Introduction to character-sheaves

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1 k-structures

In what follows K is a field and k a subfield of k. If K/k is Galois, we denote by $\mathrm{Gal}(K/k)$ the corresponding Galois group.

By a *k-variety*, we shall mean a separated reduced k-scheme of finite type.

1.1 k-structures on vector spaces

Definition 1. *(1) A k-structure on a K-vector space V is a k-subspace V_o of V such that the natural morphism*

$$K \otimes_k V_o \to V, \qquad \lambda \otimes x \mapsto \lambda x$$

is an isomorphism. This is equivalent to saying that V has a basis whose elements form a basis of V_o.

(2) Given two K-vector spaces V and W equipped with k-structures V_o and W_o, we say that K-linear map $\varphi : V \to W$ is defined over k (i.e. compatible with the k-structures) if $\varphi(V_o) \subset W_o$.

We say that a k-vector space if defined over k if it equipped with a k-structure.

Lemma 1. *(1) If φ is defined over k, then $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$ come with natural k-structures.*

(2) For V and W defined over k, a K-linear map $\varphi : V \to W$ is defined over k if and only if its graph $\{(v, \varphi(v)) \in V \oplus W\, v \in V\}$ is a subspace of $V \oplus W$ defined over k.

Assume that K/k is Galois. If V_o is a k-structure on V, then we have an action of $\mathrm{Gal}(K/k)$ on V given by $\sigma \cdot (\lambda \otimes v_o) = \sigma(\lambda) \otimes v_o$ and

$$V_o = \{v \in V \,|\, \sigma(v) = v \text{ for all } \sigma \in \mathrm{Gal}(K/k)\}.$$

Proposition 2. *(1) Assume that V is defined over k. A subspace W of V is defined over k if and only if $\sigma(W) = W$ for all $\sigma \in \mathrm{Gal}(K/k)$.*

(2) Given two K-vector spaces V and W defined over k, a K-linear map $\varphi : V \to W$ is defined over k if $\varphi \circ \sigma = \sigma \circ \varphi$ for all $\sigma \in \mathrm{Gal}(K/k)$.

Proof. Exercise.

□

Proposition 3. *[4, Chap. 5, 10.4, prop. 7] Assume that K/k is a finite Galois extension and that $\mathrm{Gal}(K/k)$ acts k-linearly on a K-vector space V such that $\sigma(\lambda v) = \sigma(\lambda)\sigma(v)$ for all $v \in V$, $\lambda \in K$ and $\sigma \in \mathrm{Gal}(K/k)$. Then the k-subspace*

$$V_o := \{v \in V \,|\, \sigma(v) = v \textit{ for all } \sigma \in \mathrm{Gal}(K/k)\}$$

defines a k-structure on V.

We have the following consequence (see [5, Corollary 3.5] for details).

Corollary 4. *Let V be an $\overline{\mathbb{F}}_q$-vector space and let Φ be an $\mathbb{F}_q$-linear endomorphism of V such that $\Phi(\lambda v) = \lambda^q \Phi(v)$ for all $v \in V$ and $\lambda \in \overline{\mathbb{F}}_q$. If for each $x \in V$ there exists an integer $n > 0$ such that $\Phi^n(x) = x$, then $V^\Phi := \{v \in V \,|\, \Phi(v) = v\}$ defines an $\mathbb{F}_q$-structure on V, i.e. $V = V^\Phi \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.*

1.2 Varieties defined over $\mathbb{F}_q$ and Frobenius

Definition 2. *A k-structure on K-variety X is a k-variety X_o such that X is obtained from X_o by extension of scalars from k to K, namely*

$$\begin{array}{ccc} X = X_o \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) & \longrightarrow & X_o \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(k). \end{array}$$

If $X = \mathrm{Spec}(A)$, this means that there exists a finitely generated k-algebra A_o such that $A = A_o \otimes_k K$.

Say that a K-variety is defined over k if it is equipped with a k-structure.

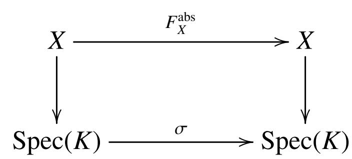
Example 1. *(1) The n-dimensional affine space $\mathbb{A}^n_K := \mathrm{Spec}(K[t_1, \ldots, t_n])$ has an obvious k-structure given by $A_o = k[t_1, \ldots, t_n]$ which by gluing provides a natural k-structure on the projective n-space $\mathbb{P}^n_K := \mathrm{Proj}\, K[x_o, \ldots, x_n]$. These k-structures are called the standard ones.*
(2) The $\mathbb{R}$-algebra $A_o = \mathbb{R}[i(x-y), x+y]$ defines an $\mathbb{R}$-structure on $\mathbb{A}^2_{\mathbb{C}}$. Similarly for $a \in \mathbb{F}_{q^2}$ such that $a^q = -a$, the $\mathbb{F}_q$-algebra $\mathbb{F}_q[a(x-y), x+y]$ defines an $\mathbb{F}_q$-structure on $\mathbb{A}^2_{\overline{\mathbb{F}}_q}$ which is not isomorphic to the standard one.

From now K will be $\overline{\mathbb{F}}_q$ and k will be $\mathbb{F}_q$, and let X be a K-variety. Note that

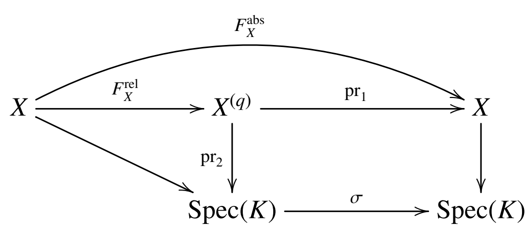
$$\mathrm{Gal}(K/k) = \varprojlim_n \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \simeq \hat{\mathbb{Z}} = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})\,.$$

Denote by σ the element of $\mathrm{Gal}(K/k)$ which acts on K as $x \mapsto x^q$. By notation abuse we will denote again σ is the induced map $\mathrm{Spec}(K) \to \mathrm{Spec}(K)$.

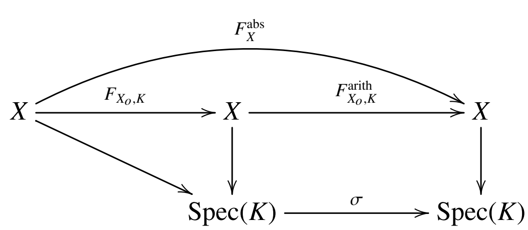
The *absolute Frobenius* $F^{\mathrm{abs}}_X : X \to X$ is the homeomorphism defined locally on $\mathrm{Spec}(A)$ by the comorphism $A \to A$, $P \mapsto P^q$. It makes the following diagram commutative



where the verticale arrows are the structure maps. Let $X^{(q)}$ be the fiber product $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^\sigma)$ where K^σ is the K-algebra with underlying ring K and structural map $K \to K^\sigma$, $x \mapsto x^q$. Then the *relative Frobenius* $F^{\mathrm{rel}}_X : X \to X^{(q)}$ is defined by the commutative diagram



A k-structure X_o on X, i.e. a decomposition $X = X_o \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$, defines a canonical isomorphism $X^{(q)} \simeq X$ and we call the composition of the relative Frobenius with this isomorphism the *geometric Frobenius* associated with X_o and denote it simply by $F_{X_o,K} : X \to X$ or simply F if there is no risk of confusion. The composition of $X \simeq X^{(q)}$ with $\mathrm{pr}_1 : X^{(q)} \to X$ is called the *arithmetic Frobenius* and is denoted by $F^{\mathrm{arith}}_{X_o,K} : X \to X$ or simply F^{arith}_X if there is no ambiguities. We thus have the following commutative diagram



The arithmetic Frobenius $F^{\mathrm{arith}}_{X_o,K}$ is exactly $\mathrm{id}_{X_o} \times \sigma$, it is an automorphism over k but not over K.

Locally, on $\mathrm{Spec}(A)$ with $A = A_o \otimes_k K$, the geometric Frobenius is given by $P \otimes \lambda \mapsto P^q \otimes \lambda$ and the arithmetic Frobenius by $P \otimes \lambda \mapsto P \otimes \lambda^q$ and the composition of the two yields the absolute Frobenius $P \otimes \lambda \mapsto (P \otimes \lambda)^q$.

Exercise 1. *Consider the projective line $\mathbb{P}^1_K$ equipped with its standard k-structure. Prove that $F^*(O(n)) \simeq O(qn)$ and that $(F^{\mathrm{arith}})^*(O(n)) \simeq O(n)$.*

Recall that for any field L, an L-point of an L-variety Y is a morphism $\mathrm{Spec}(L) \to Y$ over $\mathrm{Spec}(L)$. If $Y = \mathrm{Spec}(A)$, then an L-point is just a morphism $A \to L$ of L-algebras.

We denote by $X(K)$ the set of K-points of X.

A *k-point of a K-variety* X with k-structure X_o is a k-point of X_o. We will put $X(k) := X_o(k)$.

Example 2. *The K-points of $\mathbb{A}^n_K$ are naturally identified with K^n. Indeed*

$$\mathbb{A}^n_K(K) = \mathrm{Hom}_K(\mathrm{Spec}(K), \mathbb{A}^n_K) \simeq \mathrm{Hom}_{K-\mathrm{alg}}(K[x_1, \dots, x_n], K) \simeq K^n.$$

If X is equipped with a k-structure X_o, then the resulting (geometric) Frobenius $F : X \to X$ induces a map $F : X(K) \to X(K)$, $\varphi \mapsto F \circ \varphi$.

Lemma 5. *We have*

$$X(K)^F \simeq X(k).$$

Proof. We may assume without loss of generality that $X = \mathrm{Spec}(A)$ with $A = A_o \otimes_k K$. Then a K-point of X fixed by F corresponds to a morphism of k-algebras $f : A_o \to K$ such that $f(P^q) = f(P)^q = f(P)$, i.e. to a morphism of k-algebras $f : A_o \to k$.

□

Example 3. *If we equip $\mathbb{A}^n_K$ with its standard k-structure, the resulting Frobenius on K-points is $K^n \to K^n$, $(x_1, \dots, x_n) \mapsto (x_1^q, \dots, x_n^q)$.*

Exercise 2. *Equip $\mathbb{A}^2_K$ with the non-standard k-structure $k[a(x-y), x+y]$ where $a^q = -a$. Determine the corresponding Frobenius on K-points and describe the k-points.*

Exercise 3. *The symmetric group S_3 acts by permutation of the coordinates on K^3. Let w be an element of S_3 and F be the standard Frobenius on K^3. Show that $wF := w \circ F : K^3 \to K^3$ is a Frobenius corresponding to some k-structure on $K[x, y, z]$ that you will determine. Check that wF and $w'F$ corresponds to isomorphic k-structures if and only if w and w' leaves in the same conjugacy class of S_3.*

We have the following proposition (see [15, Proposition 11.2.8] for a more general statement and complete proof).

Proposition 6. *Assume that X is defined over k with corresponding Frobenius $F : X \to X$ and let Y be a locally closed K-subvariety of X. Then Y is defined over k if and only if Y is F-stable. In this case the corresponding Frobenius on Y is the restriction of F to Y.*

Proof. Assume $X = \mathrm{Spec}(A)$ and Y a closed subvariety of X defined by an ideal I of A. Suppose that Y is F-stable. Then Y is also stable by the arithmetic Frobenius F^{arith} and so is I. By Corollary 4 we have $I = I_o \otimes_k K$ with I_o the ideal $\{v \in I \mid F^{\mathrm{arith}}(v) = v\}$ of A_o. □

Corollary 7. *Let X and Y be two K-varieties with k-structures X_o and Y_o. A morphism $f : X \to Y$ of K-varieties is defined over k (i.e. is obtained by extension of scalars from $f_o : X_o \to Y_o$) if and only if*

$$f \circ F_{X_o,K} = F_{Y_o,K} \circ f.$$

Proof. Apply Proposition 6 to the graph of f. □

A practical proposition [5, Proposition 3.3].

Proposition 8. *Assume that* $X = \mathrm{Spec}(A)$.
(1) Assume that $F : X \to X$ *is the (geometric) Frobenius associated with a k-structure* A_o *on A, then*

$$A_o = \{x \in A \mid F(x) = x^q\}.$$

(2) Let F be a surjective morphism of K-algebras $A \to A^q$. *Then F is the geometric Frobenius associated with a k-structure if and only if for any* $x \in A$ *there exists a positive integer n such that* $F^n(x) = x^{q^n}$.

Proof. The assertion (1) follows from the fact that $F \circ F^{\mathrm{arith}}(x) = x^q$ and $A_o = \{x \in A \mid F^{\mathrm{arith}}(x) = x\}$. Let us prove (2). Asume first that F is a geometric Frobenius associated with a k-structure A_o. Let $x = \sum_i x_i \otimes \lambda_i$ with $x_i \in A_o$ and $\lambda_i \in K$. Let n be such that all $\lambda_i \in \mathbb{F}_{q^n}$. Then

$$F^n(x) = \sum_i x_i^{q^n} \otimes \lambda_i = \sum_i x_i^{q^n} \otimes \lambda_i^{q^n} = \left(\sum_i x_i \otimes \lambda_i\right)^{q^n} = x^{q^n}.$$

The assumption on F implies that it is injective. The map $G : A \to A$, $x \mapsto F^{-1}(x^q)$ satisfies the assumption of Corollary 4 and so $A_o := A^G$ defines a k-structure on A and G is the arithmetic Frobenius for that k-structure. □

We now have the following very useful proposition (see [5, Proposition 3.6]).

Proposition 9. *Let X be a K-variety defined over k with Frobenius* $F : X \to X$.

(i) If φ *is an automorphism of X such that* $(\varphi F)^n = F^n$ *for some positive integer n, then* φF *is the Frobenius associated with a k-structure on X.*
(ii) If F' *is an other Frobenius on X, then for some positive integer n we have* $F^n = F'^n$.

Proof. The assertion (ii) follows from the fact that X is of finite type. The assertion (i) reduces to the affine case which we prove from Proposition 8 (2).

□

Proposition 10. *For any k-structure on the n-dimensional affine space* $\mathbb{A}^n_K$, *we have* $|\mathbb{A}^n_K(k)| = q^n$.

Proof. This is an easy consequence of the Grothendieck trace formula for ℓ-adic cohomology (which we will see later). □

2 Review on algebraic groups

2.1 Definition (from scheme theoritical point of vue)

In what follows, K is any field. We denote by Sch_K the category of K-schemes, Alg_K the category of K-algebras, Sets the category of sets and Groups the category of groups.

The *functor of points* of a K-scheme X is the functor $h_X : \mathrm{Sch}_K^{\mathrm{op}} \to \mathrm{Sets}$, $T \to \mathrm{Mor}_K(T, X)$.

If $T = \mathrm{Spec}(A)$, we will write simply $h_X(A)$ instead of $h_X(\mathrm{Spec}(A))$. Note that $h_X(K)$ is precisely the set of K-points of X.

A functor $f : \mathrm{Sch}_K^{\mathrm{op}} \to \mathrm{Sets}$ is *representable* by a K-scheme if it is of the form h_X for some K-scheme X.

By Yoneda's lemma, if such an X representing f exists, it must be unique up to isomorphism. In fact we have more (see for instance [6, Proposition VI-2]):

Lemma 11. *The functor of points* $h_X : \mathrm{Sch}_K^{\mathrm{op}} \to \mathrm{Sets}$ *is completely determined by its restriction to affine schemes* $\mathrm{Alg}_K \to \mathrm{Sets}$, $A \mapsto \mathrm{Mor}_K(\mathrm{Spec}(A), X)$.

The point of vue of functors is very convenient for the study of algebraic groups :

Definition 3 (First definition)**.** *A K-*algebraic group *is a K-variety G whose functor of points* h_G *factorizes through* $h_G : \mathrm{Sch}_K^{\mathrm{op}} \to \mathrm{Groups}$, *i.e. for every K-scheme T, the set of* $h_G(T) = \mathrm{Mor}_K(T, G)$ *is an abstract group and for any any K-morphism* $T \to T'$, *the map* $h_G(T') \to h_G(T)$ *is an abstract group homomorphism.*

The above definition is equivalent to the following one :

Definition 4 (Second definition)**.** *A K-*algebraic group *is a K-variety G together with*

(i) a morphism $m : G \times_K G \to G$ (multiplication),

(ii) a morphism $\iota : G \to G$ (inverse),

(iii) a K-point $e : \mathrm{Spec}(K) \to G$ (the unit),

such that the following usual axioms are satisfied :

(1) (associativity) $m(m \times \mathrm{id}_G) = m(\mathrm{id}_G \times m)$,

(2) (unit) $m(\mathrm{id}_G \times e) : G \times_K \mathrm{Spec}(K) \to G$ is the projection on the first coordinate and $m(e \times \mathrm{id}_G) : \mathrm{Spec}(K) \times_K G \to G$ is the projection on the second coordinate.

(3) (inverse) $m(\mathrm{id}_G \times \iota)d = m(\iota \times \mathrm{id}_G)d$ is the composition $G \to \mathrm{Spec}(K) \xrightarrow{e} G$ where the first map is the structural map and $d : G \to G \times_K G$ is the diagonal embedding.

A morphism of K-algebraic groups is a morphism of K-varieties that is compatible with the maps (i), (ii) and (iii).

The equivalence between the two above definitions generalizes as follows :

Exercise 4. *Consider a category C with finite products and terminal object denoted by 1. A group object of C is an object of C together with morphisms $m : G \times G \to G$, $\iota : G \to G$ and $e : 1 \to G$ such that the axioms analogous to (1), (2) and (3) in the above definition hold. Prove that G is a group object if and only if $X \mapsto \mathrm{Mor}(X, G)$ is a functor $C^{\mathrm{op}} \to$ Groups.*

A K-algebraic group is *affine* if the underlying scheme is affine. Via the natural identification

$$\mathrm{Mor}_K(\mathrm{Spec}(B), \mathrm{Spec}(A)) = \mathrm{Mor}_{K-\mathrm{alg}}(A, B),$$

we will think of the functor of points of an affine K-algebraic group $G = \mathrm{Spec}(A)$ as the functor $\mathrm{Alg}_K \to$ Groups, $B \mapsto \mathrm{Mor}_K(A, B)$.

Example 4. *(1) The functor $\mathrm{Alg}_K \to$ Groups, $A \mapsto (A, +)$ is represented by the K-algebraic group $\mathbb{G}_a := \mathrm{Spec}(K[t])$. Its set of K-points is $\mathbb{G}_a(K) = K$.*
(2) The functor $A \mapsto A^\times$ is represented by the K-algebraic group $\mathbb{G}_m := \mathrm{Spec}(K[t, t^{-1}])$.
(3) The functor $A \mapsto \mathrm{GL}_n(A)$ is represented by the K-algebraic group $\mathrm{GL}_n := \mathrm{Spec}(K[t_{i,j}, \det(t_{i,j})^{-1}])$. Its set of K-points is just $\mathrm{GL}_n(K)$.

2.2 Definition (the classical way)

There are many good references [3][9][15] for a detailed exposition. From now, unless specified, K is an algebraically closed field. An affine K-algebraic variety X in the **classical sense** is a closed subset of K^n, for some positive integer n, equipped with Zariski topology. The K-algebra of *regular* functions $K[X]$ on X are the restrictions of polynomial functions $K^n \to K$, $(x_1, \ldots, x_n) \mapsto P(x_1, \ldots, x_n)$ to X. It is of the form $K[X] = K[T_1, \ldots, T_n]/I$ where I is the ideal of polynomials vanishing on X. Any reduced K-algebra of finite type arise in this form. The set of K-points of $\mathrm{Spec}(K[X])$ corresponds to the subset of maximal ideals (equipped with the induced Zariski topology) which subset is homeomorphic to X (to a point of X associate the ideal of polynomials that vanish at that point).

Therefore, over algebraically closed fields, we do not loose information if we work with algebraic varieties in the classical sense.

A K-algebraic variety X in the classical sense is a topological space obtained by gluing together a finite number of affine K-varieties (using local charts analogously to differential manifolds) and satisfying the Hausdorff axiom (i.e. the diagonal embedding $X \to X \times X$ is a closed immersion). A K-variety must be a noetherian topological space, namely a strictly decreasing sequence of closed subsets (for inclusion) must be finite.

Definition 5 (Algebraic groups in the classical sense)**.** *(1) An algebraic group over K is an algebraic variety G equipped with a group structure such that the product $m : G \times G \to G$ and $G \to G$, $x \mapsto x^{-1}$ are morphisms of algebraic varieties.*
(2) A morphism of algebraic groups $G \to G'$ is a group homomorphism which is also a morphism of algebraic varieties.

Example 5. $\mathrm{GL}_n(K) = \{x \in \mathrm{Mat}_n(K) \,|\, \det(x) \neq 0\}$ *is an open subset of the affine space* $\mathrm{Mat}_n(K)$ *and so is an algebraic variety which is compatible with the group structure. This algebraic group is affine because of the isomorphism* $\mathrm{GL}_n(K) \to \{(g, y) \in \mathrm{Mat}_n(K) \times K \,|\, y\det(g) = 1\}$ *given by*

$$g \mapsto (g, \det(g)^{-1}),$$

and the K-algebra of regular functions on $\mathrm{GL}_n(K)$ *is* $K[t_{i,j}, \det(t_{i,j})^{-1}]$.

The set of K-points of K-algebraic groups in the scheme theoritical sense are algebraic groups in the classical sense and algebraic groups in the classical sense are all obtained in this way.

It is convenient to keep in mind the definition of algebraic groups from scheme theory (when talking of k-structure for instance) but for most of our purpose of the point of view of classical algebraic geometry will suffice.

Therefore from now on, unless specified, we will use the terminology algebraic group in the context of classical algebraic geometry.

We fix an algebraic group G over K and we denote by 1 the identity element.

We have the following basic result.

Proposition 12. *(i) There exists a unique irreducible component G^o of G that contains* 1. *This is a closed normal subgroup of finite index in G.*
(ii) The irreducible components of G are also the connected components and they are all of the form gG^o for some $g \in G$.

Proof. Denote by $X_1, \ldots, X_n$ the irreducible components passing through 1. Then the image $X_1 \cdots X_n$ of $X_1 \times \cdots \times X_n$ by the multiplication in G must contain 1 and is irreducible. In particular it must be contained in some X_i but on the other hand each X_i is contained in $X_1 \cdots X_n$, and so we must have $n = 1$. Let us prove that G^o is a subgroup of G. Choose $x \in G^o$, then $x^{-1}G^o$ is an irreducible component of G containing 1 and so we must have $x^{-1}G^o = G^o$, i.e. $x^{-1} \in G^o$ and $xy \in G^o$ for any $x, y \in G^o$, hence the result. It is normal because $x^{-1}G^o x$ is an irreducible component and so equals G^o. The cosets of G^o in G are exactly the irreducible components and so G^o is of finite index in G. □

Say that G is connected if $G = G^o$.

Example 6. *The algebraic group* $\mathrm{GL}_n(K)$ *is connected as it is an open subset of* $\mathrm{Mat}_n(K) \simeq K^{n^2}$ *which is clearly irreducible.*

We have the following result (see [9, Proposition B, §7.4] for a proof).

Proposition 13. *Let $f : G \to G'$ be a morphism of algebraic groups.*
(i) $\mathrm{Ker}(f)$ *is closed normal subgroup of G.*
(ii) $\mathrm{Im}(f)$ *is a closed subgroup of G'.*
(iii) $f(G^o) = f(G)^o$.
(iv) $\dim G = \dim \mathrm{Ker}(f) + \dim \mathrm{Im}(f)$.

2.3 G-varieties

We remain in the context of classical algebraic geometry. Let X be a K-variety and G be a K-algebraic group acting (set-theoritically) on X. We say that X is a *G-variety* if this action is algebraic, namely if the map $G \times X \to X$, $(g, x) \mapsto g \cdot x$ is a morphism of algebraic varieties. If moreover this action is transitive, we say that X is an homogeneous G-variety. If $f : X \to Y$ is a morphism between two G-varieties, we say that f is *G-equivariant* if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$, $x \in X$.

Example 7. *(i) The action of G on itself by conjugation makes G a G-variety.*
(ii) The action of G on itself by left (or right) translation makes G an homogeneous G-variety.

Note that an homogeneous G-variety X must be smooth (i.e. non-singular). Indeed, the set of smooth points of X is not empty and the automorphism $X \to X$, $x \mapsto g \cdot x$, with $g \in G$, maps smooth points to smooth points. In particular, by the above example G is a smooth variety.

Lemma 14. *Let X be a G-variety.*
(i) Each G-orbit is a smooth locally closed subset of X (i.e. open in their Zariski closure).
(ii) There exists a closed G-orbit.

Proof. The image $Y = G \cdot x$ of the map $G \to X$, $g \mapsto g \cdot x$ is constructible (Chevalley's theorem) and so it contains a dense open subset U of $\overline{Y}$. The variety $\overline{Y}$ is clearly G-stable and the union of all $g \cdot U$, where g runs over G, is an open subset of $\overline{Y}$ but is also equal to Y as Y is homogeneous. Hence Y is locally closed, it is therefore an algebraic variety and it must be smooth because it is homogeneous. We now see that there exists a closed orbit in $\overline{Y}$. Indeed if Y is not closed, then take a G-orbit O_1 of the complementary $\overline{Y} - Y$. Its Zariski closure $\overline{O_1}$ is strictly contained in $\overline{Y}$. If it is not closed, take a G-orbit O_2 in $\overline{O_1} - O_1$. Repeating the process we obtain a strictly decreasing sequence of Zariski closures of G-orbits. Since X is noetherian this sequence must stop, namely we must end with a closed G-orbit. □

Exercise 5. *Consider the action of $G = \mathrm{GL}_2(K)$ on itself by conjugation.*
(1) Let

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Determine the decomposition of $\overline{G \cdot x}$ as a union of G-orbits.
(2) Determine the closed G-orbits.

2.4 Lie algebra of an algebraic group

Definition 6. *Let k be a field. A k-Lie algebra is a k-vector space L equipped with a k-bilinear map $L \times L \to L$, $(x, y) \mapsto [x, y]$ such that*
(i) $[x, y] = -[y, x]$, for all $x, y \in L$,
(ii) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in L$ (Jacobi identity).

[,] is called the *Lie bracket*.

Example 8. *Let A be an associative algebra.*
(1) The bracket $[a, b] = ab - ba$ endows A with a structure of Lie algebra.
(2) We put

$$\mathrm{Der}_k(A) := \{D \in \mathrm{End}_k(A) \,|\, D(ab) = aD(b) + D(a)b\}.$$

This is a Lie algebra for the bracket $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$.

As before K is an algebraically closed field and G is a affine K-algebraic group.
Denote by O_G the structure sheaf of regular functions on G and for $g \in G$ put

$$O_{G,g} = \varinjlim_{U \ni g} O_G(U).$$

Since G is affine, this is also the localization of the K-algebra of regular functions $K[G] = O_G(G)$ at the maximal ideal $\mathfrak{m}_g$ of functions vanishing at g, that is

$$O_{G,g} = \left\{ \frac{f}{h} \,\middle|\, f, h \in K[G], h \notin \mathfrak{m}_g \right\}.$$

We now define the Lie algebra $\mathrm{Lie}(G)$ of G as the tangent space T_1G of G at 1, namely

$$\mathrm{Lie}(G) := \mathrm{Der}_K(O_{G,1}, K) = \{D \in \mathrm{End}_K(O_{G,1}, K) \,|\, D(fg) = f(1)D(g) + g(1)D(f)\}.$$

Since G is affine, we have

$$\mathrm{Lie}(G) \simeq \mathrm{Der}_K(K[G], K) = \{D \in \mathrm{End}_K(K[G], K) \mid D(fg) = f(1)D(g) + g(1)D(f)\}.$$

We want to put a Lie algebra structure on $\mathrm{Lie}(G)$.
For $g \in G$, consider $\lambda(g) : G \to G$, $z \mapsto g^{-1}z$ and its co-morphism

$$\lambda(g)^{\#} : K[G] \to K[G], \qquad f \mapsto f \circ \lambda(g).$$

Then G acts on $\mathrm{End}_K(K[G])$ as

$$g \cdot D = \lambda(g)^{\#} \circ D \circ \lambda(g^{-1})^{\#},$$

and this action leaves stable $\mathrm{Der}_K(K[G])$. The space of invariant derivations

$$\mathrm{Der}_K(K[G])^{\lambda(G)} = \{D \in \mathrm{Der}_K(K[G]) \mid \lambda(g)^{\#} \circ D \circ \lambda(g^{-1})^{\#} = D \text{ for all } g \in G\}$$

is a Lie subalgebra of $\mathrm{Der}_K(K[G])$ which we denote by $L(G)$.

Proposition 15. *The map*

$$L(G) \to \mathrm{Lie}(G), \qquad D \mapsto \epsilon \circ D,$$

where $\epsilon : K[G] \to K$, $f \mapsto f(1)$, is an isomorphism of vector spaces.

We use the above isomorphism to put a Lie algebra structure on $\mathrm{Lie}(G)$.

Proposition 16. *If $f : G \to H$ is a morphism of algebraic groups then its differential $df : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ at 1 is an homomorphism of Lie algebras.*

Since G is smooth, notice that $\dim G = \dim \mathrm{Lie}(G)$.

Lemma 17. *Let H be a closed subgroup of G and let $f_1, \ldots, f_r$ be the generators of the ideal of H in $K[G]$. Then*

$$\mathrm{Lie}(H) = \{x \in \mathrm{Lie}(G) \mid df_1(x) = \cdots = df_r(x) = 0\}.$$

Example 9. *(1) Put $G = \mathrm{GL}_n(K)$, then $\mathrm{gl}_n(K) := \mathrm{Lie}(G) = \mathrm{Mat}_n(K)$ with Lie algebra structure given by $[X, Y] = XY - YX$.*
(2) If $G = \mathrm{SL}_n(K) = \{g \in \mathrm{GL}_n(K) \mid \det(g) = 1\}$, then

$$\mathrm{sl}_n(K) := \mathrm{Lie}(G) = \{X \in \mathrm{Mat}_n(K) \mid \mathrm{Tr}(X) = 0\}.$$

Lemma 18. *Denote the multiplication on G by $m : G \times G \to G$ and $\iota : G \to G$, $x \mapsto x^{-1}$. Then $dm(x, y) = x + y$ and $d\iota(x) = -x$ for all $x, y \in \mathrm{Lie}(G)$.*

Proof. Exercise ! □

2.5 Separable morphisms

See [15] for more details.

Consider an inclusion of fields $E \subset F$. Recall that F is a *separable algebraic* field extension of E if for each $f \in F$, there exists a polynomial $P \in E[t]$ which does not have multiple roots and $P(f) = 0$. If the characteristic of E equals 0 then, any algebraic extension of E is separable.

Assume that F is a of finite type over E, namely $F = E(x_1, \ldots, x_m)$. Recall that the *transcendence degree* $\mathrm{Trdeg}_E(F)$ of F over E is the maximal number of x_i's which are algebraically independent over E. We say that the extension $E \subset F = E(x_1, \ldots, x_m)$ is *purely transcendental* if $\mathrm{Trdeg}_E(F) = m$.

Definition 7. *We say that the field extension $E \subset F$ is* separably generated *if there exist field inclusions $E \subset E' \subset F$ such that $E' \subset F$ is separable algebraic and $E \subset E'$ is purely transcendental.*

Note that if the characteristic of E equals 0, then all field extension of finite type are separably generated.

A morphism $\varphi : X \to Y$ of irreducible K-algebraic varieties is said to be *dominant* if $\varphi(X)$ is dense in Y. In this case we have an injective map between the fields of rational functions $K(Y) \hookrightarrow K(X)$. We say that φ is *separable* if the extension $K(Y) \subset K(X)$ is separably generated.

Theorem 19. *Let G be a connected K-algebraic group, X and Y two homogeneous G-varieties and let $\varphi : X \to Y$ be a G-equivariant morphism.*

(a) φ is separable if and only if $d_x\varphi$ is surjective for some $x \in X$.

(b) φ is an isomorphism if and only if φ is separable and bijective.

Corollary 20. *Let $\varphi : G \to H$ be a bijective morphism of algebraic groups. If $d\varphi$ is surjective then it is an isomorphism.*

2.6 Linear algebraic groups and Jordan decomposition

A *linear algebraic group* over K is a closed subgroup of $\mathrm{GL}_n(K)$.

Theorem 21. *Affine linear algebraic groups are linear.*

Let G be an affine algebraic group, then it acts (abstractly) on $K[G]$ as $(gf)(x) = f(xg)$, with $g \in G$, $f \in K[G]$ and $x \in G$.

We start with the following proposition.

Proposition 22. *Let G be an affine algebraic group acting on an affine variety X, and let F be a finite dimensional subspace of $K[X]$.*

(a) There exists a finite dimensional subspace E of $K[X]$ containing F and which is G-stable.

(b) F is G-stable if and only if $\varphi^\# F \subset K[G] \otimes_K F$ where $\varphi : G \times X \to X$, $(g, x) \mapsto g \cdot x$.

Proof. See [9, §8.6]. □

Proof of Theorem 21. Let $x_1, \dots, x_n \in K[G]$ such that $K[G] = K[x_1, \dots, x_n]$ and let V be the G-submodule of $K[G]$ generated by $x_1, \dots, x_n$. It is a finite dimensional K-vector space by Proposition 22(a). It remains to see that $\rho : G \to \mathrm{GL}(V)$ is a closed immersion. Let $f_1, \dots, f_r$ be a basis of V. Applying Proposition 22(b) with $F = V$ and $\varphi : G \times G \to G$, $(x, y) \mapsto (yx)$, we see that there exist $c_{ij} \in K[G]$ such that

$$\varphi^\#(f_i) = \sum_j c_{ij} \otimes f_j. \tag{1}$$

This proves that ρ is a morphism of algebraic groups. To see that this is a closed immersion, we need to see that the co-morphism $K[\mathrm{GL}(V)] \to K[G]$ is surjectif. But this follows from (1). Indeed, we have $c_{ij} = C_{ij} \circ \rho = \rho^\#(C_{ij})$ and for all $j = 1, \dots, r$ and $g \in G$, we have

$$f_j(g) = (gf_j)(1) = \sum_{i=1}^{r} c_{ij}(g) f_i(1),$$

and so $f_j = \rho^\# \left(\sum_{i=1}^r f_i(1) C_{i,j} \right)$. □

By a *semisimple element* of gl_n, we shall mean a diagonalizable matrix, i.e. a matrix x such that for some $g \in \mathrm{GL}_n$, the matrix gxg^{-1} is diagonal. An element $x \in \mathrm{gl}_n$ is *nilpotent* if $x^n = 0$, an element $g \in \mathrm{GL}_n$ is *unipotent* if $g - 1$ is nilpotent. The Jordan decomposition in gl_n tells us that any matrix x decomposes uniquely as $x_s + x_n$ with x_s semisimple, x_n nilpotent such that $[x_s, x_n] = 0$. Any matrix $g \in \mathrm{GL}_n$ decomposes uniquely as $g = g_s g_u$ with g_s semisimple, g_u unipotent and $g_s g_u = g_u g_s$.

Let G be a linear algebraic group and consider a closed immersion $\rho : G \to \mathrm{GL}(V)$.

Proposition 23 (Jordan decomposition)**.** *(1) For all $g \in G$, there exist unique $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$ and such that $\rho(g) = \rho(g_s)\rho(g_u)$ is the multiplicative Jordan decomposition of $\rho(g)$.*
(2) For all $x \in \mathrm{Lie}(G)$, there exist unique $x_s, x_n \in \mathrm{Lie}(G)$ such that $x = x_s + x_n, [x_s, x_n] = 0$ and $d\rho(x) = d\rho(x_s) + d\rho(x_n)$ is the Jordan decomposition of $d\rho(x)$.
(3) The decompositions $g = g_s g_u$ and $x = x_s + x_n$ do not depend on the closed immersion ρ.

We call g_s and x_s the semisimple parts of g and x, g_u the unipotent part of g and x_n the nilpotent part of x. An element g of G (resp. x of $\mathrm{Lie}(G)$) is called semisimple if $g = g_s$ (resp. $x = x_s$), unipotent (resp. nilpotent) if $g = g_u$ (resp. $x = x_n$).

We denote by G_{ss}, $\mathrm{Lie}(G)_{\mathrm{ss}}$, G_{uni} and $\mathrm{Lie}(G)_{\mathrm{nil}}$ the subsets of semisimple, unipotent and nilpotent elements.

Proposition 24. *If $G \to H$ is a morphism of algebraic groups then*

$$f(g_s) = f(g)_s, \qquad f(g_u) = f(g)_u, \qquad df(x_s) = df(x)_s, \qquad df(x_n) = df(x)_n.$$

Remark 1. $\mathrm{gl}_n(K)_{\mathrm{nil}} = \{x \in \mathrm{gl}_n \,|\, x^n = 0\}$ *and* $\mathrm{GL}_n(K)_{\mathrm{uni}} = \{g \in \mathrm{GL}_n \,|\, (g-1)^n = 0\}$ *are closed subvarieties of* gl_n *and* GL_n *respectively.*

Proposition 25. G_{uni} *and* $\mathrm{Lie}(G)_{\mathrm{nil}}$ *are closed subvarieties of G and* $\mathrm{Lie}(G)$.

Proof. We have $G_{\mathrm{uni}} = \{g \in G \,|\, \rho(g) \in \mathrm{GL}_n(K)_{\mathrm{uni}}\}$ and $\mathrm{Lie}(G)_{\mathrm{nil}} = \{x \in \mathrm{Lie}(G) \,|\, d\rho(x) \in \mathrm{gl}_n(K)_{\mathrm{nil}}\}$. □

Exercise 6. *(1) Using Jordan decomposition determines the different types of conjugacy classes of* $\mathrm{GL}_2(K), \mathrm{GL}_3(K)$.
(2) Same problem with $\mathrm{GL}_i(K)$*-orbits of* $\mathrm{gl}_i(K)$*, with $i = 2, 3$, for the conjugation action.*

2.7 Quotients

Let X be a K-algebraic variety on which an affine K-algebraic group G acts.

Definition 8 (Categorical quotient)**.** *A* categorical quotient *of the G-variety X is a morphism $p : X \to Y$ constant on G-orbits such that for any morphism $f : X \to Z$ constant on G-orbits there exists a unique morphism $\overline{f} : Y \to Z$ such that $\overline{f} \circ p = f$.*

If a categorical quotient of X exists, it is unique up to isomorphism by definition but may not be surjective.

We have the more refined notion of *good quotient* defined as follows.

Definition 9 (Good quotient)**.** *Let $p : X \to Y$ be a map constant on G-orbits. It is a* good quotient *if the following assertions are satisfied :*

(i) For any G-stable closed subset W of X, the set $p(W)$ is closed in Y.

(ii) If W_1 and W_2 are G-stable closed subsets of X with $W_1 \cap W_2 = \emptyset$, then $p(W_1) \cap p(W_2) = \emptyset$.

(iii) For any open subset U of Y, the homomorphism of rings $p^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(p^{-1}(U))$ is an isomorphism onto $\mathcal{O}_X(p^{-1}(U))^G$.

Lemma 26. *Assume that $p : X \to Y$ is a good quotient.*

(i) The map p is surjective.

(ii) For any subset U of Y, U is open if and only if $p^{-1}(U)$ is open.

Proof. We have $\mathcal{O}_Y \hookrightarrow p_*\mathcal{O}_X$ hence p is dominant. It follows from Definition 9(i) that p is surjective. Let us prove (ii). Suppose that $p^{-1}(U)$ is open. Then $W = X \backslash p^{-1}(U)$ is G-stable and closed and so $p(W)$ is closed. But $p(W) = X \backslash U$ because p is surjective hence U is open in Y. □

Lemma 27. *If $p : X \to Y$ is a good quotient then Y parametrizes the closed G-orbits of X.*

Proof. If O and O' are two closed G-orbits distinct then $p(O) \cap p(O') = \emptyset$. Hence the map

$$\{\text{closed } G - \text{orbits of } X\} \to Y, \qquad O \mapsto p(O),$$

is injective. To see the surjectivity we need to see that $p^{-1}(y)$ contains a closed orbit of X. Let O a G-orbit contained in $p^{-1}(y)$, then $\overline{O} \subset p^{-1}(y)$ but $\overline{O}$ contains a closed orbit. □

Remark 2. *A quotient $p : X \to Y$ induces a map $\tilde{p} : \{G\text{–orbits of } X\} \to Y$. If p is a good quotient, we may identify Y with the closed G-orbits of X by the above lemma and describe $\tilde{p}$ as follows. Take a G-orbit O of X and let O' be the unique closed orbit contained on $\overline{O}$. Then $\tilde{p}(O) = O'$.*

Proposition 28. *A good quotient is a categorical quotient.*

Proof. Let $q : X \to Z$ be constant on G-orbits. Let $\{V_i\}$ be a covering of Z by open affine subsets. Then $q^{-1}(V_i)$ is a G-stable open subset of X. Put $W_i = X \backslash q^{-1}(V_i)$. It is a G-stable closed subset, hence $U_i = Y \backslash p(W_i)$ is open in Y. Clearly we have $p^{-1}(U_i) \subset q^{-1}(V_i)$. Since $\bigcap_i W_i = \emptyset$ we have $\bigcap_i p(W_i) = \emptyset$ and so $Y = \bigcup_i U_i$. We compose $\alpha_i : \mathcal{O}_Z(V_i) \to \mathcal{O}_Y(q^{-1}(V_i))^G$ with the restriction $\mathcal{O}_X(q^{-1}(V_i))^G \to \mathcal{O}_X(p^{-1}(U_i)^G) \simeq \mathcal{O}_Y(U_i)$ to get an homomorphism $\mathcal{O}_Z(V_i) \to \mathcal{O}_Y(U_i)$. Since V_i is affine this defines a morphism $U_i \to V_i$ whose composition with $p^{-1}(U_i) \to U_i$ is the map $q : p^{-1}(U_i) \to V_i$. By gluing together these maps we get a morphism $Y \to Z$. □

Consider

$$\Psi : G \times X \to X \times X, \qquad (g, x) \mapsto (g \cdot x, x).$$

A good quotient $p : X \to Y$ of X is called a *geometric quotient* if the image of Ψ equals $X \times_Y X$ (note that by the G-equivariance of f, the morphism Ψ factorises through $X \times_Y X$).

We usually denote by $X//G$ for a categorical quotient and by X/G when it is a geometric quotient.

Note that if a geometric quotient of X by G exists, it parametrizes all the G-orbits of X (not only the closed ones) which explains the set-theoritic notation X/G.

Exercise 7. *(i) Determine the categorical quotient of $\mathbb{G}_m$ acting on $\mathbb{A}^n$ by multiplication on coordinates. Is this quotient geometric ?*
(ii) Prove that the map $\mathrm{Mat}_2(K) \to \mathbb{A}^2$, $x \mapsto (\mathrm{Tr}(x), \det(x))$ is the categorical quotient of $\mathrm{Mat}_2(K)$ by $\mathrm{GL}_2(K)$ for the conjugation action. Prove that it is not a geometric quotient.

We need the following theorem.

Theorem 29. *Assume that G is a finite group acting on a finitely generated K-algebra A. Then the subalgebra A^G is finitely generated over K.*

Theorem 30. *Let G be a finite group acting on $X = \mathrm{Spec}(A)$ affine. Put $Y := \mathrm{Spec}(A^G)$. Then the map $X \to Y$ given by the inclusion $A^G \hookrightarrow A$ is a geometric quotient.*

Exercise 8. *(i) Assume that the characteristic of the field K is different from 2. Determine the geometric quotient of $\mathbb{A}^1$ by $\mu_2 = \{1, -1\}$.*
(ii) Let S_n be the symmetric group acting on $T = \mathbb{G}_m^n$ by permutation of the coordinates. Determine the quotient map $T \to T//S_n$.

Theorem 31 (Chevalley)**.** *Let H be a closed subgroup of G, then the geometric quotient $G \to G/H$ exists and is a separable morphism. If moreover H is a normal subgroup then G/H is an affine algebraic group.*

Example 10. *Let B_2 denotes the upper triangular matrices. Then GL_2/B_2 is isomorphic to the projective line via the map $\mathrm{GL}_2 \to \mathbb{P}^1$ given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [a, c].$$

Exercise 9. *Let $\mathrm{gl}_{2,1}$ be the 2×2-matrices of rank 1. Let $\mathbb{G}_m$ acts by scalar multiplication. Show that the quotient of $\mathrm{gl}_{2,1}$ by $\mathbb{G}_m$ exists and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

Denote by $\mathrm{Gr}_d(V)$ the grassmannian of d-dimensional subspaces in V. Then we can regard $\mathrm{Gr}_d(V)$ as a projective variety via the Plücker embedding

$$\mathrm{Gr}_d(V) \hookrightarrow \mathbb{P}(\Lambda^d V), \qquad E \mapsto \Lambda^d E.$$

A sequence $(0) = V_o \subset V_1 \subset \cdots \subset V_n = V = K^n$ of K-vector spaces with $\dim V_i = i$ is called a flag in K^n. We denote by $\mathcal{F}(V)$ the set of all flags in V. Then

$$\mathcal{F}(V) = \left\{ (U_1, \ldots, U_{n-1}) \in \prod_{i=1}^{n-1} \mathrm{Gr}_i(V) \,|\, U_1 \subset \cdots \subset U_{n-1} \right\},$$

defining a structure a projective variety on $\mathcal{F}(V)$.

Note that GL_n acts transitively on $\mathcal{F}(V)$. Consider the standard basis $e_1, \ldots, e_n$ of V and the corresponding standard flag $F = (F_i) \in \mathcal{F}(V)$ with $F_i = \langle e_1, \ldots, e_i \rangle$. Then the stabiliser of F in GL_n is exactly the closed subgroup B_n of upper triangular matrices and so

$$\mathrm{GL}_n/B_n \simeq \mathcal{F}(V),$$

from which we see that GL_n/B_n is a projective variety.

In a different direction consider a semisimple element $s \in \mathrm{GL}_n(K)$. Its stabiliser $H_s = C_{\mathrm{GL}_n}(s)$ in GL_n is isomorphic to $\prod_{i=1}^r \mathrm{GL}_{n_i}(K) \subset \mathrm{GL}_n$ and in this case we have

$$\mathrm{GL}_n/H_s = \mathrm{Spec}(K[\mathrm{GL}_n]^{H_s}).$$

We see here that the quotient is now an affine variety (by the theorem below, the K-algebra $K[\mathrm{GL}_n]^{H_s}$ is finitely generated as H_s is reductive, see below).

Say that an affine group is *reductive* if it does not have a closed connected normal unipotent subgroup, e.g. GL_n, SL_n, $\mathrm{PGL}_n = \mathrm{GL}_n/\mathbb{G}_m$, SO_N,...

By Theorem 31, we know that the group $\mathrm{PGL_n}$ is an affine algebraic group and so is linear.

Exercise 10. *Embed* $\mathrm{PGL}_n(K)$ *in some* $\mathrm{GL}(V)$ *using an appropriate action of* $\mathrm{GL}_n(K)$ *on* $\mathrm{Mat}_n(K)$.

Theorem 32 (Nagata, Haboush)**.** *If G is a reductive group acting on an affine K-variety $X = \mathrm{Spec}(A)$, then A^G is finitely generated.*

We have the following fundamental theorem.

Theorem 33. *Let G be a reductive group acting on a K-affine variety and let $Y = \mathrm{Spec}(K[X]^G)$, then $X \to Y$ is a good quotient.*

2.8 Adjoint action

Let G be a linear group. For $g \in G$ consider $\mathrm{Int}_g : G \to G$, $h \mapsto ghg^{-1}$ and its differential $d\mathrm{Int}_g : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$. This defines a representation of G

$$\mathrm{Ad} : G \to \mathrm{GL}\left(\mathrm{Lie}(G)\right), \qquad g \mapsto d\mathrm{Int}_g,$$

called the *adjoint representation*. The differential of Ad is

$$\mathrm{ad} : \mathrm{Lie}(G) \to \mathrm{gl}\left(\mathrm{Lie}(G)\right) = \mathrm{End}\left(\mathrm{Lie}(G)\right), \qquad x \mapsto (y \mapsto [x, y]).$$

Example 11. *If $G = \mathrm{GL}_n$, then $\mathrm{Lie}(G) = \mathrm{gl}_n$, $\mathrm{Ad}(g)(x) = gxg^{-1}$, $\mathrm{ad}(x)(y) = xy - yx$.*

For $x \in \mathrm{Lie}(G)$, $g \in G$ put

$$\begin{aligned} C_G(z) &:= \{g \in G \,|\, g \cdot z = z\}, \\ C_{\mathrm{Lie}(G)}(x) &:= \{y \in \mathrm{Lie}(G) \,|\, [x, y] = 0\}, \\ C_{\mathrm{Lie}(G)}(g) &:= \{y \in \mathrm{Lie}(G) \,|\, \mathrm{Ad}(g)(y) = y\}. \end{aligned}$$

We also denote by $G \cdot z$ the G-orbit of $z \in G$ (conjugation action) or $z \in \mathrm{Lie}(G)$ (adjoint action).

Proposition 34. *[3, III 9.1] For $g \in G$, $x \in \mathrm{Lie}(G)$, consider the orbit maps*

$$\pi : G \to G \cdot g, \quad h \mapsto hgh^{-1}, \qquad \rho : G \to G \cdot x, \quad h \mapsto \mathrm{Ad}(h)(x).$$

Then

$$\mathrm{Ker}(d\pi) = C_{\mathrm{Lie}(G)}(g), \qquad \mathrm{Ker}(d\rho) = C_{\mathrm{Lie}(G)}(x).$$

Recall the following general result.

Proposition 35. *For G connected acting on X consider the orbit map $\pi : G \to G \cdot x$. The following assertions are equivalent*
(1) The morphism π_x is separable.
(2) The induced morphism $G/C_G(x) \to G \cdot x$ is an isomorphism.
(3) The natural inclusion $\mathrm{Lie}(C_G(x)) \subset \mathrm{Ker}(d\pi)$ is an equality.

We have the following useful consequence.

Corollary 36. *For $g \in G$ and $x \in \mathrm{Lie}(G)$, we have*

$$\mathrm{Lie}\,(C_G(g)) = C_{\mathrm{Lie}(G)}(g), \qquad \mathrm{Lie}\,(C_G(x)) = C_{\mathrm{Lie}(G)}(x), \tag{2}$$

if and only if the corresponding orbit maps are separable.

If the characteristic of K is 0, orbit maps are separable and so the identities (2) always hold.

Exercise 11. *If $G = \mathrm{PGL}_2$ and $\mathrm{char}(K) = 2$, prove that the orbit map $G \to G \cdot x$ with x the class of*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in $\mathrm{Lie}(\mathrm{PGL}_2)$ is not separable.

If G is reductive and connected we have the notion of *very good primes* for G. We have not introduced enough material in the lecture to define it properly (see [11, Definition 2.5.5]). If $G = \mathrm{SL}_n$, PGL_n for instance the very good primes for G are the primes that do not divide n. Then we have the following result (see [11, §2.6] for more details).

Theorem 37. *The orbit map $G \to G \cdot z$ with $z \in G$ or $z \in \mathrm{Lie}(G)$ is separable in the following cases :*
(i) z is semisimple,
(ii) $G = \mathrm{GL}_n$ or the characteristic of K is a very good prime for G.

2.9 Characters and diagonalizable groups

Let G be a linear algebraic group over K. A *character* of G is a morphism of algebraic groups $G \to \mathbb{G}_m$. We denote by $X(G)$ the set of characters of G. It is equipped with an obvious structure of abelian group. For a character χ we denote by K_χ the field K equipped with the G-action defined by $g \cdot v = \chi(g)v$ for $g \in G$ and $v \in K$.

Proposition 38 (Dedekind)**.** *For any abstract group H and any field k, the subset $\mathrm{Hom}_{\mathrm{group}}(H, k^\times)$ of the k-vector space of all functions $H \to k$ consists of linearly independent elements.*

Proof. Suppose $\sum_{i=1}^n a_i\chi_i = 0$ with $\chi_1, \cdots, \chi_n$ characters and n minimal. For all $g, h \in H$, we have

$$\sum_{i=1}^n a_i\chi_i(gh) = \sum_{i=1}^n a_i\chi_i(g)\chi_i(h) = 0 = \chi_1(h)\sum_{i=1}^n a_i\chi_i(g).$$

Hence $\sum_{i=2}^n a_i(\chi_i(h) - \chi_1(h))\chi_i = 0$. □

Corollary 39. *The elements of $X(G)$ are linearly independent vectors of $K[G]$.*

Exercise 12. *(1) Show that if G is connected, then $X(G)$ is torsion free.*
(2) Prove that $X(G \times H) \simeq X(G) \times X(H)$.

We denote by D_n the subset of GL_n of diagonal matrices. Denote by $T_i : D_n \to \mathbb{G}_m$, $(t_1, \ldots, t_n) \mapsto t_i$ the i-th projection. Then $T_i \in X(D_n)$ and we have a well-defined map $\mathbb{Z}^n \to X(D_n)$, $(r_1, \ldots, r_n) \mapsto T_1^{r_1} \cdots T_n^{r_n}$.

Proposition 40. *The above map $\mathbb{Z}^n \to X(D_n)$ is an isomorphism of groups and $X(D_n)$ forms a basis of $K[D_n]$.*

Definition 10. *An algebraic group D is* diagonalizable *if there exists a closed immersion $D \hookrightarrow D_n$.*

A diagonalizable group is thus commutative and its elements are all semisimple. We have the following theorem.

Theorem 41. *Let G be a linear algebraic group. The following assertions are equivalent.*
(1) G is commutative and $G = G_{\mathrm{ss}}$.
(2) G is diagonalizable.
(3) $X(G)$ is K-basis of $K[G]$.
(4) Any rational G-module V is a direct sum of G-modules K_χ, with $\chi \in X(G)$.

Proof. We only prove (1) $\Rightarrow$ (2). Choose a closed immersion $\rho : G \hookrightarrow \mathrm{GL}_n$. Then $\rho(G)$ is a commutative subgroup of GL_n whose elements are all diagonalizable. Since $\rho(G)$ is commutative, by a basic result of linear algebra its elements are actually simultaneously diagonalizable, i.e. there exists $g \in \mathrm{GL}_n$ such that for any $x \in \rho(G)$, $gxg^{-1} \in D_n$.

□

Exercise 13. *Prove using the above theorem that if G is diagonalisable, then $X(G)$ is of finite type.*

Let G be a diagonalisable group and $\chi_1, \ldots, \chi_r \in X(G)$. This defines a morphism $f : G \to D_r$, $g \mapsto (\chi_1(g), \ldots, \chi_r(g))$.

Proposition 42. *(1) f is a closed immersion if and only if $\chi_1, \ldots, \chi_r$ generate $X(G)$.*
(2) f is surjective if and only if the χ_i's are linearly independent.

Corollary 43. *Let G be a diagonalizable algebraic group. The following are equivalent.*
(1) G is connected.
(2) $X(G)$ is torsion free.
(3) G is a torus*, i.e. it is isomorphic to some D_r.*

Proof. Only the implication (2) $\Rightarrow$ (3) remains to be proved. Since G is diagonalisable, the group $X(G)$ is of finite type (see Exercise 13) and torsion free (by assumption) and so must be free. We choose then a basis of $X(G)$ and we apply the above proposition. □

Exercise 14. *(1) Let*

$$T = \{(t_1, \ldots, t_r) \in (K^\times)^r \mid t_1 \cdots t_r = 1\}.$$

Prove that T is a torus and that the characters $\alpha_i(t_1, \ldots, t_r) = t_i/t_{i+1}$, with $i = 1, \ldots, r-1$, form a basis of $X(T)$.
(2) Prove that the subgroup of GL_2 generated by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is diagonalisable but not connected.

2.10 Solvable and unipotent groups

We say that a linear algebraic group is *unipotent* if its elements are all unipotent. A typical example of unipotent group is the subgroup $U_n \subset \mathrm{GL}_n$ of upper-triangular matrices with 1's on the diagonal.

Proposition 44. *Let G be a linear algebraic group and $\rho : G \to \mathrm{GL}_n$ a morphism of algebraic groups . If G is unipotent then $\rho(G)$ is GL_n-conjugate to a subgroup of U_n, namely, there exists $g \in \mathrm{GL}_n$ such that $g\rho(G)g^{-1} \subset U_n$.*

Proof. Follows from the fact if G is a subgroup of $\mathrm{GL}(V)$, with $V = K^n$, consisting of unipotent matrices, then there exists a basis of V such that G is triangular. □

If A and B are subgroups of G, we denote by (A, B) the subgroup of G generated by the commutators $(a, b) := aba^{-1}b^{-1}$, with $a \in A$ and $b \in B$. We denote by $G' = \mathcal{D}(G) = (G, G)$ the derived subgroup of G and for $i \in \mathbb{N}$, we put $\mathcal{D}^{i+1}(G) = (\mathcal{D}^i(G), \mathcal{D}^i(G))$.

Definition 11. *Say that G is* solvable *if $\mathcal{D}^n(G) = \{1\}$ for some $n \in \mathbb{N}$.*

Exercise 15. *Check that the group of upper triangular matrices B_n of GL_n is solvable, $\mathcal{D}(B_n) = (B_n)_{\mathrm{uni}}$ and that $B_n = D_n \ltimes U_n$.*

Proposition 45. *If G is a connected affine algebraic group, then the subgroups $\mathcal{D}^i(G)$ are closed connected subgroups.*

Theorem 46 (Lie-Kolchin)**.** *Let G be an affine algebraic group solvable and connected and let $\rho : G \to \mathrm{GL}_n$ be a morphism of algebraic groups. Then $\rho(G)$ is conjugate to a subgroup of the group of upper triangular matrices B_n.*

Exercise 16. *Compute the normalizer $N_{\mathrm{GL}_2}(D_2)$ of D_2 in GL_2 and check that it is disconnected, solvable and can not be conjugate to a subgroup of B_2.*

Noticing that $(B_n)_{\mathrm{uni}}$ is a closed normal subgroup of B_n we deduce the following corollary from the above theorem.

Corollary 47. *If G is affine solvable and connected, then*
(1) $\mathcal{D}(G)$ is connected and unipotent,
(2) G_{uni} is a normal closed subgroup of G.

We have the following important theorem.

Theorem 48. *Let B be an affine solvable connected algebraic group and let $\mathfrak{b} := \mathrm{Lie}(B)$.*
(1) $U := B_{\mathrm{uni}}$ is a connected normal closed subgroup of B.
(2) If T is a maximal torus of B, the application

$$T \times U \to B, \qquad (t, v) \mapsto tv,$$

is an isomorphism of varieties. In particular we have $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ with $\mathfrak{t} := \mathrm{Lie}(T)$ and $\mathfrak{u} := \mathrm{Lie}(U)$.
(3) The maximal tori of B are all conjugate. Moreover any subgroup of B consisting of semisimple elements is contained in some maximal torus of B.

Remark 3. *Note that if we write an element x of B in the form tv with $t \in T$ and $v \in U$, then they do not necessarily commute and so, in general, it is not the Jordan decomposition of x.*

2.11 Borel subgroups

Let G be an affine algebraic group. A *Borel subgroup* of G is a closed connected solvable subgroup of maximal dimension.

Example 12. *The Borel subgroups of GL_n are exactly the GL_n-conjugate of B_n.*

Theorem 49. *Let G be a connected affine algebraic group. All Borel subgroups of G are conjugate.*

Corollary 50. *Let G be a connected affine algebraic group.*
(a) The maximal tori of G are conjugate and each one is a maximal torus of some Borel subgroup of G.
(b) The maximal closed connected unipotent subgroups of G are conjugate and each one is of the form B_{uni} for some Borel subgroup B of G.

Proof. Let T be a maximal torus of G, then T is slovable connected and closed and so is contained in a Borel subgroup. By maximality it is a maximal torus of B. Same argument for (b). □

Theorem 51. *Let G be a connected affine algebraic group.*
(1) G is the union of its Borel subgroups.
(2) Any semisimple element of G belongs to a maximal torus of G.
(3) Any unipotent element of G belongs to a maximal closed connected unipotent subgroup of G.

We have the following difficult theorem.

Theorem 52. *Let B be a Borel subgroup of a connected affine algebraic group G, then $N_G(B) = B$.*

Exercise 17. *Check the theorem for* GL_n.

2.12 $\mathbb{F}_q$-structures on algebraic groups

In this section $K = \overline{\mathbb{F}}_q$ and $k = \mathbb{F}_q$. Assume given an affine connected algebraic group G with a k-structure and corresponding Frobenius endomorphism $F : G \to G$ (which is a morphism of algebraic groups).

Lemma 53. *If $g \in G$, then the maps $gF : G \to G$, $x \mapsto gF(x)g^{-1}$ is a geometric Frobenius.*

Proof. We need to show that there exists an integer $n \in \mathbb{N}^*$ such that $(gF)^n = F^n$. For any integer n, we have $(gF)^n = gF(g)F^2(g)\cdots F^{n-1}(g)F^n$. The element $h = gF(g)F^2(g)\cdots F^{n-1}(g)$ is an F^n-stable element. If e denotes its order, then $(gF)^{en} = F^{en}$.

□

We denote by $\mathcal{L} : G \to G$, $g \mapsto g^{-1}F(g)$ the so-called Lang map.

Theorem 54 (Lang-Steinberg)**.** *The map $\mathcal{L}$ is surjective.*

Proof (by J. Michel). The non-empty fibers of $\mathcal{L}$ are all of cardinality $|G^F|$ and so the map $\mathcal{L}$ is dominant. Therefore $\mathrm{Im}(\mathcal{L})$ contains an open subset U of G. For any $x \in G$, the map $G \to G$, $g \mapsto g^{-1}xF(g)$ has also finite fibers since xF is a Frobenius map by Lemma 53. The image of $g \mapsto g^{-1}xF(g)$ contains also an open subset V of G. Since G is connected, the two open subsets U and V intersect and so there exists g and h in G such that $g^{-1}xF(g) = h^{-1}F(h)$, i.e. $x = \mathcal{L}(hg^{-1})$, hence the surjectivity. □

Corollary 55. *Let V be an algebraic variety defined over k and assume that G acts on V and that action is defined over k. Then any F-stable G-orbit of V contains a k-point.*

Proof. Let O be an F-stable orbit of V and let $x \in O$. Then $F(v) = g \cdot v$ for some $g \in G$. By the Lang-Steinberg theorem, there exists $h \in G$ such that $F(h)^{-1}h = g$ and so $F(h \cdot v) = h \cdot v$. □

Corollary 56. *If H is a closed connected subgroup of G defined over k, then G/H is also defined over k and $(G/H)^F \simeq G^F/H^F$.*

Proof. The fact that G/H is defined over k follows from the construction of the quotient. The natural map $G^F/H^F \to (G/H)^F$ is surjective by Corollary 55. Indeed, if xH is F-stable, then it contains an F-stable point say y, and yH^F is mapped to xH by the above map. It is obviously injective. □

Corollary 57. *There exists always a Borel subgroup of G which is F-stable, and any two of them are conjugate under G^F. In an F-stable Borel subgroup, there exists always an F-stable maximal torus.*

Proof. Fix a Borel subgroup B of G. The Borel subgroups of G are all G-conjugate and so $F(B) = gBg^{-1}$ for some $g \in G$. Then apply the Lang-Steinberg theorem by choosing h such that $F(h)^{-1}h = g$. Same proof for the second assertion using the fact that maximal tori in a given Borel subgroup are all conjugate. □

Corollary 58. *Let s be a rational (i.e. fixed by F) semisimple element of G. Then it lies in an F-stable maximal torus of G.*

Proof. If T is a maximal torus of G containing s (recall that any semisimple element if contained in some maximal torus), then T is contained in $C_G(s)^o$. As s is central in $C_G(s)^o$ it is contained in all maximal tori of $C_G(s)^o$. We can now proceed as in the proof of the above corollary (using Lang-Steinberg's theorem) to prove the existence in $C_G(s)^o$ of an F-stable maximal torus containing s. □

If σ is a group automorphism of some group H, we call σ-conjugation the action of H on itself by $x \mapsto yx\sigma(y)^{-1}$. We denote by $H^1(\sigma, H)$ the set of σ-conjugacy classes of H (the notation comes from non-abelian Galois cohomology [14, Chap. I, §5]).

Proposition 59. *We have $H^1(F, G) = 1$.*

Proof. This is a reformulation of Lang-Steinberg theorem. □

Proposition 60. *Suppose that $B \subset A$ is a an inclusion of F-stable closed subgroups of G and that B is connected and normal in A. Then*

$$H^1(F, A) \xrightarrow{\sim} H^1(F, A/B).$$

Proof. We need to prove that two elements a and a' of A are F-conjugate if and only if they are F-conjugate modulo B. One direction is obvious, so we assume that $ba = xa'F(x)^{-1}$ for some $b \in B$. As B is connected we apply the Lang-Steinberg theorem to the Frobenius $F' : y \mapsto aF(y)a^{-1}$ and we write b in the form $z^{-1}F'(z)$. We then get $a = zxa'F(zx)^{-1}$. □

Proposition 61. *Consider an action of G on a variety X defined over k. Let O be an F-stable orbit of X and $x \in O^F$.*
(i) For $g \in G$, we have $gx \in O^F$ if and only if $g^{-1}F(g) \in \mathrm{Stab}_G(x)$.
(ii) There is a well-defined map sending the G^F-orbit of $gx \in O^F$ to the F-conjugacy class of the image of $g^{-1}F(g)$ in $A_G(x) = \mathrm{Stab}_G(x)/\mathrm{Stab}_G(x)^o$ and this map is a bijection.

The proposition says that the set of G^F-orbits of O^F is parametrized by the set $H^1(F, A_G(x))$.

Proof. The assertion (i) is clear. let us prove (ii). Notice first that we have a map from O^F to $H^1(F, \mathrm{Stab}_G(x))$. Indeed, if $gx = hx$ then $h^{-1}g \in \mathrm{Stab}_G(x)$ and we verify that $g^{-1}F(g)$ and $h^{-1}F(h)$ are F-conjugate. Clearly this map factorizes through a map from the set of G^F-orbits of O^F to $H^1(F, \mathrm{Stab}_G(x))$. Let us prove the injectivity of this map. If $g^{-1}F(g)$ and $h^{-1}F(h)$ are F-conjugate by an element $n \in \mathrm{Stab}_G(x)$, then $hn^{-1}g^{-1} \in G^F$ and so $h = cgn$ with $c \in G^F$ and $n \in \mathrm{Stab}_G(x)$. Hence hx and gx are in the same G^F-orbit. For the surjectivity we use the Lang-Steinberg theorem to express an element of $\mathrm{Stab}_G(x)$ as $g^{-1}F(g)$ with $g \in G$. □

Corollary 62. *Assume that G is reductive and let T be an F-stable maximal torus of G. The G^F-conjugacy classes of the F-stable maximal torus of G are parametrized by the F-conjugacy classes of the Weyl group $W(T) = N_G(T)/T$.*

Proof. Apply Proposition 61 to the set X of all maximal tori of G and recall that $N_G(T)^o = T$. □

Example 13. *Consider $G = \mathrm{GL}_n$ endowed with the standard Frobenius. The maximal torus T of diagonal matrices is clearly F-stable and $W = W(T)$ is identified with permutation matrices and so with S_n. The Frobenius acts trivially on W and so $H^1(F, W)$ corresponds to the conjugacy classes of W. The G^F-conjugacy classes of F-stable maximal tori are thus parametrized by the conjugacy classes of S_n.*

3 ℓ-adic sheaves

For a complex manifold $X(\mathbb{C})$, one can use the complex topology and the machinery of algebraic topology to study invariants such as the Betti numbers which are the dimension of the cohomology groups $H^i(X(\mathbb{C}), \mathbb{Q})$. A famous theorem of Grothendieck asserts that if X is an irreducible topologocial space and $\mathcal{F}$ a constant sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$. Because in the Zariski topology there are not enough open subsets, a complex manifold such as the projective space which is not irreducible for the complex topology becomes irreducible for the Zariski topology, and so one sees that the dimension of the cohomology groups $H^i(X_{\mathbb{C}}, \mathbb{Z})$ and $H^i(X_{\mathrm{Zar}}, \mathbb{Z})$ differ. The idea of étale topology on an algebraic variety X (or more generally on a scheme) is to add formally more objects to the category of Zariksi open subsets of X in order to have a good sheaf theory similar to the sheaf theory for complex topology. The objects we add are the so-called *étale morphisms* $U \to X$. They should be thought as an analogue of local homeomorphisms for differentiable manifolds.

We are going to give only an intuition of what étale topology is. Indeed it is not necessary to understand the details for the rest of the lecture as one can basically manipulate étale sheaves (more precisely work in the derived category of ℓ-adic sheaves) as we would do with sheaves for the complex topology. This is even less necessary since the recent work by Bhatt and Scholze [2] who introduced a new topology so-called the pro-étale topology which makes sheaf theory machinery much closer to what we would have for the complex topology.

3.1 Etale morphisms

We first give the definition for algebraic varieties over algebraically closed fields which will suffice for most of our purpose.

3.1.1 Definition for algebraic varieties over algebraically closed fields

Definition 12. *Let $f : X \to Y$ be a morphism between nonsingular algebraic varieties over an algebraically closed field K. We say that f is* étale *at a point $a \in X$ if the differential $d_a f : T_a X \to T_{f(a)} Y$ is an isomorphism.*

In particular, $f : \mathbb{A}^n \to \mathbb{A}^n$, $x \mapsto (f_1(x), \dots, f_n(x))$ is étale at $a \in K^n$ if and only if the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(a) & \cdots & \frac{\partial f_n}{\partial x_n}(a) \end{pmatrix}$$

is invertible in K.

Example 14. *The map $\mathbb{A}^1 \to \mathbb{A}^1$, $x \mapsto x^r$ is not étale at* 0 *but is étale at any point of $\mathbb{G}_m$ when the characteristic of K does not divide r.*

Exercise 18. *Consider the map* $\mathrm{GL}_2 \to \mathrm{GL}_2$*, $x \mapsto x^2$. Find the open subset of* GL_2 *at which this map is étale.*

The definition of étale maps generalizes to possibly singular algebraic varieties by considering the *tangent cone $C_x X$* of an algebraic variety X at a point $a \in X$. Since it is defined locally, taking an affine open subset of X containing a if necessary we may assume that $X = \mathrm{Spec}(K[x_1, \dots, x_n]/I)$ where I is a radical ideal. Denote by $a = (a_1, \dots, a_n)$ the coordinates of a in K^n. Let $P \in I$, then $P(a) = 0$ and so we may write P in the coordinates $(x_1 - a_1, \dots, x_n - a_n)$ so that $P = P_1 + P_2 + \cdots$ with P_r an homogeneous polynomial of degree r in the variables $x_i - a_i$. Let s be the smallest integer such that $P_s \neq 0$ and put $P_* = P_s$.

Example 15. *Consider $P = x^2 - y^3$. Then at $a = (0,0)$ we have $P_* = x^2$. One also have $P(1,1) = 0$ and we find*

$$P(x, y) = 2(x - 1) - 3(y - 1) - 3(y - 1)^2 + (x - 1)^2 - (y - 1)^3,$$

and so at $a = (1,1)$ we have $P_* = 2(x-1) - 3(y-1)$ which is the equation of the tangent space of X at $(1,1)$ which unlike $(0,0)$ is a nonsingular point.

We denote by I_* the ideal generated by the P_* where P runs over I. The *tangent cone* of X at a is the affine scheme $C_aX := \mathrm{Spec}(K[s_1,\dots,s_n]/I_*)$, with $s_i = x_i - a_i$ for all $i = 1,\dots,n$. Note that C_aX may not be reduced as we see in the example with $a = (0,0)$. Note that if a is nonsingular, then the tangent cone at a is simply the tangent space.

There is another way to defined the tangent cone. Denote by $\mathfrak{n}$ the maximal ideal of $K[x_1,\dots,x_n]/I$ corresponding to $a \in X$, it is the image of the ideal $\mathfrak{m} = (x_1 - a_1,\dots,x_n - a_n)$ via $K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]/I$. Then

$$C_aX \simeq \mathrm{Spec}\left(\bigoplus_{i\geq 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}\right),$$

with $\mathfrak{n}^0 := K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]/I$ so that $\mathfrak{n}^0/\mathfrak{n} = K$.

Indeed, the projection $K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]/I$ induces a map of K-algebras

$$K[s_1,\dots,s_n] = \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} \to \bigoplus_{i\geq 0} \mathfrak{n}^i/\mathfrak{n}^{i+1},$$

where $s_i = x_i - a_i$, whose kernel is exactly I_*.

Definition 13. *Let $f : X \to Y$ be a morphism between algebraic varieties over an algebraically closed field K. We say that f is* étale *at a point $a \in X$ if the induced map on tangent cones $C_aX \to C_{f(a)}Y$ is an isomorphism of schemes.*

Exercise 19. *Consider $X = \mathrm{Spec}\left(K[x,y]/(y^2 - x^2 - x^3)\right)$ and consider the map*

$$\mathbb{A}^1 \to X, \qquad t \mapsto ((t^2-1), t(t^2-1)).$$

Prove that this map is not étale at 1.

3.1.2 Definition for schemes

We start by defining flat morphisms. We collect the main properties, for the details we refer to the first chapter of Milne's book [13].

Let $f : A \to B$ be a ring homomorphism (A and B are commutative with a unit). This endows B with a structure of A-algebra. We say that f is *flat* if B is flat as an A-module.

Recall that an A-module M is flat if for any ideal I of A of finite type the application $I \otimes_A M \to M$, $a \otimes m \mapsto am$ is injective.

Proposition 63. *If $f : A \to B$ is flat then for all multiplicative subsets $S \subset A$ and $T \subset B$ such that $f(S) \subset T$, the induced morphism $S^{-1}A \to T^{-1}B$ is also flat. Conversely, if $A_{f^{-1}(\mathfrak{n})} \to B_{\mathfrak{n}}$ is flat for all maximal ideals $\mathfrak{n}$ of B, then $A \to B$ is flat.*

A morphism of schemes $\varphi : Y \to X$ is said to be *flat* if for all $y \in Y$, the induced ring homomorphism of local rings $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat.

Proposition 64. *The following are equivalent.*
(1) The morphism $\varphi : Y \to X$ is flat.
(2) For any affine open subset U of X and any affine open subset V of Y such that $\varphi(V) \subset U$, the ring homomorphism $\Gamma(U,\mathcal{O}_X) \to \Gamma(V,\mathcal{O}_Y)$ is flat.
(3) There exists a covering $(U_i)_{i\in I}$ of X by affine open subsets, and for all $i \in I$, there exists a covering $(V_{ij})_{j\in J}$ of $\varphi^{-1}(U_i)$ by affine open subsets such that for $i \in I$ and $j \in J$, the ring homomorphism $\Gamma(U_i,\mathcal{O}_X) \to \Gamma(V_{ij},\mathcal{O}_Y)$ is flat.

Proposition 65. *(a) An open immersion is flat.*
(b) The composite of two flat morphisms is flat.
(c) Any morphism obtained by base change of a flat morphism is flat.

Theorem 66. *Any flat morphism of schemes which is of finite type is open.*

Example 16. *(i) If R is a ring, then the n-dimensional affine space* $\mathbb{A}^n_R = \mathrm{Spec}(R[x_1, \ldots, x_n])$ *is flat over R as* $R[x_1, \ldots, x_n]$ *is free over R.*
(ii) If k is a field and $n \in \mathbb{N}^*$, *then* $k[x^n] \hookrightarrow k[x]$ *is flat because* $(1, x, \ldots, x^{n-1})$ *is a* $k[x^n]$*-basis of* $k[x]$, *and so the map* $\mathbb{A}^1_k \to \mathbb{A}^1_k$, $\zeta \mapsto \zeta^n$ *is flat.*

Proposition 67. *Let* $f : X \to Y$ *be a morphism of schemes of finite type over a field k.*
(1) If f is flat then for any point $x \in X$ *with* $y = f(x)$ *we have*

$$\dim_x(X_y) = \dim_x X - \dim_y Y. \tag{3}$$

(2) If moreover X and Y are regular and the identity (3) holds for any $y = f(x)$, *then f is flat.*

Exercise 20. *Let k be a field. Using the above proposition prove that the morphism* $\mathrm{Spec}(k[x, y]/(xy)) \to \mathrm{Spec}(k[x])$ *is not flat (describe geometrically this morphism).*

We now define the notion of unramified morphism of schemes.

A local homomorphism of local rings $A \to B$ is said to be *unramifield* if $f(\mathfrak{m}_A)B = \mathfrak{m}_B$ and $B/\mathfrak{m}_B$ is a finite separable field extension of $A/\mathfrak{m}_A$.

A morphism of schemes $X \to Y$ is said to be *unramified* if it is locally of finite type and if for any $x \in X$, the local homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{Y,y}$ is unramified.

A morphism of schemes $X \to Y$ is said to be *étale* at $x \in X$ if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is unramified and flat. A morphism of schemes is said to be étale if it is étale at any point, i.e. if it is flat and unramified.

Example 17. *Let k be a field. We already saw that the morphism* $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$, $\zeta \mapsto \zeta^2$ *is flat but not étale. Indeed consider the corresponding k-algebras homomorphism* $f^\# : k[x] \to k[x]$, $P(x) \mapsto P(x^2)$. *Then* $f^\#(xk[x]_x) = x^2 k[x]_x \neq xk[x]_x$ *and so f is ramified at* 0.

Exercise 21. *Prove that* $f : \mathbb{G}^1_k \to \mathbb{G}^1_k$, $\zeta \mapsto \zeta^2$ *is étale if the characteristic of k is different from* 2.

We now resume the properties we will need.

Proposition 68. *(a) An open immersion is étale.*
(b) The composite of two flat morphisms is étale.
(c) Any base change of an étale morphism is étale.
(d) Let $f : Y \to X$ *be an étale morphism, then*
(i) for all $y \in Y$, $\mathcal{O}_{Y,y}$ *and* $\mathcal{O}_{X,x}$ *are of same dimension,*
(ii) the morphism f is quasi-finite (i.e. its fibers are finite),
(iii) the morphism f is open,
(iv) If X is reduced (resp. regular), then Y is also reduced (resp. regular).

The following exercise relates the definition of étale morphisms we just gave in terms of algebra and the first definition we gave for algebraic varieties over algebraically closed fields.

Exercise 22. *Let Y and X be nonsingular varieties over a field. Show that a morphism* $Y \to X$ *is étale if and only if it induces an isomorphism on tangent spaces for any closed point of Y.*

3.2 The category of étale sheaves

We work in the category of algebraic varieties over a field k and we let X be an object of that category.

We consider the étale site $\mathbf{Et}/_X$ over X, namely the objects of $\mathbf{Et}/_X$ are the pairs (U, f) with $f : U \to X$ an étale morphism and the morphisms $(U, f) \to (U', f')$ are the morphisms $U \to U'$ above X, i.e. such that the following diagram commutes

$$\begin{array}{ccc} U & \longrightarrow & U' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

An *étale covering* of X is a family $(U_i, f_i)_{i\in I}$ of objects of $\mathbf{Et}/_X$ such that X is the union of the $f(U_i)$, with $i \in I$.

Definition 14. *An* étale presheaf *on X (of abelian groups) is a contravariant functor* $\mathcal{F} : \mathbf{Et}/_X \to \mathbf{Ab}$*, where* $\mathbf{Ab}$ *denotes the category of abelian groups.*

When there is no ambiguity we will often write simply U instead of (U, f). If $U \in \mathbf{Et}/_X$ and $V \in \mathbf{Et}/_U$, we call restriction of $s \in \mathcal{F}(U)$ to V the image of s by $\mathcal{F}(U) \to \mathcal{F}(V)$.

The étale presheaves on X form an abelian category denoted by $\mathbf{PreSh}(X)$ (we have the abvious notion of kernels, cokernels, products, direct sums, inverse limites, direct limites,...). Then $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ is exact if and only if for any $U \in \mathbf{Et}/_X$, the sequence $\mathcal{F}(U) \to \mathcal{F}'(U) \to \mathcal{F}''(U)$ is exact.

We now define the notion of *gluing data* : Let $U \in \mathbf{Et}/_X$ and let $(U_i)_{i\in I}$ be a covering of U. We have the following cartesian diagram where all maps are étale

$$\begin{array}{ccc} U_i \times_U U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_j & \longrightarrow & U \end{array} .$$

Let $(s_i)_{i\in I}$ be a family such that $s_i \in \mathcal{F}(U_i)$ for all $i \in I$. For all $i, j \in I$, we denote by s_{ij} the restriction of s_i to $U_i \times_U U_j$ and by s_{ji} the restriction of s_j to $U_i \times_U U_j$. We say that $(s_i)_{i\in I}$ is a *gluing data* if for all $i, j \in I$, we have $s_{ij} = s_{ji}$.

Definition 15. *We call* étale sheaf *on X any étale presheaf* $\mathcal{F}$ *on X such that for any* $U \in \mathbf{Et}/_X$*, any gluing data* $(s_i)_{i\in I}$ *on any étale covering* $(U_i)_{i\in I}$ *of U, there exists a unique* $s \in \mathcal{F}(U)$ *whose restriction to* $\mathcal{F}(U_i)$ *is* s_i*.*

As open immersions are étale, the site $\mathbf{Zar}/_X$ of open subsets of X is a subcategory of $\mathbf{Et}/_X$ and so by restriction an étale sheaf $\mathcal{F}$ on X gives a sheaf $\mathcal{F}_{\text{zar}}$ on X for the Zariski topology (note that if U and V are two open subsets of X, then $U \times_X V = U \cap V$).

Proposition 69. *An étale presheaf* $\mathcal{F}$ *is an étale sheaf if and only if the following two conditions are satisfied :*
(1) For all $U \in \mathbf{Et}/_X$ *if we let* $\mathcal{F}_U$ *be the restriction of* $\mathcal{F}$ *to U, then* $(\mathcal{F}_U)_{\text{Zar}}$ *is a sheaf.*
(2) For any surjective morphism $U' \to U$ *in* $\mathbf{Et}/_X$ *with* U, U' *affine consider*

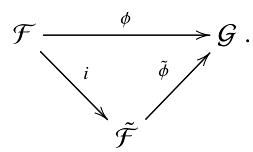
$$\begin{array}{ccc} U' \times_U U' & \xrightarrow{pr_1} & U' \\ {\scriptstyle pr_2}\downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array} .$$

Let $s' \in \mathcal{F}(U')$ *be such that* $\mathcal{F}(pr_1)(s') = \mathcal{F}(pr_2)(s')$*. Then there exists a unique* $s \in \mathcal{F}(U)$ *whose restriction to* U' *is* s'*.*

Exercise 23. *Let $\mathcal{F}$ be an O_X-quasi-coherent sheaf. For any $U \in \mathbf{Et}/_X$, put $\overline{\mathcal{F}}(U) = \mathcal{F}(X) \otimes_{O_X(X)} O_U(U)$. Prove that $\overline{\mathcal{F}}$ is an étale sheaf on X.*

Example 18. *Let F be an abelian group endowed with the discrete topology. For any $U \in \mathbf{Et}/_X$, we denote by $F(U)$ the set of all continues map $U \to F$ (i.e. functions that are constant on connected components of U). This is an étale sheaf called the constant sheaf associated with F and denoted by F.*

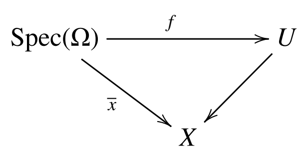
Theorem 70. *Let $\mathcal{F}$ be an étale presheaf on X. Then there exists an étale sheaf $\tilde{\mathcal{F}}$ on X and an injective morphism of étale presheaf $i : \mathcal{F} \to \tilde{\mathcal{F}}$ such that for any étale sheaf on X and any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ there exists a unique morphism of étale sheaves $\tilde{\phi} : \tilde{\mathcal{F}} \to \mathcal{G}$ such that the following diagram commutes*



Moreover the pair $(\tilde{\mathcal{F}}, i)$ is unique up to isomorphism and if ϕ is injective then so is $\tilde{\phi}$. If $\mathcal{F}$ is a sheaf then $i : \mathcal{F} \to \tilde{\mathcal{F}}$ is an isomorphism.

We denote $\mathbf{Sh}(X)$ the full subcategory of $\mathbf{PreSh}(X)$ of étale sheaves on X, it is an abelian category with enough injective.

A *geometric point* of X is a morphism $\mathrm{Spec}(\Omega) \to X$ where Ω is a separably closed field. An *étale neighbourhood* of a geometric point $\overline{x} : \mathrm{Spec}(\Omega) \to X$ is an object $U \in \mathbf{Et}/_X$ and a morphism $f : \mathrm{Spec}(\Omega) \to U$ above X, i.e. such that the following diagram commutes



Let $\mathcal{F}$ be an étale presheaf on X and let $\overline{x}$ be a geometric point. We define the stalk of $\mathcal{F}$ at $\overline{x}$ as

$$\mathcal{F}_{\overline{x}} := \varinjlim_{(U,u)} \mathcal{F}(U),$$

where the limit is over the étale neighbourhood of $\overline{x}$.

Proposition 71. *The sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact in $\mathbf{Sh}(X)$ if and only if the sequence*

$$0 \to \mathcal{F}'_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to \mathcal{F}''_{\overline{x}} \to 0$$

is exact in $\mathbf{Ab}$ for all geometric points $\overline{x}$.

Direct image

Let $f : X \to Y$ be a morphism. We define a functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ as follows. For an étale sheaf $\mathcal{F}$ on X and $U \to Y$ étale, we put

$$f_*(\mathcal{F})(U) = \Gamma(U, f_*\mathcal{F}) = \Gamma(U \times_Y X, \mathcal{F}).$$

One can verify that $f_*\mathcal{F}$ is indeed an étale sheaf on Y. We call the functor f_* the *direct image functor*.

Proposition 72. *(i) If $Z \xrightarrow{g} X \xrightarrow{f} Y$, then $(fg)_* = f_* g_*$.*
(ii) The functor is left exact, namely if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H},$$

is an exact sequence of étale sheaves on X, then

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H},$$

is an exact sequence of étale sheaves on Y.
(iii) If the morphism f is finite then f_* is exact.

Inverse image

Let $f : X \to Y$ be a morphism of algebraic varieties. We now defined the inverse image functor $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ as follows.

Let $\mathcal{F}$ be an étale sheaf on Y and let $U \to X$ be étale. We denote by C_U the class of étale maps $V \to Y$ such that there exists a morphism $U \to V$ making the following diagram commutative

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

We define an étale presheaf $f^{-1}\mathcal{F}$ on X as

$$\Gamma(U, f^{-1}\mathcal{F}) := \varinjlim_{V \in C_U} \Gamma(V, \mathcal{F}).$$

Unfortunately $f^{-1}\mathcal{F}$ need not to be a sheaf and so one define $f^*\mathcal{F}$ as the étale sheaf associated to $f^{-1}\mathcal{F}$.

Proposition 73. *(i) The functor f^* is left adjoint to f_*, namely*

$$\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}),$$

for any $\mathcal{F} \in \mathbf{Sh}(Y)$ and $\mathcal{G} \in \mathbf{Sh}(X)$.
(ii) If $Z \xrightarrow{g} X \xrightarrow{f} Y$, then $(fg)^* = g^* f^*$.
(iii) If $\overline{x}$ is a geometric point of X and $\mathcal{F}$ an étale sheaf on X, then $\overline{x}^*\mathcal{F} = \mathcal{F}_{\overline{X}}$.
(iv) For any geometric point $\overline{x}$ of X and $\mathcal{F} \in \mathbf{Sh}(Y)$ then $(f^*\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{y}}$ where $\overline{y} = f \circ \overline{x}$.
(v) The functor f^* is exact.

If $f : U \to X$ is a morphism and $\mathcal{F}$ an étale sheaf on X, sometimes, when the context is clear, we will denote by $\mathcal{F}|_U$ for $f^*\mathcal{F}$ and call it the restriction of $\mathcal{F}$ to U (along the map f).

Extension by zero

Let X be an algebraic variety and let $j : U \hookrightarrow X$ be an open immersion. Let $\mathcal{F}$ be an étale sheaf on U, the stalks of $j_*\mathcal{F}$ need not to be zero outside U. We define the functor $j_! : \mathbf{Sh}(U) \to \mathbf{Sh}(X)$ which has this property, i.e. the stalks of $j_!\mathcal{F}$ are zero on $X \backslash U$. It is defined as follows.

For any $f : V \to X$ étale, put

$$\mathcal{F}'(V) := \begin{cases} \mathcal{F}(V) & \text{if } \varphi(V) \subset U \\ 0 & \text{otherwise.} \end{cases}$$

We define $j_!\mathcal{F}$ as the étale sheaf associated to the étale presheaf $\mathcal{F}'$.

Proposition 74. *(i) The functor $j_!$ is a left adjoint to j^*, namely*

$$\mathrm{Hom}(j_!\mathcal{F},\mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, j^*\mathcal{G}).$$

(ii) For any geometric point $\overline{x}$ of X with image $x \in X$, we have

$$(j_!\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

(iii) Let Z be the complementary of U in X and $i : Z \hookrightarrow X$, then for any étale sheaf $\mathcal{F}$ on X the sequence

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0,$$

is exact.

3.3 Etale cohomology

The functor $\Gamma(X, .) : \mathbf{Sh}(X) \to \mathbf{Ab}$ and its right derived functors are written $H^i(X, .)$. The group $H^i(X,\mathcal{F})$ is called the i-th cohomology group of X with value in $\mathcal{F}$.

Remark 4. *(i) For a morphism $f : X \to Y$, the functor f_* being left exact, its right derived functors are written $R^i f_*$ and are called higher direct images. If $f : X \to \mathrm{Spec}(k)$ and $\mathcal{F} \in \mathbf{Sh}_c(X)$, we have*

$$\Gamma(\mathrm{Spec}(k), R^i f_*\mathcal{F}) = H^i(X,\mathcal{F}).$$

(ii) Let $f : X \to Y$ be a finite morphism and $\mathcal{F} \in \mathbf{Sh}(X)$. The functor f_* being exact, it transforms injectives to injectives and so injective resolutions of $\mathcal{F}$ into injective resolutions of $f_*\mathcal{F}$. Therefore

$$H^i(X,\mathcal{F}) = H^i(Y, f_*\mathcal{F}).$$

We have also the following result (particular case of [13, Theorem 1.15]).

Theorem 75. *If Ω is a separably closed field and $f : X \to \mathrm{Spec}(\Omega)$, then*

$$R^i f_*\mathcal{F} \simeq H^i(X,\mathcal{F}).$$

Let ℓ be a prime. We say that an abelian group is a torsion group (or ℓ-torsion group) if for all $x \in F$, there exists an integer n (respectively a power ℓ^r of ℓ) such that $nx = 0$ (respectively $\ell^r x = 0$). This definition extends to étale sheaves.

Definition 16. *We say that $\mathcal{F} \in \mathbf{Sh}(X)$ is a* torsion sheaf *(resp. ℓ-torsion sheaf) if for any étale map $U \to X$, the group $\Gamma(U,\mathcal{F})$ is a torsion group (resp. ℓ-torsion group).*

We have the following important theorem (see [13, Chapter VI, Theorem 1.1]).

Theorem 76. *Assume that k is separably closed. Let $\mathcal{F}$ be an ℓ-torsion étale sheaf on X. Then the group $H^i(X,\mathcal{F})$ vanishes if $i > 2\dim(X)$.*

Let $\mathcal{F} \in \mathbf{Sh}(X)$ is *locally constant* if there exists a étale covering (U_i) of X such that for all i, the restriction of $\mathcal{F}$ to U_i is constant. We say that $\mathcal{F}$ is *constructible* if for all closed irreducible subvariety Z of X, there exists a non-empty open subset U of Z such that the restriction of $\mathcal{F}$ to U is locally constant with finite stalks.

Theorem 77. *Let $\mathcal{F}$ be constructible and $f : X \to Y$ proper, then for all $i \in \mathbb{N}$, the sheaf $R^i f_*\mathcal{F}$ is constructible. Moreover if $\mathcal{F}$ is a torsion sheaf and $\overline{y}$ is a geometric point of Y and $X_{\overline{y}}$ the geometric fiber above $\overline{y}$, then there is a canonical isomorphism $(R^i f_*\mathcal{F})_{\overline{y}} \to H^i(X_{\overline{y}},\mathcal{F}|_{X_{\overline{y}}})$.*

Corollary 78. *If k is separably closed and X is proper over k, then for any constructible étale sheaf $\mathcal{F}$ the groups $H^i(X, \mathcal{F})$ are finite for all $i \in \mathbb{N}$.*

Proof. Follows from the fact that $R^i f_* \mathcal{F} = H^i(X, \mathcal{F})$ with $f : X \to \mathrm{Spec}(k)$. □

Base change

Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{4}$$

By adjunction properties we have a morphism (of functors) $a^* a_* \to \mathrm{Id}$ and a morphism $\mathrm{Id} \to a'_*(a')^*$. By the Leray spectral sequence (cf [13, Theorem 1.18]) we have morphisms $R^i(g_*) \circ (a')_* \to R^i(f_* a'_*)$. Therefore we have also morphisms

$$a^* R^i g_* \to a^*(R^i g_*)(a')_*(a')^* \to a^* R^i(ga')_*(a')^* \to a^* R^i(af)_*(a')^* \to a^* a_*(R^i f_*)(a')^* \to (R^i f_*)(a')^*.$$

Corollary 79 (Proper base change theorem)**.** *Assume that the diagram (4) is cartesian, g is proper and $\mathcal{F}$ is a torsion sheaf, then the canonical morphism*

$$a^* R^i g_*(\mathcal{F}) \longrightarrow (R^i f_*)(a')^*(\mathcal{F}),$$

is an isomorphism.

Proof. Check that it is an isomorphism on stalks by applying Theorem 77. □

3.4 Cohomology with compact support and direct image with proper support

Theorem 80 (Nagata)**.** *Let $f : X \to Y$ be a morphism of algebraic varieties. Then there exists an open immersion $j : X \hookrightarrow \overline{X}$ and a proper morphism $\overline{f} : \overline{X} \to Y$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{j} & \overline{X} \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \overline{f}} & \\ Y & & \end{array}$$

commutes.

Applying Nagata's theorem to the structure morphism $X \to \mathrm{Spec}(k)$, we find an open embedding $j : X \hookrightarrow \overline{X}$ of X in a complete variety $\overline{X}$. Let $\mathcal{F}$ be a torsion sheaf on X and define the compactly supported cohomology $H^i_c(X, \mathcal{F})$ as

$$H^i_c(X, \mathcal{F}) := H^i(\overline{X}, j_! \mathcal{F}).$$

We have the following proposition (cf [13] for a proof).

Proposition 81. *The groups $H^i_c(X, \mathcal{F})$ are independent of the choice of an embedding $j : X \to \overline{X}$ above* $\mathrm{Spec}(k)$.

To prove the above proposition, we use the proper base change theorem and so we need the assumption that $\mathcal{F}$ is a torsion sheaf.

Let $f : X \to Y$ and consider the factorisation through $\overline{X}$ as in Nagata's theorem. Denote by $\mathbf{Sh}_{\mathrm{tor}}(X)$ the full subcategory of $\mathbf{Sh}(X)$ of torsion sheaves.

We define the functor $f_! : \mathbf{Sh}_{\mathrm{tor}}(Y) \to \mathbf{Sh}_{\mathrm{tor}}(X)$ as $\overline{f}_* \circ j_!$ called the direct image with proper support. It higher direct image with proper support are defined as

$$R^i f_! = R^i(\overline{f})_* \circ j_!.$$

Remark 5. *$R^i f_!$ is not the i-th derived right functor of $f_!$.*

We can prove using proper base change theorem and spectral sequences that the functor $f_!$ does not depend on the factorisation by $j : X \to \overline{X}$. If f is proper then $f_! = f_*$ (see [13, Chapter VI, §3] for more details).

Proposition 82. *(i) If $\mathcal{F}$ is contructible, then so is $R^i f_! \mathcal{F}$.*
(ii) $R^i f_!$ commutes with base change.
(iii) If k is separably closed, for any torsion sheaf $\mathcal{F}$ we have

$$R^i f_! \mathcal{F} \simeq H^i_c(X, \mathcal{F}).$$

Proof. The first assertion follows from Theorem 77 and the fact that $j_! \mathcal{F}$ is constructible. The assertion (ii) is a consequence of the proper base change theorem and assertion (iii) is a particular case of (ii). □

3.5 Constructible $\overline{\mathbb{Q}}_\ell$-sheaves

Let X be an algebraic variety over a field k and let ℓ be a prime which does not divide the characteristic of k.

We denote by $\mathbf{Sh}_c(X, \mathbb{Z}/\ell^n\mathbb{Z}) = \mathbf{Sh}_c(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ the category of constructible étale sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on X.

Remark 6. *If $k = \mathbb{C}$, we have a morphism of topos $\epsilon : X(\mathbb{C}) \to X_{\acute{e}t}$ and the functor ϵ^* induces an equivalence*

$$\mathbf{Sh}_c(X_{\acute{e}t}, \mathbb{Z}/\ell^n\mathbb{Z}) \simeq \mathbf{Sh}_c(X(\mathbb{C}), \mathbb{Z}/\ell^n\mathbb{Z}).$$

Definition 17. *A* constructible étale sheaf of $\mathbb{Z}_\ell$-modules *on X is a family $(\mathcal{F}_n, f_{n+1} : \mathcal{F}_{n+1} \to \mathcal{F}_n)_n$ such that*

(a) for all n, $\mathcal{F}_n \in \mathbf{Sh}_c(X, \mathbb{Z}/\ell^n\mathbb{Z})$,

(b) for all n, f_{n+1} factorizes via un isomorphism

$$\mathcal{F}_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \to \mathcal{F}_n.$$

We define morphims of constructible étale $\mathbb{Z}_\ell$-sheaves as

$$\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{F}, \mathcal{G}) := \varprojlim_n \mathrm{Hom}_{\mathbb{Z}/\ell^n\mathbb{Z}}(\mathcal{F}_n, \mathcal{G}_n).$$

3.5.1 The catgeory $\mathbf{Sh}_c(X, E)$ of constructible E-sheaves

The objects of $\mathbf{Sh}_c(X, \mathbb{Q}_\ell)$ are the constructible $\mathbb{Z}_\ell$-sheaves, but the morphisms are defined as

$$\mathrm{Hom}_{\mathbb{Q}_\ell}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

For a finite field extension E of $\mathbb{Q}_\ell$ with valuation ring $\mathfrak{o}$ and with π a generator of the maximal ideal of $\mathfrak{o}$ we can defined the category $\mathbf{Sh}_c(X, E)$ of constructible E-sheaves as we just did replacing $\mathbb{Z}/\ell^n\mathbb{Z}$ by $\mathfrak{o}/\pi^i\mathfrak{o}$.

For an inclusion of finite field extensions $E \subset E'$ we have a functor $\mathbf{Sh}_c(X, E) \to \mathbf{Sh}_c(X, E'), \mathcal{F} \mapsto \mathcal{F} \otimes_E E'$ (one has

$$\mathrm{Hom}_{E'}(\mathcal{F} \otimes_E E', \mathcal{G} \otimes_E E') = \mathrm{Hom}_E(\mathcal{F}, \mathcal{G}) \otimes_E E').$$

3.5.2 The category $\mathbf{Sh}_c(X) = \mathbf{Sh}_c(X, \overline{\mathbb{Q}}_\ell)$ of constructible $\overline{\mathbb{Q}}_\ell$-sheaves

The objects of $\mathbf{Sh}_c(X)$ are the constructible E-sheaves for some finite filed extension $\mathbb{Q}_\ell \subset E \subset \overline{\mathbb{Q}}_\ell$ and the morphisms are defined as follows.

For $\mathcal{F} \in \mathbf{Sh}_c(X, E)$, $\mathcal{G} \in \mathbf{Sh}_c(X, E')$ and F a common finite field extension of E and E', define

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_F(\mathcal{F} \otimes_E F, \mathcal{G} \otimes_{E'} F) \otimes_F \overline{\mathbb{Q}}_\ell.$$

The category $\mathbf{Sh}_c(X)$ is abelian.

Definition 18 (Local systems)**.** *A constructible $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{F} = (\mathcal{F}_n)_n$ is said to be a* local system*, or a* smooth sheaf*, if for each n the sheaf $\mathcal{F}_n$ is locally constant.*

If $\mathcal{F} \in \mathbf{Sh}_c(X)$ and $\overline{x}$ is a geometric point of X, then

$$\mathcal{F}_{\overline{x}} = \left(\varprojlim_n \mathcal{F}_{n,\overline{x}} \right) \otimes \overline{\mathbb{Q}}_\ell.$$

These are finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces.

If $f : X \to Y$ is a morphism, then for any $\mathcal{E} = (\mathcal{E}_n)_n \in \mathbf{Sh}_c(X)$ define

$$R^i f_* \mathcal{E} := (R^i f_* \mathcal{E}_n)_n, \qquad R^i f_! \mathcal{E} := (R^i f_! \mathcal{E}_n)_n,$$

and for $\mathcal{G} = (\mathcal{G}_n)_n \in \mathbf{Sh}_c(Y)$, define $f^*\mathcal{G} = (f^*\mathcal{G}_n)_n$.

Theorem 83. *$R^i f_*$ and $R^i f_!$ are well-defined functors $\mathbf{Sh}_c(X) \to \mathbf{Sh}_c(Y)$ and f^* is a functor $\mathbf{Sh}_c(Y) \to \mathbf{Sh}_c(X)$.*

Proof. The second assertion is clear because f^* is exact. For the first assertion, see details in [7, Chapter 1, Theorem 12.5]. □

For $\mathcal{F} \in \mathbf{Sh}_c(X)$, define

$$H^i(X, \mathcal{F}) = \left(\varprojlim_n H^i(X, \mathcal{F}_n) \right) \otimes \overline{\mathbb{Q}}_\ell, \qquad H^i_c(X, \mathcal{F}) = \left(\varprojlim_n H^i_c(X, \mathcal{F}_n) \right) \otimes \overline{\mathbb{Q}}_\ell.$$

Proposition 84. *If k is separably closed, then $H^i(X, \mathcal{F})$ and $H^i_c(X, \mathcal{F})$ are finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces and vanish for $i > 2 \dim X$.*

Proof. Let $f : X \to \mathrm{Spec}(k)$ the structure map. Since k is separably closed, we have $R^i f_* \mathcal{F}_n = H^i(X, \mathcal{F}_n)$ and $R^i f_! \mathcal{F}_n = H^i_c(X, \mathcal{F}_n)$. The $\mathbb{Z}/\ell^n\mathbb{Z}$-modules $H^i(X, \mathcal{F}_n)$ and $H^i_c(X, \mathcal{F}_n)$ are finite and vanish for $i > 2\dim(X)$. Since the projective system $(\mathcal{F}_n)$ satifies the condition (a) and (b) fo definition 17, their projective limit is finitely generated (see [7, Chapter 1, Proposition 12.4]) and vanish for $i > 2\dim(X)$.

□

3.6 The “derived category” $\mathcal{D}_c^b(X)$ of constructible $\overline{\mathbb{Q}}_\ell$-sheaves

In this section k will be an algebraically closed field or a finite field.

We have a triangulated category $\mathcal{D}_c^b(X)$ (see [10, II. 5]) which should be thought as the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on X (see [2]).

The objects of $\mathcal{D}_c^b(X)$ can be thought as bounded complexes

$$K^\bullet : \cdots \to K^{-2} \to K^{-1} \to K^0 \to \cdots,$$

such that the $\mathcal{H}^i(K^\bullet)$ are constructible $\overline{\mathbb{Q}}_\ell$-sheaves.

We will regard a constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{E}$ as an object $K^\bullet$ of $\mathcal{D}_c^b(X)$ by setting $K^i = 0$ if $i \neq 0$ and $K^0 = \mathcal{E}$, and we will denote by $\overline{\mathbb{Q}}_\ell$ the constant sheaf with stalks $\overline{\mathbb{Q}}_\ell$.

For a morphism of algebraic varieties $f : X \to Y$ we have the functors (derived)

$$f_* : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(Y), \qquad f_! : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(Y) \qquad (f_* = f_! \text{ if } f \text{ is proper}),$$

$$f^* : \mathcal{D}_c^b(Y) \to \mathcal{D}_c^b(X), \qquad f^! : \mathcal{D}_c^b(Y) \to \mathcal{D}_c^b(X) \qquad (f^* = f^! \text{ if } f \text{ open immersion}),$$

with adjunctions

$$\mathrm{Hom}(K, f^! L) \simeq \mathrm{Hom}(f_! K, L), \qquad \mathrm{Hom}(f^* L, K) \simeq \mathrm{Hom}(L, f_* K).$$

$\mathcal{D}_c^b(X)$ is also equipped with a (derived) tensor product which we denote by $\otimes$ (see [7, Appendix II]). If d is an integer we have the shift operation $K^\bullet \mapsto K^\bullet[d]$ such that $\mathcal{H}^i(K^\bullet[d]) = \mathcal{H}^{i+d}K^\bullet$; it commutes with the functors $f_!, f_*, f^*, f^!$.

Remark 7 (Analogue of these functors for functions)**.** *If $f : E \to F$ is a map of finite sets and $C(E)$ the k-vector space of all functions on E, then the map $f_! : C(E) \to C(F)$ is defined by*

$$f_!(h)(y) = \sum_{x \in f^{-1}(y)} h(x),$$

and we also have the map $f^* : C(F) \to C(E)$ defined by

$$f^*(h)(x) = h(f(x)).$$

Theorem 85 (Base change theorem)**.** *Assume that the following diagram is cartesian*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then, for any $K^\bullet \in \mathcal{D}_c^b(Y')$ the canonical morphism $b^* g_! K^\bullet \to f_! b^* K^\bullet$ is an isomorphism.

Proposition 86 (Projection formula)**.** *If $f : X \to Y$ is a morphism then for any $K^\bullet \in \mathcal{D}_c^b(Y)$ and $G^\bullet \in \mathcal{D}_c^b(X)$, the morphism*

$$K^\bullet \otimes f_! G^\bullet \to f_!(f^* K^\bullet \otimes G^\bullet)$$

is an isomorphism.

Hypercohomology. Assume that k is algebraically closed and let $f : X \to \mathrm{Spec}(k)$ the structural map. Then for $K^\bullet \in \mathcal{D}_c^b(X)$ we put

$$\mathbb{H}^i(X, K^\bullet) := \mathcal{H}^i(f_* K^\bullet), \qquad \mathbb{H}_c^i(X, K^\bullet) = \mathcal{H}^i(f_! K^\bullet).$$

These are respectively called the i-th *hypercohomology groups* and i-th *hypercohomology groups with compact support* with coefficients in $K^\bullet$.

Remark 8. *The hypercohomology groups can be defined in a similar fashion as the ℓ-adic cohomology in terms of projective limits in which case we do not need to assume that k is separably closed (see [13] for instance).*

Verdier duality. There exists a functor $D_X : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(X)$ such that
(i) $D_X \circ D_X \simeq \mathrm{Id}$.
(ii) If $f : X \to Y$ is a morphism, then $D_Y \circ f_! \simeq f_* \circ D_X$ and $f^! \circ D_Y \simeq D_X \circ f^*$.
(iii) If k is algebraically closed, $\mathbb{H}_c^{-i}(X, K^\bullet)^\vee \simeq \mathbb{H}^i(X, D_X K^\bullet)$.
(iv) If $\mathcal{E}$ is a local system on X smooth equidimensional, then

$$D_X(\mathcal{E}[\dim X]) \simeq \mathcal{E}^\vee[\dim X],$$

where $\mathcal{E}^\vee$ is the dual sheaf $\mathcal{H}om(\mathcal{E}, \overline{\mathbb{Q}}_\ell)$.

Exercise 24. *Prove (iii) from (ii).*

3.7 Perverse sheaves

For simplicity we assume that k is algebraically closed (we could assume that it is a finite field with some modifications).

Let Y be an irreducible smooth locally closed subvariety of X.

Goresky-MacPherson-Deligne and Gabber defined for any local system $\mathcal{E}$ on Y a complex $\mathrm{IC}(\overline{Y}, \mathcal{E}) \in \mathcal{D}_c^b(\overline{Y})$, called the intersection cohomology complex on $\overline{Y}$ with coefficients in $\mathcal{E}$, such that

$$D_{\overline{Y}}\left(\mathrm{IC}(\overline{Y}, \mathcal{E})[\dim Y]\right) \simeq \mathrm{IC}(\overline{Y}, \mathcal{E}^\vee)[\dim Y].$$

In particular, the complex $\mathrm{IC}(\overline{Y}, \overline{\mathbb{Q}}_\ell)[\dim Y]$ is auto-dual.

The complex $K^\bullet = \mathrm{IC}(\overline{Y}, \mathcal{E})[\dim Y]$ is characterized (up to isomorphism) by the following conditions

$$\begin{aligned}
&\text{(i) } \mathcal{H}^i K^\bullet = 0, && \text{if } i < -\dim Y,\\
&\text{(ii) } \mathcal{H}^{-\dim Y} K^\bullet|_Y \simeq \mathcal{E}, &&\\
&\text{(iii) } \dim\left(\mathrm{Supp}\,\mathcal{H}^i K^\bullet\right) < -i, && \text{if } i > -\dim Y,\\
&\text{(iv) } \dim\left(\mathrm{Supp}\,\mathcal{H}^i D K^\bullet\right) < -i, && \text{if } i > -\dim Y.
\end{aligned}$$

Remark that $\mathrm{IC}(Y, \mathcal{E}) \simeq \mathcal{E}$.

The *intersection cohomology* of $\overline{Y}$ (and its compactly supported version) is defined by

$$IH^i(\overline{Y}, \mathcal{E}) := \mathbb{H}^i(\overline{Y}, \mathrm{IC}(\overline{Y}, \mathcal{E})), \qquad IH_c^i(\overline{Y}, \mathcal{E}) := \mathbb{H}_c^i(\overline{Y}, \mathrm{IC}(\overline{Y}, \mathcal{E})).$$

Exercise 25. *Assume that Z is a smooth open subset of $\overline{Y}$ and let ξ be a local system on Z such that $\xi|_{Z\cap Y} \simeq \mathcal{E}|_{Z\cap Y}$. Verify that $\mathrm{IC}(\overline{Z}, \xi) \simeq \mathrm{IC}(\overline{Y}, \mathcal{E})$.*

Definition 19 (Perverse sheaves)**.** *An object $K^\bullet \in \mathcal{D}_c^b(X)$ is a* perverse sheaf *if*
(i) $\dim\left(\mathrm{Supp}\,\mathcal{H}^i K^\bullet\right) \leq -i$ *for all i,*
(ii) $\dim\left(\mathrm{Supp}\,\mathcal{H}^i D K^\bullet\right) \leq -i$ *for all i.*

We denote by $\mathcal{M}(X)$ the full subcategory of $\mathcal{D}_c^b(X)$ of perverse sheaves. This is an abelian category that contains all complexes $\mathrm{IC}(\overline{Y}, \mathcal{E})[\dim Y]$ extended by zero on $X \backslash \overline{Y}$.

We have the following theorem that characterizes the simple objects of $\mathcal{M}(X)$ (see [1, §4.3]).

Theorem 87. *The simple objects of $\mathcal{M}(X)$ are all obtained as the extension by zero of some intersection cohomology complex* $\mathrm{IC}(\overline{Y}, \mathcal{E})[\dim Y]$ *for some irreducible local system $\mathcal{E}$ on some smooth open subset Y of X.*

The functors $f_!$, f_*, f^* and $f^!$ do not preserve perverse sheaves in general. We have the following proposition (see [1, 4.2.5, 4.2.6]).

Proposition 88. *Let $f : X \to Y$ be a smooth morphism with connected fibers of dimension d.*

(a) Assume that X is irreducible (and so Y) and let V be a smooth open subset of Y. Denote by $f_V : f^{-1}(V) \to V$ the restriction of f to $f^{-1}(V)$. Then

$$\mathrm{IC}(\overline{f^{-1}(V)}, f_V^*\mathcal{E})[\dim X] \simeq f^*[d]\left(\mathrm{IC}(\overline{V}, \mathcal{E})[\dim Y]\right).$$

(b) The functor $f^*[d]$ induces a fully faithful functor $\mathcal{M}(Y) \to \mathcal{M}(X)$.

We now define the external tensor product $\boxtimes$ on $X \times Y$ as

$$K \boxtimes K' = \mathrm{pr}_1^* K \otimes \mathrm{pr}_2^* K',$$

where $K \in \mathcal{D}_c^b(X)$, $K' \in \mathcal{D}_c^b(Y)$ and pr_1 and pr_2 are the two projections. It commutes with the usual operations, namely if $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ then $(f_1)_* K \boxtimes (f_2)_* K' \xrightarrow{\sim} (f_1 \times f_2)_*(K \boxtimes K')$.

We have the following proposition (see [1, Proposition 4.2.8]).

Proposition 89. *If $K \in \mathcal{M}(X)$ and $K' \in \mathcal{M}(Y)$, then $K \boxtimes K' \in \mathcal{M}(X \times Y)$.*

3.8 Equivariance : The finite groups case

Let W be a finite group (which we regard as an algebraic group) together with an action of W on a k-variety X (with k-algebraically closed) on the right. We thus have morphisms $\sigma : X \to X$ for all $\sigma \in W$ such that

$$(\sigma\tau)(x) = \tau(\sigma(x)), \qquad x \in X,\ \sigma, \tau \in W.$$

Definition 20. *By an action of W on $K \in \mathcal{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ we shall mean a family $(\varphi_\sigma)_{\sigma \in W}$ of isomorphisms*

$$\varphi_\sigma : \sigma^*(K) \simeq K$$

satisfying the following conditions

(a) $\varphi_{\sigma\tau} = \varphi_\sigma \circ \sigma^*(\varphi_\tau)$ for all $\sigma, \tau \in W$.

(b) We have $\varphi_1 = \mathrm{Id}$.

If W acts trivially on X, then an action of W on $K \in \mathcal{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ is a group homomorphism

$$W \to \mathrm{Aut}_{\mathcal{D}_c^b(X, \overline{\mathbb{Q}}_\ell)}(K).$$

Proposition 90. *Let $f : X \to Y$ be a W-equivariant morphism. Then*

(i) If W acts on $K \in \mathcal{D}_c^b(Y, \overline{\mathbb{Q}}_\ell)$, then W acts on f^*K.

(ii) If W acts on $K \in \mathcal{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$, then W acts on $f_!K$.

Proof. Use base change theorems. □

Proposition 91. *If U is a smooth irreducible W-equivariant open subset of X and if W acts on a local system $\mathcal{E}$ on U, then W acts on* $\mathrm{IC}(\overline{U}, \mathcal{E})[\dim U]$.

Theorem 92. *Assume that $\pi : X \to Y$ is an étale Galois covering with Galois group W and X smooth. Then $\pi_* \overline{\mathbb{Q}}_\ell$ is a local system on which W acts and decomposes as*

$$\pi_* \overline{\mathbb{Q}}_\ell = \bigoplus_{\varphi \in \mathrm{Irr}(W)} V_\varphi \otimes \mathcal{E}_\varphi,$$

where $\mathcal{E}_\varphi$ are irreducible local systems of rank $\deg(\varphi)$ *on Y and V_φ is an irreducible $\overline{\mathbb{Q}}_\ell[W]$-module with character φ.*

Proof. Decomposes $\pi_*\overline{\mathbb{Q}}_\ell$ into φ-isotipic components where φ runs over $\mathrm{Irr}(W)$,

$$\pi_*\overline{\mathbb{Q}}_\ell = \bigoplus_{\varphi\in\mathrm{Irr}(W)} V_\varphi \otimes \mathcal{E}_\varphi,$$

with V_φ irreducible $\overline{\mathbb{Q}}_\ell[W]$-module with character φ. We want to prove that $\mathcal{E}_\varphi$ are irreducible (hence non-zero).

Notice that $(\pi_*\overline{\mathbb{Q}}_\ell)_y \simeq \overline{\mathbb{Q}}_\ell[W]$ is the regular representation of W and so every irreducible representation of W appears, hence $\mathcal{E}_\varphi \neq 0$ for all $\varphi \in \mathrm{Irr}(W)$ and

$$|W| = \sum_{\varphi\in\mathrm{Irr}(W)} r(\varphi)\cdot\deg(\varphi), \tag{5}$$

where $r(\varphi)$ is the rank of $\mathcal{E}_\varphi$. Moreover

$$\pi^*\mathcal{E}_\varphi \simeq (\overline{\mathbb{Q}}_\ell)^{r(\varphi)}.$$

By the adjunction properties, we have

$$\mathrm{Hom}(\overline{\mathbb{Q}}_\ell, \pi^*(C)) = \mathrm{Hom}(\pi_*\overline{\mathbb{Q}}_\ell, C).$$

For $C = \pi_*\overline{\mathbb{Q}}_\ell$ we deduce that

$$|W| = \dim \mathrm{End}(\pi_*\overline{\mathbb{Q}}_\ell),$$

and so

$$|W| = \sum_{\varphi_1,\varphi_2} \deg(\varphi_1)\deg(\varphi_2)\dim\mathrm{Hom}(\mathcal{E}_{\varphi_1}, \mathcal{E}_{\varphi_2}). \tag{6}$$

For $C = \mathcal{E}_\varphi$, we have

$$r(\varphi) = \sum_\chi \deg(\chi)\dim\mathrm{Hom}(\mathcal{E}_\chi, \mathcal{E}_\varphi) \geq \deg(\varphi).$$

From Formula (5), the above inequality and $|W| = \sum_{\chi\in\mathrm{Irr}(W)}\deg(\chi)^2$, we deduce that $r(\varphi) = \deg(\varphi)$. From Formula (6) we deduce that $\mathcal{E}_{\varphi_1}$ and $\mathcal{E}_{\varphi_2}$ are isomorphic if and only if $\varphi_1 = \varphi_2$ and that $\dim\mathrm{Hom}(\mathcal{E}_\varphi, \mathcal{E}_\varphi) = 1$, i.e. $\mathcal{E}_\varphi$ is irreducible. □

3.9 Equivariance : The case of connected groups

Assume k algebraically closed and let G be a connected affine algebraic group over k acting on a k-algebraic variety X.

Consider the maps

$$\pi = \pi_X : G\times X \to X,\ \ (g,x)\mapsto x, \qquad \rho = \rho_X : G\times X \to X,\ \ (g,x)\mapsto g\cdot x.$$

A sheaf $\mathcal{E} \in \mathbf{Sh}_c(X)$ is said to be *G-equivariant* if

$$\pi^*\mathcal{E} \simeq \rho^*\mathcal{E}.$$

Similarly we say that $K \in \mathcal{M}(X)$ is G-equivariant if $\pi^*K \simeq \rho^*K$.

Lemma 93. *Let $f : X \to Y$ be a G-equivariant morphism between two G-varieties.*

(i) If $\mathcal{E} \in \mathbf{Sh}_c(Y)$ is G-equivariant, then so is $f^*\mathcal{E}$; if $K \in \mathcal{M}(Y)$ is G-equivariant and if $f^*[d](K) \in \mathcal{M}(X)$ for some $d \in \mathbb{Z}$, then $f^*[d](K)$ is G-equivariant.

(ii) If $\mathcal{E} \in \mathbf{Sh}_c(X)$ is G-equivariant, then so is $f_!\mathcal{E}$; if $K \in \mathcal{M}(X)$ is G-equivariant and if for some $d \in \mathbb{Z}$, $f_![d](K)$ is G-equivariant, then $f_![d](K)$ is also G-equivariant.

Proof. Exercise ! □

From now consider

$$\alpha : G\times G \to G,\ \ (g,h)\mapsto gh, \qquad p_2 : G\times G\times G,\ \ (x,y)\mapsto y, \qquad i : X\to G\times X,\ \ x\mapsto (1,x).$$

We start with the following lemma (see [11, Lemma 4.2.3] for a proof).

Lemma 94. *Let G acts on $G\times X$ by left translation on the first coordinate and trivially on X. Then $\pi = \pi_X$ is G-equivariant. The functor $\pi^* : \mathbf{Sh}_c(X) \to \mathbf{Sh}_c(G\times X)$ induces an equivalence of categories from $\mathbf{Sh}_c(X)$ to the full subcategory of $\mathbf{Sh}_c(G\times X)$ of G-equivariant objects. The inverse functor is given by i^*.*

Theorem 95. *Let $\mathcal{E}$ be a G-equivariant sheaf on X (resp. a G-equivariant perverse sheaf). There exists a unique isomorphism*

$$\phi_{\mathcal{E}} : \pi^*\mathcal{E} \to \rho^*\mathcal{E}$$

such that

(i) $i^*(\phi_{\mathcal{E}}) : \mathcal{E}\to\mathcal{E}$ is the identity,

(ii)

$$(\alpha\times \mathrm{Id}_X)^*(\phi_{\mathcal{E}}) = (\mathrm{Id}_G\times\rho)^*(\phi_{\mathcal{E}})\circ(p_2\times\mathrm{Id}_X)^*(\phi_{\mathcal{E}}).$$

Proof. We only prove (i). The proof of (ii) maybe found in [11]. Let $h : \pi^*\mathcal{E}\to\rho^*\mathcal{E}$ be an isomorphism. Let $\phi_{\mathcal{E}} = h\circ\pi^*(i^*h)^{-1} : \pi^*\mathcal{E}\to\rho^*\mathcal{E}$. Then $i^*(\phi_{\mathcal{E}}) = \mathrm{Id}_{\mathcal{E}}$.

Let f_1, f_2 be two isomorphisms $\pi^*\mathcal{E}\to\rho^*\mathcal{E}$ such that $i^*(f_1) = i^*(f_2) = \mathrm{Id}_{\mathcal{E}}$. Then $f_1^{-1}\circ f_2 : \pi^*\mathcal{E}\to\pi^*\mathcal{E}$ is an isomorphism. By Lemma 94, there exists $g : \mathcal{E}\to\mathcal{E}$ such that $\pi^*(g) = f_1^{-1}\circ f_2$. Hence $i^*\pi^*(g) = i^*(f_1^{-1})\circ i^*(f_2)$ and so $g = \mathrm{Id}_{\mathcal{E}}$, i.e. $f_1 = f_2$. □

We are now in position to define the categories $\mathcal{M}_G(X)$ and $\mathbf{Sh}_c(X)_G$ of G-equivariant sheaves. The objects are the G-equivariant sheaves and the morphisms are defined as follows :

A morphism $h : \mathcal{E}\to\mathcal{L}$ in $\mathbf{Sh}_c(X)_G$ (resp. in $\mathcal{M}_G(X)$) is a morphism in $\mathbf{Sh}_c(X)$ (resp. in $\mathcal{M}(X)$) such that the following diagram commutes

$$\begin{array}{ccc} \pi^*\mathcal{E} & \xrightarrow{\ \pi^*(h)\ } & \pi^*\mathcal{L} \\ \downarrow{\scriptstyle \phi_{\mathcal{E}}} & & \downarrow{\scriptstyle \phi_{\mathcal{L}}} \\ \rho^*\mathcal{E} & \xrightarrow[\ \rho^*(h)\ ]{} & \rho^*\mathcal{L} \end{array}$$

Proposition 96. $\mathbf{Sh}_c(X)_G$ *(resp. $\mathcal{M}_G(X)$) are full subcategories of* $\mathbf{Sh}_c(X)$ *(resp. $\mathcal{M}_G(X)$).*

Proof. Let $h : \mathcal{E}\to\mathcal{L}$ be a morphism in $\mathbf{Sh}_c(X)$. We want to prove that

$$\phi_{\mathcal{L}}\circ\pi^*(h) = \rho^*(h)\circ\phi_{\mathcal{E}}. \tag{7}$$

We have $i^*(\phi_{\mathcal{L}}\circ\pi^*(h)) = h = i^*(\rho^*(h)\circ\phi_{\mathcal{E}})$. Moreover ρ and π are both G-equivariant and so the sheaves $\pi^*\mathcal{E}, \pi^*\mathcal{L}, \rho^*\mathcal{E}, \rho^*\mathcal{L}$ are G-equivariants sheaves on $G\times X$. By Lemma 94 we deduce the equality (7). □

Proposition 97. *Let $\mathcal{E}$ be a G-equivariant sheaf on X. For all $g\in G$, we define $i_g : X\to G\times X$, $(x\mapsto(g,x)$. Let G_1 be a closed subgroup of G acting trivially on X. Then the application $G_1\to\mathrm{Aut}(\mathcal{E})$, $h\mapsto(i_h)^*(\phi_{\mathcal{E}})$ is a group homomorphism that factorizes through*

$$G_1/G_1^o \to \mathrm{Aut}(\mathcal{E}).$$

Proof. Exercise ! □

Remark 9. *If $\mathcal{E}$ is an irreducible local system then $\mathrm{Aut}(\mathcal{E})\simeq\overline{\mathbb{Q}}_\ell^\times$ and so it defines a character of the finite group G_1/G_1^o.*

Remark 10. *Since π and ρ are smooth with connected fibers (all of same dimension), the Verdier dual D_X induces an equivalence of categories $\mathcal{M}_G(X) \to \mathcal{M}_G(X)$.*

Proposition 98. *If $K \in \mathcal{M}_G(X)$ then any sub-quotients of K in $\mathcal{M}(X)$ is also G-equivariant.*

Proposition 99. *The simple objects of $\mathcal{M}_G(X)$ are the perverse extensions of G-equivariant irreducible local systems on G-stable smooth locally closed irreducible subsets of X.*

3.10 Local systems on homogeneous varieties

Let G be a connected affine algebraic group and let X be an homogeneous G-variety. We fix $x \in X$ and recall that $A_G(x) = \mathrm{Stab}_G(x)/\mathrm{Stab}_G(x)^o$.

Then there is a bijection between the set of irreducible G-equivariant local systems on X and the irreducible (complex) characters of the finite group $A_G(x)$.

This bijection is constructed as follows.

The bijective morphism $f : G/\mathrm{Stab}_G(x) \to X$, $g \mapsto g \cdot x$ induces an equivalence of categroies

$$f^* : \mathbf{Sh}_c(X) \to \mathbf{Sh}_c(G/\mathrm{Stab}_G(x)).$$

Consider then the morphism $\overline{f} : G/\mathrm{Stab}_G(x)^o \to X$. When f is separable, this is a Galois covering with Galois group $A_G(x)$. Since $\overline{f}$ is G-equivariant we thus have

$$\overline{f}_*\left(\overline{\mathbb{Q}}_\ell\right) = \bigoplus_{\rho \in \mathrm{Irr}(A_G(x))} V_\rho \otimes \mathcal{E}_\rho,$$

where $\mathcal{E}_\rho$ are G-equivariant irreducible local systems on X.

We thus got a map from $\rho \mapsto \mathcal{E}_\rho$ from the set of irreducible characters of $A_G(x)$ to the set of irreducible G-equivariant local systems on X.

Conversely, if $\mathcal{E}$ is a G-equivariant irreducible local system on X, then $\rho : A_G(x) \to \mathrm{GL}(\mathcal{E}_x)$ is an irreducible representation.

3.11 F-equivariance and the sheaf-function correspondence

Suppose now that X is defined over $\mathbb{F}_q$ with geometric Frobenius $F : X \to X$. We consider the functor $F^* : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(X)$.

We say that $K \in \mathcal{D}_c^b(X)$ is F-stable if $F^*K \simeq K$. An F-equivariant complex $K \in \mathcal{D}_c^b(X)$ is a pair (K, φ) with $\varphi : F^*K \simeq K$.

A morphism $(K, \varphi) \to (K', \varphi')$ is a morphism $f : K \to K'$ such that the following diagram commutes

$$\begin{array}{ccc} F^*K & \xrightarrow{F^*(f)} & F^*K' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ K & \xrightarrow{f} & K' \end{array}$$

We denote by $\mathcal{D}_c^b(X)_F$ the category of F-equivariant complexes on X.

We define the *characteristic function* $\mathcal{X}_{K,\varphi} : X^F = X(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$ of $(K, \varphi) \in \mathcal{D}_c^b(X)_F$ by

$$\mathcal{X}_{K,\varphi}(x) = \sum_i (-1)^i \mathrm{Tr}\left(\varphi_x^i, \mathcal{H}_x^i K\right).$$

Then if $(K, \varphi) \simeq (K', \varphi')$, we have $\mathcal{X}_{K,\varphi} = \mathcal{X}_{K',\varphi'}$.

Lemma 100. *If $K \simeq K'$ are two isomorphic simple perverse sheaves then for all $\varphi : F^*K \to K$ and $\varphi' : F^*K' \to K'$, there exists a unique $c_{\varphi,\varphi'} \in \overline{\mathbb{Q}}_\ell^\times$ such that*

$$\mathcal{X}_{K,\varphi} = c_{\varphi,\varphi'} \mathcal{X}_{K',\varphi'}.$$

If $c_{\varphi,\varphi'} = 1$ then $(K, \varphi) \simeq (K', \varphi')$.

Proof. Let $\alpha : K \simeq K'$ then $F^*(\alpha) : F^*(K) \simeq F^*(K')$. Consider

$$\alpha \circ \varphi \circ F^*(\alpha)^{-1} : F^*(K') \to K'.$$

Since K' is simple, there exists $c \in \overline{\mathbb{Q}}_\ell^\times$ such that $\alpha \circ \varphi \circ F^*(\alpha)^{-1} = c\varphi'$ from which we have

$$X_{K',\alpha\circ\varphi\circ F^*(\alpha)^{-1}} = cX_{K',\varphi'}.$$

But the left hand side is exactly $X_{K,\varphi}$. □

Proposition 101. *Let $f : X \to Y$ be a morphism defined over $\mathbb{F}_q$ (i.e. which commutes with the Frobenius endomorphisms). Then $f_!$ (resp. f^*) induces functors $\mathcal{D}^b_c(X)_F \to \mathcal{D}^b_c(Y)_F$ (resp. $\mathcal{D}^b_c(Y)_F \to \mathcal{D}^b_c(X)_F$).*

Proof. Exercise.

□

Theorem 102. *Let $f : X \to Y$ be defined over $\mathbb{F}_q$ and let $f^F : X^F \to Y^F$ be the induces map on $\mathbb{F}_q$-points. Then*

(1) $(f^F)^*\left(X_{K',\varphi'}\right) = X_{f^*(K',\varphi')}$,

(2) $(f^F)_!\left(X_{K,\varphi}\right) = X_{f_!(K,\varphi)}$.

The proof of (1) in the above theorem is straightforward, the proof of (2) is equivalent to the so-called Grothendieck-Deligne trace formula:

Let $\varphi : F^*K \simeq K$. The Frobenius $F : X \to X$ induces a map $\mathbb{H}^i_c(X, K) \to \mathbb{H}^i_c(X, F^*K)$ which composed with φ induces $F^*_\varphi : \mathbb{H}^i_c(X, K) \to \mathbb{H}^i_c(X, K)$. Then

$$\sum_{x\in X^F} X_{K,\varphi}(x) = \sum_i (-1)^i \mathrm{Tr}\left(F^*_\varphi, \mathbb{H}^i_c(X, K)\right). \tag{8}$$

Remark 11. *If K is the constant sheaf $\overline{\mathbb{Q}}_\ell$ and $\varphi : F^*\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell$ induces the identity on stalks at $x \in X^F$, the formula (8) becomes*

$$|X^F| = \sum_i (-1)^i \mathrm{Tr}\left(F^*, H^i_c(X, \overline{\mathbb{Q}}_\ell)\right).$$

If X is pure *and has* polynomial-count *then X has vanishing odd cohomology and*

$$|X^F| = \sum_i \dim H^{2i}_c(X, \overline{\mathbb{Q}}_\ell) q^i.$$

(e.g. GL_n/B_n, $\mathbb{P}^n$*, quiver varieties,...)*

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