Subtriviality of continuous fields of nuclear $C^*$-algebras

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Abstract

We extend in this paper the characterisation of a separable nuclear $C^*$-algebra given by Kirchberg proving that given a unital separable continuous field of nuclear $C^*$-algebras $A$ over a compact metrizable space $X$, the $C(X)$-algebra $A$ is isomorphic to a unital $C(X)$-subalgebra of the trivial continuous field $O_2 \otimes C(X)$, image of $O_2 \otimes C(X)$ by a norm one projection.

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0 Introduction

In order to study deformations in the $C^*$-algebraic framework, Dixmier introduced the notion of continuous field of $C^*$-algebras over a locally compact space ([7]). In the same way as there is a faithful representation in a Hilbert space for any $C^*$-algebra thanks to the Gelfand–Naimark–Segal construction, a separable continuous field of $C^*$-algebras $A$ over a compact metrizable space $X$ always admits a continuous field of faithful representations $\pi$ in a Hilbert $C(X)$-module, i.e. there exists a family of representations $\{\pi_x, x \in X\}$, in a separable Hilbert space $H$ which factorize through a faithful representation of the fibre $A_x$ such that for each $a \in A$, the map $x \mapsto \pi_x(a)$ is strongly continuous ([4, théorème 3.3]).

In a work on tensor products over $C(X)$ of continuous fields of $C^*$-algebras over $X$ ([16]), Kirchberg and Wassermann raised the question of whether the continuous field of $C^*$-algebras $A$ could be subtrivialized, i.e. whether one could find a continuous field of faithful representations $\pi$ such that the map $x \mapsto \pi_x(a) \in L(H)$ is actually norm continuous for all $a$ in $A$. In this case, given any $C^*$-algebra $B$, the minimal tensor product $A \otimes B$ is a $C(X)$-subalgebra of the trivial continuous field $[L(H) \otimes B] \otimes C(X)$ and is therefore a continuous field with fibres $(A \otimes B)_x = A_x \otimes B$. They proved that a non-exact continuous field with exact fibres cannot be subtrivialized and they constructed such examples.

The non-trivial example of the continuous field of rotation algebras over the unit circle $\mathbb{T}$ had already been studied by Haagerup and Rørdam in [10]. More precisely, they constructed continuous functions $u, v$ from $\mathbb{T}$ to the unitary group $U(H)$ of the infinite-dimensional separable Hilbert space $H$ satisfying the commutation relation $u_t v_t = t v_t u_t$. 
for all $t \in T$ and the uniform continuity condition $\max \{\|u_t - u_{t'}\|, \|v_t - v_{t'}\|\} < C'|t - t'|^{1/2}$ where $C'$ is a computable constant.

Our purpose in the present paper is to show that the subtrivialization is always possible in the nuclear separable case through a generalisation of the following theorem of Kirchberg using $RKK$-theory arguments:

**Theorem 0.1** ([15]) A unital separable C*-algebra $A$ is exact if and only if it is isomorphic to a C*-subalgebra of $O_2$. Moreover the C*-algebra $A$ is nuclear if and only if $A$ is isomorphic to a C*-subalgebra of $O_2$ containing the unit $1_{O_2}$ of $O_2$, image of $O_2$ by a unital completely positive projection.

As a matter of fact, we get an equivalent characterisation of nuclear separable continuous fields of C*-algebras (theorem 3.2) which is made possible thanks to $C(X)$-linear homotopy invariance of the bifunctor $RKK(X; -, -)$ (theorem 1.6) and $C(X)$-linear Weyl-von Neumann absorption results (proposition 2.5). This also enables us to have a better understanding of the characterisation of separable continuous fields of nuclear C*-algebras given by Bauval in [2].

In an added appendix, the corresponding characterisation of exact separable continuous fields of C*-algebras as $C(X)$-subalgebras of $O_2 \otimes C(X)$ given by Eberhard Kirchberg is described (theorem A.1).

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1 Preliminaries

1.1 $C(X)$-algebras

Let $X$ be a compact Hausdorff space and $C(X)$ be the C*-algebra of continuous functions on $X$ with complex values. We start by recalling the following definition.

**Definition 1.1** ([13]) A $C(X)$-algebra is a C*-algebra $A$ endowed with a unital morphism from $C(X)$ in the centre of the multiplier algebra $M(A)$ of $A$.

**Remark:** We do not assume that $C(X)$ embeds into $M(A)$. For instance, there is a natural structure of $C([0, 2])$-algebra on the C*-algebra $C([0, 1])$.

For $x \in X$, define the kernel $C_x(X)$ of the evaluation map $ev_x : C(X) \to \mathbb{C}$ at $x$; denote by $A_x$ the quotient of a $C(X)$-algebra $A$ by the closed ideal $C_x(X)A$ and by $a_x$ the image of an element $a \in A$ in the fibre $A_x$. Then the function

$$x \mapsto \|a_x\| = \inf \{\|[1 - f + f(x)]a\|, f \in C(X)\}$$

is upper semi-continuous for any $a \in A$ and the $C(X)$-algebra $A$ is said to be a continuous field of C*-algebras over $X$ if the function $x \mapsto \|a_x\|$ is actually continuous for every $a \in A$ ([7]).
Examples 1. If $A$ is a $C(X)$-algebra and $D$ is a $C^*$-algebra, the spatial tensor product $B = A \otimes D$ is naturally endowed with a structure of $C(X)$-algebra through the map $f \in C(X) \mapsto f \otimes 1_{M(D)} \in M(A \otimes D)$. In particular, if $A = C(X)$, the tensor product $B$ is a trivial continuous field over $X$ with constant fibre $B_x \simeq D$

2. Given a $C(X)$-algebra $A$, define the unital $C(X)$-algebra $A$ generated by $A$ and $u[C(X)]$ in $M[A \oplus C(X)]$ where $u(g)(a + f) = ga + gf$ for $a \in A$ and $f, g \in C(X)$. It defines a continuous field of $C^*$-algebras over $X$ if and only if the $C(X)$-algebra $A$ is continuous ([4, proposition 3.2]).

Remark: If $A$ is a separable continuous field of non-zero $C^*$-algebras (not necessarily unital) over the compact Hausdorff space $X$, the positive cone $C(X)_+$ and so the $C^*$-algebra $C(X)$ are separable. Hence, the topological space $X$ is metrizable.

Definition 1.2 ([4, 5]) Given a continuous field of $C^*$-algebras $A$ over the compact Hausdorff space $X$, a continuous field of representations of a $C(X)$-algebra $D$ in the multiplier $C^*$-algebra $M(A)$ of $A$ is a $C(X)$-linear morphism $\pi : D \to M(A)$, i.e. for each $x \in X$, the induced representation $\pi_x$ of $D$ in $M(A_x)$ factorizes through the fibre $D_x$.

If the $C(X)$-algebra $D$ admits a continuous field of faithful representations $\pi$ in the $C(X)$-algebra $M(A)$ where $A$ is a continuous field over $X$, i.e. the induced representation of the fibre $D_x$ in $M(A_x)$ is faithful for every point $x \in X$, the function

$$x \mapsto \|\pi_x(d)\| = \sup\{\|\pi(d)a_x\|, a \in A \text{ such that } \|a\| \leq 1\}$$

is lower semi-continuous for all $d \in D$ and the $C(X)$-algebra $D$ is therefore continuous.

In particular a separable $C(X)$-algebra $D$ is continuous if and only if there exists a Hilbert $C(X)$-module $E$ such that $D$ admits a continuous field of faithful representations in the multiplier algebra $M(K(E)) = L(E)$ of the continuous field over $X$ of compact operators $K(E)$ acting on $E$ ([4, théorème 3.3]).

Let us also mention the characterisation of separable continuous fields of nuclear $C^*$-algebras over a compact metrizable space $X$ given by Bauval in [2] using a natural $C(X)$-linear version of nuclearity introduced by Kasparov and Skandalis in [14][§6.2 : a $C(X)$-linear completely positive $\sigma$ from a $C(X)$-algebra $A$ into a $C(X)$-algebra $B$ is said to be $C(X)$-nuclear if and only if given any compact set $F$ in $A$ and any strictly positive real number $\varepsilon$, there exist an integer $k$ and $C(X)$-linear completely positive contractions $T : A \to M_k(\mathbb{C}) \otimes C(X)$ and $S : M_k(\mathbb{C}) \otimes C(X) \to B$ such that for all $a \in F$, one has the inequality

$$\|\sigma(a) - (S \circ T)(a)\| < \varepsilon.$$ 

One can then state the following results. The first assertion is a simple $C(X)$-linear reformulation of the Choi-Effros theorem and the second one is due to Bauval.

Proposition 1.3 Let $X$ be a compact metrizable space and $A$ be a separable $C(X)$-algebra.
1. ([14]§6.2) Given a \( C(X) \)-algebra \( B \) and a closed ideal \( J \subset B \), any contractive \( C(X) \)-nuclear map \( A \to B/J \) admits a contractive \( C(X) \)-linear completely positive lift \( A \to B \).

2. ([2, théorème 7.2]) The \( C(X) \)-algebra \( A \) is a continuous fields of nuclear \( C^* \)-algebras over \( X \) if and only if the identity map \( \text{id}_A : A \to A \) is \( C(X) \)-nuclear.

Remark: In assertion 1., the ideal \( J = (C(X)B)J = C(X)J \) is a \( C(X) \)-algebra.

### 1.2 \( C(X) \)-extensions

Given a compact Hausdorff space \( X \), we introduce a natural \( C(X) \)-linear version of the semi-group \( \text{Ext}(-,-) \) defined by Kasparov ([12, 13]).

Call a morphism of \( C(X) \)-algebras a \(*\)-homomorphism between \( C(X) \)-algebras which is \( C(X) \)-linear.

**Definition 1.4** A \( C(X) \)-extension of a \( C(X) \)-algebra \( A \) by a \( C(X) \)-algebra \( B \) is a short exact sequence

\[
0 \to B \to D \xrightarrow{\pi} A \to 0
\]

in the category of \( C(X) \)-algebras. The \( C(X) \)-extension is said to be trivial if the map \( \pi \) admits a cross section \( s : A \to D \) which is a morphism of \( C(X) \)-algebras.

As in the \( C^* \)-algebraic case a \( C(X) \)-extension \( 0 \to B \to D \to A \to 0 \) of \( A \) by \( B \) defines unambiguously an homomorphism from \( D \) to the multiplier algebra \( M(B) \) of \( B \), which gives a morphism of \( C(X) \)-algebras \( \sigma : A \to M(B)/B \) (called the Busby invariant of the extension) and the \( C(X) \)-extension is trivial if and only if the map \( \sigma \) lifts to a morphism of \( C(X) \)-algebras \( A \to M(B) \). Conversely, given a morphism of \( C(X) \)-algebras \( \sigma : A \to M(B)/B \), setting \( D = \{(a, m) \in A \times M(B), \sigma(a) = q(m)\} \) where \( q \) is the quotient map \( M(B) \to M(B)/B \), one has a \( C(X) \)-extension \( 0 \to B \to D \to A \to 0 \) (see [12]§7).

Remark: A \( C(X) \)-extension \( 0 \to B \to D \to A \to 0 \) induces for every \( x \in X \) a \( C^* \)-extension \( 0 \to B_x \to D_x \to A_x \to 0 \).

In order to define the sum of two \( C(X) \)-extensions, recall that the Cuntz algebra \( \mathcal{O}_2 \) is the unital \( C^* \)-algebra generated by two orthogonal isometries \( s_1 \) and \( s_2 \) subject to the relation \( 1 = s_1 s_1^* + s_2 s_2^* \) ([6]). Then if \( \mathcal{K} \) is the \( C^* \)-algebra of compact operators on the infinite-dimensional separable Hilbert space, one defines the sum of two \( C(X) \)-extensions \( \sigma_1 \) and \( \sigma_2 \) of the \( C(X) \)-algebra \( A \) by the stable \( C(X) \)-algebra \( \mathcal{K} \otimes B \) through the choice of a unital copy of \( \mathcal{O}_2 \) in the multiplier algebra \( M(\mathcal{K}) \) of \( \mathcal{K} \) to be the \( C(X) \)-extension

\[
\sigma_1 \oplus \sigma_2 : a \mapsto q(s_1 \otimes 1)\sigma_1(a)q(s_1^* \otimes 1) + q(s_2 \otimes 1)\sigma_2(a)q(s_2^* \otimes 1) \in M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B),
\]

where \( q \) is the quotient map \( M(\mathcal{K} \otimes B) \to M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B) \).
Definition 1.5 Given a compact Hausdorff space $X$ and two $C(X)$-algebras $A$ and $B$, $\Ext(X; A, B)$ is the semi-group of $C(X)$-extensions of $A$ by $K \otimes B$ divided by the equivalence relation $\sim$ where $\sigma_1 \sim \sigma_2$ if there exist a unitary $U \in M(K \otimes B)$ of image $q(U)$ in the quotient $M(K \otimes B)/(K \otimes B)$ and two trivial $C(X)$-extensions $\pi_1$ and $\pi_2$ such that for all $a \in A$,

$$(\sigma_2 \oplus \pi_2)(a) = q(U)^* (\sigma_1 \oplus \pi_1)(a) q(U) \ (\text{in } M(K \otimes B)/(K \otimes B)).$$

Let $\Ext(X; A, B)^{-1}$ be the group of invertible elements of $\Ext(X; A, B)$, i.e. the group of classes of $C(X)$-extension $\sigma$ such that there exists a $C(X)$-extension $\theta$ with $\sigma \oplus \theta$ trivial. One can generalise Kasparov's theorem of homotopy invariance of the group $\Ext(X; A, B)^{-1}$ to the framework of $C(X)$-algebras as follows.

Theorem 1.6 ([12]) Assume that $A$ is a separable $C(X)$-algebra and that $B$ is a $\sigma$-unital $C(X)$-algebra. Then the group $\Ext(X; A, B)^{-1}$ is isomorphic to the group $\KK(X; A, B)$ and is therefore $C(X)$-linear homotopy invariant in both entries $A$ and $B$.

Proof: Let us first make the following observation. Given a $C(X)$-algebra $B$ and a Hilbert $B$-module $E$, denote by $\mathcal{L}(E)$ the set of bounded $B$-linear operators on $E$ which admit an adjoint ([11]). Then any operator $T \in \mathcal{L}(E)$ is $B$-linear and so $C(X)$-linear. This argument provides a natural extension of the Stinespring-Kasparov theorem ([12]) to the framework of $C(X)$-algebras. Consequently, if $A$ is a separable $C(X)$-algebra and $B$ is a $\sigma$-unital $C(X)$-algebra, the class of a $C(X)$-extension $\sigma : A \to M(K \otimes B)/(K \otimes B)$ is invertible in $\Ext(X; A, B)$ if and only if there is a $C(X)$-linear completely positive contractive lift $A \to M(K \otimes B)$.

Let $\mathcal{R}X(E; A, B)$ be the set of Kasparov $C(X)$–$A$, $B$-bimodules ([13], definition 2.19), i.e. the set of Kasparov $A$, $B$ bimodules $(\mathcal{E}, F)$ such that the representation $A \to \mathcal{L}(\mathcal{E})$ is a $C(X)$-representation. Call a $C(X)$-linear operator homotopy an element $\{F_t, 0 \leq t \leq 1\} \in \mathcal{R}X(E; A, B \otimes C([0, 1]))$ such that $t \mapsto F_t$ is norm continuous and define on $\mathcal{R}X(E; A, B)$ the equivalence relation corresponding to the one defined by Skandalis in [18, definition 2]. The constructions given by Kasparov in [12, section 7] imply that, if the $C(X)$-algebra $B$ is $\sigma$-unital, the group of equivalence classes $\KK(X; A, B \otimes C_1)$ is isomorphic to $\Ext(X; A, B)^{-1}$, where $C_1$ is the first (graded) Clifford algebra.

On the other hand, given two graded $C(X)$-algebras $A$ and $B$ with $A$ separable, the different steps of the demonstration of [18, theorem 19] provide us with an isomorphism between the two groups $\KK(X; A, B)$ and $\KK(X; A, B)$ since proposition 2.21 of [13] defines an intersection product in $\KK(X)$-theory and lemma 18 of [18] gives us the equality

$$(ev_0 \otimes id_{C(X)})^* (1_{C(X)}) = (ev_1 \otimes id_{C(X)})^* (1_{C(X)}) \in \KK(X; C([0, 1]) \otimes C(X), C(X)),$$

where $1_{C(X)}$ is the Kasparov $C(X), C(X)$-bimodule $(C(X), 0)$ and $ev_t : C([0, 1]) \to C$ is the evaluation map at $t \in [0, 1]$. □

Remarks: 1. Kuiper's theorem implies that the law of addition on the abelian group $\Ext(X; A, B)^{-1}$ is independent of the choice of the unital copy of $O_2$ in $M(K)$. 


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2. If $A$ is a separable nuclear continuous field of $C^*$-algebras over $X$ and $B$ is a $C(X)$-algebra, every $C(X)$-linear morphism from $A$ to the quotient $M(K \otimes B)/(K \otimes B)$ is $C(X)$-nuclear and therefore admits a $C(X)$-linear completely positive lifting $A \to M(K \otimes B)$ thanks to proposition 1.3. Accordingly one has the equality

$$\text{Ext}(X; A, B)^{-1} = \text{Ext}(X; A, B).$$

2 An absorption result

In this section we prove a continuous generalisation of a statement contained in [15] which will enable us to get a $C(X)$-linear Weyl-von Neumann type result (proposition 2.5). Let us start with the following definition of Cuntz.

**Definition 2.1** ([6]) A simple $C^*$-algebra $A$ distinct from $\mathbb{C}$ is said to be purely infinite if and only if for any non-zero $a, b \in A$, there exist elements $x, y \in A$ such that $a = xby$.

Then, we can state a proposition from Kirchberg’s classification work, based on Glimm’s lemma ([7], § 11.2). A sketch of proof can also be found in [1, proposition 5.1].

**Proposition 2.2** ([15]) Let $A$ be a purely infinite simple $C^*$-algebra and $D$ be a separable $C^*$-subalgebra of $M(A)$. Assume that $V : D \to A$ is a nuclear contraction.

Then there exists a sequence $(a_n)$ of elements in $A$ of norm less than $1$ such that $V(d) = \lim_{n \to \infty} a_n^* da_n$ for all $d \in D$.

**Remark:** A simple ring has by definition exactly two distinct two sided ideals and is therefore non-zero.

**Corollary 2.3** Let $A$ be a continuous field of purely infinite simple $C^*$-algebras over a compact Hausdorff space $X$ and assume that $D$ is a separable $C(X)$-subalgebra of the multiplier algebra $M(A)$ such that there is a unital $C(X)$-embedding of the $C(X)$-algebra $O_\infty \otimes C(X)$ in the commutant $D'$ of $D$ in $M(A)$ and the identity map $id_D : D \to M(A)$ is a continuous field of faithful representations.

If $V : D \to A$ is a $C(X)$-nuclear contraction, there exists a sequence $(a_n)$ in the unit ball of $A$ with the property that for all $d \in D$,

$$V(d) = \lim_{n \to \infty} a_n^* da_n.$$

**Proof:** If $F$ is a compact generating set for $D$, it is enough to prove that given a strictly positive real number $\varepsilon > 0$, there exists an element $a$ in the unit ball of $A$ such that $\|V(d) - a^* da\| < \varepsilon$ for all $d \in F$.

For $x \in X$, the fibre $A_x$ is a purely infinite simple $C^*$-algebra and the map $d \mapsto V(d)_x \in A_x$ factorizes through $D_x \simeq (id_D)_x(D) \subset M(A_x)$ since $id_D$ is a continuous field of faithful representations. As a consequence, the previous proposition implies that we can find an element $g \in A$ with $\|g\| \leq 1$ satisfying for all $d \in F$ the inequality

$$\|\left[V(d) - g^* dg\right]_x\| < \varepsilon.$$
Thus, by upper semi-continuity and compactness, there exist a finite open covering \( \{U_1, \ldots, U_n\} \) of the space \( X \) and elements \( g_1, \ldots, g_n \) in the unit ball of \( A \) such that for all \( d \in F \) and \( x \in U_i, 1 \leq i \leq n, \)

\[
\| \left[ V(d) - g_i^* d g_i \right] \| < \varepsilon.
\]

Choose \( n \) orthogonal isometries \( w_1, \ldots, w_n \) in the \( C^* \)-algebra \( O_\infty \otimes 1_{C(X)} \subset D' \) and let \( \{\phi_i\} \) be a partition of the unit \( 1_{C(X)} \) subordinate to the covering \( \{U_i\} \) of \( X \). The element \( a = \sum_i \phi_i^{1/2} w_i g_i \in A \) verifies:

1. \( a^* a = \sum_{i,j} \sqrt{\phi_i \phi_j} g_i^* w_i^* w_j g_j = \sum_i \phi_i g_i^* g_i \leq 1_{M(A)}, \)

2. for \( d \in F \) and \( x \in X, \| \left[ V(d) - a^* da \right] \| \leq \sum_i \phi_i(x) \| \left[ V(d) - g_i^* d g_i \right] \| < \varepsilon. \text{ \( \Box \)}

Let us mention the following technical corollary which will be needed in theorem 3.2.

**Corollary 2.4** If \( p \in O_2 \otimes C(X) \) is a projection such that for all points \( x \in X, p_x \) is non-zero, then there exists an isometry \( u \in O_2 \otimes C(X) \) such that \( p = uu^* \).

**Proof:** Let \( D_2 = \lim_{n \to \infty} O_2 \otimes O_2 \) be the infinite tensor product of \( O_2 \).

Given a projection \( q \in D_2 \otimes C(X) \) such that \( \| q_x \| = 1 \) for all \( x \in X \), we first show that there exists an element \( v \in D_2 \otimes C(X) \) satisfying \( 1_{D_2 \otimes C(X)} = v^*qv \). Namely, by density of the algebraic tensor product

\[
\left[ \bigcup_n O_2 \otimes O_2 \right] \otimes C(X) = \bigcup_n \left[ O_2 \otimes O_2 \otimes C(X) \right]
\]

in the \( C^* \)-algebra \( D_2 \otimes C(X) \) and functional calculus one can find an integer \( n > 0 \) and a projection \( r \in O_2 \otimes O_2 \otimes C(X) \subset D_2 \otimes C(X) \) such that \( \| q - r \| < 1 \), which implies that \( r = s^* q s \) for some element \( s \in D_2 \otimes C(X) \). Take then a faithful state \( \varphi \) on \( O_2 \otimes O_2 \) and consider the \( C(X) \)-linear completely positive map

\[
V : [O_2 \otimes O_2 \otimes 1_{O_2}] \otimes C(X) \to O_2 \otimes O_2 \otimes O_2 \otimes O_2 \otimes 1_{O_2} \otimes 1_{O_2} \otimes C(X)
\]

defined by the formula \( V(d) = (\varphi \otimes id_{C(X)})(d)1_{O_2 \otimes O_2} \otimes C(X) \) for \( d \in [O_2 \otimes O_2 \otimes O_2 \otimes O_2] \otimes C(X) \). According to corollary 2.3, there exists an element \( t \in O_2 \otimes O_2 \otimes C(X) \) such that

\[
1_{D_2 \otimes C(X)} = 1_{O_2 \otimes O_2} \otimes C(X) = t^* r t = (st)^* q(st).
\]

Consider now the set \( P \) of projections \( p \in O_2 \otimes C(X) \) such that \( p_x \neq 0 \) for all points \( x \in X \). If \( p \) belongs to \( P \), there exists an isometry \( v \in O_2 \otimes C(X) \) such that \( p \geq vv^* \) since the \( K \)-trivial purely infinite separable unital nuclear \( C^* \)-algebra \( D_2 \) satisfying the U.C.T. is isomorphic to \( O_2 \) ([15]). As a consequence, if \( t \) is the isometry \( t = v(s_1 \otimes 1)^* v^* \), the projection \( r = tt^* \) (Murray-von Neumann equivalent to \( 1_{O_2 \otimes C(X)} \)) verifies

\[
p - r \geq r' = v(s_2 s_2^* \otimes 1) v^* \in P.
\]
The non-empty set $\mathcal{P}$ therefore satisfies the conditions $(\pi_1)-(\pi_4)$ defined by Cuntz in [6]. But the $C^*$-algebra $O_2 \otimes C(X)$ is $K_0$-triviality thanks to [6, theorem 2.3] and the theorem 1.4 of [6] enables us to conclude. □

One now deduces from corollary 2.3 the following absorption results ([21, 12, 15]):

**Proposition 2.5** Let $A$ be a $\sigma$-unital continuous field of purely infinite simple nuclear $C^*$-algebras over a compact Hausdorff space $X$ and let $K$ be the $C^*$-algebra of compact operators on the separable Hilbert space $H$. Denote by $q$ the quotient map $M(K \otimes A) \to M(K \otimes A)/(K \otimes A)$.

1. Assume that $D$ is a unital separable $C(X)$-subalgebra of the multiplier algebra $M(K \otimes A)$ with same unit such that there is a unital $C(X)$-embedding of the $C(X)$-algebra $O_\infty \otimes C(X)$ in the commutant of $D$ in $M(K \otimes A)$ and the identity map $id_D$ is a continuous field of faithful representations of $D$ in $M(K \otimes A)$.

(a) If $V$ is a unital $C(X)$-linear completely positive map from $D$ in $M(K \otimes A)$ which is zero on the intersection $D \cap (K \otimes A)$, there exists a sequence of isometries $s_n$ in $M(K \otimes A)$ such that for every $d \in D$,

$$V(d) - s_n^*ds_n \in K \otimes A \text{ and } V(d) = \lim_n s_n^*ds_n.$$  

(b) If $\pi$ is a unital morphism of $C(X)$-algebras from $D$ into $M(K \otimes A)$ which is zero on the intersection $D \cap (K \otimes A)$, there exists a sequence of unitaries $u_n$ in $M(K \otimes A)$ such that for every $d \in D$,

$$(d \oplus \pi(d)) - u_n^*du_n \in K \otimes A \text{ and } (d \oplus \pi(d)) = \lim_n u_n^*du_n.$$  

(c) Let $B$ be a $C(X)$-algebra and assume that the quotient $D/(D \cap (K \otimes A))$ is isomorphic to the $C(X)$-algebra $B$, where $B$ is the unital $C(X)$-algebra generated by $C(X)$ and $B$ in $M[B \oplus C(X)]$ ([4, définition 2.7]).

Then, if $\pi : B \to M(K \otimes A)$ is a $C(X)$-linear homomorphism, there exists a unitary $U \in M(K \otimes A)$ such that for all $b \in B \subset M(K \otimes A)/(K \otimes A)$,

$$b \oplus (q \circ \pi)(b) = q(U)^*bq(U).$$

2. Assume that the continuous field $A$ is separable and let $D$ be a separable $C(X)$-subalgebra of $M(A)$ containing $A$ such that the identity representation $D \to M(A)$ is a continuous field of faithful representations and there is a unital $C(X)$-embedding of the $C(X)$-algebra $O_\infty \otimes C(X)$ in the commutant of $D$ in $M(A)$. Define the quotient $C(X)$-algebra $B = D/A$.

If $\pi : K \otimes B \to M(K \otimes A)$ is a morphism of $C(X)$-algebras, there exists a unitary $U \in M(K \otimes A)$ such that for all $b \in K \otimes B \subset M(K \otimes A)/(K \otimes A)$,

$$b \oplus (q \circ \pi)(b) = q(U)^*bq(U).$$
Proof : 1. It derives from corollary 2.3 by the same method as the one developed by Kasparov in [11, theorem 5 and 6]. Nevertheless, for the convenience of the reader we describe the different steps of the demonstration.

1.a) Let $F$ be a compact generating set for $D$ containing the unit $1_{M(K \otimes A)}$. Then given a real number $\varepsilon > 0$, it is enough to find an element $a \in M(K \otimes A)$ such that $V(d) - a^*da \in K \otimes A$ and $\|V(d) - a^*da\| < 3\varepsilon$ for all $d \in F$.

Let $\{e_n\}$ be an increasing, positive, quasicentral, countable approximate unit in the ideal $K \otimes A$ of the C*-algebra generated by $K \otimes A + V(D)$. If we set $f_0 = (e_0)^{1/2}$ and $f_k = (e_k - e_{k-1})^{1/2}$ for $k \geq 1$, we can then assume, passing to a subsequence of $(e_n)$ if necessary, that $\|V(d_k) - f_k V(d_k)\| < 2^{-k}\varepsilon$ for all $k \in \mathbb{N}$ and $d \in F$. This implies that the series $\sum_k [V(d_k) - f_k V(d_k)] f_k$ is convergent in $K \otimes A$ and its norm is less than $\varepsilon$. Furthermore, the series $\sum_k [f_k V(d_k)] f_k$ is strictly convergent in $M(K \otimes A)$ for all $d \in F$ since $\sum_k f_k^2$ is strictly convergent to $1$.

Notice now that the maps $V_k(d) = f_k V(d) f_k$ are all $C(X)$-nuclear since the separable continuous field $K \otimes A$ is nuclear. The corollary 2.3 therefore enables us to choose by induction $a_k \in K \otimes A$ satisfying the following conditions:

1. $\forall d \in F$, $\|V_k(d) - a_k^*da_k\| < 2^{-k}\varepsilon$,
2. $\forall d \in F, \forall l < k$, $\|a_l^*da_k\| < 2^{-l-k}\varepsilon$,
3. $\sum_k a_k$ is strictly convergent toward an element $a \in M(A)$.

One then checks as in [11, theorem 5] that the limit $a$ satisfies the desired properties.

1.b) Take a compact generating $F$ for $D$ containing $1_{M(K \otimes A)}$ and consider the homomorphism $\pi' = 1 \otimes \pi : D \to M(K \otimes (K \otimes A)) \simeq M(K \otimes A)$. Given $\delta > 0$, one can find, thanks to the previous assertion, an isometry $s \in M(K \otimes A)$ such that

$$s^*ds - \pi'(d) \in K \otimes A$$

and $\|s^*ds - \pi'(d)\| < \delta$ for all $d \in K^*K$.

As a consequence, if we fix $\varepsilon > 0$, the choice of $\delta$ small enough gives us the inequality $\|pd - dp\| < \varepsilon$, and so $\|d - [pd + p^d + dp]\| < 2\varepsilon$ for all $d \in F$, where $p = ss^*$ and $p^+ = 1 - p$.

Define the unital map $\Theta : D \to M(p^+ (K \otimes A)p^+)$ by the formula $\Theta(d) = p^+ dp^+$. According to the stabilisation theorem of Kasparov ([11, theorem 2]), one can construct a unitary $w \in M(K \otimes A)$ verifying for all $d \in F$ the inequality

$$\|d - w^*[\pi'(d) \oplus \Theta(d)] w\| < 3\varepsilon.$$ 

To finish the demonstration, notice that the two homomorphisms $\pi'$ and $\pi' \oplus \pi$ are unitarily equivalent.

1.c) Consider the unital extension $\tilde{\pi}$ of $\pi$ to $B$. Then, the morphism $\tilde{\pi} \circ q : D \to M(K \otimes A)$ reduces the demonstration to the previous assertion.

2. The identity representation of the unital $C(X)$-algebra $D = (K \otimes D) + C(X) \subset M(K \otimes A)$ is clearly a continuous field of faithful representations since the unital $C(X)$-representation $C(X) \to M(A)$ is a continuous field of faithful representations. Extend
the map $\pi : \mathcal{K} \otimes B = (\mathcal{K} \otimes D)/(\mathcal{K} \otimes A) \to M(\mathcal{K} \otimes A)$ to a unital morphism of
$C(X)$-algebras $\tilde{\pi} : D/(\mathcal{K} \otimes A) \to M(\mathcal{K} \otimes A)$. Applying assertion $1.b)$ to the unital
homomorphism $d \mapsto (\tilde{\pi} \circ q)(d)$ from the $C(X)$-subalgebra $D \subset M(\mathcal{K} \otimes A)$ to the
multiplier algebra $M(\mathcal{K} \otimes A)$ now leads to the desired conclusion. □

3 The subtriviality

Given a separable continuous field of nuclear $C^*$-algebras $A$ over $X$, the strategy to
prove the subtriviality of the $C(X)$-algebra $A$ will be the same as the one developed
by Kirchberg in [15] to prove theorem 0.1 whose main ideas of demonstration are also
explained in [1, Théorème 6.1]. We associate to $A$ a $C(X)$-extension by an hereditary
$C^*$-subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$ (propo-sition 3.1) and then
prove that after stabilisation, this $C(X)$-extension splits by $\text{RKK}$-theory arguments
(theorem 3.2).

3.1 Let us construct the fundamental $C(X)$-extension associated to an exact separable
continuous field of $C^*$-algebras.

**Proposition 3.1** Given a compact Hausdorff space $X$ and a non-zero separable unital
exact $C(X)$-algebra $A$, there exist a unital $C(X)$-subalgebra $F$ of $\mathcal{O}_2 \otimes C(X)$ with same
unit and an hereditary subalgebra $I$ of $\mathcal{O}_2 \otimes C(X)$ such that $I$ is an ideal in $F$ and the
$C(X)$-algebra $A$ is isomorphic to the quotient $C(X)$-algebra $F/I$.

Furthermore, if the topological space $X$ is perfect (i.e. without any isolated point)
and the $C(X)$-algebra $A$ is continuous, the canonical map $F \to M(I)$ is a continuous
field of faithful representations.

**Proof**: Thanks to the characterisation of separable exact $C^*$-algebras obtained by
Kirchberg (theorem 0.1), one may assume that the $C^*$-algebra $A$ is a $C^*$-subalgebra of
$\mathcal{O}_2$ containing the unit of $\mathcal{O}_2$.

Let $G \subset \mathcal{O}_2 \otimes C(X)$ be the trivial continuous field $A \otimes C(X)$ over $X$. Then the
kernel of the $C(X)$-linear morphism $\pi : G \to A$ defined by $\pi(a \otimes f) = fa$ is the ideal
$J = C_\Delta(X \times X)G$ where $C_\Delta(X \times X)$ is the ideal in $C(X \times X)$ of functions which are
zero on the diagonal. Indeed suppose that $T \in G$ verifies $\pi(T) = 0$. Then given $\varepsilon > 0$, take a
finite number of elements $a_i \in A$, $f_i \in C(X)$ such that $\|T - \sum_i a_i \otimes f_i\| < \varepsilon$; one has $\|T - \sum_i (1 \otimes f_i - f_i \otimes 1)(a_i \otimes 1)\| < \varepsilon + \|\pi(\sum_i a_i \otimes f_i)\| < 2\varepsilon$.

Define then the hereditary subalgebra $I = J[\mathcal{O}_2 \otimes C(X)]J$ in $\mathcal{O}_2 \otimes C(X)$ generated
by $J$. It is a $C(X)$-algebra since it is closed by Cohen theorem (see e.g. [4, proposition
1.8]) and the product $(1 \otimes f)(bc) = b(1 \otimes f)c$ belongs to $I$ for all $f \in C(X)$ and
$b, c \in I$. If we set $F = I + G$, the intersection $G \cap I$ is reduced by construction to the
subalgebra $J$, and so we have a $C(X)$-extension

$$0 \to I \to F \to A \to 0.$$  

Assume now that the space $X$ is perfect and that the $C(X)$-algebra $A$ is continuous.
We need to prove that the map $F_x \to M(I_x)$ is injective for each $x \in X$. Let $a \in G$
and $b \in I$ be two elements such that the sum $d = a + b \in F$ verifies for a given point
$x \in X$ the equality

$$
To end the proof, we have to show that \( d_x \) is zero. For every \( f \in C_\Delta(X \times X) \), one has \( (bf)_x = -(af)_x \in J_x \), whence \( b_x \in J_x \) and so \( d_x \in G_x \). But the representation of \( G_x \) is perfect in \( M(J_x) \approx M(C_\Delta(X)) \) is injective since \( X \) is perfect and \( A \) is continuous, from which we deduce that \( d_x = 0 \). \( \square \)

Remark: With the previous notations, if the \( C(X) \)-algebra \( A \) is nuclear and \( \psi \) is a unital completely positive projection from \( O_2 \) onto \( A \), the map \( \pi \circ (\psi \otimes \text{id}_{C(X)}) \) is a unital \( C(X) \)-linear completely positive map from \( O_2 \otimes C(X) \) onto the \( C(X) \)-subalgebra \( A \) which is zero on the nuclear hereditary \( C(X) \)-subalgebra \( I \).

3.2 We can now state the main theorem:

**Theorem 3.2** Let \( X \) be a compact metrizable space and \( A \) be a unital separable \( C(X) \)-algebra with a unital embedding of the \( C(X) \)-algebra \( C(X) \) in \( A \).

The following assertions are equivalent:

1. \( A \) is a continuous field of nuclear \( C^* \)-algebras over \( X \);

2. there exist a unital monomorphism of \( C(X) \)-algebras \( \alpha : A \rightarrow O_2 \otimes C(X) \) and a unital \( C(X) \)-linear completely positive map \( E : O_2 \otimes C(X) \rightarrow A \) such that \( E \circ \alpha = \text{id}_A \).

**Proof:** \( 2 \Rightarrow 1 \) By assumption the identity map \( \text{id}_A = E \circ \text{id}_{O_2 \otimes C(X)} \circ \alpha : A \rightarrow A \) is nuclear since the \( C^* \)-algebra \( O_2 \otimes C(X) \) is nuclear and so the \( C^* \)-algebra \( A \) is nuclear. Besides, the \( C(X) \)-algebra \( A \) is isomorphic to the \( C(X) \)-subalgebra \( \alpha(A) \) of the continuous field \( O_2 \otimes C(X) \) and is therefore continuous.

\( 1 \Rightarrow 2 \) Let us first deal with the case where the space \( X \) is perfect.

Given a unital nuclear separable continuous fields \( A \) over \( X \) which is unitally embedded in the \( C^* \)-algebra \( O_2 \), consider the \( C(X) \)-extension

\[
0 \rightarrow I \rightarrow F \xrightarrow{\pi} A \rightarrow 0
\]

constructed in proposition 3.1 and take the associated \( C(X) \)-extension

\[
0 \rightarrow K \otimes I \otimes O_2 \rightarrow D = (K \otimes F \otimes 1_{O_2}) + (K \otimes I \otimes O_2) \rightarrow K \otimes A \rightarrow 0.
\]

The \( C(X) \)-nuclear quotient map \( \sigma = \sigma \circ \text{id}_{K \otimes A} \) from the separable nuclear continuous field \( K \otimes A \) to the quotient \( D/(K \otimes I \otimes O_2) \subset M(K \otimes I \otimes O_2)/(K \otimes I \otimes O_2) \) admits a \( C(X) \)-linear completely positive lifting \( K \otimes A \rightarrow D \subset K \otimes [O_2 \otimes C(X)] \otimes O_2 \) thanks to proposition 1.3. This means that the class of \( \sigma \) is invertible in \( Ext(X; K \otimes A, I \otimes O_2) \) (see the second remark following theorem 1.6).

But the group \( Ext(X; K \otimes A, I \otimes O_2)^{-1} \) is \( C(X) \)-linear homotopy invariant (theorem 1.6), hence zero since the endomorphism \( \varphi_2(a) = s_1a^*s_1 + s_2a^*s_2 \) of \( O_2 \) is homotopic to the identity map \( \text{id}_{O_2} \) ([6, proposition 2.2]) and so \( [\theta] = 2[\theta] \) in \( Ext(X; K \otimes A, I \otimes O_2)^{-1} \) for any invertible extension \( \theta \) of \( K \otimes A \) by \( I \otimes O_2 \). As a consequence, the \( C(X) \)-extension defined by \( \sigma \) is stably trivial. Furthermore, the identity representation of
$D \subset M(K \otimes I \otimes O_2)$ is a continuous field of faithful representations (proposition 3.1) and the assertion 2. of proposition 2.5 implies that the quotient morphism $(id_K \otimes \pi \otimes id_{O_2})$ from $D$ to $K \otimes A$ admits a cross section $\alpha$ which is a morphism of $C(X)$-algebras.

This monomorphism $\alpha$ is going to enable us to conclude by standard arguments, using theorem 0.1 and the result of Elliott ([9]) that the $C^*$-algebra $O_2$ is isomorphic to $O_2 \otimes O_2$.

Choose a non-zero minimal projection $e_{11}$ in the $C^*$-algebra $K$ of compact operators that we embed in $O_2$ and let $\varphi$ be a state on $O_2$ such that $\varphi(e_{11}) = 1$. If we take a unital completely positive projection $\psi$ of $O_2$ onto the nuclear $C^*$-subalgebra $A \subset O_2$ (theorem 0.1), the composed map

$$E = (\varphi \otimes id_A) \circ (id_{O_2} \otimes [\pi \circ (\psi \otimes id_{C(X)})] \otimes \varphi)$$

is a unital $C(X)$-linear completely positive map from $O_2 \otimes [O_2 \otimes C(X)] \otimes O_2$ onto $A$. Take also an isometry $u \in O_2 \otimes C(X)$ such that $\alpha(e_{11} \otimes 1_A) = uu^*$ (corollary 2.4) and define the unital $C(X)$-algebra morphism

$$\beta : A \rightarrow O_2 \otimes [O_2 \otimes C(X)] \otimes O_2 \simeq O_2 \otimes C(X)$$

by the formula $\beta(a) = uu^* \alpha(e_{11} \otimes a)u$. If $\tilde{E} : O_2 \otimes C(X) \rightarrow A$ is the completely positive unital map $d \mapsto E(udu^*)$, one gets for all $a \in A$ the equality

$$(\tilde{E} \circ \beta)(a) = (E \circ \alpha)(e_{11} \otimes a) = a$$

Let us now come back to the general case of a compact space $X$.

Define the continuous field $B = A \otimes C([0,1])$ over the perfect compact space $Y = X \times [0,1]$. According to the previous discussion, there exist a unital completely positive map $\tilde{E} : O_2 \otimes C(Y) \rightarrow B$ and a $C(X) \otimes C([0,1])$-linear monomorphism $\tilde{\alpha} : B \rightarrow O_2 \otimes C(Y)$ such that $\tilde{E} \circ \tilde{\alpha} = id_B$. If $ev_1 : C([0,1]) \rightarrow \mathbb{C}$ is the evaluation map at $x = 1 \in [0,1]$, define the two maps $E : O_2 \otimes C(X) \rightarrow A$ and $\alpha : A \rightarrow O_2 \otimes C(X)$ by

$$E(d) = (id_A \otimes ev_1) \circ \tilde{E}(d \otimes 1_{C([0,1])})$$

and $\alpha(a) = (id_{O_2 \otimes C(X)} \otimes ev_1) \circ \tilde{\alpha}(a \otimes 1_{C([0,1])})$.

Then $E$ is a unital $C(X)$-linear completely positive map, $\alpha$ is a unital $C(X)$-linear monomorphism and one has the identity $E \circ \alpha = id_A$. □

**Remark:** Assume that $X$ is a locally compact metrizable space and that the $C_0(X)$-algebra $A$ is a nuclear continuous field of $C^*$-algebras over $X$, where $C_0(X)$ denotes the algebra of continuous functions on $X$ vanishing at infinity. If $\tilde{X}$ is the Alexandroff compactification of $X$, the unital $C(\tilde{X})$-algebra $A$ generated by $A$ and $C(\tilde{X})$ in the multiplier algebra $M[A \oplus C(\tilde{X})]$ is a separable unital continuous field of $C^*$-algebras over $\tilde{X}$ ([4, proposition 3.2]). By theorem 3.2, there exists therefore a $C(\tilde{X})$-linear monomorphism $\alpha : A \hookrightarrow O_{2} \otimes C(\tilde{X})$ and the $C_0(X)$-algebra $A$ is isomorphic to the $C_0(X)$-subalgebra $\alpha(A)$ of $O_{2} \otimes C_0(X)$.  

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4 Concluding remarks

4.1 A $C(X)$-subalgebra of $O_2 \otimes C(X)$ is by construction exact and continuous. Conversely, if $A$ is a non-zero exact separable unital continuous field of C*-algebras over a perfect metrizable compact space $X$, one has by proposition 3.1 a $C(X)$-extension

$$0 \to I \to F \to A \to 0$$

where $F$ is a $C(X)$-subalgebra of $O_2 \otimes C(X)$. If the identity map $A \to A = F/I$ admits a $C(X)$-linear completely positive lifting $A \to F$, the same method as the one used in theorem 3.2 will imply that the exact continuous field $A$ is isomorphic to a $C(X)$-subalgebra of the trivial continuous field $O_2 \otimes C(X)$.

It is therefore interesting to know whether this map admits a $C(X)$-linear completely positive lifting in the not discrete case.

4.2 Let us have a look at one of the technical problems involved, the Hahn-Banach type extension property in the continuous field framework for finite type $C(X)$-submodules.

Let $A$ be a separable unital continuous field of C*-algebras over a compact metrizable space $X$ and let $D$ be a finitely generated $C(X)$-submodule which is an operator system. Assume that $\phi : D \to C(X)$ is a $C(X)$-linear unital completely positive map. Then for $x \in X$, there exists, thanks to [4, proposition 3.13], a continuous field of states $\Phi_x$ on $A$, i.e. a $C(X)$-linear unital positive map from $A$ to $C(X)$, such that for all $d \in D$,

$$\Phi_x(d)(x) = \phi(d)(x).$$

As a consequence, given $\varepsilon > 0$ and a finite subset $F$ of $D$, one can build by continuity and compactness a continuous field of states $\Phi$ on $A$ such that

$$\max\{\|\Phi(d) - \phi(d)\|, d \in F\} < \varepsilon.$$

But one cannot find in general any continuous field of states on $A$ whose restriction to $D$ is $\phi$. Indeed, consider the $C(\tilde{N})$-algebra $A = C^2 \otimes C(\tilde{N})$ where $\tilde{N} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of the space $\mathbb{N}$ of positive integers. Define the positive element $a \in C_{\infty}(\tilde{N})A \subset A$ by the formulas

$$a_n = \begin{cases} \left(\frac{1}{n+1}, 0\right) & \text{if } n \text{ even} \\ (0, \frac{1}{n+1}) & \text{if } n \text{ odd} \end{cases}$$

and let $\phi$ be the $C(\tilde{N})$-linear unital completely positive map with values in $C(\tilde{N})$ defined on the $C(\tilde{N})$-submodule generated by the two $C(\tilde{N})$-linearly independent elements $1_A$ and $a$ through the formula

$$\phi(a)(n) = \frac{1}{n+1} \text{ if } n < \infty \text{ and } \phi(a)(\infty) = 0.$$

Suppose that the continuous field of states $\Phi$ is a $C(\tilde{N})$-linear extension of $\phi$ to $A$. Then as A. Bauval already noticed it, one has the contradiction

$$1 = \Phi(1_A)(\infty) = \Phi((1, 0) \otimes 1)(\infty) + \Phi((0, 1) \otimes 1)(\infty) = \lim_{n \to \infty} \Phi((1, 0) \otimes 1)(2n + 1) + \lim_{n \to \infty} \Phi((0, 1) \otimes 1)(2n) = 0 + 0 = 0.$$
Appendix by Eberhard Kirchberg
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In this appendix, we solve in proposition A.3 the lifting question raised in paragraph 4.1 through a continuous generalisation of joint work of E.G. Effros and U. Haagerup on lifting problems for C*-algebras ([8], see also [22]). This result enables us to state the following characterisation of separable exact continuous fields of C*-algebras:

**Theorem A.1** Let \(X\) be a compact metrizable space and \(A\) be a (unital) separable continuous field of C*-algebras over \(X\).

Then the C*-algebra \(A\) is exact if and only if there exists a (unital) monomorphism of \((X)\)-algebras \(A \hookrightarrow \mathcal{O}_2 \otimes \mathcal{C}(X)\).

Let us start with a technical C(\(X\))-linear version of Auerbach’s theorem ([17, proposition 1.c.3]) for a continuous field of C*-algebras \(A\) over \(X\) which gives us local bases over \((X)\) with continuous coordinate maps for particular free \((X)\)-submodules of finite type in \(A\).

Define a \((X)\)-operator system in \(A\) to be a \((X)\)-submodule which is an operator system.

**Lemma A.2** ([8, lemma 2.4]) Let \(A\) be a separable unital continuous field of C*-algebras over a compact metrizable space \(X\), \(E \subset A\) be a \((X)\)-operator system and assume that there exists an integer \(n \in \mathbb{N}^*\) such that for all \(x \in X\), the dimension \(\dim E_x\) of the operator system \(E_x \subset A_x\) equals \(n\). Then the following holds.

Given any point \(x \in X\), there exist an open neighbourhood \(U\) of \(x\) in \(X\), self-adjoint \((X)\)-linear contractions \(\varphi_i : A \to (X)\) and self-adjoint elements \(f_i \in E\) with \(\|f_i\| \leq 2\) for \(1 \leq i \leq n\) such that

\[
\forall a \in C_0(U)E, \quad a = \sum_i \varphi_i(a)f_i.
\]

Furthermore, there exists a continuous field of states \(\Psi : A \to (X)\) such that the restriction of the map \(2n\Psi - id_A\) to the operator system \(E\) is completely positive.

**Proof:** Let us fix a point \(x \in X\). Then there exist, thanks to Auerbach’s theorem, a normal basis \(\{r_1, \ldots, r_n\}\) of the fibre \(E_x\) where each \(r_i\) is self adjoint and norm one hermitian functionals \(\phi_j : A_x \to \mathbb{C}\), \(1 \leq j \leq n\), such that \(\phi_j(r_i) = \delta_{i,j}\).

Consider the polar decomposition \(\phi_j = \phi_j^+ - \phi_j^-\) where \(\phi_j^+\) and \(\phi_j^-\) are positive functionals such that \(1 = \|\phi_j\| = \|\phi_j^+\| + \|\phi_j^-\|\). By [4] lemme 3.12, there exist \((X)\)-linear positive maps \(\varphi_j^+ : A \to (X)\) which extend the functionals \(\phi_j^+\) and \(\varphi_j^- : A \to (X)\) which extend the functionals \(\phi_j^-\) on the fibre \(A_x\) to the \((X)\)-algebra \(A\) with the property that \(\varphi_j^+(1) = \|\phi_j^+\|\) and \(\varphi_j^-(1) = \|\phi_j^-\|\). Take also \(n\) norm 1 self-adjoint elements \(e_i \in E\) satisfying the equality \((e_i) = r_i\) and define the matrix \(T = [\varphi_j(e_i)]_{i,j} \in \mathcal{M}_n(\mathbb{R}) \otimes (X)\).

One has by construction \(T_x = 1_{\mathcal{M}_n(\mathbb{C})}\); the set \(U_1 \subset X\) of points \(y \in X\) for which the spectrum of \(T_y \in \mathcal{M}_n(\mathbb{R})\) is included in the open set \(\{z \in \mathbb{C}, |z| > 1/2\}\) is therefore an open neighbourhood of \(x\) in \(X\) ([4, proposition 2.4 b)]. If \(\eta\) is a continuous function on \(X\) with values in \([0,1]\) which is 0 outside \(U_1\) and 1 on an open neighbourhood \(U\) of the point \(x \in X\), the self-adjoint elements \(f_1, \ldots, f_n\) of norm less than 2 are then well defined in \(C_0(U_1)E\) by the formula
Then the \step is the following. Given the equality $\dim(E_n)$ satisfies the equality $\dim(E_n) = \frac{1}{n} \sum_i (\varphi_i^+ + \varphi_i^-)$. Then one gets for all $a \in C_0(\mathcal{U})E$ the equality:

$$
(2n\Phi - id_A)(a) = \sum_{1 \leq i \leq n} \left[ \varphi_i^+(a)(2 - f_i) + \varphi_i^-(a)(2 + f_i) \right].
$$

The \restriction of the map $(2n\Phi - id_A)$ to $C_0(\mathcal{U})E$ is therefore completely positive and an appropriate \partition of the unit $1_{C(X)}$ enables us to conclude. \hfill \Box

Noticing that a $C(X)$-linear map $\sigma : A \to B$ between $C(X)$-algebras is completely positive if and only if each induced map $\sigma_x : A_x \to B_x$ is completely positive (see for instance [4, proposition 2.9]), the lemma A.2 allows us to state a continuous version of theorem 2.5 of [8]. Replacing then the continuous field $A$ by $A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$ (where $M_{2^\infty}(\mathbb{C}) = \lim_{n \to \infty} M_{2^n}(\mathbb{C})$) and working with finitely generated $C(X)$-operator systems $E_k \subset A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X)$ for which the function $x \mapsto \dim(E_k)_x$ is continuous, one derives the following desired $C(X)$-linear completely positive lifting result.

**Proposition A.3** ([8, theorem 3.4]) Suppose that $A$ and $B$ are two unital \separable \exact \continuous fields of $C^*$-algebras over a compact space $X$ with $A = B / J$ for some nuclear ideal $J$ in $B$.

*Then there exists a $C(X)$-linear unital completely positive lifting $A \to B$ of $id_A$.\*

**Proof:** Let us define the two continuous fields $\mathcal{A} = A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$ and $\mathcal{B} = B \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$. It is clearly enough to find a $C(X)$-linear unital completely positive cross section $\theta$ of the quotient morphism $\mathcal{B} \to \mathcal{A}$ (by [4, theorem 3.3]).

Consider a dense sequence $\{a_k\}$ in the self-adjoint part of $\mathcal{A}$ where each $a_k$ belongs to the dense subalgebra $A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X)$ of $\mathcal{A}$ and $a_1 = 1$. Let us show that we may assume inductively that $C(X)$-operator system $E_n$ generated by the $a_k$, $1 \leq k \leq n$, satisfies the equality $\dim(E_n)_x = n$ for every $n \in \mathbb{N}^*$ and every $x \in X$. The inductive step is the following. Given $n \geq 2$, there exists by construction an integer $l$ such that $E_n \subset A \oplus M_{2l}(\mathbb{C}) \otimes C(X)$. Set $a'_n = a_n + 2^{-n-1} d_l \otimes 1_{C(X)}$ where

$$
d_l = 1_{M_{2l}(\mathbb{C})} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2^{l+1}}(\mathbb{C}) \subset M_{2^\infty}(\mathbb{C}).
$$

Then the $C(X)$-module $E'_n = E_{n-1} + C(X)a'_n$ verifies for each $x \in X$ the equality $\dim(E'_n)_x = \dim(E_{n-1})_x + 1$.

Using proposition 1.3, one can now finish the proof by the same method as the one developed by E.G. Effros and U. Haagerup in [8],3 (see also [22, theorem 6.10]). \hfill \Box
References


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