Coverings in p-adic analytic geometry and log coverings II: Cospecialization of the (p')-tempered fundamental group in higher dimensions

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Abstract

This paper constructs cospecialization homomorphisms between the (p') versions of the tempered fundamental group of the fibers of a smooth morphism with polystable reduction (the tempered fundamental group is a sort of analog of the topological fundamental group of complex algebraic varieties in the p-adic world). We studied the question for families of curves in another paper. To construct them, we will start by describing the pro-(p') tempered fundamental group of a smooth and proper variety with polystable reduction in terms of the reduction endowed with its log structure, thus defining tempered fundamental groups for log polystable varieties.

Introduction

This paper is a sequel to [9]. In that article we studied the behavior of the tempered fundamental groups of the fibers of a p-adic family of curves. More precisely we proved the following:

Theorem 0.1 ([9, th. 0.1]). *Let K be a complete discretely valued field. Let $\mathbb{L}$ be a set of primes that does not contain the residual characteristic of K. Let $Y \to O_K$ be a morphism of log schemes. Let $Y_0 = Y_{\mathrm{tr}} \cap \mathfrak{Y}_\eta \subset Y^{\mathrm{an}}$ where $\mathfrak{Y}$ is the completion of Y along its closed fiber. Let $X \to Y$ be a proper semistable curve with compatible log structure. Let $U = X_{\mathrm{tr}}$. Let η_1 and η_2 be two Berkovich points of Y_0 whose residue fields have discrete valuation, and let $\bar{\eta}_1, \bar{\eta}_2$ be geometric points above them. Let $\bar{s}_2 \to \bar{s}_1$ be a log specialization of their log reductions such that there exists a compatible specialization $\bar{\eta}_2 \to \bar{\eta}_1$. Then, there is a cospecialization homomorphism $\pi_1^{\mathrm{temp}}(U_{\bar{\eta}_1})^{\mathbb{L}} \to \pi_1^{\mathrm{temp}}(U_{\bar{\eta}_2})^{\mathbb{L}}$. Moreover, it is an isomorphism if $\overline{M}_{Y,\bar{s}_1} \to \overline{M}_{Y,\bar{s}_2}$ is an isomorphism.*

The aim of this paper is to generalize this result in higher dimension. However, in this paper, we will only consider the case of vertical semistable morphisms $X \to Y$ (which means mainly that $U_{\bar{\eta}_i} = X_{\bar{\eta}_i}$).

Recall that, if $\mathbb{L}$ is a set of primes, the $\mathbb{L}$-*tempered fundamental group* is the prodiscrete group that classifies the $\mathbb{L}$-tempered coverings, which are étale coverings in the sense of A.J. de Jong (that is to say that locally on the Berkovich topology, it is a direct sum of finite étale coverings) such that, after pulling back by some $\mathbb{L}$-finite étale covering, they become topological coverings (for the Berkovich topology).

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In this article, we shall study the following situation. Let K be a discretely valued field, O_K be its valuation ring, k be its residue field and p its characteristics (which can be 0). Let $X \to Y$ be a proper pluristable (for example semistable) morphism of schemes over O_K with geometrically connected fibers.

Let $\mathbb{L}$ be a set of prime that does not contain p. If η_1 is a (Berkovich) point of the generic fiber of Y, we first want to describe the geometric $\mathbb{L}$-tempered fundamental group of X_{η_1} in terms of X_{s_1} where s_1 is the reduction of η_1. To be sure that this reduction exists we have to assume η_1 is in the tube $\mathfrak{Y}_\eta$ of the special fiber of Y. Let us make sure at first that we can get such a description for the pro-$\mathbb{L}$ completion, *i.e.* the algebraic fundamental group. One cannot apply directly Grothendieck's specialization theorems since the special fiber is not smooth but only pluristable. Indeed, a pro-$\mathbb{L}$ geometric covering of the generic fiber will in generally only induce a Kummer covering on the special fiber. These are more naturally described in terms of *log geometry* and of the log fundamental group. The log fundamental group classifies Kummer log étale coverings (or, equivalently finite log étale coverings) : étale locally, these coverings are pullbacks of a morphism $\operatorname{Spec} \mathbf{Z}[Q] \to \operatorname{Spec} \mathbf{Z}[P]$ of a morphism of monoids $P \to Q$ where Q is the saturation of P in an extension of P^{gp} of finite index invertible on the log scheme. For a proper and log smooth log scheme over a complete discrete valuation ring, there is, as in the proper and smooth case for Grothendieck's fundamental group, a specialization morphism from the pro-$\mathbb{L}$ log fundamental group of the generic fiber (which is isomorphic to the pro-$\mathbb{L}$ algebraic fundamental group of the maximal open subset of the generic fiber where the log structure is trivial) to the pro-$\mathbb{L}$ log fundamental group of the closed fiber. We will have to assume the field $\mathcal{H}(\eta_1)$ to be with discrete valuation in order to get log schemes with good finiteness properties (more precisely to be fs). Then, one can endow X_{s_1} with a natural log structure. The pro-$\mathbb{L}$ fundamental group of X_{η_1} is isomorphic to the pro-$\mathbb{L}$ log fundamental group of X_{s_1}. To try to describe the $\mathbb{L}$-tempered fundamental group, one has to describe the topological behavior of any $\mathbb{L}$-algebraic covering of X_{η_1}. Berkovich, in [3], constructed a combinatorial object (more precisely a *polysimplicial set*) depending only on X_{s_1}, such that the Berkovich generic fiber X_{η_1} is naturally homotopically equivalent to the geometric realization of this combinatorial object, thus generalizing the case of curves with semistable reduction, where the homotopy type of the generic fiber can be naturally described in terms of the graph of this semistable reduction. We will extend such a description to our log coverings: for every log covering $Y \to X_{O_{\mathcal{H}(\eta_1)}}$, we will construct a combinatorial object $\mathrm{C}(Y)$, depending only on Y_{s_1}, such that its geometric realization $|\,\mathrm{C}(Y)|$ is naturally homotopically equivalent to the Berkovich generic fiber Y_{η_1}. This will enable us to define a $\mathbb{L}$-tempered fundamental group of our log reduction, which is isomorphic to the tempered fundamental group of the generic fiber: for any Galois két covering $f : Y \to X_{s_1}$, there is an action of $\mathrm{Gal}(Y/X_{s_1})$ on $\mathrm{C}(Y)$. Such an action defines an extension G_Y of $\mathrm{Gal}(Y/X_{s_1})$ by $\pi_1^{\mathrm{top}}(|\,\mathrm{C}(Y)|)$: $G_Y = \{(g_1, g_2) \in \mathrm{Aut}(|\,\mathrm{C}(Y)|^\infty) \times \mathrm{Gal}(Y/X_{s_1}) | \pi g_1 = g_2 \pi\}$, where $\pi : |\,\mathrm{C}(Y)|^\infty \to |\,\mathrm{C}(Y)|$ is the universal topological covering of $|\,\mathrm{C}(Y)|$. The $\mathbb{L}$-tempered fundamental group of X_{s_1} is the projective limits of these extensions G_Y, where Y runs through pointed két Galois coverings of X of $\mathbb{L}$ order. In particular, one gets:

Theorem 0.2 (see th. 3.2). *The $\mathbb{L}$-tempered fundamental group of X_{η_1} only depends on the log reduction X_{s_1}.*

Once we have a definition for the log geometric tempered fundamental group $\pi_1^{\text{temp-geom}}(X_{s_1})$ of the log fibers in the special locus of Y, one can reformulate our cospecialization problem only in terms of this special locus.

We will prove the following:

Theorem 0.3 (th. 4.8). *Let η_1 and η_2 be two Berkovich points with discrete valuation fields of*

$Y_0 = Y_{\mathrm{tr}}^{\mathrm{an}} \cap \mathfrak{Y}_\eta$. Let $\bar\eta_1, \bar\eta_2$ be geometric points above them. Let $\bar s_2 \to \bar s_1$ be a specialization of their log reductions such that there exists a compatible specialization $\bar\eta_2 \to \bar\eta_1$. Then there is a cospecialization homomorphism $\pi_1^{\text{temp-geom}}(X_{\bar\eta_1})^{\mathbb{L}} \to \pi_1^{\text{temp-geom}}(X_{\bar\eta_2})^{\mathbb{L}}$.

Moreover, one can give a criterion for this cospecialization homomorphism to be an isomorphism. To do this, we will have to make an assumption on the combinatorial behavior of the geometric fibers of $X \to Y$. More precisely, the polysimplicial set associated with those geometric fibers will be assumed to be interiorly free (this is for example the case if $X \to Y$ is strictly polystable or if $X \to Y$ is of relative dimension 1, which explains why such a condition did not appear in [9]). If the morphism of monoids $\overline{M}_{Y,\bar s_1} \to \overline{M}_{Y,\bar s_2}$ is an isomorphism, then the cospecialization homomorphism $\pi_1^{\text{temp-geom}}(X_{\bar\eta_1})^{\mathbb{L}} \to \pi_1^{\text{temp-geom}}(X_{\bar\eta_2})^{\mathbb{L}}$ is an isomorphism.

The first thing we need to construct the cospecialization homomorphism for tempered fundamental groups is a specialization morphism for the $\mathbb{L}$-log geometric fundamental groups of $X_{\bar s_1}$ and $X_{\bar s_2}$. More precisely we would like to extend any $\mathbb{L}$-log geometric covering of X_{s_1} to a két neighborhood of s_1. By restricting this extension to $X_{\bar s_2}$, one obtains a functor from $\mathbb{L}$-log coverings of $X_{\bar s_1}$ to $\mathbb{L}$-log coverings of $X_{\bar s_2}$, this functor induces the wanted specialization morphism of $\mathbb{L}$-log geometric fundamental groups. If one has such a specialization morphism, by comparing it to the fundamental groups of $X_{\bar\eta_1}$ and $X_{\bar\eta_2}$ and using Grothendieck's specialization theorem, we will easily get that it must be an isomorphism). These specialization morphisms has already been constructed in [9, prop. 2.10].
Then we have to study the combinatorial behavior of a két covering with respect to cospecialization. By étale localization, one can assume that Y is strictly local with special point $\bar s_1$. Up to két localization of Y, any két covering $U_{\bar s_1}$ of $Y_{\bar s_1}$ extends to a két covering U of Y, and $U \to Y$ is saturated. For a stratum u of $U_{\bar s_1}$, there is among the strata of U_{s_2} whose closure contains u a stratum u' with smallest closure (*i.e.* a biggest stratum for specialization): it defines a map $\mathrm{Str}(U_{\bar s_1}) \to \mathrm{Str}(U_{s_2})$. The fact that $U \to Y$ is saturated implies that the closure of the strata of U are flat over their image in Y and have geometrically reduced fibers. Thanks to [6, cor. 18.9.8]), this implies that u' is geometrically connected, whence a cospecialization map $\mathrm{Str}(U_{\bar s_1}) \to \mathrm{Str}(U_{\bar s_2})$. This cospecialization map can be extended into a morphism of polysimplicial sets. One gets by pullback a specialization functor between the category of topological coverings of the polysimplicial sets $U_{\bar s_2}$ and $U_{\bar s_1}$. Since The cospecialization morphisms of polysimplicial sets commute with két coverings, the specialization functor can be seen as a functor of fibered categories over the category of $\mathbb{L}$-log coverings of $X_{\bar s_1}$ (or equivalently of $\mathbb{L}$-finite étale coverings of $X_{\bar\eta_1}$). But the fibered category of tempered coverings over the category of $\mathbb{L}$-finite étale coverings of $X_{\bar\eta_1}$ is naturally equivalent to the stack associated to the fibered categories of topological coverings over the category of $\mathbb{L}$-finite étale coverings of $X_{\bar\eta_1}$. Thus the topological specialization functor gives us the wanted tempered specialization functor.

Let us now discuss the organization of the paper.
The first section of this paper will be devoted to recall the main tools we will need later. We will recall the definition of the tempered fundamental group and its basic properties. We will also consider an $\mathbb{L}$-version of the tempered fundamental group, where $\mathbb{L}$ is a set of prime numbers ($\mathbb{L}$-tempered fundamental groups were already introduced in [10] in the case of curves). We will then recall the basics of log geometry, especially the theory of két coverings and log fundamental groups. We will end this part by recalling the topological structure of the generic fiber (considered as a Berkovich space) of a pluristable formal scheme, as studied in [3] and in [4].

In §2, we define the tempered fundamental group of a connected pluristable log scheme X over a log point. To do this, we define a functor C from the Kummer étale site of our pluristable log scheme X to the category of polysimplicial sets (which extends the definition of the polysimplicial set associated to a pluristable scheme defined by Berkovich in [3]). We also defines a log geometric version by taking the projective limit under connected két extensions of the base log point.

In §3, for a connected, proper, generically smooth and pluristable scheme X over a complete discretely valued ring O_K (thus endowed with a canonical log structure), we construct a specialization morphism between the $\mathbb{L}$-tempered fundamental group of the generic fiber, considered as a Berkovich space, and the $\mathbb{L}$-tempered fundamental group of the special fiber endowed with the inverse image log structure, which is an isomorphism if the residual characteristic of K is not in $\mathbb{L}$.
This specialization morphism is induced by the specialization morphism from the algebraic fundamental group of the generic fiber to the log fundamental group of the special fiber, and by the fact that the geometric realization of the polysimplicial set $|\,\mathrm{C}(Y)|$ of a két covering of the special fiber of X is canonically homotopically equivalent to the Berkovich space Y_η^{an} of the corresponding étale covering of the generic fiber. This homotopy equivalence is obtained by extending the strong deformation retraction of X_η^{an} to a strong deformation retraction of Y_η^{an} onto a subset canonically homeomorphic to $|\,\mathrm{C}(Y)|$.

In §4, we construct cospecialization morphisms between the polysimplicial sets of the geometric fibers of a polystable fibration. To do so, we first prove that, up to étale localization of Y at $\bar{s}_1$, for any stratum x of $X_{\bar{s}_1}$, the set of strata of X_{s_2} whose closure contains x has a biggest element (for the order induced by existence of specialization), and this biggest stratum is geometrically irreducible. This will induce cospecialization morphisms on the set of strata of the geometric fibers of $X \to Y$. Up to két localization, the same result is also true for két coverings of Y. This cospecialization maps of set of strata in fact come from maps of polysimplicial sets. If we identify the categories of $\mathbb{L}$-két coverings of $X_{\bar{s}_1}$ and $X_{\bar{s}_2}$ thanks to specialization of ket coverings, one gets, for U in this category, a map $|\,\mathrm{C}(U_{\bar{s}_1})| \to |\,\mathrm{C}(U_{\bar{s}_2})|$ functorially in U (and in particular, when U is Galois, compatibly with the action of the Galois group of U). We get from this cospecialization morphisms between the $\mathbb{L}$-geometric tempered fundamental groups of the fibers of our strictly polystable log fibration.
Thanks to the isomorphisms between the $\mathbb{L}$-geometric tempered fundamental group of the fiber over a discretely valued Berkovich point of the generic part of our base log scheme and the $\mathbb{L}$-geometric tempered fundamental group of the fiber over the reduction log point, we will get theorem 0.3.

1. Reminder on the skeleton of a Berkovich space with pluristable reduction

1.1 Polystable morphisms

Let K be a complete nonarchimedean field and let O_K be its ring of integers.
If $\mathfrak{X}$ is a locally finitely presented formal scheme over O_K, $\mathfrak{X}_\eta$ will denote the generic fiber of $\mathfrak{X}$ in the sense of Berkovich ([2, section 1]).
Recall the definition of a polystable morphism of formal schemes:

Definition 1.1 ([3, def. 1.2], [4, section 4.1]). Let $\phi : \mathfrak{Y} \to \mathfrak{X}$ be a locally finitely presented morphism of formal schemes over O_K.

(i) ϕ is said to be *strictly polystable* if, for every point $y \in \mathfrak{Y}$, there exists an open affine neighborhood $\mathfrak{X}' = \mathrm{Spf}(A)$ of $x := \phi(y)$ and an open neighborhood $\mathfrak{Y}' \subset \phi^{-1}(\mathfrak{X}')$ of y such that the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$ factors through an étale morphism $\mathfrak{Y}' \to \mathrm{Spf}(B_0) \times_{\mathfrak{X}'} \cdots \times_{\mathfrak{X}'} \mathrm{Spf}(B_p)$ where each B_i is of the form $A\{T_0, \cdots, T_{n_i}\}/(T_0 \cdots T_{n_i} - a_i)$ with $a \in A$ and $n \geqslant 0$. It is said to be *nondegenerate* if one can choose X', Y' and (B_i, a_i) such that $\{x \in (\mathrm{Spf}(A)_\eta) | a_i(x) = 0\}$ is nowhere dense.

(ii) ϕ is said to be *polystable* if there exists a surjective étale morphism $\mathfrak{Y}' \to \mathfrak{Y}$ such that $\mathfrak{Y}' \to \mathfrak{X}$ is strictly polystable. It is said to be *nondegenerate* if one can choose $\mathfrak{Y}'$ such that $\mathfrak{Y}' \to \mathfrak{X}$ is nondegenerate.

Then a (*nondegenerate*) *polystable fibration* of length l over $\mathfrak{S}$ is a sequence of (nondegenerate) polystable morphisms $\underline{\mathfrak{X}} = (\mathfrak{X}_l \to \cdots \to \mathfrak{X}_1 \to \mathfrak{S})$.
Then K-$\mathcal{P}stf_l^{\text{ét}}$ (resp. K-$\mathcal{P}stf_l^{\text{sm}}$, K-$\mathcal{P}stf_l^{\text{tps}}$)will denote the category of polystable fibrations of length l over O_K, where a morphism $\underline{\mathfrak{X}} \to \underline{\mathfrak{Y}}$ is a collection of étale (resp. smooth, resp. trivially polystable) morphisms $(\mathfrak{X}_i \to \mathfrak{Y}_i)_{1 \leqslant i \leqslant l}$ which satisfies the natural commutation assumptions.
$\mathcal{P}stf_l^{\text{ét}}$ (resp. $\mathcal{P}stf_l^{\text{sm}}$, $\mathcal{P}stf_l^{\text{tps}}$) will denote the category of couples $(\underline{\mathfrak{X}}, K_1)$ where K_1 is a complete non archimedean field and $\underline{\mathfrak{X}}$ is a polystable fibration over O_{K_1}, and a morphism $(\underline{\mathfrak{X}}, K_1) \to (\underline{\mathfrak{Y}}, K_2)$ is a couple (ϕ, ψ) where ϕ is an isometric extension $K_2 \to K_1$ and ψ is a morphism $\underline{\mathfrak{X}} \to \underline{\mathfrak{Y}} \otimes_{O_{K_2}} O_{K_1}$ in K_1-$\mathcal{P}stf_l^{\text{ét}}$ (resp. K_1-$\mathcal{P}stf_l^{\text{sm}}$, K_1-$\mathcal{P}stf_l^{\text{tps}}$).

Let k be a field.
Let X be a k-scheme locally of finite type.
The normal locus $\mathrm{Norm}(X^{\text{red}})$ is a dense open subset of X. Let us define inductively $X^{(0)} = X^{\text{red}}$, $X^{(i+1)} = X^{(i)} \setminus \mathrm{Norm}(X^{(i)})$. The irreducible components of $X^{(i)} \setminus X^{(i+1)}$ are called the strata of X (of rank i). This gives a partition of X. The set of the generic points of the strata of X is denoted by $\mathrm{Str}(X)$ (This set is in natural bijection with the set of strata of X). There is a natural partial order on $\mathrm{Str}(X)$ defined by $x \leqslant y$ if and only if $y \in \overline{\{x\}}$.

Berkovich defines another filtration $X = X_{(0)} \supset X_{(1)} \supset \cdots$ such that $X_{(i+1)}$ is the closed subset of points contained in at least two irreducible components of $X_{(i)}$. X is said to be *quasinormal* if all of the irreducible components of each $X_{(i)}$, endowed with the reduced subscheme structure, is normal (this property is local for the Zariski topology and remains true after étale morphisms). If X is quasinormal, then $X_{(i)} = X^{(i)}$. X is quasinormal if and only if the closure of every stratum is normal. A strictly plurinodal scheme over a field is quasinormal ([3, prop. 2.1]).
There is a natural partial order on $\mathrm{Str}(X)$ defined by $x \leqslant y$ if and only if $y \in \overline{\{x\}}$.

We say that a strictly plurinodal scheme X over a field K is *elementary* if $\mathrm{Str}(X)$ has a biggest element; we say that it is *geometrically elementary* if it is elementary and all the strata are geometrically irreducible. Finally, a strictly pluristable morphism $Y \to X$ is *geometrically elementary* if all the fibers are geometrically elementary.

1.2 Polysimplicial sets

Berkovich defines *polysimplicial sets* in [3, section 3] as follows.
For an integer n, let $[n]$ denote $\{0, 1, \cdots, n\}$.
For a tuple $\mathbf{n} = (n_0, \cdots, n_p)$ with either $p = n_0 = 0$ or $n_i \geqslant 1$ for all i, let $[\mathbf{n}]$ denote the set $[n_0] \times \cdots \times [n_p]$ and $w(\mathbf{n})$ denote the number p.
Berkovich defines a category $\mathbf{\Lambda}$ whose objects are $[\mathbf{n}]$ and morphisms are maps $[\mathbf{m}] \to [\mathbf{n}]$ associated with triples (J, f, α), where:

- J is a subset of $[w(\mathbf{m})]$ assumed to be empty if $[\mathbf{m}] = [0]$,
- f is an injective map $J \to [w(\mathbf{n})]$,
- α is a collection $\{\alpha_l\}_{0 \leqslant l \leqslant p}$, where α_l is an injective map $[m_{f^{-1}(l)}] \to [n_l]$ if $l \in \mathrm{Im}(f)$, and α_l is a map $[0] \to [n_l]$ otherwise.

The map $\gamma : [\mathbf{m}] \to [\mathbf{n}]$ associated with (J, f, α) takes $\mathbf{j} = (j_0, \cdots, j_{w(\mathbf{m})}) \in [\mathbf{m}]$ to $\mathbf{i} = (i_0, \cdots, i_{w(\mathbf{n})})$ with $i_l = \alpha_l(j_{f^{-1}(l)})$ for $l \in \mathrm{Im}(f)$, and $i_l = \alpha_l(0)$ otherwise.
A polysimplicial set is a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathrm{Set}$. Polysimplicial sets form a category denoted by $\mathbf{\Lambda}^\circ\,\mathrm{Set}$. One considers $\mathbf{\Lambda}$ as a full subcategory of $\mathbf{\Lambda}^\circ\,\mathrm{Set}$ by the Yoneda functor. If C is a polysimplicial set $\mathbf{\Lambda}/\,\mathrm{C}$ is the category whose objects are morphisms $[\mathbf{n}] \to \mathrm{C}$ in $\mathbf{\Lambda}^\circ\,\mathrm{Set}$ and morphisms from $[\mathbf{n}] \to \mathrm{C}$ to $[\mathbf{m}] \to \mathrm{C}$ are morphisms $[\mathbf{n}] \to [\mathbf{m}]$ that make the triangle commute. Objects of $\mathbf{\Lambda}/\,\mathrm{C}$ are called *polysimplices* of C, and if $x : [\mathbf{n}] \to \mathrm{C}$ is a polysimplex, $\mathbf{n}$ will be denoted by $\mathbf{n}_x$.
A polysimplex x of a polysimplicial set C is said to be *degenerate* if there is a non isomorphic surjective map f of $\mathbf{\Lambda}$ such that x is the image by f of a polysimplex of C. Let $\mathrm{C}^{\mathrm{nd}}_{\mathbf{n}}$ be the subset of non degenerate polysimplices of $\mathrm{C}_{\mathbf{n}}$.
Thanks to an analog of Eilenberg-Zilber lemma for polysimplicial sets ([3, lem. 3.2]), a morphism $\mathrm{C}' \to \mathrm{C}$ is bijective if and only if it maps non degenerate polysimplices to nondegenerate polysimplices and $(\mathrm{C}')^{\mathrm{nd}}_{\mathbf{n}} \to \mathrm{C}^{\mathrm{nd}}_{\mathbf{n}}$ is bijective for any $\mathbf{n}$.
There is a functor $O : \mathbf{\Lambda}^\circ\,\mathrm{Set} \to \mathrm{Poset}$ where $O(\mathrm{C})$ is the partially ordered set associated to $\mathrm{Ob}(\mathbf{\Lambda}/\,\mathrm{C})$ endowed with the preorder where $x \leqslant y$ if there is a morphism $x \to y$ in $\mathbf{\Lambda}/\,\mathrm{C}$. If one sees $O(\mathrm{C})$ as a category, there is an obvious functor $\mathbf{\Lambda}/\,\mathrm{C} \to O(\mathrm{C})$. As a set, $O(\mathrm{C})$ coincides with the set of equivalence classes of nondegenerate polysimplices.
A polysimplicial set C is said *interiorly free* if $\mathrm{Aut}(\mathbf{n})$ acts freely on $\mathrm{C}^{\mathrm{nd}}_{\mathbf{n}}$. If $\mathrm{C}_1 \to \mathrm{C}_2$ is a morphism of polysimplicial sets mapping nondegenerate polysimplices to nondegenerate polysimplices such that $O(\mathrm{C}_1) \to O(\mathrm{C}_2)$ is an isomorphism and C_2 is interiorly free, then $C_1 \to \mathrm{C}_2$ is an isomorphism.
Berkovich also defines a *strictly polysimplicial category* Λ whose objects are those of $\mathbf{\Lambda}$, but with only injective morphisms between them. The functor $\Lambda \to \mathbf{\Lambda} \to \mathbf{\Lambda}^\circ\,\mathrm{Set}$ extends to a functor $\Lambda^\circ\,\mathrm{Set} \to \mathbf{\Lambda}^\circ\,\mathrm{Set}$ which commutes with direct limits (the objects of $\Lambda^\circ\,\mathrm{Set}$ will be called *strictly polysimplicial sets*).
Berkovich then considers a functor $\Sigma : \mathbf{\Lambda} \to \mathcal{K}e$ to the category of Kelley spaces, *i.e.* topological spaces X such that a subset of X is closed whenever its intersection with any compact subset of X is closed. This functor takes $[\mathbf{n}]$ to $\Sigma_{\mathbf{n}} = \{(u_{il})_{0\leqslant i\leqslant p, 0\leqslant l\leqslant n_i} \in [0,1]^{[\mathbf{n}]} | \sum_l u_{il} = 1\}$, and takes a map γ associated to (J, f, α) to $\Sigma(\gamma)$ that maps $\mathbf{u} = (u_{jk})$ to $\mathbf{u}' = (u'_{il})$ defined as follows: if $[\mathbf{m}] \neq [0]$ and $i \notin \mathrm{Im}(f)$ or $[\mathbf{m}] = [0]$ then $u'_{il} = 1$ for $l = \alpha_i(0)$ and $u'_{il} = 0$ otherwise; if $[\mathbf{m}] \neq [0]$ and $i \in \mathrm{Im}(f)$, then $u'_{il} = u_{f^{-1}(i), \alpha_i^{-1}(l)}$ for $l \in \mathrm{Im}(\alpha_i)$ and $u'_{il} = 0$ otherwise.
This induces a functor, the *geometric realization*, $|\ | : \mathbf{\Lambda}^\circ\,\mathrm{Set} \to \mathcal{K}e$ (by extending Σ in such a way that it commutes with direct limits). If $O(\mathrm{C})$ is finite (resp. locally finite), then $|\,\mathrm{C}\,|$ is compact (resp. locally compact).
There is also a bifunctor $\Box : \mathbf{\Lambda}^\circ\,\mathrm{Set} \times \mathbf{\Lambda}^\circ\,\mathrm{Set} \to \mathbf{\Lambda}^\circ\,\mathrm{Set}$ which commutes with direct limits and defined by $[(n_0, \cdots, n_p)] \Box [(n'_0, \cdots, n'_{p'})] = [(n_0, \cdots, n_p, n'_0, \cdots, n'_{p'})]$. Thus $|\,\mathrm{C} \Box \mathrm{C}'\,| = |\,\mathrm{C}\,| \times |\,\mathrm{C}'\,|$ where the product on the right is the product of Kelley spaces (which is the same as the product of topological spaces whenever C and C' are locally finite).

1.3 Polysimplicial set of a polystable fibration

If X is strictly polystable over k and $x \in \mathrm{Str}(X)$, $\mathrm{Irr}(X, x)$ will denote the metric space of irreducible components of X passing through x where $d(X_1, X_2) = \mathrm{codim}_x(X_1 \cap X_2)$. On a tuple $[\mathbf{n}]$, one can consider the metric d defined by $d((n_0, \cdots, n_p), (n'_0, \cdots, n'_p)) = |\{i \in [[0, p]] | n_i \neq n'_i\}|$. Then there is a unique tuple $[\mathbf{n}]$ such that $\mathrm{Irr}(X, x)$ is bijectively isometric to $[\mathbf{n}]$. If $[\mathbf{m}] \to [\mathbf{n}]$ is isometric, there exists a unique $y \in \mathrm{Str}(X)$ with $y \leqslant x$ and a unique isometric bijection $[\mathbf{m}] \to \mathrm{Irr}(X, y)$ such that

$$\begin{array}{ccc} [\mathbf{n}] & \to & \mathrm{Irr}(X, x) \\ \uparrow & & \uparrow \\ [\mathbf{m}] & \to & \mathrm{Irr}(X, y) \end{array}$$

commutes.
The functor which to $[\mathbf{n}]$ associates the set of couples (x,μ) where $x\in \mathrm{Str}(X)$ and μ is a isometric bijection $[\mathbf{n}]\to \mathrm{Irr}(X,x)$ defines a strict polysimplicial set $C(X)$ (and thus a polysimplicial set $\mathrm{C}(X)$).
There is a functorial isomorphism of partially ordered sets $O(\mathrm{C}(X))\simeq \mathrm{Str}(X)$.

Proposition 1.2 ([3, prop. 3.14]). *One has a functor* $\mathrm{C}:\mathcal{P}st^{\mathrm{sm}}\to \mathbf{\Lambda}^\circ\,\mathrm{Set}$*, such that* $\mathrm{C}(X)$ *is as previously defined if* X *is strictly polystable and, for every étale surjective morphism* $X'\to X$*:*

$$\mathrm{C}(X)=\mathrm{Coker}(\mathrm{C}(X'\times_X X')\rightrightarrows \mathrm{C}(X')).$$

This functor extends to a functor C for strictly polystable fibrations over K of length l.
Let us assume we already constructed C for strictly polystable fibrations of length $l-1$ such that $O(\mathrm{C}(\underline{X}))=\mathrm{Str}(X_{l-1})$. Let $\underline{X}:X_l\to X_{l-1}\to\cdots\to \mathrm{Spec}\,k$ be a strictly polystable fibration, and let $\underline{X}_{l-1}:X_{l-1}\to\cdots\to\mathrm{Spec}\,k$. Then for every $x'\leqslant x\in \mathrm{Str}(X_{l-1})$, one has:

Lemma 1.3 ([3, cor.6.2]). *There is a canonical cospecialization morphism* $\mathrm{C}(X_{l,x})\to \mathrm{C}(X_{l,x'})$ *and if* $x''\leqslant x'\leqslant x$*, the morphism* $\mathrm{C}(X_{l,x})\to\mathrm{C}(X_{l,x''})$ *coincides with the composition* $\mathrm{C}(X_{l,x})\to \mathrm{C}(X_{l,x'})\to \mathrm{C}(X_{l,x''})$.

This gives a functor $\mathrm{Str}(X_{l-1})^{\mathrm{op}}\to\mathbf{\Lambda}^\circ\,\mathrm{Set}$ that maps an object x in $\mathrm{Str}(X_{l-1})^{\mathrm{op}}$ to $\mathrm{C}(X_{l,x})$ and an arrow $x'\to x$ to the cospecialization morphism $\mathrm{C}(X_{l,x'})\to\mathrm{C}(X_{l,x})$ given by lemma 1.3. If one composes this functor with $(\mathbf{\Lambda}/(\mathrm{C}(\underline{X}_{l-1})))^{\mathrm{op}}\to O(\mathrm{C}(\underline{X}_{l-1}))^{\mathrm{op}}=\mathrm{Str}(X_{l-1})^{\mathrm{op}}$, one gets a functor

$$D:(\mathbf{\Lambda}/(\mathrm{C}(\underline{X}_{l-1})))^{\mathrm{op}}\to\mathbf{\Lambda}^\circ\,\mathrm{Set}\,.$$

Berkovich then defines a polysimplicial set (where we set $\mathrm{C}=\mathrm{C}(\underline{X}_{l-1})$):

$$\mathrm{C}(\underline{X})=\mathrm{C}\,\square\, D:=\mathrm{Coker}(\coprod_{N_1(\mathbf{\Lambda}/\,\mathrm{C})}[\mathbf{n}_y]\,\square\, D_x\rightrightarrows\coprod_{N_0(\mathbf{\Lambda}/\,\mathrm{C})}[\mathbf{n}_x]\,\square\, D_x),$$

where, for $f:y\to x\in N_1(\mathbf{\Lambda}/\,\mathrm{C})$, the upper arrow sends $[\mathbf{n}_y]\,\square\, D_x$ to $[\mathbf{n}_x]\,\square\, D_x$ by the morphism $[f]\,\square\,\mathrm{id}_{D_x}$ and the lower arrow sends $[\mathbf{n}_y]\,\square\, D_x$ to $[\mathbf{n}_y]\,\square\, D_y$ by the morphism $\mathrm{id}_{[\mathbf{n}_y]}\,\square\, D_f$. This construction extends to (non necessarily strictly) polystable fibrations:

Proposition 1.4 ([3, prop 6.9]). *There is a functor* $\mathrm{C}:\mathcal{P}st_l^{\mathrm{tps}}\to\mathbf{\Lambda}\,\mathrm{Set}$ *such that:*

(i) for every étale surjective morphism of polystable fibrations $X'\to X$*:*

$$\mathrm{C}(X)=\mathrm{Coker}(\mathrm{C}(X'\times_X X')\rightrightarrows\mathrm{C}(X')).$$

(ii) $O(\mathrm{C}(\underline{X}))\simeq\mathrm{Str}(X)$.

1.4 Skeleton of a Berkovich space

Berkovich attaches to a polystable fibration $\underline{\mathfrak{X}}=(\mathfrak{X}_l\to\mathfrak{X}_{l-1}\to\cdots\to\mathrm{Spf}(O_K))$ a subset of the generic fiber $\mathfrak{X}_{l,\eta}$ of $\mathfrak{X}_l$, the *skeleton* $S(\underline{\mathfrak{X}})$ of $\underline{\mathfrak{X}}$, which is canonically homeomorphic to $|\,\mathrm{C}(\mathfrak{X}_s)|$ (see [3, th. 8.2]). In fact, when $\underline{\mathfrak{X}}$ is non degenerate—for example generically smooth (we will only use the results of Berkovich to such polystable fibrations)—the skeleton of $\underline{\mathfrak{X}}$ depends only on $\mathfrak{X}_l$ according to [4, prop. 4.3.1.(ii)]; such a formal scheme that fits into a polystable fibration will be called *pluristable*, and we will note $S(\mathfrak{X}_l)$ this skeleton.
In this case [4, prop. 4.3.1.(ii)] gives a description of $S(\mathfrak{X}_l)$, which is independant of the retraction. For any $x,y\in\mathfrak{X}_{l,\eta}$, we write $x\preceq y$ if for every étale morphism $\mathfrak{X}'\to\mathfrak{X}_l$ and any x' over x, there exists y' over y such that for any $f\in O(\mathfrak{X}_\eta)$, $|f(x')|\leqslant|f(y')|$ ($\preceq$ is a partial order on $\mathfrak{X}_{l,\eta}$). Then $S(\mathfrak{X}_l)$ is just the set of maximal points of $\mathfrak{X}_{l,\eta}$ for $\preceq$.

Moreover there is a strong deformation retraction of $\mathfrak{X}_{l,\eta}$ to $S(\underline{\mathfrak{X}})$ and this construction is compatible with étale morphisms; more precisely, one has the following theorem:

THEOREM 1.5 ([3, th. 8.1]). *One can construct, for every polystable fibration* $\underline{\mathfrak{X}} = (\mathfrak{X}_l \overset{f_{l-1}}{\to} \cdots \overset{f_1}{\to} \mathfrak{X}_1 \to \mathrm{Spf}(O_K))$, *a proper strong deformation retraction* $\Phi^l : \mathfrak{X}_{l,\eta} \times [0,l] \to \mathfrak{X}_{l,\eta}$ *of* $\mathfrak{X}_{l,\eta}$ *onto the skeleton* $S(\underline{\mathfrak{X}})$ *of* $\underline{\mathfrak{X}}$ *such that:*

- *(i)* $S(\underline{\mathfrak{X}}) = \bigcup_{x \in S(\underline{\mathfrak{X}}_{l-1})} S(\mathfrak{X}_{l,x})$ *(set-theoretic disjoint union), where* $\underline{\mathfrak{X}}_{l-1} = (\mathfrak{X}_{l-1} \to \cdots \to \mathrm{Spf}(O_K))$*;*
- *(ii) if* $\phi : \mathfrak{Y} \to \mathfrak{X}$ *is a morphism of fibrations in* $\mathcal{P}stf_l^{\text{ét}}$*, one has* $\phi_{l,\eta}(y_t) = \phi_{l,\eta}(y)_t$ *for every* $y \in \mathfrak{Y}_{l,\eta}$*.*

Let us describe more precisely how the retraction is defined.
If $\mathfrak{X} = \mathrm{Spf}\, O_K\{P\}/(p_i - z_i)$ where P is isomorphic to $\oplus_{0 \leqslant i \leqslant p} \mathbf{N}^{n_i+1}$, $p_i = (1, \cdots, 1) \in \mathbf{N}^{n_i+1}$ and $z_i \in O_K$, let $\mathfrak{G}_{\mathrm{m}}$ be the formal multiplicative group $\mathrm{Spf}\, O_K\{T, \frac{1}{T}\}$ over O_K, let us denote for any n by ${\mathfrak{G}_{\mathrm{m}}}^{(n)}$ the kernel of the multiplication ${\mathfrak{G}_{\mathrm{m}}}^{n+1} \to \mathfrak{G}_{\mathrm{m}}$ and let $\mathfrak{G}$ be the formal completion at the identity of $\prod_i {\mathfrak{G}_{\mathrm{m}}}^{(n_i)}$ (it is a formal group). Then $\mathfrak{G}$ acts on $\mathfrak{X}$. The group $G = \mathfrak{G}_\eta$ acts then on $\mathfrak{X}_\eta$. G has canonical subgroups G_t for $t \in [0,1]$ defined by the inequalities $|T_{ij} - 1| \leqslant t$ where T_{ij} are the coordinates in G. G_t has a maximal point g_t. Similarly, for any complete extension K'/K, $G_t \otimes_K K'$ has a maximal point $g_{t,K'}$. If $x \in X$, one defines $x_t := g_t * x$ to be the image of $g_{t,\mathcal{H}(x)}$ by the map $G_t \otimes_K K' = (G_t \times X)_x \subset G_t \times X \to X$.
If $\mathfrak{X}$ is étale over $\mathrm{Spf}\, O_K\{P\}/(p_i - z_i)$, the action of $\mathfrak{G}$ extends in a unique way to an action on X, and x_t is still defined by $g_t * x$. For any $\mathfrak{X}$ polystable over O_K, one has thus defined the strong deformation locally for the quasi-étale topology of $\mathfrak{X}_\eta^{\mathrm{an}}$, and Berkovich verifies that it indeed descends to a strong deformation on $\mathfrak{X}$.
For a polystable fibration $\mathfrak{X} \to \mathfrak{X}_{l-1} \to \cdots \to \mathrm{Spf}\, O_K$, we first assume that $\mathfrak{X} \to \mathfrak{X}_{l-1}$ is of the kind $\mathrm{Spf}\, B \to \mathrm{Spf}\, A$ with $B = A\{P\}/(p_i - a_i)$ (this will be called a *standard* polystable morphism), one first retracts fiber by fiber on $S(\mathfrak{X}/\mathfrak{X}_{l-1})$, which are strictly polystable. The image obtained can be identified with $S = \{(x, \mathbf{r}_0, \cdots, \mathbf{r}_p) \in \mathfrak{X}_{l-1,\eta}, r_{i0} \cdots r_{in_i} = |a_i(x)|\}$, one then has a homotopy $\Psi : S \times [0,1] \to S$ by $\Psi(x, \mathbf{r}_0, \cdots, \mathbf{r}_p, t) = (x_t, \psi_{n_0}(\mathbf{r}_0, |a_0(x_t)|), \cdots, \psi_{n_p}(\mathbf{r}_p, |a_p(x_t)|))$, where ψ_n is some strong deformation of $[0,1]^{n+1}$ to $(1, \cdots, 1) \in [0,1]^{n+1}$ defined by Berkovich (we will just need that $\psi_n(r_i, t)_k^\lambda = \psi_n(r_i^\lambda, t^\lambda)_k$ for any $\lambda \in \mathbf{R}^{*+}$ and any $k \in [[0,n]]$) , and x_t is defined by the strong deformation of $\mathfrak{X}_{l-1,\eta}$.
If $\mathfrak{X} \to \mathfrak{X}' \to \mathfrak{X}_{l-1}$ is a geometrically elementary composition of an étale morphism and a standard polystable morphism, $S(\mathfrak{X}/\mathfrak{X}_{l-1}) \to S(\mathfrak{X}'/\mathfrak{X}_{l-1})$ is an isomorphism, so that we deform $\mathfrak{X}'$ fiber by fiber onto $S(\mathfrak{X}/\mathfrak{X}_{l-1})$, then we just do the same retraction as for $S(\mathfrak{X}'/\mathfrak{X}_{l-1})$. For an arbitrary polystable fibration $X \to \cdots \to O_K$, this defines the retraction locally for the quasi-étale topology of $\mathfrak{X}_\eta$, and Berkovich verifies that it descends to a deformation retraction on X.

Berkovich deduces from (1.5.(ii)) the following corollary:

COROLLARY 1.6 ([3, cor. 8.5]). *Let* K' *be a finite Galois extension of* K *and let* $\underline{\mathfrak{X}}$ *be a polystable fibration over* $O_{K'}$ *with a normal generic fiber* $\mathfrak{X}_{l,\eta}$*. Suppose we are given an action of a finite group* G *on* $\underline{\mathfrak{X}}$ *over* O_K *and a Zariski open dense subset* U *of* $\mathfrak{X}_{l,\eta}$ *which is stable under the action of* G*. Then there is a strong deformation retraction of the Berkovich space* $G\backslash U$ *to a closed subset homeomorphic to* $G\backslash|\mathrm{C}(\underline{\mathfrak{X}})|$*.*

More precisely, in this corollary, the closed subset in question is the image of $S(\underline{\mathfrak{X}})$ (which is G-equivariant and contained in U) by $U \to G\backslash U$.

Theorem 1.5 also implies that the skeleton is functorial with respect to pluristable morphisms:

PROPOSITION 1.7 [4, prop. 4.3.2.(i)]. *If $\phi : \mathfrak{X} \to \mathfrak{Y}$ is a pluristable morphism between nondegenerate pluristable formal schemes over O_K, $\phi_\eta(S(\mathfrak{X})) \subset S(\mathfrak{Y})$.*

In fact, more precisely, from the construction of S, $S(\mathfrak{X}) = \bigcup_{y \in S(\mathfrak{Y})} S(\mathfrak{X}_y)$.

2. Tempered fundamental group of a polystable log scheme

In this section we define a tempered fundamental group for a polystable fibration over a field, endowed with some compatible log structure (we will call this a polystable log fibration). To define our tempered fundamental group, we will need a notion of "topological covering" of a két covering Z of our polystable log fibration $X \to \cdots \to k$. To do this we will define for any Z a polysimplicial set $\mathrm{C}(Z)$ over the polysimplicial set $\mathrm{C}(X)$, functorially in Z. Thus if Z is a finite Galois covering of X with Galois group G, there is an action of G on $\mathrm{C}(Z)$ which defines an extension of groups:

$$1 \to \pi_1^{\mathrm{top}}(|\,\mathrm{C}(Z)|) \to \Pi_Z \to G \to 1.$$

Our tempered fundamental group will be the projective limits of Π_Z when Z runs through pointed Galois coverings of X.

2.1 Polystable log schemes

All monoids are assumed to be commutative. We will use multiplicative notations. If X is an fs log scheme, we will denote by $\mathring{X}$ the underlying scheme, by M_X the étale sheaf of monoids on $\mathring{X}$ defining the log structure, and by X_{tr} the open subset of X where the log structure is trivial.
A strict étale morphism of fs log scheme $Y \to X$ is a strict morphism of log schemes such that $\mathring{Y} \to \mathring{X}$ is étale. If we talk about étale topology on X, it will mean strict étale topology on X (or equivalently étale topology on $\mathring{X}$), and not log étale topology.

Let S be a fs log scheme.

DEFINITION 2.1. A morphism $\phi : Y \to X$ of fs log schemes will be said:

- *standard nodal* if X has an fs chart $X \to \operatorname{Spec} P$ and Y is isomorphic to $X \times_{\operatorname{Spec} \mathbf{Z}[P]} \mathbf{Z}[Q]$ with $Q = (P \oplus u\mathbf{N} \oplus v\mathbf{N})/(u \cdot v = a)$ with $a \in P$.
- a *strictly plurinodal morphism of log schemes* if for every point $y \in Y$, there exists a Zariski open neighborhood X' of $\phi(y)$ and a Zariski open neighborhood Y' of y in $Y \times_X X'$ such that $Y' \to X'$ is a composition of strict étale morphisms and standard nodal morphisms.
- a *plurinodal morphism of log schemes* if, locally for the étale topology of X and Y, it is strictly plurinodal.
- a *strictly polystable morphism of log schemes* if for every point $y \in Y$, there exists aa affine Zariski open neighborhood $X' = \operatorname{Spec} A$ of $\phi(y)$, an fs chart $P \to A$ of the log structure of X' and a Zariski open neighborhood Y' of y in $Y \times_X X'$ such that $Y' \to X$ factors through a strict étale morphism $Y' \to X' \times_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ where $Q = (P \oplus \bigoplus_{i=0}^{p} < T_{i0}, \cdots, T_{in_i} >)/(T_{i0} \cdots \cdots T_{in_i} = a_i)$ with $a_i \in P$.
- a *polystable morphism of log schemes* if, locally for the étale topology of Y and X, it is a strict polystable morphism of log schemes.

A *polystable log fibration* (resp. *strictly polystable log fibration*) $\underline{X}$ over S of length l is a sequence of polystable (resp. strictly polystable) morphism of log schemes $X_l \to \cdots \to X_1 \to X_0 = S$.
A morphism of polystable log fibrations of length l $\underline{f} : \underline{Y} \to \underline{X}$ is given by morphisms $f_i : Y_i \to X_i$

of fs log schemes for every i such that the obvious diagram commutes.
A morphism $\underline{f}$ of polystable fibrations will be said két (resp. strict étale) if f_i is két (resp. strict étale) for all i.

A polystable (resp. strictly polystable) morphism of log schemes is plurinodal (resp. strictly plurinodal).
A plurinodal morphism is log smooth and saturated.

Remark. If $\phi : X \to Y$ is a stricly polystable morphism of log schemes, then for any $y \in Y$, for any Zariski open neighborhood X' of $\phi(y)$ and *any* chart $X' \to \operatorname{Spec} P$, there is a Zariski open neighborhood $X'' \subset X'$ of $\phi(y)$ and a Zariski open neighborhood Y' of y in $Y \times_X X''$ such that $Y' \to X$ factors through a strict étale morphism $Y' \to X' \times_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ where $Q = (P \oplus \bigoplus_{i=0}^{p} < T_{i0}, \cdots, T_{in_i} >)/(T_{i0} \cdot \cdots \cdot T_{in_i} = a_i)$ with $a_i \in P$.

LEMMA 2.2. *Let $\phi : Y \to X$ be a plurinodal (resp. strictly plurinodal, resp. polystable, resp. strictly pluristable) morphism of schemes, such that X has a log regular log structure M_X and ϕ is smooth over X_{tr}. Then $(Y, O_Y \cap j_* O^*_{Y_{X_{tr}}}) \to (X, M_X)$ is a plurinodal (resp. strictly plurinodal, resp. polystable, resp. strictly pluristable) morphism of log schemes.*

Proof. Let us prove it for the case of a stricly polystable morphism.
One can assume that $X = \operatorname{Spec}(A)$ has a chart $\psi : P \to A$ and that $Y = B_0 \times_X \cdots \times_X B_p$ with $B_i = \operatorname{Spec} A[T_{i0}, \cdots, T_{in_i}]/T_{i0} \cdots T_{in_i} - a_i$ with $a_i \in A$. Since ϕ is smooth over X_{tr}, a_i is invertible over X_{tr}, thus after multiplying a_i by an element of A^* (we can do that by also multiplying T_{i0} by this element), we may assume that $a_i = \psi(b_i)$ for some $b_i \in P$. Thus $Y = X \times_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ where $Q = (P \oplus \bigoplus_{i=0}^{p} < T_{i0}, \cdots, T_{in_i} >)/(T_{i0} \cdot \cdots \cdot T_{in_i} = b_i)$ with $b_i \in P$. If we endow Y with the log structure M_Y associated with Q, $Y \to X$ becomes a strict polystable morphism of log schemes. In particular Y is log regular ([7, th. 8.2]). Since, the set of points of Y where M_Y is trivial is $Y_{X_{\mathrm{tr}}}$, $M_Y = O_Y \cap j_* O^*_{Y_{X_{\mathrm{tr}}}}$ according to [11, prop. 2.6]. □

2.2 Strata of log schemes

For a polystable (log) fibration $\underline{X} : X \to \cdots \to \operatorname{Spec} k$, Berkovich defines a polysimplicial set $\mathrm{C}(\underline{X})$. In this part we want to generalize this construction to any két log scheme Z over X. To do this we will study the stratification of an fs log scheme defined by $\mathrm{rk}(z) = \mathrm{rk}(\overline{M}_z^{\mathrm{gp}})$, which corresponds to Berkovich stratification for plurinodal schemes, and we will show that étale locally a két morphism $X \to Y$ induces an isomorphism between the posets of the strata of X and Y. This will enables us to define the polysimplicial set of Z étale locally. We will then descend it so that it satisfies the same descent property as in proposition 1.4.

Let Z be an fs log scheme, one gets a stratification on Z by saying that a point z of Z is of rank r if $\mathrm{rk}^{\log}(z) = \mathrm{rk}(M_{\bar{z}}^{\mathrm{gp}}/\mathcal{O}_{\bar{z}}^*) = r$ (where $\bar{z}$ is some geometric point over z and where rk is the rank of an abelian group of finite type).
The subset of points of Z such that the rank is $\leqslant r$ is an open subset of Z ([12, cor. II.2.3.5]). We thus get a good stratification.
The strata of rank r of Z are then the connected components of the subset of points z of rank r. This is a partition of Z, and a strata of rank r is open in the closed subset of points x of rank $\geqslant r$. It is endowed with the reduced subscheme structure of Z.
The set of strata is partially ordered by $x \leqslant y$ if and only if $y \subset \bar{x}$. One denotes by $\mathrm{Str}_x(Z)$ the poset of strata below x. More generally, if z is a point of Z, we denote by $\mathrm{Str}_z(Z)$ the set of strata y of Z such that $z \in \bar{y}$ ($\mathrm{Str}_z(Z)$ is simply $\mathrm{Str}_x(Z)$ where x is the stratum of z containing x). If $\bar{z}$ is

a geometric point of Z, let $\mathrm{Str}^{\mathrm{geom}}_{\bar z}(Z) = \varprojlim_{(U,\bar u)} \mathrm{Str}_{\bar u}(U)$ where $(U,\bar u)$ goes through étale neighborhoods of $\bar z$; it can be identified with $\mathrm{Str}(Z(\bar z))$ where $Z(\bar z)$ is the strict localization of Z at $\bar z$.
If $f : Z' \to Z$ is a két morphism, then $\mathrm{rk}^{\log}(x) = \mathrm{rk}^{\log}(f(x))$, so the strata of Z' are the connected components of the preimages of the strata of Z.

If $f : P \to O_Z$ is a chart of Z, it induces a continuous map $f^* : Z \to \mathrm{Spec}\, P$ that maps a point z to the prime $\mathfrak{p}_z = P\backslash f^{-1}(O*_{Z,z})$ of P. Let $F_z = P\backslash\mathfrak{p}_z$ be the corresponding face. Then $\overline{M}_{Z,z} = P/F$. One deduces from it that the strata of Z are exactly the connected components of the preimages by f^* of points in $\mathrm{Spec}\, P$. In particular one gets a map $\mathrm{Str}(Z) \to \mathrm{Spec}\, P$. If z is a point of Z, the map $Z(z) \to \mathrm{Spec}\, P$ factorizes through a map $Z(z) \to \mathrm{Spec}\, M_{Z,z}$, which does not depend on the choice of the chart. One gets a map $\mathrm{Str}_z(Z) \to \mathrm{Spec}\, M_{Z,z}$. For a general log scheme Z, if $\bar z$ is a geometric point of Z, one gets a map $\mathrm{Str}^{\mathrm{geom}}_{\bar z}(Z) \to \mathrm{Spec}\, M_{Z,\bar z}$.
Let us look at the structure of the strata of $\mathrm{Spec}\, k[P]$ endowed with the log structure for which $f : P \to k[P]$ is a chart. Let $f^* : \mathrm{Spec}\, k[P] \to \mathrm{Spec}\, P$ and let $\mathfrak{p}$ be a prime of P and let $F = P\backslash\mathfrak{p}$ be thecorresponding face of P. Then $f^{*,-1}(\overline{\{\mathfrak{p}\}})$ is a closed subset of $\mathrm{Spec}\, k[P]$ which, endowed with its structure of reduced closed subscheme, is $\mathrm{Spec}\, k[P]/(\mathfrak{p})$ where $(\mathfrak{p}) = \bigoplus_{p_i\in\mathfrak{p}} k.p_i \subset k[P]$ ($(\mathfrak{p})$ is a prime ideal of $k[P]$). Moreover, the obvious morphism of rings $k[F] \to k[P]/(\mathfrak{p})$ is an isomorphism, inducing thus an isomorphism of schemes $f^{*,-1}(\overline{\{\mathfrak{p}\}}) = \mathrm{Spec}\, k[P]/(\mathfrak{p}) \simeq \mathrm{Spec}\, k[F]$. However the log structure on $\mathrm{Spec}\, k[F]$ for which F is a chart is not correspond in general with the log structure on $\mathrm{Spec}\, k[P]/(\mathfrak{p})$ for which P is a chart. The open immersion $f^{*,-1}(\{\mathfrak{p}\}) \subset f^{*,-1}(\overline{\{\mathfrak{p}\}})$ corresponds then to the open immersion $\mathrm{Spec}\, k[F^{\mathrm{gp}}] \to \mathrm{Spec}\, k[F]$. In particular, since $\mathrm{Spec}\, k[F^{\mathrm{gp}}]$ is connected, there is a unique stratum of $\mathrm{Spec}\, k[P]$ above $\mathfrak{p}$ and thus $\mathrm{Str}(\mathrm{Spec}\, k[P]) \to \mathrm{Spec}\, P$ is bijective.

Let Z be a plurinodal log scheme over some log point (k, M_k) of characteristic p and of rank r_0 and let z be a point of Z. One has $\mathrm{rk}^{\log}(z) = r_0 + \mathrm{rk}(z)$ where $\mathrm{rk}(z)$ is the codimension of the strata containing z in Z for the Berkovich stratification of plurinodal schemes. Thus the strata are the same for this stratification and the stratification of Berkovich. The strata of Z are normal.
We will often denote abusively in the same way a stratum and its generic point.

Recall that Z is said to be quasinormal if the closure of any stratum endowed with its reduced scheme structure is normal.

Lemma 2.3. *Let $f : Z \to S = \mathrm{Spec}\, k$ be a log smooth morphism. Let $\bar z$ be a geometric point. Let $f_* : \mathrm{Spec}\, M_{Z,\bar z} \to \mathrm{Spec}\, M_{S,\bar s}$. Then $\phi_{Z,\bar z} : \mathrm{Str}^{\mathrm{geom}}_{\bar z}(Z) \to \mathrm{Spec}\, M_{Z,\bar z}$ is injective and its image is $f_*^{-1}(M^*_{S,\bar s})$. Moreover $Z(\bar z)$ is quasinormal. In particular, every stratum of Z is normal.*

Proof. Since the unique stratum of S is mapped to $M^*_{S,\bar s}$ by the map $\mathrm{Str}^{\mathrm{geom}}_{\bar s}(S) \to \mathrm{Spec}\, M_{S,\bar s}$, one has $\mathrm{Im}\, \phi_{Z,\bar z} \subset f_*^{-1}(M^*_{S,\bar s})$.
The lemma can be proven étale locally: one can assume that S has a chart $S \to \mathrm{Spec}\, k[P]$ where P is sharp, and that $Z = S \times_{\mathrm{Spec}\, k[P]} \mathrm{Spec}\, k[Q]$ where $\psi : P \to Q$ is injective and the torsion part of $\mathrm{Coker}\, \psi^{\mathrm{gp}}$ are finite. Let $\mathfrak{q}' \in f_*^{-1}(M^*_{S,\bar s})$ and let $\mathfrak{q}$ be its image in $\mathrm{Spec}\, Q$. The image of $\mathfrak{q}$ in $\mathrm{Spec}\, P$ is the image $\mathfrak{p}$ of $M^*_{S,\bar s}$. Let $F = Q\backslash\mathfrak{q}$ and $F_0 = P\backslash\mathfrak{p}$. The morphism $S \to \mathrm{Spec}\, k[P]$ factors through $\mathrm{Spec}\, k[F_0^{\mathrm{gp}}]$. Let $\phi : Z \to \mathrm{Spec}\, Q$ and let Z_F be the closed subset $\psi^{-1}(\overline{\{\mathfrak{q}\}})$ of Z ($\bar z$ lies in Z_F). Then Z_F is the support of the closed subscheme $Z \times_{\mathrm{Spec}\, k[Q]} \mathrm{Spec}\, k[Q]/(\mathfrak{q})$, which we also denote by Z_F. Then,

$$Z_F = Z\times_{\mathrm{Spec}\, k[Q]}\mathrm{Spec}\, k[F] = S\times_{\mathrm{Spec}\, k[P]}\mathrm{Spec}\, k[F] = S\times_{\mathrm{Spec}\, k[F_0]}\mathrm{Spec}\, k[F] = S\times_{\mathrm{Spec}\, k[F_0^{\mathrm{gp}}]}\mathrm{Spec}\, k[FF_0^{\mathrm{gp}}].$$

Let T_0 be the saturation of F_0^{gp} in F^{gp} and let T_1 be a subgroup of F^{gp} such that $F^{\mathrm{gp}} = T_0 \oplus T_1$. The morphism $S \times_{\mathrm{Spec}\, k[F_0^{\mathrm{gp}}]} \mathrm{Spec}\, k[T_0] \to S$ is étale, so up to replacing k by a finite extension, one can assume $F_0^{\mathrm{gp}} = T_0$. Then $Z_F = S \times_{\operatorname{Spec} k[T_0]} \operatorname{Spec} k[FF_0^{\mathrm{gp}}] = \operatorname{Spec} k[FF_0^{\mathrm{gp}} \cap T_1]$. But $FF_0^{\mathrm{gp}} \cap T_1$ is a saturated monoid, hence Z_F is normal. Thus $Z_F(\bar{z})$ is irreducible. Moreover, if $F' \subsetneq F$, then $Z_{F'} \subsetneq Z_F$: the generic point of Z_F lies above $\mathfrak{q}$. One thus obtains that there is a unique stratum of $Z(\bar{z})$ lying above $\mathfrak{q}$. □

LEMMA 2.4. *Let Z be a Zariski log scheme, let $Z \to \operatorname{Spec} k$ be a log smooth morphism and let $Z' \to Z$ be a két morphism, then $\operatorname{Str}_{z'}(Z') \to \operatorname{Str}_z(Z)$ is an isomorphism of posets.*

Proof. There is a commutative diagram:

$$\begin{array}{ccccc} \operatorname{Str}^{\mathrm{geom}}_{\bar{z}'}(Z') & \longrightarrow & \operatorname{Str}^{\mathrm{geom}}_{\bar{z}}(Z) & \hookrightarrow & \operatorname{Spec} M_{Z,\bar{z}} \\ \downarrow & & \downarrow & & \| \\ \operatorname{Str}_{z'}(Z') & \longrightarrow & \operatorname{Str}_z(Z) & \longrightarrow & \operatorname{Spec} M_{Z,z} \end{array}$$

Since $\operatorname{Str}^{\mathrm{geom}}_{\bar{z}}(Z) \to \operatorname{Spec} M_{Z,z}$ is injective, $\operatorname{Str}^{\mathrm{geom}}_{\bar{z}}(Z) \to \operatorname{Str}_z(Z)$ must be bijective. The morphism $\operatorname{Str}^{\mathrm{geom}}_{\bar{z}'}(Z') \to \operatorname{Str}^{\mathrm{geom}}_{\bar{z}}(Z)$ is bijective thanks to lemma 2.3 because $\operatorname{Spec} \overline{M}_{Z',z'} \to \operatorname{Spec} \overline{M}_{Z,z}$ is bijective since $\overline{M}_{Z,z} \to \overline{M}_{Z',z'}$ is Kummer. Hence $\operatorname{Str}_{z'}(Z') \to \operatorname{Str}_z(Z)$ must also be bijective.
If z'_1 and z'_2 are elements of $\operatorname{Str}_{z'}(Z')$, then $\operatorname{Str}_{z'_1}(Z'_1) \to \operatorname{Str}_{z_1}(Z_1)$ is also bijective, so that $z'_2 \in \operatorname{Str}_{z'_1}(Z'_1)$ if and only if $z_2 \in \operatorname{Str}_{z_1}(Z_1)$, *i.e.* $z'_2 \leqslant z'_1$ if and only if $z_2 \leqslant z_1$. □

In particular, one can apply lemma 2.4 if Z is strictly plurinodal.

2.3 Polysimplicial set of a két log scheme over a polystable log scheme

Let $\mathrm{C} \to \mathrm{C}'$ be a morphism of polysimplicial sets. Let $\alpha : S \to O(\mathrm{C})$ (resp. $\alpha' : S' \to O(\mathrm{C}')$) be a morphism of posets such that $S_{\leqslant x} \overset{\simeq}{\to} O(\mathrm{C})_{\leqslant \alpha(x)}$ (resp. $S'_{\leqslant x} \overset{\simeq}{\to} O(\mathrm{C}')_{\leqslant \alpha'(x)}$ for any x). Then α defines a functor $O(\mathrm{C})^{\mathrm{op}} \to \mathrm{Set}$ by sending c to $\alpha^{-1}(c)$ and if $c \leqslant c'$, then the map $\alpha^{-1}(c') \to \alpha^{-1}(c)$ sends $x' \in \alpha^{-1}(c')$ to the unique preimage of c by the map $S_{\leqslant x'} \to O(\mathrm{C})_{\leqslant c'}$. One gets a functor $F : (\mathbf{\Lambda}/\,\mathrm{C})^{\mathrm{op}} \to O(\mathrm{C})^{\mathrm{op}} \to \mathrm{Set}$ (resp. $F' : (\mathbf{\Lambda}/\,\mathrm{C}')^{\mathrm{op}} \to O(\mathrm{C}')^{\mathrm{op}} \to \mathrm{Set}$), which defines a polysimplicial set $D = \mathrm{C} \times F$ (resp. $D' = \mathrm{C}' \times F'$):

$$D = \operatorname{Coker}(\coprod_{x \to y} \coprod_{F(x)} [\mathbf{n}_y] \rightrightarrows \coprod_{x} \coprod_{F(x)} [\mathbf{n}_x]).$$

If we consider F as a functor $(\mathbf{\Lambda}/\,\mathrm{C})^{\mathrm{op}} \to \mathbf{\Lambda}^{\circ}\,\mathrm{Set}$, then D is nothing else than $\mathrm{C} \,\square\, F$ (but this is a very simple case of $\square$-product where all the fibers are discrete). To give a slightly more explicit description of D, $D_{\mathbf{n}} = \coprod_{x \in \mathrm{C}_{\mathbf{n}}} F(x)$ and if $f : \mathbf{m} \to \mathbf{n}$ is a morphism of $\mathbf{\Lambda}$ and $z \in F(x)$ with $x \in \mathrm{C}_{\mathbf{n}}$, $f^*(z) = F(\bar{f}) \in F(f^*(x))$ where $\bar{f}$ is the morphism $f^*(x) \to x$ in $\mathbf{\Lambda}/\,\mathrm{C}$. Since F maps surjective morphisms to isomorphisms, a polysimplex $z \in F(x)$ of D is nondegenerate if and only if x is nondegenerate. One gets that $O(D) = S$ and that D is interiorly free if C is.
Then any morphism of posets $f : S \to S'$ such that

$$\begin{array}{ccc} S & \to & S' \\ \downarrow & & \downarrow \\ O(\mathrm{C}) & \to & O(\mathrm{C}') \end{array}$$

is commutative induces a unique morphism of polysimplicial sets $\underline{f} : D \to D'$ over $\mathrm{C} \to \mathrm{C}'$ such that $O(\underline{f}) = f$.

Let us consider now a strictly polystable log fibration $\underline{X} : X \to X_{l-1} \to \cdots \to s$ where s is an fs log point. If $f : Z \to X$ is két, the map of posets $\operatorname{Str}(f) : \operatorname{Str}(Z) \to \operatorname{Str}(X) = O(\mathrm{C}(\underline{X}))$ is such that $\operatorname{Str}(Z)_{\leqslant z} \simeq \operatorname{Str}(X)_{\leqslant f(z)}$ for any $z \in \operatorname{Str}(Z)$ according to lemma 2.4. Thus one gets a functor $D_Z = (\mathbf{\Lambda}/\,\mathrm{C}(\underline{X}))^{\mathrm{op}} \to \mathrm{Set}$ and a polysimplicial set $\mathrm{C}_{\underline{X}}(Z) = \mathrm{C}(\underline{X}) \,\square\, D_Z$ (we will often write

$\mathrm{C}(Z)$ instead of $\mathrm{C}_{\underline{X}}(Z)$). This polysimplicial set is still interiorly free and $O(\mathrm{C}(Z))$ is functorially isomorphic to $\mathrm{Str}(Z)$.

Lemma 2.5. *If $\underline{X} \to \underline{X}'$ is a két morphism of strictly polystable log fibrations, then there is a canonical isomorphism $\mathrm{C}_{\underline{X}'}(X_l) \simeq \mathrm{C}(\underline{X})$ such that $\mathrm{Str}(X_l) = O(\mathrm{C}_{\underline{X}'}(X_l)) \to \mathrm{Str}(X_l) = O(\mathrm{C}(\underline{X}))$ is the identity of $\mathrm{Str}(X_l)$.*

Proof. Assume we already construct the isomorphism $\mathrm{C}_{\underline{X}'_{l-1}}(X_{l-1}) \simeq \mathrm{C}(\underline{X}_{l-1})$. Then, $\mathrm{C}_{\underline{X}'}(X_l) = D_1 \,\square\, \mathrm{C}(X'_{l-1})$ and $\mathrm{C}(\underline{X}) = D_2 \,\square\, \mathrm{C}(X'_{l-1})$ where if x is the generic point of a stratum of X'_{l-1}, $D_1(x) = \mathrm{C}_{X'_{l,x}}(X_{l,x})$ and $D_2(x) = \mathrm{C}(X_{l,x})$. By induction on l, the problem is thus reduced to the case where $l = 1$ and $X \to X'$ is a két morphism of strictly polystable objects over $\mathrm{Spec}\, k$.
We have $\mathrm{C}_{X'}(X) = D_X \times \mathrm{C}(X')$ where D_X maps $x' \in \mathrm{Str}(X')$ to the set of strata of X above x'. Then $\mathrm{C}_{X'}(X)$ is associated to the strictly polysimplicial set $C' = D_X \times C(X')$. Then

$$C'_{\mathbf{n}} = \{(x, x', \mu), x \in \mathrm{Str}(X), x' = f(x), \mu : \mathbf{n} \simeq \mathrm{Irr}(X', x')\} = \{(x, \mu), x \in \mathrm{Str}(X), \mu : \mathbf{n} \simeq \mathrm{Irr}(X, x)\}$$

because $\mathrm{Irr}(X, x) \to \mathrm{Irr}(X', x')$ is an isomorphism. Thus $C'_{\mathbf{n}} \simeq C_{\mathbf{n}}$ (and the bijection is compatible with maps of Λ), which gives the wanted isomorphism. □

Let us consider a commutative diagram

$$\begin{array}{ccc} Z & \to & Z' \\ \downarrow & & \downarrow \\ \underline{X} & \to & \underline{X}' \end{array}$$

where $\underline{X} \to \underline{X}'$ is a két morphism of strictly polystable log fibration. Then

$$\mathrm{C}_{\underline{X}}(Z) = D_{Z/X} \times \mathrm{C}(\underline{X}) \simeq D_{Z/X} \times \mathrm{C}_{\underline{X}'}(X) = D_{Z/X} \times (D_{X/X'} \times \mathrm{C}(\underline{X}')) = D_{Z/X'} \times \mathrm{C}(\underline{X}') = \mathrm{C}_{\underline{X}'}(Z)$$

where $D_{Z/X}(x) = \mathrm{Str}(Z \to X)^{-1}(x)$, $D_{X/X'}(x') = \mathrm{Str}(X \to X')^{-1}(x')$ and $D_{Z/X'}(x') = \mathrm{Str}(Z \to X')^{-1}(x')$. There is a morphism of functors $D_{Z/X'} \to D_{Z'/X'}$ which induces a morphism of polysimplicial sets

$$\mathrm{C}_{\underline{X}}(Z) = D_{Z/X'} \times \mathrm{C}(\underline{X}') \to D_{Z'/X'} \times \mathrm{C}(\underline{X}') = \mathrm{C}_{\underline{X}'}(Z').$$

This morphism is an isomorphism if and only if $\mathrm{Str}(Z) \to \mathrm{Str}(Z')$ is bijective.

Let $Z' \to Z$ be a két covering, let $Z'' = Z' \times_Z Z'$ and let x be a stratum of X_s, then $D_Z(x) = \mathrm{Coker}(D_{Z''}(x) \rightrightarrows D_{Z'}(x))$. We deduce from it that

$$\mathrm{C}(Z'') = \mathrm{Coker}(\mathrm{C}(Z') \rightrightarrows \mathrm{C}(Z)).$$

One may also define $\mathrm{C}_{\underline{X}}(Z)$ for $\underline{X}$ a general polystable fibration. Let $\underline{X}' \to \underline{X}$ be an étale covering where $\underline{X}'$ is strictly polystable, let $\underline{X}'' = \underline{X}' \times_{\underline{X}} \underline{X}'$ and let Z' and Z'' the pullbacks of Z to X' and X''. then one defines $\mathrm{C}_{\underline{X}}(Z) = \mathrm{Coker}(\mathrm{C}_{\underline{X}''}(Z'') \rightrightarrows \mathrm{C}_{\underline{X}'}(Z'))$ (it does not depend of the choice of $\underline{X}'$).

If $Z' \to Z$ is a surjective két morphism over $\underline{X}$ and $Z'' = Z' \times_Z Z'$, $\mathrm{Str}(Z) = \mathrm{Coker}(\mathrm{Str}(Z'') \rightrightarrows \mathrm{Str}(Z'))$.
One thus gets (két(X) denotes the category of két log schemes over X):

Proposition 2.6. *Let $\underline{X}$ be a polystable log fibration, one has a functor $\mathrm{C}_X : \text{két}(X) \to (\Lambda)^\circ\,\mathrm{Set}$ such that:*

– *if $Z' \to Z$ is a két covering of* két(X),

$$\mathrm{C}(Z) = \mathrm{Coker}(\mathrm{C}(Z' \times_Z Z') \rightrightarrows \mathrm{C}(Z')).$$

– $O(\mathrm{C}(Z))$ *is functorially isomorphic to* $\mathrm{Str}(Z)$.

Remark. If one has a két morphism $\underline{Y} \to \underline{X}$ of polystable fibrations of length l, the polysimplicial complex $\mathrm{C}(Y_l)$ we have just define by considering Y_l as két over X_l is canonically isomorphic to the polysimplicial complex of the polystable fibration $\mathrm{C}(\underline{Y})$ defined by Berkovich.

We say that a fs log scheme Z over a log point s is log geometrically irreducible if the underlying scheme of $Z \times_s s'$ is irreducible for any morphism $s' \to s$ of log points. If $\mathring{Z}/\mathring{s}$ is geometrically irreducible and $Z \to s$ is saturated, then Z/s is log geometrically irreducible since the underlying scheme of $Z \times_s s'$ is $\mathring{Z} \times_{\mathring{s}} \mathring{s}'$.
If Z is quasicompact, then there is a connected két covering $s' \to s$ such that all the strata of $Z_{s'}$ are geometrically irreducible and $Z_{s'} \to s'$ is saturated. Then all the strata of $Z_{s'}$ are log geometrically irreducible. In particular, for any morphism of fs log points $s'' \to s'$, $\mathrm{C}(Z_{s''}) \to \mathrm{C}(Z_{s'})$ is an isomorphism. The polysimplicial complex $\mathrm{C}(Z_{s'})$ for such an s' is denoted by $\mathrm{C}_{\mathrm{geom}}(Z/s)$.

Let $\bar{z}$ be a geometric point of Z. Let U be an étale neighborhood of $\bar{z}$ such that $\mathrm{Str}^{\mathrm{geom}}_{\bar{z}}(Z) \to \mathrm{Str}(U)$ is an isomorphism. One defines $\mathrm{C}(Z)_{\bar{z}} := \mathrm{C}(U)$ (it does not depend on the choice of U. If $Z \to X$ is két, $\mathrm{C}(Z)_{\bar{z}} \to \mathrm{C}(X)_{\bar{x}}$ is an isomorphism of polysimplicial sets.

LEMMA 2.7. *The space* $|\,\mathrm{C}(Z)_{\bar{z}}|$ *is contractible.*

Proof. Let $\Phi_{\mathbf{n}} : |[\mathbf{n}]| \times [0,1] \to |[\mathbf{n}]|$ be defined by $\Phi_{\mathbf{n}}((u_{il}),t)_{il} = (1-t)u_{il} + \frac{t}{n_i}$. This is a deformation retraction to a point. These deformation retractions are compatible with surjective maps $\mathbf{m} \to \mathbf{n}$.

One can assume that $X \xrightarrow{\psi} X_{l-1} \to \cdots \to s$ is a strictly polystable fibration of length l and that $Z = X$. Let $\bar{x}'$ be the image of $\bar{x} := \bar{z}$ in X_{l-1}. One can also assume that $\mathrm{Str}^{\mathrm{geom}}_{\bar{x}}(X) \to \mathrm{Str}(X)$ and $\mathrm{Str}^{\mathrm{geom}}_{\bar{x}'}(X_{l-1}) \to \mathrm{Str}(X_{l-1})$ are bijections. By induction on l, one can assume that $|\,\mathrm{C}(X_{l-1})|$ is contractible.
If y' is a stratum of X_{l-1}, $X_{y'}$ has a biggest stratum y and $\mathrm{C}(X_{y'}) \simeq [\mathbf{n}_y]$. Then

$$|\,\mathrm{C}(X)| = \mathrm{Coker}(\coprod_{\substack{f:y_1 \to y_2 \in \\ \mathbf{\Lambda}/\,\mathrm{C}(X_{l-1})}} |[\mathbf{n}_{y'_1}]| \times |[\mathbf{n}_{y_2}]| \overset{a,b}{\rightrightarrows} \coprod_{y' \in \mathbf{\Lambda}/\,\mathrm{C}} |[\mathbf{n}_{y'}]| \times |[\mathbf{n}_y]|),$$

where a maps $|[\mathbf{n}_{y'_1}]| \times |[\mathbf{n}_{y_2}]|$ to $|[\mathbf{n}_{y'_1}]| \times |[\mathbf{n}_{y_1}]|$ by $\mathrm{id} \times f_0$ where f_0 is the cospecialization map $\mathrm{C}(X_{y_2}) \to \mathrm{C}(X_{y_1})$ given by lemma 1.3 and b maps $|[\mathbf{n}_{y'_1}]| \times |[\mathbf{n}_{y_2}]|$ to $|[\mathbf{n}_{y'_2}]| \times |[\mathbf{n}_{y_2}]|$ $f^* \times \mathrm{id}$.

One defines a deformation retraction Φ of $\coprod_{y' \in \mathbf{\Lambda}/\,\mathrm{C}(X_{l-1})} |[\mathbf{n}_{y'}]| \times |[\mathbf{n}_y]|$ by $\Phi(u,v,t) = (u, \Phi_{\mathbf{n}_y}(v,t))$. Moreover, if $(z_1, z_2) \in |[\mathbf{n}_{y'_1}]| \times |[\mathbf{n}_{y_2}]|$,

$$\Phi(a(z_1,z_2),t) = (z_1, \Phi_{\mathbf{n}_{y_1}}(f_0(z_2),t)) = (z_1, f_0(\Phi_{\mathbf{n}_{y_2}}(z_2,t))) = a(z_1, \Phi_{\mathbf{n}_{y_2}}(z_2,t))$$

because the map $\mathbf{n}_{y_2} \to \mathbf{n}_{y_1}$ inducing f_0 is surjective, and

$$\Phi(b(z_1,z_2),t) = (f^* z_1, \Phi_{\mathbf{n}_{y_2}}(z_2,t)) = b(z_1, \Phi_{\mathbf{n}_{y_2}}(z_2,t)).$$

Thus Φ induces a deformation retraction of $\mathrm{C}(X)$, also denoted by Φ by abuse of notation. This retraction is compatible with $\psi : |\,\mathrm{C}(X)| \to |\,\mathrm{C}(X_{l-1})|$ in the sense that $\psi(\Phi(z,t)) = \psi(z)$ for every $t \in [0,1]$.Let S be the image of this retraction. Let $u \in |\,\mathrm{C}(X_{l-1})|$ and let y' be the stratum of X_{l-1} corresponding to the cell of $|\,\mathrm{C}(X_{l-1})|$ containing u. then $\psi^{-1}(u)$ is canonically homeomorphic to $|[\mathbf{n}_y]|$ (cf. [3, cor. 6.6]), and the deformation retraction of $\psi^{-1}(u)$ induced by Φ is just $\Phi_{\mathbf{n}_y}$. Thus $S \cap \psi^{-1}(u)$ is reduced to a point: the map $S \to |\,\mathrm{C}(X_{l-1}|$ is bijective. Since $\mathrm{Str}(X)$ is finite, $|\,\mathrm{C}(X)|$ is compact and S is also compact since it is the image of $|\,\mathrm{C}(X)|$ by a continuous map. The map $S \to \mathrm{C}(X_{l-1})$ is thus an homeomorphism, and $\mathrm{C}(X_{l-1})$ is contractible by induction. Thus $\mathrm{C}(X)$ is contractible. □

2.4 Tempered fundamental group of a polystable log fibration

Here we define the tempered fundamental group of a log fibration $\underline{X}$ over an fs log point. If T is a két covering of X, the topological coverings of $|\mathrm{C}(T)|$ will play the role of the topological coverings of T.

Let us start by a categorical definition of tempered fundamental groups that we will use later in our log geometric situation.
Consider a fibered category $\mathcal{D} \to \mathcal{C}$ such that:

- $\mathcal{C}$ is a Galois category,
- for every connected object U of $\mathcal{C}$, $\mathcal{D}_U$ is a category equivalent to Π_U-Set for some discrete group Π_U,
- if U and V are two objects of $\mathcal{C}$, the functor $\mathcal{D}_{U \coprod V} \to \mathcal{D}_U \times \mathcal{D}_V$ is an equivalence,
- if $f : U \to V$ is a morphism in $\mathcal{C}$, $f^* : \mathcal{D}_V \to \mathcal{D}_U$ is exact.

Then, one can define a fibered category $\mathcal{D}' \to \mathcal{C}$ such that the fiber in U is the category of descent data of $\mathcal{D} \to \mathcal{C}$ with respect to the morphism $U \to e$ (where e is the final element of $\mathcal{C}$).
Let U be a connected Galois object of $\mathcal{C}$ and let G be the Galois group of U/e. Then $\mathcal{D}'_U$ can be described in the following way:

- its objects are couples $(S_U, (\psi_g)_{g\in G})$, where S_U is an object of $\mathcal{D}_U$ and $\psi_g : S_U \to g^* S_U$ is an isomorphism in $\mathcal{D}_U$ such that for any $g, g' \in G$, $(g^*\psi'_g) \circ \psi_g = \psi_{g'g}$ (after identifying $(g'g)^*$ and $g^* g'^*$ by the canonical isomorphism to lighten the notations).
- a morphism $(S_U, (\psi_g)) \to (S'_U, \psi'_g)$ is a morphism $\phi : S_U \to S'_U$ in $\mathcal{D}_U$ such that for any $g \in G$, $\psi'_g \phi = (g^*\phi)\psi_g$.

There is a natural functor $F_0 : \mathcal{D}'_U \to \mathcal{D}_U$, which maps $(S_U, (\psi_g))$ to S_U. Let F_U be a fundamental functor $\mathcal{D}_U \to \mathrm{Set}$, such that $\mathrm{Aut}\, F_U = \Pi_U$.
Let $F = F_U F_0$, and $\Pi'_U = \mathrm{Aut}\, F$.

Proposition 2.8. *(i) The natural functor $\mathcal{F} : \mathcal{D}'_U \to \Pi'_U$-Set is an equivalence.*
(ii) There is a natural exact sequence

$$1 \to \Pi_U \to \Pi'_U \to G \to 1.$$

Proof. First notice that $\mathcal{D}'_U$ is a boolean topos and that F is faithful and exact.
A *pointed object* of $\mathcal{D}'_U$ is by definition a pair (S, s) with S an object of S, and $s \in F(S)$. Let us show that, to prove (i), it is enough to show that there exists a pointed object (T^∞, t^∞) of $\mathcal{D}'_U$ such that for every pointed object (S, s) of $\mathcal{D}'_U$, the map $\mathrm{Hom}(T^\infty, S) \to F(S)$ that maps f to $F(f)(t^\infty)$ is bijective (*i.e.* T^∞ represents the functor F).
The group $\mathrm{Aut}(T^\infty)$ acts on $\mathrm{Hom}(T^\infty, S) = F(S)$ by action on the left compatibly for every S: one gets a morphism $a : \mathrm{Aut}(T^\infty) \to \mathrm{Aut}(F)$, which is bijective by Yoneda's lemma.
If $\underline{S}_0 \subset F(S)$ is stable by $\mathrm{Aut}\, F$, then the subobject S_0 of S defined as the unions of the images of morphisms $\phi : T^\infty \to S$ such that $F(\phi)(t) \in \underline{S}_0$ satisfies $F(S_0) = \underline{S}_0$. Thus if S, S' are objects of $\mathcal{D}'_U$,

$$\begin{aligned} \mathrm{Hom}(S, S') &= \{S_0 \hookrightarrow S \times S' | S_0 \xrightarrow{\sim} S\} \\ &= \{\underline{S}_0 \subset F(S) \times F(S') \text{ stable by the action of } \mathrm{Aut}\, F | \underline{S}_0 \xrightarrow{\sim} F(S)\} \\ &= \mathrm{Hom}_{\Pi'_U}(F(S), F(S')). \end{aligned}$$

Thus $\mathcal{F}$ is fully faithful. Let $\underline{S}$ be a Π'_U-set. There exists an epimorphism $\underline{S}' \to \underline{S}$ such that Π'_U acts freely on $\underline{S}'$ and on $\underline{S}'' := \underline{S}' \times_{\underline{S}} \underline{S}'$. Thus there exists S'' and S' such that $\mathcal{F}(S') = \underline{S}'$ and

$\mathcal{F}(S'') = \underline{S}''$ (S' and S'' are direct sums of copies of T^∞). Let $S = \mathrm{Coker}(S'' \rightrightarrows S')$, where the two morphisms are defined thanks to the full faithfulness of $\mathcal{F}$. Then $\mathcal{F}(S) = \underline{S}$. Thus $\mathcal{F}$ is an equivalence.

Let us construct T^∞. If S is an object of $\mathcal{D}_U$ let $\widetilde{S} = \coprod_{g \in G} g^* S$, et

$$\psi_h : \widetilde{S} = \coprod_{g\in G} g^*S = \coprod_{gh\in G} (gh)^*S \xrightarrow{\simeq} \coprod_{g\in G} h^*g^*S = h^*(\coprod_{g\in G} g^*S) = h^*\widetilde{S}.$$

This defines an object $\widetilde{S}$ of $\mathcal{D}'_U$. Then, for any object S' of $\mathcal{D}'_U$, there is a natural map

$$\mathrm{Hom}_{\mathcal{D}'_U}(\widetilde{S}, T) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{D}_U}(S, F_0(T))$$

that maps ψ to the restriction of $F_0(\psi)$ to the subobject S of $F_0(\widetilde{S})$.
The restriction of $F_0(\psi)$ to $g^*S \subset F_0(\widetilde{S})$ is $\psi_g^{-1} g^*\alpha(\psi)$. Hence $F_0(\psi)$ only depends on $\alpha(\psi)$, which shows the injectivity of α since F is faithful. Conversely, if $\beta \in \mathrm{Hom}_{\mathcal{D}_U}(S, F_0(T))$, one defines $\beta_0 : F_0(\widetilde{S}) = \coprod_g g^*S \to F_0(T)$ by gluing the composite morphisms $g^*S \xrightarrow{g^*\beta} g^*F_0(T) \xrightarrow{\psi_g^{-1}} F_0(T)$. The following diagram is commutative:

$$\begin{array}{ccccc}
F_0(\widetilde{S}) = \coprod g^*S & \longrightarrow & \coprod g^*F_0(T) & \xrightarrow{\coprod \psi_g^{-1}} & F_0(T) \\
\Big\| \psi_h & & \Big\| & & \downarrow \psi_h \\
h^*F_0(\widetilde{S}) = \coprod h^*g^*S & \longrightarrow & \coprod h^*g^*F_0(T) & \xrightarrow{\coprod h^*\psi_g^{-1}} & h^*F_0(T)
\end{array}$$

and thus β_0 defines a morphism $\psi \in \mathrm{Hom}_{\mathcal{D}'_U}(\widetilde{S}, T)$ such that $\alpha(\psi) = \beta$. Thus α is bijective.
If (S^∞, s^∞) is a universal pointed object of $\mathcal{D}_U$, then, for every T,

$$\mathrm{Hom}(\widetilde{S}^\infty, T) \xrightarrow{\sim} \mathrm{Hom}(S^\infty, F_0(T)) \xrightarrow{\sim} F(T).$$

Thus $(\widetilde{S}^\infty, s^\infty)$ is a universal pointed object of $\mathcal{D}'_U$.

The functor F_0 induces a morphism $\Pi_U \to \Pi'_U$. There is also a natural exact functor F_1 : H-Set $\to \mathcal{D}'_U$ which maps a H-set Y to $(Y = \coprod_{y\in Y}\{y\}, (\psi_h))$ where Y is a constant object of $\mathcal{D}_U$ and ψ_h maps y to $h \cdot y$. FF_1 is canonically isomorphic to the forgetful functor H-Set $\to$ Set, the functor F_1 thus induces a morphism $\Pi'_U \to H$. Since $\Pi_U = F_U(S^\infty)$ and $\Pi'_U = F(\widetilde{S}^\infty)$, one only has to see that the following exact sequence of pointed sets is exact:

$$1 \to F_U(S^\infty) \to F(\widetilde{S}^\infty) = \coprod_g F_U(g^*S^\infty) \to G \to 1$$

where the map $\coprod_g F_U(g^*S^\infty) \to G$ maps $F_U(g^*S^\infty)$ to g. □

If $(U_i, u_i)_{i\in I}$ is a cofinal projective system of pointed Galois objects (and let P be the corresponding object of pro-$\mathcal{C}$), one may define $\mathcal{B}^{\mathrm{temp}}(\mathcal{D}/\mathcal{C}, P)$ to be the category $\varinjlim_i \mathcal{D}'_{U_i}$. An isomorphism of pro-objects $P \to P'$ induces an equivalence $\mathcal{B}^{\mathrm{temp}}(\mathcal{D}/\mathcal{C}, P') \to \mathcal{B}^{\mathrm{temp}}(\mathcal{D}/\mathcal{C}, P)$, so that $\mathcal{B}^{\mathrm{temp}}(\mathcal{D}/\mathcal{C}, P)$ does not depend up to equivalence on the choice of $(U_i)_i$. Moreover, if $h \in G_i = \mathrm{Gal}(U_i/e)$ the endofunctor $h^* : \mathcal{D}'_{U_i} \to \mathcal{D}'_{U_i}$ maps $S = (S_{U_i}, \psi_g)$ to $h^*S = (h^*S_{U_i}, \psi_{hg}\psi_h^{-1})$. Then $\psi_h : S_{U_i} \to h^*S_{U_i}$ defines an isomorphism $S \to h^*S$ functorially in S. Thus $h^* : \mathcal{D}'_{U_i} \to \mathcal{D}'_{U_i}$ is canonically isomorphic to the identity of $\mathcal{D}'_{U_i}$. Thus every automorphism of the pro-object P induces an endofunctor of $\mathcal{B}^{\mathrm{temp}}(\mathcal{D}/\mathcal{C}, P)$ which is canonically isomorphic to the identity (functorially on $\mathrm{Aut}\, P$).

Let $(F_i)_{i\in I}$ be a family of fundamental functors $F_i : \mathcal{D}_{U_i} \to$ Set and assume one has a family $(\alpha_f)_{f:U_i \to U_j}$, indexed on the set of morphisms in I, of isomorphisms of functors $F_i f^* \to F_j$ such that for any $U_i \xrightarrow{f} U_j \xrightarrow{g} U_k$, $\alpha_g(\alpha_f \cdot g^*) = \alpha_{gf}$ (after identifying $(gf)^*$ and f^*g^* to lighten the notations). Such a family exists if I is just $\mathbf{N}$. Then, this induces a projective system $(\Pi'_{U_i})_{i\in I}$ (unique up to isomorphism independantly of (α_f) if I=$\mathbf{N}$ and the functors $\mathcal{D}'_{U_i} \to \mathcal{D}'_{U_j}$ are fully faithful), so that one can define

$$\pi_1^{\text{temp}}(\mathcal{D}/\mathcal{C}, (F_i)) = \varprojlim \Pi'_{U_i}$$

Assume one has a 2-commutative diagram with fibered vertical arrows:

$$\begin{array}{ccc} \mathcal{D}_1 & \to & \mathcal{D}_2 \\ \downarrow & & \downarrow \\ \mathcal{C}_1 & \xrightarrow{f} & \mathcal{C}_2 \end{array}$$

such that $f : \mathcal{C}_1 \to \mathcal{C}_2$ is exact, and $\mathcal{D}_{1,U} \to \mathcal{D}_{2,f(U)}$ is exact for every object U of $\mathcal{C}_1$.
One then gets a functor $\mathcal{B}^{\text{temp}}(\mathcal{D}_1/\mathcal{C}_1) \to \mathcal{B}^{\text{temp}}(\mathcal{D}_2/\mathcal{C}_2)$.

For example, let X be a K-manifold, $\mathcal{C}$ be the category of finite étale covering of X and $\mathcal{D} \to \mathcal{C}$ be the fibered category such that $\mathcal{D}_U$ is the category of topological coverings of U. Then, since finite étale coverings are morphisms of effective descent for tempered coverings, $\mathcal{D}'_U$ can be identified functorially with the full subcategory of $\mathrm{Cov}^{\text{temp}}(X)$ of tempered coverings S such that S_U is a topological covering of U. If (U_i, u_i) is a cofinal system of pointed Galois cover of (X, x), then $\mathcal{B}^{\text{temp}}(\mathcal{C}/\mathcal{D})$ becomes canonically equivalent with $\mathrm{Cov}^{\text{temp}}(X)$.

Let us apply our categorical definition of tempered fundamental groups to our log geometrical case.
Let $\underline{X} : X \to X_{l-1} \to \cdots \to \mathrm{Spec}(k)$ be a polystable log fibration, and assume that X is connected. Then one has a functor $\mathrm{C}_{\text{top}} : \mathrm{KCov}(X) \to \mathcal{K}\mathrm{e}$ obtained by composing the functor C of proposition 2.6 with the geometric realization functor.
One can thus define a fibered category $\mathcal{D}_{\text{top}} \to \mathrm{KCov}(X)$ such that the fiber of a két covering of Y of X is the category of topological coverings of $\mathrm{C}_{\text{top}}(Y)$ (which is equivalent to $\pi_1^{\text{top}}(\mathrm{C}_{\text{top}}(Y))$-Set).
One defines a fibered category $\mathrm{DD}_{\text{temp}} \to \mathrm{KCov}(X)$ such that the fiber of a két covering $f : Y \to X$ is the category of descent data of $\mathcal{D}_{\text{top}} \to \mathrm{KCov}(X)$ with respect to $Y \to X$ (this corresponds heuristically to the "tempered" coverings of X that become topological after pullback by $Y \to X$).
Let x be a log geometric point of X and let (Y, y) be a log geometrically pointed connected Galois két covering of (X, x). Let $\tilde{y} := |\,\mathrm{C}(Y)_y| \to |\,\mathrm{C}(Y)|$. The space $\tilde{y}$ is contractible according to lemma 2.7. Then one has a fundamental functor $F_y : \mathcal{D}_{\text{top}Y} \to \mathrm{Cov}^{\text{top}}(\tilde{y}) = \mathrm{Set}$ that corresponds to the base point $\tilde{y}$ ($F_y(S)$ is the set of connected components of $S \times_{|\,\mathrm{C}(Y)|} \tilde{y}$). Moreover, for any morphism $f : (Y', y') \to (Y, y)$, the two functors $F_{y'} f^*$ and F_y are canonically isomorphic.
Then one can consider the functor $F_{(Y,y)} : \mathrm{DD}_{\text{temp}Y} \to \mathrm{Set}$ which associates to a descent datum T the set $F_y(T_Y)$. The induced functor $\mathrm{DD}_{\text{temp}Y} \to \mathrm{Aut}(F_{(Y,y)})$-Set is an equivalence of categories.
One has an exact sequence:

$$1 \to \pi_1^{\text{top}}(|\,\mathrm{C}(Y)|, \tilde{y}) \to \mathrm{Aut}(F_{(Y,y)}) \to \mathrm{Gal}(Y/X) \to 1.$$

Then one defines

$$\pi_1^{\text{temp}}(X, x)^{\mathbb{L}} = \varprojlim_{(Y,y)} \mathrm{Aut}(F_{(Y,y)}),$$

where the projective limit is taken over the directed category $\mathbb{L}$- $\mathrm{GalKCov}(X, x)$ of pointed connected Galois $\mathbb{L}$-finite két coverings of (X, x).

If $x_2 \to x_1$ is a specialization of log geometric points of X, it induces a natural equivalence between the category of pointed coverings of (X, x_2) and the category of pointed coverings of (X, x_1)

(we thus identify the two categories). If Y is a pointed covering (Y,y_1) of (X,x_1), the corresponding pointed covering of (X,x_2) is (Y,y_2) where y_2 is the unique log geometric point above x_2 such that there is a specialization $y_2 \to y_1$ (and this specialization is unique). There is a commutative diagram

$$\begin{array}{ccc} \tilde{y}_1 & \longrightarrow & \tilde{y}_2 \\ & \searrow & \downarrow \\ & & |\,\mathrm{C}(Y)| \end{array}$$

This induces a canonical isomorphism $F_{y_1} \simeq F_{y_2}$, functorial in Y, so that one gets a canonical isomorphism $\pi_1^{\text{temp}}(X,x_1)^{\mathbb{L}} \to \pi_1^{\text{temp}}(X,x_2)^{\mathbb{L}}$. If X is connected and x_1,x_2 are two log geometric points of X, there exists a sequence of specializations and cospecializations joining x_1 to x_2, so that $\pi_1^{\text{temp}}(X,x_1)^{\mathbb{L}}$ and $\pi_1^{\text{temp}}(X,x_2)^{\mathbb{L}}$ are isomorphic.

One has an equivalence of categories between

$$\mathcal{B}^{\text{temp},\mathbb{L}}{}_{(X,x)} = \varinjlim \mathrm{DD}_{\text{temp}Y} \,/\mathbb{L}\text{- GalKCov}(X,x)$$

and the category $\pi_1^{\text{temp}}(X,x)^{\mathbb{L}}$-Set of sets with an action of $\pi_1^{\text{temp}}(X,x)^{\mathbb{L}}$ that goes through a discrete quotient of $\pi_1^{\text{temp}}(X,x)^{\mathbb{L}}$.

Assume now that X is log geometrically connected, *i.e.* that $X_{k'}$ is connected for any két extension k' of k. Let $\bar{k}$ be a log geometric point on k, let $\bar{x} = (\bar{x}_{k'})$ be a compatible system of log geometric points of $X_{k'}$ where k' runs through két extensions of $(k,\bar{k})$ (for every k', the set of geometric points above $\bar{x}_k$ is a non empty finite set and thus the set of compatible systems of log geometric points is a non empty profinite set).
Then, one defines $\pi_1^{\text{temp-geom}}(X,\bar{x})^{\mathbb{L}} = \varprojlim_{k'} \pi_1^{\text{temp}}(X_{k'},\bar{x}_{k'})^{\mathbb{L}}$, where k' runs through két extensions of k in a log geometric point $\bar{k}$. Let $\mathrm{KCov}_{\text{geom}}(X) = \varinjlim \mathrm{KCov}(X_{k'})$ where k' runs through két extensions of k in $\bar{k}$. It is the category of log geometric coverings of X.

If $Y \to X$ is a log geometric covering, defined over k', $\mathrm{C}_{\text{geom}}(Y_{k'})$ does not depend of k', so that one gets a functor $\mathrm{KCov}_{\text{geom}}(X) \to \mathcal{K}\mathrm{e}$ which maps Y to $|\,\mathrm{C}_{\text{geom}}(Y)|$. If $\bar{x}$ is a compatible system of points, for any pointed log geometric covering $(Y,\bar{y})$ of $(X,\bar{x})$, $\bar{y}$ defines a fundamental functor $F_{\bar{y}}$ of $\mathcal{D}_{\text{top-geom}Y}$ which are canonically isomorphic for any morphism $(Y',\bar{y}') \to (Y,\bar{y})$. One thus get a fibered category $\mathcal{D}_{\text{top-geom}} \to \mathrm{KCov}_{\text{geom}}(X)$, whose fiber in Y is the category of topological coverings of $|\,\mathrm{C}_{\text{geom}}(Y)|$. Then

$$\pi_1^{\text{temp-geom}}(X,\bar{x})^{\mathbb{L}} := \pi_1^{\text{temp}}(\mathcal{D}_{\text{top-geom}} \,/\, \mathrm{KCov}_{\text{geom}}(X), (F_{\bar{y}}))^{\mathbb{L}}.$$

3. Comparison result for the pro-(p') tempered fundamental group

If $\underline{X} : X \to \cdots \to \mathrm{Spec}(O_K)$ is a proper polystable log fibration, we want to compare the tempered fundamental group of the generic fiber X_η with the tempered fundamental group of the special fiber endowed with its natural log structure. The specialization theory of the log fundamental group already gives us a functor from két coverings of the special fiber and algebraic coverings of the generic fiber. To extend this to tempered fundamental groups, one has to compare, for any két covering T_s of the special fiber, the topological space $\mathrm{C}(T_s)$ with the Berkovich space of the corresponding covering T_η of the generic fiber. Thus we will define, as in [3], a strong deformation retraction of T_η^{an} to a subset canonically homeomorphic to $|\,\mathrm{C}(T_s)|$. We will construct this retraction étale locally, where T has a Galois covering V' by some polystable log fibration over a finite tamely ramified extension of O_K. Then the retraction of the tube of T_s is obtained by descending the retraction of the tube of V'_s, defined in [3]. We will then verify that the retraction does not depend on the choice of V' so that we can descend the retraction we defined étale locally.

3.1 Skeleton of a két log scheme over a pluristable log scheme

If $X \to \operatorname{Spec} O_K$ is a morphism of finite type, we denote by $\mathfrak{X}$ the completion of X along the closed fiber X_s. The generic fiber, in the sense of Berkovich, of a locally topologically finitely generated formal scheme $\mathfrak{X}$ over $\operatorname{Spf} O_K$ will be denoted by $\mathfrak{X}_\eta$.
Let $\underline{X} : X \to \cdots \to \operatorname{Spec}(O_K)$ be a polystable log fibration over $\operatorname{Spec}(O_K)$.

Proposition 3.1. *For every két morphism $T \to X$, let $\mathfrak{T}_\eta$ be the generic fiber, in the sense of Berkovich, of the formal completion of T along its special fiber. Then, there is a functorial map $|\operatorname{C}(T)_s)| \to \mathfrak{T}_\eta$, which identifies, $|\operatorname{C}(T_s)|$ with a subset $S(T)$ of $\mathfrak{T}_\eta$ on which $\mathfrak{T}_\eta$ retracts by strong deformation.*

Remark. $\mathfrak{T}_\eta$ is naturally an analytic subdomain of T_η^{an}. Moreover if T is proper over O_K (for example if X is proper, and T is a finite két covering), then $\mathfrak{T}_\eta \to T_\eta^{\mathrm{an}}$ is an isomorphism.

Proof. Let $f : T \to X$ be a két morphism. Let $x \in T_s$. Let $\underline{U} : U_l \to \cdots \to U_0$ be a polystable fibration étale over $\underline{X}$ such that (U_l, x_l) is an étale neighborhood of $f(x)$, such that, for every i, U_i has an exact chart $P_i \to A_i$ and compatible morphisms $P_i \to P_{i+1}$ such that the induced morphism $U_{i+1} \to U_i \times_{\operatorname{Spec} \mathbf{Z}[P_i]} \operatorname{Spec} \mathbf{Z}[P_{i+1}]$ is étale. One has an étale neighborhood $i : (V, x') \to (T, x)$ of x, a (p')-Kummer morphism $P_l \to Q$ such that $V \to X$ factors through an étale morphism $V \to U_l \times_{\operatorname{Spec} \mathbf{Z}[P_l]} \operatorname{Spec} \mathbf{Z}[Q]$. By definition of a (p')-Kummer morphism, there exists n prime to p such that $P_l \to \frac{1}{n} P_l$ factors through $P_l \to Q$. Thus V has a két Galois covering that comes from a polystable fibration $\underline{U}' = V' \to U'_{l-1} \to \cdots \to \operatorname{Spec} O_{K'}$, where $U'_i = U_i \times_{\operatorname{Spec} Z[P_i]} \operatorname{Spec} Z[\frac{1}{n} P_i]$ for $i \leqslant l$ and $V' = V \times_{\mathbf{Z}[Q]} \mathbf{Z}[\frac{1}{n} P_l]$ (so that there is a strict étale morphism $V' \to U'_l$) over $O_{K'}$ for some finite tamely ramified extension $K' = K[\pi^{1/n}]$ of K . Let us call $G = (\frac{1}{n} P^{\mathrm{gp}}/Q^{\mathrm{gp}})^\vee$ the Galois group of this két covering.
The deformation retraction of $\mathfrak{V}'_\eta$ defined in theorem 1.5 is G-equivariant, so that it defines a deformation retraction of $\mathfrak{V}_\eta$. Let $S(\ )$ denote the image of the retraction of $(\ )_\eta$. Then $S(\mathfrak{V}_\eta) = G \backslash S(\mathfrak{V}'_\eta) = G \backslash |\operatorname{C}(V'_s)| = |G \backslash \operatorname{C}(V'_s)| = |\operatorname{C}(V_s)|$ (corollary 1.6).

Let us show that the previously defined retraction of $\mathfrak{U}_\eta$ does not depend on n. Let us start by the case of a polystable morphism.
Let

$$\psi : Z_1 = \operatorname{Spec} A[P]/(p_i - \lambda_i) \to Z_2 = \operatorname{Spec} A[P]/(p_i - \lambda_i^s)$$

where $P = \mathbf{N}^{|\mathbf{r}|} = \oplus_{(i,j) \in \mathbf{r}} \mathbf{N} e_{ij}$ and $p_i = \sum_j e_{ij}$ induced by the multiplication by s on P, where s is an integer prime to p and where $\lambda \in A$.
Let G be the generic fiber of the formal completion of $\mathbf{G}_m^{(\mathbf{r})}$ at the identity; it acts on Z_1 and Z_2. One has $\psi(g \cdot x) = g^s \cdot \psi(x)$.
Let T_{ij} be the coordinates of G. Then $|T_{ij}^s - 1| = |T_{ij} - 1|$ if $|T_{ij} - 1| < 1$. Thus, for $t < 1$, $(\ )^s : G \to G$ induces an isomorphism $(\ )^s : G_t \to G_t$, and $g_t^s = g_t$.
Thus, if $t < 1$ (and also for $t = 1$ by continuity),

$$\psi(x_t) = \psi(g_t * x) = g_t^s * \psi(x) = g_t * \psi(x) = \psi(x)_t.$$

For a standard polystable fibration, the same result will easily follow by induction using that $\psi_n(r_i, t)^{1/s} = \psi_n(r_i^{1/s}, t^{1/s})$ (we kept the notations from the sketch of the proof of theorem 1.5).

More precisely, suppose we have the diagram:

$$\begin{array}{ccc} B = B'[Y_{ij}]/(Y_{i0}\cdots Y_{in_i} - b_i) & \longleftarrow & B' \\ \phi\uparrow & & \uparrow\phi' \\ A = A'[X_{ij}]/(X_{i0}\cdots X_{in_i} - a_i) & \longleftarrow & A' \end{array}$$

where $\phi(X_{ij}) = Y_{ij}^s$ and thus $\phi'(a_i) = b_i^s$, and $\tilde{\phi}' := \mathrm{Spf}\,\phi' : \mathrm{Spf}\,B' \to \mathrm{Spf}\,A'$ is a két morphism of polystable log fibrations and assume by induction that we already know that $\tilde{\phi}(x_t) = \tilde{\phi}(x)_t$.
Let $\mathfrak{X}$ (resp. $\mathfrak{X}'$, $\mathfrak{Y}$, $\mathfrak{Y}'$) denote Spf A (resp. Spf A', Spf B, Spf B').
The first part of the retraction of $\mathfrak{X}_\eta^{\mathrm{an}}$ and $\mathfrak{Y}_\eta^{\mathrm{an}}$ (consisting of the retraction fiber by fiber) commutes with $\tilde{\phi} := \mathrm{Spf}\,\phi$ according to the previous case. We thus just have to study the second part of the retraction.
$\tilde{\phi}$ induces a map:

$$\begin{array}{ccl} S_A & = & \{(x, r_{ij}) \in (\mathfrak{X}')_\eta^{\mathrm{an}} \times [0,1]^{[\mathbf{n}]} | r_{i0}\cdots r_{in_i} = |a_i(x)|\} \subset \mathfrak{X}_\eta^{\mathrm{an}} \\ \downarrow & & \\ S_B & = & \{(y, r_{ij}) \in (\mathfrak{Y}')_\eta^{\mathrm{an}} \times [0,1]^{[\mathbf{n}]} | r_{i0}\cdots r_{in_i} = |b_i(y)|\} \subset \mathfrak{Y}_\eta^{\mathrm{an}} \end{array}$$

which maps (x, r_{ij}) to $(\tilde{\phi}'(x), r_{ij}^{1/s})$ (remark that $|a_i(x)| = |b_i(\tilde{\phi}'(x))|^s$).
Then, if $(x, r_{ij}) \in S_A$ (we will write $y := \tilde{\phi}'(x)$; by induction assumption, $\tilde{\phi}'(x_t) = y_t$)

$$\begin{array}{rcl} \tilde{\phi}((x, r_{ij})_t) & = & \tilde{\phi}((x_t, \psi_{n_i}(r_{ij}, |a_i(x_t)|)_k)) \\ & = & (y_t, \psi_{n_i}(r_{ij}, |a_i(x_t)|)_k^{1/s}) \\ & = & (y_t, \psi_{n_i}(r_{ij}^{1/s}, |a_i(x_t)|^{1/s})_k) \\ & = & (y_t, \psi_{n_i}(r_{ij}^{1/s}, |b_i(y_t)|)_k) \\ & = & (y, r_{ij}^{1/s})_t \\ & = & \tilde{\phi}(x, r_{ij})_t \end{array}$$

Thus we get that the retraction of $\mathfrak{U}_\eta$ does not depend on n.

Let $W \to T$ be another neighborhood of x satisfying the same properties as V, and W' defined in the same way. One may assume by the previous remark that we chose the same n. Let $W'' = V' \times_T W'$. We have a commutative diagram

$$\begin{array}{ccc} W'' & \xrightarrow{p'} & W' \\ \downarrow p & & \downarrow i' \\ V' & \xrightarrow{i} & T \end{array}$$

where $W'' = V' \times_T W'$. . Let us show that $p : W'' \to V'$ is étale (symmetrically, p' is étale too). Since p is két, it is enough to prove that p is strict, *i.e.* that for any geometric point $z \in W''$, $\overline{M}_{V',p(z)} \to \overline{M}_{W'',z}$ is an isomorphism. Let $v = p(z)$, $w = p'(z)$, $\tau = i(v) = i'(w)$ and $\xi = f(\tau) \in X$. Then $\overline{M}_{X,\xi} = P_l/F$ where F is a face of P_l. Then $\overline{M}_{V',v} = \frac{1}{n}P_l/F_n = \frac{1}{n}\overline{M}_{X,\xi}$ where F_n is the saturation of F in $\frac{1}{n}P$. Symmetrically, one also has $\overline{M}_{W',w} = \frac{1}{n}\overline{M}_{X,\xi}$. Thus,

$$\begin{array}{rcl} \overline{M}_{W'',z} & = & \overline{M_{V',v} \oplus_{M_{T,\tau}} M_{W',w}} \\ & = & \overline{\overline{M}_{V',v} \oplus_{\overline{M}_{T,\tau}} \overline{M}_{W',w}} \\ & = & \overline{\frac{1}{n}\overline{M}_{X/\xi} \oplus_{\overline{M}_{T,\tau}} \frac{1}{n}\overline{M}_{X/\xi}} \\ & = & \overline{\frac{1}{n}\overline{M}_{X/\xi} \oplus \frac{1}{n}\overline{M}_{X/\xi}^{\mathrm{gp}}/\overline{M}_{T,\tau}^{\mathrm{gp}}} \\ & = & \frac{1}{n}\overline{M}_{X/\xi}, \end{array}$$

where the sums are sums in the category of fs monoids. Thus p is strict, and therefore étale.
Let thus $v \in \mathfrak{V}'_\eta$ and $w \in \mathfrak{W}'_\eta$ with same image τ in $\mathfrak{T}_\eta$. Let $z \in \mathfrak{W}''_\eta$ be above v and w. Then, for every $t \in [0,1]$, $v_t = p(z_t)$ and $w_t = p'(z_t)$ according to theorem 1.5.(ii). Thus $i(v_t) = ip(z_t) = i'p'(z_t) = i'(y'_t)$. Thus, the retractions of the different $\mathfrak{V}_\eta$ are compatible and define a map $\mathfrak{T}_\eta \times [0,1] \to \mathfrak{T}_\eta$. This map is continuous since, $\coprod \mathfrak{V}_i$ is a covering of $\mathfrak{T}$, $\coprod \mathfrak{V}_{i,\eta} \to \mathfrak{T}_\eta$ is quasi-étale and surjective and thus a topological factor map (as in the proof of theorem 1.5 of Berkovich; cf. [3, lem. 5.11]). Moreover, if $\phi : T_1 \to T_2$ is a két morphism of két log schemes over X, $\phi(x_t) = \phi(x)_t$. As in theorem 1.5.(vi), it is also compatible with isometric extensions of K.

Let $\widetilde{V} = \bigcup_i V_i$ be a covering of T such that every V_i satisfies the same property as V. Since $f : \widetilde{\mathfrak{V}}_\eta \to \widetilde{T}_\eta$ is a topological factor map, $S(\widetilde{\mathfrak{V}}_\eta) = f^{-1}(S(\widetilde{T}_\eta)) \to S(\widetilde{T}_\eta)$ is also a topological factor map. Thus one gets an isomorphism, functorial in T,

$$S(\mathfrak{T}_\eta) = \mathrm{Coker}(S(\widetilde{\mathfrak{V}}_\eta) \times_{S(\mathfrak{T}_\eta)} S(\widetilde{\mathfrak{V}}_\eta) \rightrightarrows S(\widetilde{\mathfrak{V}}_\eta))) = \mathrm{Coker}(|\,\mathrm{C}(V_s)| \times_{|\,\mathrm{C}(T_s)|} |\,\mathrm{C}(V_s)| \rightrightarrows |\,\mathrm{C}(V_s)|) = |\,\mathrm{C}(T_s)|.$$

□

3.2 Comparison theorem

Let K be a complete discrete valuation field. Let p be the residual characteristic (which can be 0).
Let $\underline{X} : X \to \cdots \to \operatorname{Spec} O_K$ be a proper polystable log fibration.
Let us now compare the tempered fundamental group of the generic fiber, as a K-manifold, and the tempered fundamental group of its special fiber as defined in §2.4.
A geometric point $\bar{x}$ of X_η^{an} is given by a algebraically closed complete nonarchimedean extension Ω of K and a K-morphism $\bar{x} : \operatorname{Spec} \Omega \to X$. Since $X \to \operatorname{Spec} O_K$ is proper, $\bar{x}$ extends uniquely to a morphism $\operatorname{Spec} O_\Omega \to X$. If one endows $\operatorname{Spec} O_\Omega$ of the log structure induced by $O_\Omega \backslash \{0\}$, one can extend $\operatorname{Spec} O_\Omega \to X$ in a morphism of log schemes. By looking at the closed fiber, one gets a morphism of log schemes $\tilde{x} : \operatorname{Spec} k_\Omega \to X_s$, where $\operatorname{Spec} k_\Omega$ has the log structure induced by $O_\Omega \backslash \{0\}$ (it is a log geometric point). The log geometric point $\tilde{x}$ is called the log reduction of $\bar{x}$.

Theorem 3.2. *Let $\bar{x}$ be a geometric point of X_η^{an}, and let $\tilde{x}$ be its log reduction. One has a morphism $\pi_1^{\mathrm{temp}}(X_\eta^{\mathrm{an}}, \bar{x})^{\mathbb{L}} \to \pi_1^{\mathrm{temp}}(X_s, \tilde{x})^{\mathbb{L}}$ which is an isomorphism if $p \notin \mathbb{L}$.*

These morphisms are compatible with finite extensions of K.

Proof. One has two functors $\mathbb{L}\text{-}\mathrm{KCov}(X) \to \mathbb{L}\text{-}\mathrm{Cov}^{\mathrm{alg}}(X_\eta)$, which is an equivalence of categories if $p \notin \mathbb{L}$, and $\mathbb{L}\text{-}\mathrm{KCov}(X) \to \mathbb{L}\text{-}\mathrm{KCov}(X_s)$ which is an equivalence of categories (theorem [9, th. 2.4]). One has a fibered category $\mathcal{D}_{\mathrm{top}}^{\mathrm{an}}(X_\eta)$ over $\mathbb{L}\text{-}\mathrm{KCov}(X_\eta)$ whose fiber at a $\mathbb{L}$-finite két covering T of X_η is the category of topological coverings of T^{an}. Let us call $\mathcal{D}_{\mathrm{top}}^{\mathrm{an}}(X)$ the pullback of $\mathcal{D}_{\mathrm{top}}^{\mathrm{an}}(X_\eta)/\mathbb{L}\text{-}\mathrm{KCov}(X_\eta)$ to $\mathbb{L}\text{-}\mathrm{KCov}(X)$: the fiber at a $\mathbb{L}$-finite két covering T of X is the category of topological coverings of T_η^{an}. One has also another fibered category $\mathcal{D}_{\mathrm{top}}^{\mathrm{sp}}(X)$ over $\mathbb{L}\text{-}\mathrm{KCov}(X)$ obtained by pulling back the fibered category $\mathcal{D}_{\mathrm{top}}(X_s) \to \mathbb{L}\text{-}\mathrm{KCov}(X_s)$ defined in part 2.4 along $\mathbb{L}\text{-}\mathrm{KCov}(X) \to \mathbb{L}\text{-}\mathrm{KCov}(X_s)$: the fiber at a $\mathbb{L}$-finite két covering T of X is the category of topological coverings of $|\,\mathrm{C}(T_s)|$. Proposition 3.1 induces an equivalence of fibered categories $\mathcal{D}_{\mathrm{top}}^{\mathrm{an}}(X) \to \mathcal{D}_{\mathrm{top}}^{\mathrm{sp}}(X)$, and thus an isomorphism $\pi_1^{\mathrm{temp}}(\mathcal{D}_{\mathrm{top}}^{\mathrm{an}}(X)/\mathbb{L}\text{-}\mathrm{KCov}(X)) \simeq \pi_1^{\mathrm{temp}}(\mathcal{D}_{\mathrm{top}}^{\mathrm{sp}}(X)/\mathbb{L}\text{-}\mathrm{KCov}(X))$.

The 2-commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{an}}_{\mathrm{top}}(X) & \longrightarrow & \mathcal{D}^{\mathrm{an}}_{\mathrm{top}}(X_\eta) \\ \downarrow & & \downarrow \\ \mathbb{L}\text{- KCov}(X) & \longrightarrow & \mathbb{L}\text{- Cov}^{\mathrm{alg}}(X_\eta) \end{array}$$

induces a morphism

$$\pi_1^{\mathrm{temp}}(X_\eta^{\mathrm{an}})^{\mathbb{L}} = \pi_1^{\mathrm{temp}}(\mathcal{D}^{\mathrm{an}}_{\mathrm{top}}(X_\eta)/\mathbb{L}\text{- Cov}^{\mathrm{alg}}(X_\eta)) \to \pi_1^{\mathrm{temp}}(\mathcal{D}^{\mathrm{an}}_{\mathrm{top}}(X)/\mathbb{L}\text{- KCov}(X))$$

which is an isomorphism if $p \notin \mathbb{L}$. Similarly,

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{sp}}_{\mathrm{top}}(X) & \longrightarrow & \mathcal{D}^{\mathrm{sp}}_{\mathrm{top}}(X_s) \\ \downarrow & & \downarrow \\ \mathbb{L}\text{- KCov}(X) & \longrightarrow & \mathbb{L}\text{- Cov}^{\mathrm{alg}}(X_s) \end{array}$$

induces an isomorphism

$$\pi_1^{\mathrm{temp}}(X_s)^{\mathbb{L}} \to \pi_1^{\mathrm{temp}}(\mathcal{D}^{\mathrm{sp}}_{\mathrm{top}}(X)/\mathbb{L}\text{- KCov}(X))$$

since $\mathbb{L}\text{- KCov}(X) \to \mathbb{L}\text{- KCov}(X_s)$ is an equivalence of categories. □

3.3 Geometric comparison theorem

We will assume in this section that $p \notin \mathbb{L}$.

THEOREM 3.3. *There is a natural isomorphism*

$$\pi_1^{\text{temp-geom}}(X_s)^{\mathbb{L}} \simeq \pi_1^{\mathrm{temp}}(X_{\bar\eta})^{\mathbb{L}}.$$

Proof. One knows, according to [1, prop 5.1.1], that

$$\pi_1^{\mathrm{temp}}(X_{\bar\eta}) \simeq \varprojlim_{K_i} \pi_1^{\mathrm{temp}}(X_{K_i}),$$

where K_i runs through the finite extensions of K in $\overline{K}$.
This induces an analog result for the $\mathbb{L}$-version.
However, we would like to know, in the case where $p \notin \mathbb{L}$, if one can only take the projective limit over tamely ramified extensions of K (*i.e.* to két extensions of O_k). Then the isomorphism we want would simply be obtained from theorem 3.2 by taking the projective limit over két extensions of O_k.

We have to show that if T' is a $\mathbb{L}$-finite két geometric covering of X (which is defined over a finite tamely ramified extension of K according to [8, prop. 1.15]: one can thus assume that T' is defined over K), the universal topological covering $\widetilde{T}'_\eta$ of T'_η is defined over some tamely ramified extension of K.
By changing $\operatorname{Spec} O_K$ by some két covering (which amounts to changing K by some tamely ramified extension) one may assume that $T' \to \operatorname{Spec} O_K$ is saturated.
One already knows that $\widetilde{T}'_\eta$ is defined over some finite extension K_2 of K ([1, lem 5.1.3]). Let K_1 be the maximal unramified extension of K in K_2. As $T' \to O_K$ is saturated, the underlying scheme of $T'_{O_{K_2}}$ is obtained by the base change of schemes $\operatorname{Spec} O_{K_2} \to \operatorname{Spec} O_{K_1}$ of the underlying scheme of $T'_{O_{K_1}}$. By looking at the special fiber, as $K_1 = K_2$ (as schemes), the morphism $T'_{K_2} \to T'_{K_1}$ induces an isomorphism between the underlying schemes, thus a bijection between their strata, and thus an isomorphism $|C(T'_{K_2})| \to |C(T'_{K_1})|$ and $S(T'_{K_2}) \to S(T'_{K_1})$.

Thus $\widetilde{T}'_\eta$ is defined over K_1.

□

This isomorphism is $\mathrm{Gal}(\bar{K}, K)$-equivariant (since the isomorphism for each Galois extension K_i of K is $\mathrm{Gal}(K_i/K)$-equivariant).

4. Cospecialization of pro-(p') tempered fundamental group

Let $X \to Y$ be a proper polystable log fibration, such that Y is log smooth and proper over O_K (the properness of $Y \to O_K$ is only assumed so that every point of Y_η has a reduction in Y_s, but the cospecialization morphisms we will construct only depend of Y locally). In this section we will construct the cospecialization morphisms for the (p')-tempered fundamental group of the geometric fibers of $X_\eta \to Y_\eta$. Thanks to theorem 3.3 we will be reduced to construct cospecialization morphisms for the (p')-tempered fundamental group of the log geometric fibers of $X_s \to Y_s$. Let thus $\bar{s}_2 \to \bar{s}_1$ be a specialization of log geometric points of Y, where $\bar{s}_1$ and $\bar{s}_2$ are the reductions of geometric points $\bar{\eta}_1, \bar{\eta}_2$ of Y_η.

We constructed in [9, th. 0.2] an equivalence of geometric (p')-két coverings of X_{η_1} and X_{η_2}. Now we must compare, for any such két covering Z_{s_1} corresponding to Z_{s_2} (which extends over the preimage X_U of some két neighborhood U of s_1 in Y), their polysimplicial sets as defined in proposition 2.6. First assume that s_2 is the generic point of its stratum. We will construct the cospecialization morphism of polysimplicial set étale locally, so that we can assume X to be strictly polystable (the properness will not be used for this). This cospecialization morphism of polysimplicial set will be constructed in the following way. Let z be a geometric stratum of Z_{s_1}. After some két localization of the base so that Z_U becomes saturated. Then the set of strata z_2 of Z_{s_2} such that z is in the closure of z_2 has a unique minimal element (as in lemma 1.3), which we call z'. Then, thanks to the fact that $Z_U \to U$ is saturated, the closure of z' in the strict localization of the generic point of z is separable onto its image. According to [6, cor. 18.9.8], z' is geometrically connected, thus defining a geometric stratum of Z_{s_2}. One thus obtains a map from the set of geometric strata of Z_{s_1} to the set of geometric strata of Z_{s_2}; this map induces a morphism of polysimplicial sets. In the case where polysimplicial sets of the geometric fibers of $Y \to X$ are interiorly free, the cospecialisation morphism of polysimplicial sets is an isomorphism if s_1 and s_2 are in the same stratum. We will end this article by glueing our specialization isomorphism of (p')-log tempered fundamental group with our cospecialization morphisms of polysimplicial sets in a cospecialization morphism of tempered fundamental groups.

4.1 Cospecialization of polysimplicial sets

In this section, we construct a cospecialization map of polysimplicial set for a composition of a két morphism and of a log polystable fibration.

LEMMA 4.1. *If $\phi : P \to Q$ is an integral (resp. saturated) morphism of fs monoids and F' is a face of Q, let $F = \phi^{-1}(F')$. Then $F \to F'$ is also integral (resp. saturated).*

Proof. To prove that $F \to F'$ is integral, thanks to [12, prop. I.4.3.11], one only has to prove that if $f'_1, f'_2 \in F'$ and $f_1, f_2 \in F$ are such that $f'_1\phi(f_1) = f'_2\phi(f_2)$, there are $g' \in F'$ and $g_1, g_2 \in F$ such that $f'_1 = g'\phi(g_1)$ and $f'_2 = g'\phi(g_2)$.
But there exists $g' \in Q$ and $g_1, g_2 \in P$ that satisfies those properties since $P \to Q$ is integral. But, since F' is a face of Q, $g', \phi(g_1), \phi(g_2)$ must be in F', and thus g_1 and g_2 are in F.

Thanks to a criterion of T. Tsuji ([13, prop. 4.1]), an integral morphism of fs monoids $f : P_0 \to Q_0$ is saturated if and only if for any $a \in P_0, b \in Q_0$ and any prime number p such that $f(a)|b^p$, there exists $c \in P_0$ such that $a|c^p$ and $f(c)|b$. Let $a \in F, b \in F'$ and p be a prime such that $\phi(a)|b^p$. Then since $\phi : P \to Q$ is saturated, there exists $c \in P$ such that $a|c^p$ and $f(c)|b$. But $f(c)|b$ implies that $f(c) \in F'$, whence $c \in F$. $\square$

PROPOSITION 4.2. *Let $f : X \to Y$ be a saturated log smooth morphism of fs log schemes. Assume $\overset{\circ}{Y}$ is strictly henselian of special point $\bar{y}_1$ and let $y_2 \in Y$. Let $x \in X_{\bar{y}_1}$. The set $A := \{Z \in \mathrm{Str}(X_{y_2}) | x \in \overline{Z}\}$ has a biggest element Z_0. Moreover, Z_0 is geometrically connected.*

Proof. Up to replacing $\overset{\circ}{Y}$ by a closed subscheme, one can assume that $\overset{\circ}{Y}$ is integral and y_2 is the generic point of Y. One can assume that f has a chart:

$$\begin{array}{ccc} X' & \longrightarrow & \operatorname{Spec} \mathbf{Z}[Q] \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \operatorname{Spec} \mathbf{Z}[P] \end{array}$$

where P is sharp, $\phi : P \to Q$ is an injective saturated morphism of fs monoids, $X' \to Y_Q = Y \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$ is étale, $X' \to Y$ factorizes through f and $g : X' \to X$ is étale. One also assumes that X' has a unique point x' above x. If $A' := \{Z' \in \mathrm{Str}(X'_{y_2}) | x' \in \overline{Z'}\}$ has a biggest element Z'_0, $g(Z_0)$ is a biggest element of A. Moreover, if Z'_0 is geometrically connected, $g(Z'_0)$ is also geometrically connected. One can thus assume $X' = X$.
Let $F'_2 = P \backslash \mathfrak{p}_2$ be the kernel of $P \to M_{Y,y_2}$; since y_2 is a generic point of Y, $Y \to \operatorname{Spec} \mathbf{Z}[P]$ factorizes through $Y \to \operatorname{Spec} \mathbf{Z}[P]/(\mathfrak{p}_2) \simeq \mathbf{Z}[F'_2]$. Let $F_1 = Q \backslash \mathfrak{q}_1$ be the kernel of $Q \to M_{X,x}$. Let $F =< F_1, \phi(F'_2) >$ be the face of Q generated by F_2 and $\phi(F_1)$, and let $\mathfrak{q}_2 = Q \backslash F$. Then $\mathfrak{q}_2$ is the biggest element of $\operatorname{Spec} Q$ above $\mathfrak{q}_1$ contained in $\mathfrak{p}_2$. Let $X_0 := X \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[Q]/(\mathfrak{q}_2)$: it is a closed subscheme of X. Set-theoretically it is the union of the strata of X whose image in Q contains $\mathfrak{q}_2$ (it contains x since $\mathfrak{q}_2 \subset \mathfrak{q}_1$).

Let us show that $X_0 \to Y$ is separable (*i.e.* flat with geometrically reduced fibers). Since $X_0 \to Y_F = Y \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]/(\mathfrak{q}_2) \simeq Y \times_{\operatorname{Spec} \mathbf{Z}[F'_2]} \operatorname{Spec} \mathbf{Z}[F]$ is étale, it is enough to show that $\operatorname{Spec} \mathbf{Z}[F'_2] \to \operatorname{Spec} \mathbf{Z}[F]$ is separable. But $F \to F'_2$ is saturated thanks to lemma 4.1; this implies that $\operatorname{Spec} \mathbf{Z}[F'_2] \to \operatorname{Spec} \mathbf{Z}[F]$ is separable. According to [6, cor. 18.9.8], for everx $y \in Y$, $X_0(x)_y$ is geometrically connected (where $X_0(x)$ denotes the localization of X_0 at x). Set-theoretically $X_0(x)_{y_2}$ is the subset of X_{y_2} consisting of points z which specialize to x and such that the kernel F_z of $M_{X,z} \to \operatorname{Spec} Q$ is contained in F. For every point of $z = X_0(x)_{y_2}$, F_z is contained in F, contains F_1 because x is a specialization of z and contains $\phi^{-1}(F_z) = F'_2$ because the face corresponding to y_2 is F'_2: thus $F_z = F$. Thus $X_0(x)_{y_2}$ is contained in a single stratum Z_0 of X_{y_2} (Z_0 is an element of A). Since the generic point z_0 of Z_0 is in $X_0(x)_{y_2}$, Z_0 must also be geometrically connected.

Let $Z \neq Z_0$ be in A a maximal element and let z be its generic point. Let $\mathfrak{q}_Z$ be the corresponding face of Q then $\mathfrak{q}_Z \subset \mathfrak{q}_1$ and $\phi^*(\mathfrak{q}_Z) = \mathfrak{p}_2$. Thus $\mathfrak{q}_Z \subset \mathfrak{q}_2$. Let $X_{\mathfrak{q}_Z} = X \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[Q]/(\mathfrak{q}_Z)$ (this is union of the strata of X' yhose image in Q contains $\mathfrak{q}_Z$). As previously, $X_{\mathfrak{q}_Z}(x)_{y_2}$ is geometrically connected and contains z as a generic point. It also contains z_0. Since Z is open in $(X_{\mathfrak{q}_Z})_{y_2}$, and $Z \cap X_{\mathfrak{q}_Z}(x)_{y_2} \subsetneq X_{\mathfrak{q}_Z}(x)_{y_2}$, z must specialize in $X'_{\mathfrak{q}_Z}(x')_{y_2}$ to an element z' that is not in Z. The stratum containing z' is in A and is bigger than Z. Thus A has no maximal element other than Z_0. Since A is locally finite, Z_0 must be the biggest element of A.

$\square$

If $f : X \to Y$ is a saturated log smooth morphism of fs log schemes and $\bar{y}_2 \to \bar{y}_1$ is a specialization

of geometric points, one can apply proposition 4.2 to the pullback of f to the strict henselization of $\bar{y}_1$: one gets a nondecreasing map $\mathrm{Str}(X_{\bar{y}_1}) \to \mathrm{Str}(X_{\bar{y}_2})$.
If $Z \to X$ is két and $X \to Y$ is asaturated log smooth morphism of fs log schemes and $\bar{y}_2 \to \bar{y}_1$ is a két specialization of log geometric points, there exists a két neighborhood U of $\bar{y}_1$ such that $X_U := X \times_Y U \to U$ is saturated. One thus gets a cospecialization map

$$\mathrm{Str}(Z_{\bar{y}_1}) \to \mathrm{Str}(Z_{\bar{y}_2}).$$

Proposition 4.3. *If $X \to Y$ is proper and $\overline{M}_{Y,\bar{y}_1} \to \overline{M}_{Y,\bar{y}_2}$ is an isomorphism, then the cospecialization map $\mathrm{Str}(X_{\bar{y}_1}) \to \mathrm{Str}(X_{\bar{y}_2})$ is bijective.*

Proof. Assume $\mathring{Y} = \mathrm{Spec}\, A$ is strictly local with special point $\bar{y}_1$, integral with generic point $\bar{y}_2$ and $X \to Y$ is saturated. By pulling back along the normalization of $\mathring{Y}$, one can also assume that A is normal.
Let Z be a stratum of $\mathring{X}_{y_2}$ and let z be its generic point. Let $\widetilde{Z}$ be the normalization of the closure $\overline{Z}$ of Z (endowed with the pullback log structure). Let $v : V \to X$ be an étale morphism such that $V \to Y$ has a global chart:

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{Spec}\, \mathbf{Z}[Q] \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{Spec}\, \mathbf{Z}[P] \end{array}$$

such that $V \to Y_Q = \mathrm{Spec}\, \mathbf{Z}[Q] \times_{\mathrm{Spec}\, \mathbf{Z}[P]} Y$ is étale and $P \to Q$ is injective and saturated.
Let $\mathfrak{p} \in \mathrm{Spec}\, P$ be the image of $\bar{y}_1$ by the map $Y \to \mathrm{Spec}\, P$ and let $F = P \backslash \mathfrak{p}$. The morphism $Y \to \mathrm{Spec}\, \mathbf{Z}[P]$ factorizes through $Y \to \mathrm{Spec}\, \mathbf{Z}[F]$, where $\mathrm{Spec}\, \mathbf{Z}[F]$ is the closure of the stratum of $\mathrm{Spec}\, \mathbf{Z}[P]$ corresponding to $\mathfrak{p}$. Since $\overline{M}_{Y,\bar{y}_1} \to \overline{M}_{Y,\bar{y}_2}$ is in an isomorphism, it even factorizes through $Y \to \mathrm{Spec}\, \mathbf{Z}[F^{\mathrm{gp}}]$, where $\mathrm{Spec}\, \mathbf{Z}[F^{\mathrm{gp}}]$ is the stratum of $\mathrm{Spec}\, \mathbf{Z}[P]$ corresponding to $\mathfrak{p}$. If Z_1 is a stratum of $V_{\bar{y}_1}$ above a face F_0 of Q, the corresponding stratum Z_2 of $V_{\bar{y}_2}$ by the cospecialization map is also above F_0: the map $\overline{M}_{V,\bar{z}_1} \to \overline{M}_{V,\bar{z}_2}$ is an isomorphism.
Let $(z_i)_{i \in I}$ be the family of preimages of z in V. Let $\mathfrak{q}_i \in \mathrm{Spec}\, Q$ be the image of z_i by the map $V \to \mathrm{Spec}\, Q$. Let $F_i = \mathrm{Spec}\, Q \backslash \mathfrak{q}_i$. According to lemma 4.1, $F \to F_i$ is a saturated morphism of fs monoids. Then $\overline{\{z_i\}}$ is an irreducible component of $V_{F_i} = V \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[F_i]$, which is étale above $Y_Q \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[F_i] = \mathrm{Spec}\, A \otimes_{\mathbf{Z}[F^{\mathrm{gp}}]} \mathbf{Z}[F^{-1}F_i] = \mathrm{Spec}\, A[F^{-1}F_i \cap T]$, where T is a direct summand of F^{gp} in Q^{gp}. $F^{-1}F_i \cap T$ is saturated: according to [12, prop.I.3.3.1], $\mathrm{Spec}\, A[F^{-1}F_i \cap T]$ is normal. Hence $\overline{\{z_i\}}$ is a connected component of V_{F_i} and is normal. Thus $\widetilde{Z} \times_X V = \coprod \overline{\{z_i\}}$. Since, the geometric fibers of $\mathrm{Spec}\, A[F^{-1}F_i \cap T] \to \mathrm{Spec}\, A$ are normal for any choice of V, the geometric fibers of $\widetilde{Z} \to Y$ are also normal.
The morphism $\widetilde{Z} \to Y$ is proper. Let $\widetilde{Z} \to W \to Y$ be its Stein factorization. Since $\widetilde{Z} \to Y$ is separable, according to [5, prop X.1.2], $W \to Y$ is an étale covering. Since Y is strictly henselian W is a direct sum of copies of Y. Since $\widetilde{Z}_{y_2}$ is connected, $W = Y$. Thus all the fibers of $\widetilde{Z} \to Y$ are geometrically connected. Since they are normal, they are also geometrically irreducible. Since $\widetilde{Z} \to \overline{Z}$ is surjective, $\overline{Z}_{\bar{y}_1}$ is also irreducible. Let z_1 be the generic point of $\overline{Z}_{\bar{y}_1}$. Then for any specialization of geometric points $\bar{z} \to \bar{z}_1$, $\overline{M}_{X,\bar{z}_1} \to M_{X,\bar{z}}$ is an isomorphism (this can be checked on $\widetilde{Z} \times_X V$ if V is a neighborhood of $\bar{z}_1$). The strata Z_1 of $X_{\bar{y}_1}$ containing z_1 should cospecialize to a stratum Z' bigger than Z, but then$\overline{M}_{X,\bar{z}'} \to M_{X,\bar{z}}$ should also be an isomorphism and thus $Z' = Z$. This shows the surjectivity of $\mathrm{Str}(X_{y_1}) \to \mathrm{Str}(X_{y_2})$.
If $Z'_1 \in \mathrm{Str}(X_{y_1})$ cospecializes to Z, then $Z'_1 \subset \overline{Z}_{\bar{y}_1}$ and thus Z'_1 is bigger than Z_1 but then the morphism $\overline{M}_{X,\bar{z}'_1} \to \overline{M}_{X,\bar{z}_1}$ is also an isomorphism, and thus $Z'_1 = Z_1$, which shows the injectivity of the cospecialization map. □

We now want to define cospecialization maps of polysimplicial complexes. Let us begin with an analog of [3, lem. 6.1].

Lemma 4.4. *Let $X \to Y$ be a strictly polystable morphism of log schemes and let $\bar{y}_2 \to \bar{y}_1$ be a specialization of geometric points of Y. Let x_1 be a stratum of $X_{\bar{y}_1}$ and let x_2 be its image in $\mathrm{Str}(X_{\bar{y}_2})$ by the cospecialization map. Then, given an isometric bijection $\mu : [\mathbf{n}] \to \mathrm{Irr}(X_{\bar{y}_1}, x_1)$, there exists a unique couple (I, μ') consisting of a subset $I \subset [w(\mathbf{n})]$ and of an isometric bijection $\mu' : [\mathbf{n}_I] \to \mathrm{Irr}(X_{\bar{y}_2}, x_2)$ such that*

$$\begin{array}{ccc} [\mathbf{n}] & \to & \mathrm{Irr}(X_{\bar{y}_1}, x_1) \\ \downarrow & & \downarrow \\ [\mathbf{n}_I] & \to & \mathrm{Irr}(X_{\bar{y}_2}, x_2) \end{array}$$

If moreover $\overline{M}_{Y,\bar{y}_1} \to \overline{M}_{Y,\bar{y}_2}$ is an isomorphism, then $I = [w(\mathbf{n})]$.

Proof. The uniqueness is obvious since there is no isometric bijection $[\mathbf{n}_I] \to [\mathbf{n}_J]$ for $I \neq J$ and $[\mathbf{n}] \to [\mathbf{n}_I]$ is surjective. One can replace Y by its strict henselization at $\bar{y}_1$ and assume $Y = \mathrm{Spf}\, A$. Let $\pi : M_{Y,\bar{y}_1} \to A$. Thanks to [3, lem. 2.10], the proposition is local on the étale topology of X so that one can assume $X = \mathrm{Spec}\, B$ where $B = B_1 \otimes_A \cdots \otimes_A B_p \otimes_A C$ and

$$B_i = A[T_{i0}, \cdots, T_{in_i}]/(T_{i0} \cdots T_{in_i} - \pi(m_i))$$

with $\pi(m_i)(\bar{y}_1) = 0$ and C smooth over A. Let $I = \{i \in [p] | \pi(m_i)(\bar{y}_2)\} = 0$. Then one gets an isometric bijection $\mathrm{Irr}(X_{\bar{y}_2}, x_2) \simeq [\mathbf{n}_I]$. □

Thus, if $X \to Y$ is strictly polystable, $\mathrm{Str}(X_{\bar{y}_1}) \to \mathrm{Str}(X_{\bar{y}_2})$ induces a natural cospecialization morphism of polysimplicial sets $\mathrm{C}(X_{\bar{y}_1}) \to \mathrm{C}(X_{\bar{y}_2})$. If $\overline{M}_{Y,\bar{y}_1} \to \overline{M}_{Y,\bar{y}_2}$ is an isomorphism, $\mathrm{C}(X_{\bar{y}_1}) \to \mathrm{C}(X_{\bar{y}_2})$ maps nondegenerate polysimplices to nondegenerate polysimplices.

Let $\underline{X} : X = X_l \to \cdots \to Y$ be a strictly polystable fibration. Assume we constructed a cospecialization morphism of polysimplicial sets $\psi_{l-1} : \mathrm{C}(X_{l-1,\bar{y}_2}) \to \mathrm{C}(X_{l-1,\bar{y}_1})$ such that the induced map $\mathrm{Str}(X_{l-1,\bar{y}_2}) \to \mathrm{Str}(X_{l-1,\bar{y}_1})$ obtained by applying O is the cospecialization map already defined.One has $\mathrm{C}(X_{\bar{y}_1}) = \mathrm{C}(X_{l-1,\bar{y}_1}) \square D_1$ and $\mathrm{C}(X_{\bar{y}_2}) = \mathrm{C}(X_{l-1,\bar{y}_2}) \square D_2$. Assume Y to be strictly local. Let y_2 be the point of Y where $\bar{y}_2$ lies. Let $x_1 \in \mathrm{Str}(X_{l-1,\bar{y}_1})$. Let x_2 be the image of x_1 by the cospecialization map $\mathrm{Str}(X_{l-1,\bar{y}_1}) \to \mathrm{Str}(X_{l-1,\bar{y}_2})$. Let $\tilde{x}_2$ be the image of x_2 in $\mathrm{Str}(X_{l-1,y_2})$. If $z_1 \in \mathrm{Str}(X_{x_1})$, then the set $\{Z \in \mathrm{Str}(X_{y_2}) | z_1 \subset \overline{Z}\}$ has a biggest element $\tilde{z}_2$ according to proposition 4.2 and is geometrically irreducible. Since $\{Z \in \mathrm{Str}(X_{\tilde{x}_2}) | z_1 \subset \overline{Z}\}$ is nonempty, one has $\tilde{z}_2 \in \mathrm{Str}(X_{\tilde{x}_2}) \subset \mathrm{Str}(X_{y_2})$. Since $\tilde{z}_2$ is geometrically irreducible, it defines a stratum z_2 of $\mathrm{Str}(X_{x_2})$. Thus one gets a map $\mathrm{Str}(X_{x_1}) \to \mathrm{Str}(X_{x_2})$. Moreover, if $x'_1 \leqslant x_1 \in \mathrm{Str}(X_{l-1,\bar{y}_1})$ and x'_2 is the image of x'_1 by the cospecialization map $\mathrm{Str}(X_{l-1,\bar{y}_1}) \to \mathrm{Str}(X_{l-1,\bar{y}_2})$ (thus $x'_2 \leqslant x_2$), then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Str}(X_{x_1}) & \longrightarrow & \mathrm{Str}(X_{x'_1}) \\ \downarrow & & \downarrow \\ \mathrm{Str}(X_{x_2}) & \longrightarrow & \mathrm{Str}(X_{x'_2}) \end{array}$$

where the horizontal arrows are given by lemma 1.3. Let us choose geometric points $\bar{x}_1$ and $\bar{x}_2$ of X_{l-1} above x_1 and x_2. Let us choose a specialization $\bar{x}_2 \to \bar{x}_1$. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Str}(X_{\bar{x}_1}) & \longrightarrow & \mathrm{Str}(X_{\bar{x}_2}) \\ \downarrow & & \downarrow \\ \mathrm{Str}(X_{x_1}) & \longrightarrow & \mathrm{Str}(X_{x_2}) \end{array}$$

Let $\bar{z}_1$ be a preimage of z_1 in $\mathrm{Str}(X_{\bar{x}_1})$ and let $\bar{z}_2$ be the image of z_1 in $\mathrm{Str}(X_{\bar{x}_2})$. Then $\mathrm{Irr}(X_{\bar{x}_1}) \to \mathrm{Irr}(X_{x_1})$ and $\mathrm{Irr}(X_{\bar{x}_2}) \to \mathrm{Irr}(X_{x_2})$ are isomorphisms. One thus gets a statement similar to lemma 4.4 for z_1 and z_2: one gets a morphism $\mathrm{C}(X_{x_1}) \to \mathrm{C}(X_{x_2})$, compatible with morphisms of lemma 1.3. One thus gets a morphism of functors $D_1 \to D_2\psi_{l-1,*}$.
This induces a morphism $\mathrm{C}(X_{\bar{y}_1}) \to \mathrm{C}(X_{\bar{y}_2})$. If $\overline{M}_{Y,\bar{y}_1} \to \overline{M}_{Y,\bar{y}_2}$ is an isomorphism and x_2 is the image of x_1 by the cospecialization map, $\overline{M}_{X_{l-1},\bar{x}_1} \to \overline{M}_{X_{l-1},\bar{x}_2}$ is also an isomorphism. By induction on l, one gets that $\mathrm{C}(X_{\bar{y}_1}) \to \mathrm{C}(X_{\bar{y}_2})$ maps nondegenerate polysimplices to nondegenerate polysimplices.

If $Z \to X$ is a két morphism, the morphism $\mathrm{Str}(Z_{\bar{y}_2}) \to \mathrm{Str}(Z_{\bar{y}_1})$ above $\mathrm{Str}(X_{\bar{y}_1}) \to \mathrm{Str}(X_{\bar{y}_2})$ induces a morphism $\mathrm{C}(Z_{\bar{y}_1}) \to \mathrm{C}(Z_{\bar{y}_2})$.

Assume now $\underline{X}$ is a polystable fibration over Y and $Z \to X$ is két. Let $\underline{X}' \to \underline{X}$ be étale and surjective such that $\underline{X}$ isa strictly polystable fibration over Y Let $\underline{X}'' = \underline{X}' \times_{\underline{X}} \underline{X}'$, $Z' = Z \times_X X'$, $Z'' = Z \times_X X''$. Then the commutative diagram

$$\begin{array}{ccc} \mathrm{C}(X''_{\bar{y}_1}) & \rightrightarrows & \mathrm{C}(X'_{\bar{y}_1}) \\ \downarrow & & \downarrow \\ \mathrm{C}(X''_{\bar{y}_2}) & \rightrightarrows & \mathrm{C}(X'_{\bar{y}_2}) \end{array}$$

induces a cospecialization morphism of polysimplicial sets $\mathrm{C}(X_{\bar{y}_1}) \to \mathrm{C}(X_{\bar{y}_2})$. One gets the following result:

Proposition 4.5. *Let $\underline{X}$ be a polystable log fibration over Y. Let $\bar{y}_1 \to \bar{y}_2$ be a két specialization of log geometric points. There is, for every két morphism $Z \to X$ a cospecialization map $\mathrm{C}_{\mathrm{geom}}(Z_{y_1}/\bar{y}_1) \to \mathrm{C}_{\mathrm{geom}}(Z_{y_2}/\bar{y}_2)$ functorial in Z.*

Let us assume now that $Z \to Y$ is proper and that $\overline{M}_{Y,\bar{y}_1} \to \overline{M}_{Y,\bar{y}_2}$ is an isomorphism. The morphism $\mathrm{C}(Z_{y_2}/\bar{y}_1) \to \mathrm{C}(Z_{y_2}/\bar{y}_2)$ maps nondegenerate polysimplices to nodegenerate polysimplices and, according to proposition 4.3, $\mathrm{Str}(Z_{\bar{y}_1}) \to \mathrm{Str}(Z_{\bar{y}_2})$ is bijective.
Therefore, if one assumes moreover that $\mathrm{C}_{\mathrm{geom}}(Z_{y_2}/\bar{y}_2)$ is interiorly free (this is the case if $\mathrm{C}_{\mathrm{geom}}(Z_{y_2}/\bar{y}_2)$ is interiorly free), then

$$\mathrm{C}_{\mathrm{geom}}(Z_{y_2}/\bar{y}_1) \to \mathrm{C}_{\mathrm{geom}}(Z_{y_2}/\bar{y}_2)$$

is also an isomorphism.

4.2 Specialization of tempered fundamental groups of log schemes

First, recall the result we proved in [9, §2.4] about specialization of log fundamental groups.
Let $X \to Y$ be a proper and saturated morphism of log schemes. Assume moreover $X \to Y$ to have log geometrically connected fibers. Let $\bar{y}_2 \to \bar{y}_1$ be a specialization of log geometric points of Y.
Let T be the strictly local scheme of Y at $\bar{y}_1$ endowed with the inverse image log structure, and let z be its closed point, endowed with the inverse image log structure.
One has the following arrows (defined up to inner homomorphisms):

$$\pi_1^{\text{log-geom}}(X_{y_2}/y_2)^{(p')} \to \pi_1^{\text{log-geom}}(X_z/z)^{(p')} \overset{\simeq}{\leftarrow} \pi_1^{\text{log-geom}}(X_T/T)^{(p')} \leftarrow \pi_1^{\text{log-geom}}(X_{y_1}/y_1)^{(p')}.$$

Theorem 4.6 [9, prop. 2.4]. *One has a specialization morphism*

$$\pi_1^{\text{log-geom}}(X_{\bar{y}_2}/y_2)^{(p')} \to \pi_1^{\text{log-geom}}(X_{\bar{y}_1}/y_1)^{(p')}$$

that factors through $\pi_1^{\text{log-geom}}(X_T/T)^{(p')}$.

We can now use this with our cospecialization morphism of polysimplicial sets when these are isomorphisms.

PROPOSITION 4.7. *Let Y be a fs log scheme, let $X \to Y$ be a proper polystable log fibration with geometrically connected fibers. Assume moreover that the polysimplicial set $\mathrm{C}_{\mathrm{geom}}(X_{\bar{s}})$ of any geometric fiber is interiorly free. Let $\bar{y}_2 \to \bar{y}_1$ be a specialization of log geometric points over fs log points $y_2 \to y_1$ of Y such that $\overline{M}_{Y,\bar{y}_1} \to \overline{M}_{\bar{y}_2}$ is an isomorphism. Let $\mathbb{L}$ be a set of primes which does not contain the residual characteristic of y_1 One has a specialization morphism defined up to inner automorphism:*

$$\pi_1^{\text{temp-geom}}(X_{\bar{y}_2})^{\mathbb{L}} \to \pi_1^{\text{temp-geom}}(X_{\bar{y}_1})^{\mathbb{L}}.$$

Proof. One can assume that $\mathring{Y}$ is strictly local with closed point y_1. There is a functor

$$F : \mathrm{KCov}_{\mathrm{geom}}(X_{y_1}/\bar{y}_1)^{\mathbb{L}} \to \mathrm{KCov}_{\mathrm{geom}}(X_{y_2}/\bar{y}_2)^{\mathbb{L}}.$$

According to theorem 4.6, if $Z_{\bar{y}_1}$ is some geometric két covering of X_{y_1}/y_1, it extends to a geometric két covering of X/Y: there is a connected finite pointed két covering $(U, \bar{u}_1)$ of $(Y, \bar{y}_1)$ such that $Z_{\bar{y}_1}$ extends to a két covering $Z_U \to X_U := X \times_Y U$. This extension becomes unique after replacing U by some bigger covering. If $\bar{u}_2 \to \bar{u}_1$ is the két specialization of log geometric points lifting $\bar{y}_2 \to \bar{y}_1$, then $(Z_U)_{\bar{u}_2}$ is nothing but the geometric két covering $F(Z_{\bar{y}_1})$ of $X_{\bar{y}_2}$. We will simply denote it by $Z_{\bar{y}_2}$. One has an isomorphism $\mathrm{C}_{\mathrm{geom}}(Z_{\bar{y}_1}) \simeq \mathrm{C}_{\mathrm{geom}}(Z_{\bar{y}_2})$ functorially in $Z_{\bar{y}_1}$. One gets a cospecialization functor of fibered categories:

$$\begin{array}{ccc} \mathcal{D}_{\text{top-geom}}(X_{\bar{y}_1}) & \to & \mathcal{D}_{\text{top-geom}}(X_{\bar{y}_2}) \\ \downarrow & & \downarrow \\ \mathrm{KCov}_{\mathrm{geom}}(X_{y_1}/y_1)^{\mathbb{L}} & \to & \mathrm{KCov}_{\mathrm{geom}}(X_{y_2}/y_2)^{\mathbb{L}} \end{array}$$

and thus a specialization morphism $\pi_1^{\text{temp-geom}}(X_{\bar{y}_2})^{\mathbb{L}} \to \pi_1^{\text{temp-geom}}(X_{\bar{y}_1})^{\mathbb{L}}$. □

4.3 Cospecialization morphisms of pro-(p') tempered fundamental groups

Let K be a discrete valuation field, and $\operatorname{Spec} O_K$ is endowed with its usual log structure, and assume that the residual characteristic p of K is not in $\mathbb{L}$. Let $Y \to \operatorname{Spec} O_K$ be a morphism of fs log schemes such that $\mathring{Y}$ is locally noetherian. Let $\mathfrak{Y}$ be the formal completion of Y along its closed fiber. Then $\mathfrak{Y}_\eta$ is an analytic domain of Y_K^{an}. Let $Y_0 = \mathfrak{Y}_\eta \cap Y_{\mathrm{tr}}^{\mathrm{an}} \subset Y_K^{\mathrm{an}}$.
Let $X \to Y$ be a proper and polystable log fibration with geometrically connected fibers.
Let $\tilde{y}$ be a K'-point of Y_0 where K' is a complete extension of K. One has canonical morphism of log schemes $\operatorname{Spec} O_{K'} \to Y$ where $\operatorname{Spec} O_{K'}$ is endowed with the log structure given by $O_{K'} \backslash \{0\} \to O_{K'}$. The *log reduction* $\tilde{s}$ of $\tilde{y}$ is the log point of Y corresponding to the special point of $\operatorname{Spec} O_{K'}$ with the inverse image of the log structure of $\operatorname{Spec} O_{K'}$. If K' has discrete valuation, then $\tilde{s}$ is a fs log point. If K' is algebraically closed, $\tilde{s}$ is a geometric log point.

Let $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$ be the category whose objects are geometric points $\bar{y}$ of Y_0, such that $\mathcal{H}(y)$ is discretely valued (where y is the underlying point of $\bar{y}$) and $\mathrm{Hom}(\bar{y}, \bar{y}')$ is the set of két specializations from $\bar{s}$ to $\bar{s}'$, where $\bar{s}$ and $\bar{s}'$ are the log reductions of $\bar{y}$ and $\bar{y}'$, such that there exists some specialization $\bar{y} \to \bar{y}'$ of geometric points in the sense of algebraic étale topology for which the following diagram commutes:

$$\begin{array}{ccc} \bar{y} & \longrightarrow & \bar{s} \\ \downarrow & & \downarrow \\ \bar{y}' & \longrightarrow & \bar{s}' \end{array}$$

Let $\mathrm{Pt}^{\mathrm{an}}(Y)$ be the category defined from $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$ by inverting the class of morphisms $\bar{y} \to \bar{y}'$ for which $\bar{s} \to \bar{s}'$ is a cospecialization isomorphism.
Let $\mathrm{Pt}_0^{\mathrm{an}}(Y)$ be the category obtained from $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$ by inverting the class of morphisms $\bar{y} \to \bar{y}'$ such that $\overline{M}_{Y,\bar{s}'} \to \overline{M}_{Y,\bar{s}}$ is an isomorphism.
Let $\mathrm{OutGp}_{\mathrm{top}}$ be the category of topological groups with outer morphisms.

THEOREM 4.8. *There is a functor* $\pi_1^{\mathrm{temp}}(X_{(\cdot)}) : \mathrm{Pt}^{\mathrm{an}}(Y)^{\mathrm{op}} \to \mathrm{OutGp}_{\mathrm{top}}$ *sending* $\bar{y}$ *to* $\pi_1^{\mathrm{temp}}(X_{\bar{y}})$.
If, for every geometric point bar s of Y*, the polysimplicial set* $\mathrm{C}(X_{\bar{s}})$ *is interiorly free, then the functor* $\pi_1^{\mathrm{temp}}(X_{(\cdot)})$ *factors through* $\mathrm{Pt}_0^{\mathrm{an}}(Y)^{\mathrm{op}}$.

Proof. Let $\bar{y}_2 \to \bar{y}_1$ be a morphism $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$. One has to construct a cospecialization morphism $\pi_1^{\mathrm{temp}}(X_{\bar{y}_1}) \to \pi_1^{\mathrm{temp}}(X_{\bar{y}_2})$.
One has a cospecialization functor

$$F : \mathrm{KCov}_{\mathrm{geom}}(X_{s_1}/s_1)^{\mathbb{L}} \to \mathrm{KCov}_{\mathrm{geom}}(X_{s_2}/s_2)^{\mathbb{L}}$$

which factors through $\mathrm{KCov}_{\mathrm{geom}}(X_T/T)^{\mathbb{L}}$ where T is the strict localization at s_1.
The cospecialization functor $\mathrm{KCov}_{\mathrm{geom}}(X_{s_i}/s_i)^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(X_{\bar{y}_i})$ is an equivalence since $y_i \in Y_{\mathrm{tr}}$ ([8, th. 1.4]). If one choses a specialization $\bar{y}_2 \to \bar{y}_1$ above $\bar{s}_2 \to \bar{s}_1$, the functor $\mathrm{Cov}^{\mathrm{alg}}(X_{\bar{y}_i})^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(X_{\bar{y}_2})^{\mathbb{L}}$ is also an equivalence. One gets that F is an equivalence.

If Z_{s_1} is some geometric két covering of X_{s_1}, it extends thanks to corollary 4.6 to some két neighborhood $(U, \bar{u}_1)$ of $\bar{s}_1$ in T. Let $Z_U \to U$ be this extension (unique after replacing U by some smaller neighborhood of $\bar{s}_1$). Let $\bar{u}_2 \to \bar{u}_1$ be the unique lifting of $\bar{s}_2 \to \bar{s}_1$. Then $Z_{\bar{s}_2} := F(Z_{\bar{s}_1})$ is nothing but $Z_{\bar{u}_2}$. One has a cospecialization morphism $\mathrm{C}_{\mathrm{geom}}(Z_{\bar{s}_1}) \to \mathrm{C}_{\mathrm{geom}}(Z_{\bar{s}_2})$, which induces a specialization functor

$$\mathcal{D}_{\text{top-geom}\,X_{s_2}}(Z_{s_2}) \to \mathcal{D}_{\text{top-geom}\,X_{s_2}}(Z_{s_1}).$$

It is an equivalence of categories if $\bar{s}_2 \to \bar{s}_1$ is a cospecialization isomorphism or if $\overline{M}_{Y,\bar{s}_1} \to \overline{M}_{Y,\bar{s}_2}$ is an isomorphism and all the geometric fibers of $X \to Y$ have interiorly free polysimplicial sets.
Thus we have a 2-commutative diagram:

$$\begin{array}{ccc} \mathcal{D}_{\text{top-geom}\,X_{s_1}} & \to & \mathcal{D}_{\text{top-geom}\,X_{s_2}} \\ \downarrow & & \downarrow \\ \mathrm{KCov}_{\mathrm{geom}}(X_{s_1}/s_1)^{\mathbb{L}} & \overset{F^{-1}}{\to} & \mathrm{KCov}_{\mathrm{geom}}(X_{s_2}/s_2)^{\mathbb{L}} \end{array}$$

where F^{-1} is some quasi inverse of F. This induces a cospecialization outer morphism

$$\pi_1^{\text{temp-geom}}(X_{s_1}/s_1)^{\mathbb{L}} \to \pi_1^{\text{temp-geom}}(X_{s_2}/s_2)^{\mathbb{L}}.$$

The comparison morphisms of theorem 3.3 gives us the wanted morphism, which is an isomorphism if $\bar{s}_2 \to \bar{s}_1$ is a cospecialization isomorphism or if $\overline{M}_{Y,\bar{s}_1} \to \overline{M}_{Y,\bar{s}_2}$ is an isomorphism and all the geometric fibers of $X \to Y$ have interiorly free polysimplicial sets.

□

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