [w(n)]$ and of an isometric bijection $\mu' : [n_I] \to \text{Irr}(X_{\bar{y}_2}, x_2)$ such that

$$
[n] \to \text{Irr}(X_{\bar{y}_1}, x_1) \\
\downarrow \downarrow \\
[n_I] \to \text{Irr}(X_{\bar{y}_2}, x_2)
$$

If moreover $\overline{M}_{Y, \bar{y}_1} \to \overline{M}_{Y, \bar{y}_2}$ is an isomorphism, then $I = [w(n)]$.

Proof. The uniqueness is obvious since there is no isometric bijection $[n_I] \to [n_I]$ for $I \neq J$ and $[n] \to [n_I]$ is surjective. One can replace $Y$ by its strict henselization at $\bar{y}_1$ and assume $Y = \text{Spf} A$. Let $\pi : M_{Y, \bar{y}_1} \to A$. Thanks to [3, lem. 2.10], the proposition is local on the étale topology of $X$ so that one can assume $X = \text{Spec} B$ where $B = B_1 \otimes_A \cdots \otimes_A B_p \otimes_A C$ and

$$
B_i = A[T_{i_0}, \ldots, T_{i_{m_i}}]/(T_{i_0} \cdots T_{i_{m_i}} - \pi(m_i))
$$

with $\pi(m_i)(\bar{y}_1) = 0$ and $C$ smooth over $A$. Let $I = \{i \in [p] | \pi(m_i)(\bar{y}_2)) = 0\}$. Then one gets an isometric bijection $\text{Irr}(X_{\bar{y}_2}, x_2) \simeq [n_I]$. 

Thus, if $X \to Y$ is strictly polystable, $\text{Str}(X_{\bar{y}_1}) \to \text{Str}(X_{\bar{y}_2})$ induces a natural cospecialization morphism of polysimplicial sets $C(X_{\bar{y}_1}) \to C(X_{\bar{y}_2})$. If $\overline{M}_{Y, \bar{y}_1} \to \overline{M}_{Y, \bar{y}_2}$ is an isomorphism, $C(X_{\bar{y}_1}) \to C(X_{\bar{y}_2})$ maps nondegenerate polysimplices to nondegenerate polysimplices.

Let $X : X = X_I \to \cdots \to Y$ be a strictly polystable fibration. Assume we constructed a cospecialization morphism of polysimplicial sets $\psi_{l-1} : C(X_{l-1, \bar{y}_2}) \to C(X_{l-1, \bar{y}_1})$ such that the induced map $\text{Str}(X_{l-1, \bar{y}_2}) \to \text{Str}(X_{l-1, \bar{y}_1})$ obtained by applying $O$ is the cospecialization map already defined. One has $C(X_{\bar{y}_1}) = C(X_{l-1, \bar{y}_1}) \square D_1$ and $C(X_{\bar{y}_2}) = C(X_{l-1, \bar{y}_2}) \square D_2$. Assume $Y$ to be strictly local. Let $y_2$ be the point of $Y$ where $\bar{y}_2$ lies. Let $x_1 \in \text{Str}(X_{l-1, \bar{y}_1})$. Let $x_2$ be the image of $x_1$ by the cospecialization map $\text{Str}(X_{l-1, \bar{y}_1}) \to \text{Str}(X_{l-1, \bar{y}_2})$. Let $\tilde{x}_2$ be the image of $x_2$ in $\text{Str}(X_{l-1, \bar{y}_2})$. If $z_1 \in \text{Str}(X_{x_1})$, then the set $\{Z \in \text{Str}(X_{\bar{y}_2}) | z_1 \subset Z\}$ has a biggest element $\tilde{z}_2$ according to proposition 4.2 and is geometrically irreducible. Since $\{Z \in \text{Str}(X_{\tilde{x}_2}) | z_1 \subset \overline{Z}\}$ is nonempty, one has $\tilde{z}_2 \in \text{Str}(X_{\tilde{x}_2}) \subset \text{Str}(X_{y_2})$. Since $\tilde{z}_2$ is geometrically irreducible, it defines a stratum $z_2$ of $\text{Str}(X_{x_2})$. Thus one gets a map $\text{Str}(X_{x_1}) \to \text{Str}(X_{x_2})$. Moreover, if $x'_1 \leq x_1 \in \text{Str}(X_{l-1, \bar{y}_1})$ and $x'_2$ is the image of $x'_1$ by the cospecialization map $\text{Str}(X_{l-1, \bar{y}_1}) \to \text{Str}(X_{l-1, \bar{y}_2})$ (thus $x'_2 \leq x_2$), then the following diagram is commutative:

$$
\begin{array}{ccc}
\text{Str}(X_{x_1}) & \longrightarrow & \text{Str}(X_{x'_1}) \\
\downarrow & & \downarrow \\
\text{Str}(X_{x_2}) & \longrightarrow & \text{Str}(X_{x'_2})
\end{array}
$$

where the horizontal arrows are given by lemma 1.3. Let us choose geometric points $\tilde{x}_1$ and $\tilde{x}_2$ of $X_{l-1}$ above $x_1$ and $x_2$. Let us choose a specialization $\bar{x}_2 \to \bar{x}_1$. The following diagram commutes:

$$
\begin{array}{ccc}
\text{Str}(X_{\bar{x}_1}) & \longrightarrow & \text{Str}(X_{\bar{x}_2}) \\
\downarrow & & \downarrow \\
\text{Str}(X_{x_1}) & \longrightarrow & \text{Str}(X_{x_2})
\end{array}
$$

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Let $\tilde{z}_1$ be a preimage of $z_1$ in $\text{Str}(X_{x_1})$ and let $\tilde{z}_2$ be the image of $z_1$ in $\text{Str}(X_{x_2})$. Then $\text{Irr}(X_{x_1}) \to \text{Irr}(X_{x_2})$ and $\text{Irr}(X_{x_1}) \to \text{Irr}(X_{x_2})$ are isomorphisms. One thus gets a statement similar to lemma 4.4 for $z_1$ and $z_2$: one gets a morphism $C(X_{x_1}) \to C(X_{x_2})$, compatible with morphisms of lemma 1.3. One thus gets a morphism of functors $D_1 \to D_2\psi_{i-1,*}$. This induces a morphism $C(X_{y_1}) \to C(X_{y_2})$. If $\overline{M}_{Y,y_1} \to \overline{M}_{Y,y_2}$ is an isomorphism and $x_2$ is the image of $x_1$ by the cospecialization map, $\overline{M}_{X_{x_2},x_1} \to \overline{M}_{X_{x_1},x_2}$ is also an isomorphism. By induction on $I$, one gets that $C(X_{y_1}) \to C(X_{y_2})$ maps nondegenerate polysimplices to nondegenerate polysimplices.

If $Z \to X$ is a két morphism, the morphism $\text{Str}(Z_{y_2}) \to \text{Str}(Z_{y_1})$ above $\text{Str}(X_{y_1}) \to \text{Str}(X_{y_2})$ induces a morphism $C(Z_{y_1}) \to C(Z_{y_2})$.

Assume now $X$ is a polystable fibration over $Y$ and $Z \to X$ is két. Let $X' \to X$ be étale and surjective such that $X$ is a strictly polystable fibration over $Y$. Let $X'' = X' \times_X X'$, $Z'' = Z \times_X X''$, $Z'' = Z \times_X X''$. Then the commutative diagram

\[
\begin{array}{ccc}
C(X''_{y_1}) & \to & C(X''_{y_1}) \\
\downarrow & & \downarrow \\
C(X''_{y_2}) & \to & C(X''_{y_2})
\end{array}
\]

induces a cospecialization morphism of polysimplicial sets $C(X_{y_1}) \to C(X_{y_2})$. One gets the following result:

**Proposition 4.5.** Let $X$ be a polystable log fibration over $Y$. Let $\tilde{y}_1 \to \tilde{y}_2$ be a két specialization of log geometric points. There is, for every két morphism $Z \to X$ a cospecialization map $C_{\text{geom}}(Z_{y_1}/\tilde{y}_1) \to C_{\text{geom}}(Z_{y_2}/\tilde{y}_2)$ functorial in $Z$.

Let us assume now that $Z \to Y$ is proper and that $\overline{M}_{Y,\tilde{y}_1} \to \overline{M}_{Y,\tilde{y}_2}$ is an isomorphism. The morphism $C(Z_{y_2}/\tilde{y}_1) \to C(Z_{y_2}/\tilde{y}_2)$ maps nondegenerate polysimplices to nodegenerate polysimplices and, according to proposition 4.3, $\text{Str}(Z_{\tilde{y}_1}) \to \text{Str}(Z_{\tilde{y}_2})$ is bijective. Therefore, if one assumes moreover that $C_{\text{geom}}(Z_{y_2}/\tilde{y}_2)$ is interiorly free (this is the case if $C_{\text{geom}}(Z_{y_2}/\tilde{y}_2)$ is interiorly free), then

$C_{\text{geom}}(Z_{y_2}/\tilde{y}_1) \to C_{\text{geom}}(Z_{y_2}/\tilde{y}_2)$

is also an isomorphism.

### 4.2 Specialization of tempered fundamental groups of log schemes

First, recall the result we proved in [9, §2.4] about specialization of log fundamental groups.

Let $X \to Y$ be a proper and saturated morphism of log schemes. Assume moreover $X \to Y$ to have log geometrically connected fibers. Let $\tilde{y}_2 \to \tilde{y}_1$ be a specialization of log geometric points of $Y$.

Let $T$ be the strictly local scheme of $Y$ at $\tilde{y}_1$ endowed with the inverse image log structure, and let $z$ be its closed point, endowed with the inverse image log structure. One has the following arrows (defined up to inner homomorphisms):

$\pi_1^{\log-\text{geom}}(X_{y_2}/\tilde{y}_2)(p') \to \pi_1^{\log-\text{geom}}(X_{z}/z)(p') \cong \pi_1^{\log-\text{geom}}(X_T/T)(p') \leftarrow \pi_1^{\log-\text{geom}}(X_{y_1}/y_1)(p')$.

**Theorem 4.6** [9, prop. 2.4]. One has a specialization morphism

$\pi_1^{\log-\text{geom}}(X_{\tilde{y}_2}/\tilde{y}_2)(p') \to \pi_1^{\log-\text{geom}}(X_{\tilde{y}_1}/\tilde{y}_1)(p')$

that factors through $\pi_1^{\log-\text{geom}}(X_T/T)(p')$. 27
We can now use this with our cospecialization morphism of polysimplicial sets when these are isomorphisms.

**Proposition 4.7.** Let $Y$ be a fs log scheme, let $X \to Y$ be a proper polystable log fibration with geometrically connected fibers. Assume moreover that the polysimplicial set $C_{geom}(X_s)$ of any geometric fiber is interiorly free. Let $\bar{y}_2 \to y_1$ be a specialization of log geometric points over fs log points $y_2 \to y_1$ of $Y$ such that $\overline{M}_{Y,y_1} \to \overline{M}_{y_2}$ is an isomorphism. Let $\mathbb{L}$ be a set of primes which does not contain the residual characteristic of $y_1$ One has a specialization morphism defined up to inner automorphism:

$$\pi_{1}^{temp-geom}(X_{\bar{y}_2})^\mathbb{L} \to \pi_{1}^{temp-geom}(X_{y_1})^\mathbb{L}.$$ 

**Proof.** One can assume that $\bar{Y}$ is strictly local with closed point $y_1$. There is a functor

$$F :KCov_{geom}(X_{y_1}/\bar{y}_1)^\mathbb{L} \to KCov_{geom}(X_{y_2}/\bar{y}_2)^\mathbb{L}.$$ 

According to theorem 4.6, if $Z_{\bar{y}_2}$ is some geometric kēt covering of $X_{y_1}/y_1$, it extends to a geometric kēt covering of $X/Y$; there is a connected finite pointed kēt covering $(U, \bar{u}_1)$ of $(Y, \bar{y}_1)$ such that $Z_{\bar{y}_1}$ extends to a kēt covering $Z_U \to X_U := X \times_Y U$. This extension becomes unique after replacing $U$ by some bigger covering. If $\bar{u}_2 \to \bar{u}_1$ is the kēt specialization of log geometric points lifting $\bar{y}_2 \to y_1$, then $(Z_U)_{\bar{u}_2}$ is nothing but the geometric kēt covering $F(Z_{\bar{y}_1})$ of $X_{y_2}$. We will simply denote it by $Z_{\bar{y}_2}$. One has an isomorphism $C_{geom}(Z_{\bar{y}_2}) \simeq C_{geom}(Z_{\bar{y}_1})$ functorially in $Z_{\bar{y}_1}$. One gets a cospecialization functor of fibered categories:

$$D_{top-geom}(X_{\bar{y}_1}) \longrightarrow D_{top-geom}(X_{y_2})$$

$$\downarrow \quad \quad \quad \quad \downarrow$$

$$KCov_{geom}(X_{y_1}/y_1)^\mathbb{L} \to KCov_{geom}(X_{y_2}/y_2)^\mathbb{L}$$

and thus a specialization morphism $\pi_{1}^{temp-geom}(X_{\bar{y}_2})^\mathbb{L} \to \pi_{1}^{temp-geom}(X_{y_1})^\mathbb{L}$. \(\square\)

**4.3 Cospecialization morphisms of pro-$p$' tempered fundamental groups**

Let $K$ be a discrete valuation field, and $\text{Spec } O_K$ be endowed with its usual log structure, and assume that the residual characteristic $p$ of $K$ is not in $\mathbb{L}$. Let $Y \to \text{Spec } O_K$ be a morphism of fs log schemes such that $\bar{Y}$ is locally noetherian. Let $\mathcal{Y}$ be the formal completion of $Y$ along its closed fiber. Then $\mathcal{Y}_{\eta}$ is an analytic domain of $Y_{K}^{an}$. Let $Y_0 = \mathcal{Y}_{\eta} \cap Y_{tr}^{an} \subset Y_{K}^{an}$.

Let $X \to Y$ be a proper and polystable log fibration with geometrically connected fibers. Let $\bar{y}$ be a $K'$-point of $Y_0$ where $K'$ is a complete extension of $K$. One has canonical morphism of log schemes $\text{Spec } O_{K'} \to Y$ where $\text{Spec } O_{K'}$ is endowed with the log structure given by $O_{K'} \setminus \{0\} \to O_{K'}$.

The log reduction $\tilde{s}$ of $\bar{y}$ is the log point of $Y$ corresponding to the special point of $\text{Spec } O_{K'}$ with the inverse image of the log structure of $\text{Spec } O_{K'}$. If $K'$ has discrete valuation, then $\tilde{s}$ is a fs log point. If $K'$ is algebraically closed, $\tilde{s}$ is a geometric log point.

Let $\hat{\text{Pt}}_{X}^{an}(Y)$ be the category whose objects are geometric points $\bar{y}$ of $Y_0$, such that $\mathcal{H}(y)$ is discretely valued (where $y$ is the underlying point of $\bar{y}$) and $\text{Hom}(\bar{y}, \bar{y}')$ is the set of kēt specializations from $\tilde{s}$ to $\tilde{s}'$, where $\tilde{s}$ and $\tilde{s}'$ are the log reductions of $\bar{y}$ and $\bar{y}'$, such that there exists some specialization $\bar{y} \to \bar{y}'$ of geometric points in the sense of algebraic étale topology for which the following diagram commutes:

$$\begin{array}{ccc}
\bar{y} & \longrightarrow & s \\
\downarrow & & \downarrow \\
\bar{y}' & \longrightarrow & \tilde{s}'
\end{array}$$

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Let $\tilde{\text{Pt}}^\text{an}(Y)$ be the category defined from $\tilde{\text{Pt}}^\text{an}(Y)$ by inverting the class of morphisms $\tilde{y} \to \tilde{y}'$ for which $\tilde{s} \to \tilde{s}'$ is a cospecialization isomorphism.

Let $\text{Pt}^\text{an}_0(Y)$ be the category obtained from $\tilde{\text{Pt}}^\text{an}(Y)$ by inverting the class of morphisms $\tilde{y} \to \tilde{y}'$ such that $\overline{M}_{Y,s'} \to \overline{M}_{Y,s}$ is an isomorphism.

Let $\text{OutG}_{\text{top}}$ be the category of topological groups with outer morphisms.

**Theorem 4.8.** There is a functor $\pi_1^\text{temp}(X_{(\cdot)}) : \text{Pt}^\text{an}(Y)^{\text{op}} \to \text{OutG}_{\text{top}}$ sending $\tilde{y}$ to $\pi_1^\text{temp}(X_{\tilde{y}})$.

If, for every geometric point $\bar{s}$ of $Y$, the polysimplicial set $C(X_{\bar{s}})$ is interiorly free, then the functor $\pi_1^\text{temp}(X_{(\cdot)})$ factors through $\text{Pt}^\text{an}_0(Y)^{\text{op}}$.

**Proof.** Let $\tilde{y}_2 \to \tilde{y}_1$ be a morphism $\tilde{\text{Pt}}^\text{an}(Y)$. One has to construct a cospecialization morphism $\pi_1^\text{temp}(X_{\tilde{y}_1}) \to \pi_1^\text{temp}(X_{\tilde{y}_2})$.

One has a cospecialization functor

$$F : \text{Kov}_{\text{geom}}(X_{s_1}/s_1)^L \to \text{Kov}_{\text{geom}}(X_{s_2}/s_2)^L,$$

which factors through $\text{Kov}_{\text{geom}}(X_T/T)_L$ where $T$ is the strict localization at $s_1$. The cospecialization functor $\text{Kov}_{\text{geom}}(X_{s_1}/s_1)^L \to \text{Cov}_{\text{alg}}^\text{temp}(X_{\bar{s}_1})$ is an equivalence since $\bar{y}_i \in Y_{tr}$ ([8, th. 1.4]). If one choses a specialization $\tilde{y}_2 \to \tilde{y}_1$ above $\tilde{s}_2 \to \tilde{s}_1$, the functor $\text{Cov}_{\text{alg}}^\text{temp}(X_{\tilde{y}_1}) \to \text{Cov}_{\text{alg}}^\text{temp}(X_{\tilde{y}_2})$ is also an equivalence. One gets that $F$ is an equivalence.

If $Z_{s_1}$ is some geometric két covering of $X_{s_1}$, it extends thanks to corollary 4.6 to some két neighborhood $(U, \tilde{u}_1)$ of $s_1$ in $T$. Let $Z_U \to U$ be this extension (unique after replacing $U$ by some smaller neighborhood of $s_1$). Let $\tilde{u}_2 \to \tilde{u}_1$ be the unique lifting of $\tilde{s}_2 \to \tilde{s}_1$. Then $Z_{\tilde{u}_2} := F(Z_{\tilde{u}_1})$ is nothing but $Z_{\tilde{u}_2}$. One has a cospecialization morphism $\text{Cov}_{\text{geom}}(Z_{\tilde{s}_1}) \to \text{Cov}_{\text{geom}}(Z_{\tilde{s}_2})$, which induces a specialization functor

$$\mathcal{D}_{\text{top-geom}}(Z_{s_2}) \to \mathcal{D}_{\text{top-geom}}(Z_{s_1}).$$

It is an equivalence of categories if $\tilde{s}_2 \to \tilde{s}_1$ is a cospecialization isomorphism or if $\overline{M}_{Y,\tilde{s}_1} \to \overline{M}_{Y,\tilde{s}_2}$ is an isomorphism and all the geometric fibers of $X \to Y$ have interiorly free polysimplicial sets. Thus we have a 2-commutative diagram:

$$\begin{array}{ccc}
\mathcal{D}_{\text{top-geom}}(X_{s_1}) & \to & \mathcal{D}_{\text{top-geom}}(X_{s_2}) \\
\downarrow & & \downarrow \\
\text{Kov}_{\text{geom}}(X_{s_1}/s_1)^L & \xrightarrow{F^{-1}} & \text{Kov}_{\text{geom}}(X_{s_2}/s_2)^L
\end{array}$$

where $F^{-1}$ is some quasi inverse of $F$. This induces a cospecialization outer morphism

$$\pi_1^\text{temp-geom}(X_{s_1}/s_1) \to \pi_1^\text{temp-geom}(X_{s_2}/s_2).$$

The comparison morphisms of theorem 3.3 gives us the wanted morphism, which is an isomorphism if $\tilde{s}_2 \to \tilde{s}_1$ is a cospecialization isomorphism or if $\overline{M}_{Y,\tilde{s}_1} \to \overline{M}_{Y,\tilde{s}_2}$ is an isomorphism and all the geometric fibers of $X \to Y$ have interiorly free polysimplicial sets.

\[\square\]

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