Local Systems and Constructible Sheaves

Fouad El Zein and Jawad Snoussi

Abstract. The article describes Local Systems, Integrable Connections, the equivalence of both categories and their relations to Linear differential equations. We report in details on Regular Singularities of Connections and on Singularities of local systems which leads to the theory of Intermediate extensions and the Decomposition theorem.

Mathematics Subject Classification (2000). Primary 32S60,32S40, Secondary 14F40.

Keywords. Algebraic geometry, Analytic geometry, Local systems, Linear Differential Equations, Connections, constructible sheaves, perverse sheaves, Hard Lefschetz theorem.

Introduction

The purpose of these notes is to indicate a path for students which starts from a basic theory in undergraduate studies, namely the structure of solutions of *Linear Differential Equations* which is a classical subject in mathematics (see Ince [9]) that has been constantly enriched with developments of various theories and ends in a subject of research in contemporary mathematics, namely perverse sheaves.
We report in these notes on the developments that occurred with the introduction of sheaf theory and vector bundles in the works of Deligne [4] and Malgrange [3,2)].
Instead of continuing with differential modules developed by Kashiwara and explained in [12], a subject already studied in a Cimpa school, we shift our attention to the geometrical aspect represented by the notion of *Local Systems* which describe on one side the structure of solutions of linear differential equations and on the other side the cohomological higher direct image of a constant sheaf by a proper smooth differentiable morphism.
Then we introduce the theory of *Connections* on vector bundles generalizing to analytic varieties the theory of linear differential equations on a complex disc.
The definition of local systems is easily extended to varieties of dimension n while it is more elaborate to extend the notion of differential equations into the concept

of connections. Deligne establish an equivalence of categories between local systems and integrable connections.
In particular this point of view explains how the connections, named after Gauss and Manin, are defined by the cohomology of families of algebraic or analytic varieties (precisely by a smooth proper morphism). A background to this result is the classical construction of solutions of differential equations as integrals along cycles of relative differential forms on algebraic families of varieties defined by a smooth proper morphism. DeRham resolutions of local systems are obtained via the associated integrable connections.

Singularities in the fields of algebraic and analytic geometry appear in the study of linear differential equations with meromorphic coefficients on the punctured complex disc. In particular a basic result of Fuchs on equations with *Regular Singularity* is at the origin of the theory and leads to the notion of meromorphic connections with regular singularity.
The work of P. Deligne in 1970 [4] pointed out to the developments of this theory to higher dimensional varieties in algebraic and analytic geometry.

Constructible sheaves. Singularity theory in mathematics which arise for example with the vanishing of the differential of a morphism, has had important developments in algebraic geometry; in particular Whitney's and Thom's stratification theory [10] contributed to a further generalization of local systems, namely the concept of constructible sheaves which appear in the study of cohomology theory of the fibers of any algebraic morphism. This concept is used in séminaire de géométrie algébrique [6] by Grothendieck's school and in an important article [11] on Chern classes for singular algebraic varieties by MacPherson.
Among the complexes of sheaves with constructible cohomology, the perverse sheaves have important special properties since they are related to the theory of differential modules in the sense that the DeRham complex defined by an holonomic differential module is a complex with constructible cohomology sheaves which is in fact a perverse sheaf.
Complexes with constructible cohomology sheaves are preserved by derived direct image by a proper algebraic morphism (and in general by the six classical operations). The concept and the proofs are based on Thom-whitney stratification of varieties and morphisms and a result proved by Mather known as Thom - Mather isotopy lemma describing local topological triviality along strata.

Decomposition theorem. This theorem is stated here to illustrate how it is possible to develop a basic classical result such as Lefschetz theorem via the above tools. The proof is beyond the scope of this exposition. The reader don't see here the use of regularity necessary in the proof, neither we can present Hodge theory which is hidden in the hypothesis of geometric local system. In fact we mention further references where it is possible to find more results on the subject.

Family of Elliptic Curves. The appendix gives explicit computation of the monodromy of the local system and the Gauss-Manin connection defined by the family of elliptic curves, a mathematical subject that should serve as a test example for every mathematician.

Contents.

The reader can expect to learn from this expository paper various points of view of the subject in topology, geometry and analysis in the direction of the decomposition theorem.
To cover recent developments in the theory, the reader dispose of various books listed after the references.
Finally, the theory of Local Systems and Constructible Sheaves play an important role in the theory of Arrangement of Hyperplanes and we refer the reader to expository and research articles by experts on this subject in this summer school.

1. Local Systems

We study here sheaves of groups with topological interest known as local systems or locally constant sheaves. They arise in mathematics as solutions of linear differential equations, as higher direct image of a constant sheaf by a proper differentiable submersive morphism of manifolds and as representations of the fundamental group of a topological space. Local systems can be enriched with structures reflecting geometry like the notion of Hodge structures.

1.1. Background in undergraduate studies

The affine differential equation $zu'(z) = 1$ with complex variable z, well known by students, is singular at the origin, since we can apply Cauchy's theorem on

the exitence of a unique solution with given initial condition only for $z \neq 0$. We put $u'(z) = \frac{1}{z}$, then for any $a \neq 0$ there exists an analytic solution near a for $|z-a| < |a|$, $u(z) = \sum_{n\geq 0} \frac{(-1)^n}{(n+1)a^{n+1}}(z-a)^{n+1}$. In particular, for $a = 1$ we define in this way the function $u(z) = logz$ solution of the equation satisfying the condition $u(1) = 0$. Then we can extend the above local solution into the global function $log\, z = r + i\theta, \theta \in]-\pi, \pi[$ for $z = re^{i\theta}$. The main point of interest in our study, due to the singularity of the equation at zero, comes down in this case to the fact that $logz$ cannot be extended in a continuous function beyond the above domain in the complex plane, since its limit near a negative real number $-r$ along a path in the upper half plane is $log\, r + i\pi$ and differs by $2i\pi$ with its limit $log\, r - i\pi$ along a path in the opposite half plane.
Such function is an inverse to the exponential map e^z, but other inverse maps can be written as $logz + 2ki\pi$ and are always defined on $\mathbb{C}-$ { ray }. They are called various determinations of the logarithm. The exponential map $e^z : \mathbb{C} \to \mathbb{C}^*$ is said to be a covering and a determination of $log\, z$ is a section of such covering.
However our interest is in linear differential equations, for example $zu'(z) - \alpha u(z) = 0$, for $\alpha \in \mathbf{R}$, with solutions $z^\alpha = r^\alpha e^{i\alpha\theta}$. When we cross the negative reals the solution is multiplied by $e^{i2\pi\alpha}$. We express this property by introducing the one dimensional vector space $\mathbb{C}z^\alpha$ of all the solutions defined on a simply connected open subset of $\mathbb{C}^*$ and the linear endomorphism $T : \mathbb{C}z^\alpha \to \mathbb{C}z^\alpha$ called monodromy, acting as $e^{i2\pi\alpha} Id$ on this linear space.
In another point of view, the monodromy extends to a morphism from $\mathbb{Z}$ to the group of linear automorphisms of the one dimensional vector space $\mathbb{C}z^\alpha$ defined by $n \mapsto T^n$. We obtain in this special case the representation of the fundamental group $\pi_1(\mathbb{C}^*)$ identified with $\mathbb{Z}$, defined by the differential equation.

1.2. Definition and properties

To define local systems we use the language of sheaf theory for which basic references are Godement [5] and Warner [14], then we describe here the relation with the topology of the base space, precisely the fundamental group.

The constant sheaf. On a topological space M, an abelian group G defines a constant sheaf denoted by G_M (or also by G), whose sections on a connected open subset is the group itself with the identity as a restriction morphism to smaller connected open subsets.

Definition 1.1. Let A be a ring, a local system $\mathcal{L}$ on a connected topological space M with fiber an $A-$module L, is a sheaf locally isomorphic to the constant sheaf defined by L i.e at each point v in M there exists an open neighbourhood U of v in M and an isomorphism of $A-$modules on the restriction of $\mathcal{L}$ to the constant sheaf L_U on U: $\mathcal{L}_{|U} \simeq L_U$.

There exists a covering U_i of M and isomorphisms of modules $\varphi_{i,j} : L_{U_{i,j}} \to L_{U_{i,j}}$ constant on each connected component of $U_{i,j} = U_i \cap U_j$, called transition transformations, whose restrictions to triple intersections $U_{i,j,k} = U_i \cap U_j \cap U_k$ satisfy $\varphi_{i,j}{}_{|U_{i,j,k}} \circ \varphi_{j,k}{}_{|U_{i,j,k}} = \varphi_{i,k}{}_{|U_{i,j,k}}$.

We will be mainly interested by $\mathbb{Q}-$local systems $\mathcal{L}$ with finite dimensional $\mathbb{Q}-$vector spaces as fiber (said to be of finite rank), then the transition morphisms $\varphi_{i,j}$ are defined by matrices in $GL(n, \mathbb{Q})$ constant on each connected component of $U_{i,j} = U_i \cap U_j$.

We will be concerned with local systems arising in two natural subjects. The first will consist of the higher direct image cohomology sheaves by a proper submersive morphism and the second is defined by the solutions of linear differential systems.

Properties. The inverse image of a local system $\mathcal{L}$ by a continuous map $f : N \to M$ is defined as the locally constant sheaf $f^{-1}(\mathcal{L})$ on N.

Lemma 1.2. *A local system $\mathcal{L}$ on the interval $[0,1]$ is constant.*

Proof. There exists a finite number of intervals $[t_i, t_{i+1}]$ s.t. the restriction of $\mathcal{L}$ is constant on each interval. Each element a_i in the fiber L at a point $t \in [t_i, t_{i+1}]$ defines a unique section on $[t_i, t_{i+1}]$ which extends to sections on $[t_{i-1}, t_i]$ and $[t_{i+1}, t_{i+2}]$ and successively to a section on $[0,1]$. The extension operation has an inverse defined by the restriction of global sections to the point $t \in [0,1]$, hence it is an isomorphism. □

From now on we suppose the topological space M *locally path connected and locally simply connected* (each point has a basis of connected neighbourhoods $(U_i)_{i \in I}$ with trivial fundamental groups i.e $\pi_0(U_i) = e$ and $\pi_1(U_i) = e$).

Remark 1.3. On complex algebraic varieties, we refer to the transcendental topology and not the Zariski topology to define local systems.

1.2.1. Monodromy. Let $\gamma : [0,1] \to M$ be a loop in M with origin a point v and let $\mathcal{L}$ be a $\mathbb{Q}-$local system on M with fiber L at v. The inverse image $\gamma^{-1}(\mathcal{L})$ of the local system is isomorphic to the constant sheaf defined by L on $[0,1]$: $\gamma^{-1}\mathcal{L} \simeq L_{[0,1]}$.

Definition 1.4 (Monodromy). The composition of the linear isomorphisms

$$L \simeq \mathcal{L}_v = \mathcal{L}_{\gamma(0)} \simeq \Gamma([0,1], \mathcal{L}) \simeq \mathcal{L}_{\gamma(1)} = \mathcal{L}_v \simeq L$$

is denoted by T and called the monodromy along γ. It depends only on the homotopy class of γ.

Proof. Given an homotopy H defined on $[0,1]^2$ between two loops γ and γ' we lift $\mathcal{L}$ by H to $[0,1]^2$ where we apply an argument similar to the interval case, by covering the square with products $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ s.t. the restrictions of the inverse image of $\mathcal{L}$ are constant (proofs using homotopy are standard and must be worked once in detail, see for instance the invariance of the primitive of an analytic function constructed along two homotopic loops in Cartan [1] p. 59). □

Proposition 1.5. *Let M be a topological space connected, simply connected, locally path connected and locally simply connected, then a local system $\mathcal{L}$ on M is isomorphic to a constant sheaf L_M.*

Proof. Let a be a fixed point in M and let $L = \mathcal{L}_a$ denotes the stalk of the sheaf $\mathcal{L}$ at a. For any point $x \in M$ two paths γ and γ' from a to x define equal isomorphisms $\gamma_* = \gamma'_* : \mathcal{L}_a \to \mathcal{L}_x$ since $(\gamma'.\gamma^{-1})$ is homotopic to the identity, hence $(\gamma'.\gamma^{-1})_* = Id$. We define an isomorphism of sheaves $\varphi : L_M \to \mathcal{L}$ s.t. for all point $x \in M$, $\varphi_x : (L_M)_x \simeq \mathcal{L}_a \to \mathcal{L}_x$ is equal to γ_*, then φ is well defined since φ_x is independent of the choice of the path. □

1.3. Local systems and Representations of the fundamental group

The notion of local system can be introduced as the theory of representations of the fundamental group of a topological space. This fact will be presented in terms of equivalence of categories that we recall now.

1.3.1. An equivalence of two categories $\mathcal{C}$ and $\mathcal{D}$, consists of a functor $F : \mathcal{C} \to \mathcal{D}$, a functor $G : \mathcal{D} \to \mathcal{C}$, and two natural isomorphisms $a : F \circ G \to Id_{\mathcal{D}}$ and $b : Id_{\mathcal{C}} \to G \circ F$.
An interesting criteria states that a functor $F : \mathcal{C} \to \mathcal{D}$ defines an equivalence of categories if and only if it is:
1) full, i.e. for any two objects A_1 and A_2 of $\mathcal{C}$, the map $Hom_{\mathcal{C}}(A_1, A_2) \to Hom_{\mathcal{D}}(F(A_1), F(A_2))$ induced by F is surjective
2) faithful, i.e. for any two objects A_1 and A_2 of $\mathcal{C}$, the map $Hom_{\mathcal{C}}(A_1, A_2) \to Hom_{\mathcal{D}}(F(A_1), F(A_2))$ induced by F is injective and
3) essentially surjective, i.e. each object B in $\mathcal{D}$ is isomorphic to an object of the form $F(A)$, for A in $\mathcal{C}$.

Definition 1.6. Let L be a $\mathbb{Q}-$vector space. A representation of a group G is a homomorphism of groups

$$G \xrightarrow{\rho} Aut_{\mathbb{Q}}(L)$$

from G to the group of $\mathbb{Q}-$linear automorphisms of L or equivalently a linear action of G on L.

The monodromy of a local system $\mathcal{L}$ defines a representation of the fundamental group $\pi_1(M, v)$ of a topological space M on the stalk at v, $\mathcal{L}_v = L$

$$\pi_1(M, v) \xrightarrow{\rho} Aut_{\mathbb{Q}}(\mathcal{L}_v)$$

which characterizes local systems on connected spaces in the following sense

Proposition 1.7. *Let M be a connected topological space. The above correspondence is an equivalence between the following categories*
i) $\mathbb{Q}-$local systems with fiber a vector space L on M
ii) Representations of the fundamental group $\pi_1(M, v)$ by linear automorphisms of a $\mathbb{Q}-$vector space L.

Proof. The representation associated in (i) to the local system is defined by the monodromy along a path as we have seen above and this correspondence is functorial.
ii) To simplify the proof we use the existence of a universal covering $P : \tilde{M} \to M$ of M. The group $\pi_1(M, v)$ acts on $\tilde{M}$ and can be identified with the group of

covering transformations. Given a representation $\rho : \pi_1(M, v) \to Aut\, L$, we define its associated local system $\mathcal{L}$ via the introduction of the constant sheaf $L_{\tilde{M}}$ on $\tilde{M}$; we put:

$$\Gamma(U, \mathcal{L}) : = \{s \in \Gamma(P^{-1}(U), L_{\tilde{M}}) : \forall u \in P^{-1}(U), \forall \gamma \in \pi_1(M, v), s(\gamma.u) = \rho(\gamma).s(u)\}$$

i.e the sections of $\mathcal{L}$ on U are the equivariant sections of $L_{\tilde{M}}$ on $P^{-1}(U)$ under the action of $\pi_1(M, v)$. Then the functoriality of the construction can be checked. □

Given a linear automorphism $T \in Aut_{\mathbb{Q}} L$, we define an action of T on a representation ρ by conjugation as

$$T.\rho : \pi_1(M, v) \to Aut_{\mathbb{Q}}(\mathcal{L}_v) : \gamma \in \pi_1(M, v) \mapsto T \circ \rho(\gamma) \circ T^{-1}.$$

then we deduce from the proposition a correspondence between:

i) Isomorphisms classes of $\mathbb{Q}-$*local systems on* M *with fiber* L *at a fixed point* v *and*

ii) Classes of representations of the fundamental group $\pi_1(M, v)$ *in finite dimensional* $\mathbb{Q}-$*vector spaces* L *modulo the above action of the group of linear automorphisms* $Aut_{\mathbb{Q}} L$ *on the representations*

Let $L = \mathcal{L}_v$, then for each isomorphism $T : L \to L'$ we deduce a local system $\mathcal{L}'$ isomorphic to $\mathcal{L}$ as shown in the following diagram where $\gamma \in \pi_1(M, v)$

$$\begin{array}{ccc} L & \xrightarrow{\rho(\gamma)} & L \\ T \downarrow & & T \downarrow \\ L' & \xrightarrow{T \circ \rho(\gamma) \circ T^{-1}} & L' \end{array}$$

Reciprocally a linear automorphism $\mathcal{T}$ of $\mathcal{L}$ is defined by the action of $T \in Aut_{\mathbb{Q}} L$ where T is induced by $\mathcal{T}$ at v.

Remark 1.8. In the above equivalence the vector space L is viewed as the fiber of the local system at the reference point v for the fundamental group. In another point of view the vector space L is identified with the space of global sections of the constant inverse image of the local system $\mathcal{L}$ on the universal covering of M, in which case L is called the space of multivalued sections of $\mathcal{L}$ [4].

Corollary 1.9. *The group of global sections of the local system* $\mathcal{L}$ *is isomorphic to the invariant subspace of the fiber* L *at the reference point* v *under the action of the representation* ρ

$$H^0(M, \mathcal{L}) \simeq L^\rho : = \{a \in L \,|\, \rho(\alpha)(a) = a, \forall \alpha \in \pi_1(M, v)\}$$

Proof. The above proposition applied to the constant sheaf $\mathbb{Z}_M$ states that the space of morphisms $\varphi \in Hom(\mathbb{Z}, \mathcal{L})$ is isomorphic to the space of morphisms of the trivial representation $\mathbb{Z}$ into ρ. On one side $Hom(\mathbb{Z}, \mathcal{L})$ is isomorphic to the space of global section $H^0(M, \mathcal{L})$ via $\varphi \mapsto s = \varphi(1)$ and on the other side the morphisms from the trivial representation $\mathbb{Z}$ to $\mathcal{L}$ are defined by elements $a \in L$ satisfying the above formula where $\rho(\alpha) = Id$ since it is the image by φ of the trivial action on $\mathbb{Z}$. □

Example. i) Let D_r be an open complex disc centered at 0 of radius r, then a local system on $D_r^* = D_r - \{0\}$ is defined by a vector space L and a linear automorphism T on L.
ii) More generally for $M = D_r - \{x_1, \dots, x_n\}$ a disc with n points deleted, $\pi_1(M)$ is the free group on n generators corresponding to a loop around each point; hence the representations of $\pi_1(M)$ are defined by the choice of n linear automorphisms T_i on L.

1.3.2. Cohomology. Let $\mathcal{L}$ be a local system on D^*, with fiber L at some point and monodromy $T : L \to L$, we prove that its cohomology is as follows
$H^0(D^*, \mathcal{L}) \simeq ker\,(T - Id)$, $H^1(D^*, \mathcal{L}) \simeq coker\,(T - Id)$
and $H^i(D^*, \mathcal{L}) \simeq 0$ for $i > 1$,
hence it is defined as the cohomology of the complex

$$L \xrightarrow{\partial} L : \partial(b) = (T - Id)(b)$$

The cohomology is computed via Cĕch definition; we consider the covering of D^* by the two open sets, north $U_n = \{D - D \cap i\mathbf{R}^-\}$(complement of negative imaginary numbers) and south $U_s = \{D - D \cap i\mathbf{R}^+\}$ (complement of positive imaginary numbers) and their intersections $U_n \cap U_s = U^+ \cup U^-$ where $U^+ = \{a + ib \in D : a > 0\}, U^- = \{a + ib \in D : a < 0\}$. The associated Cĕch complex C_1 is defined as (C_1)

$$H^0(U_n, \mathcal{L}) \oplus H^0(U_s, \mathcal{L}) \xrightarrow{\partial_1} H^0(U^+, \mathcal{L}) \oplus H^0(U^-, \mathcal{L}) : \partial_1(a_1 + a_{-1}) = (a_{-1} - a_1)_{|U_n \cap U_s}$$

The fundamental group of D^* is generated by the loop γ defined by $e^{2i\pi t}$ for $t \in [0, 1]$, hence according to the definition as a representation, the local system is determined by its stalk $L = \mathcal{L}_1$ at 1 and the monodromy T image of γ. Since all the open subsets are simply connected we have isomorphisms: $\varphi_1 : H^0(U_n, \mathcal{L}) \simeq \mathcal{L}_1 = L$, $\psi_1 : H^0(U^+, \mathcal{L}) \simeq \mathcal{L}_1 = L$, $\varphi_{-1} : H^0(U_s, \mathcal{L}) \simeq \mathcal{L}_{-1}$, $\psi_{-1} : H^0(U^{-1}, \mathcal{L}) \simeq \mathcal{L}_{-1}$ with the fibers L at $1 \in U_n$ and $\mathcal{L}_{-1}$ at $-1 \in U_s$. Moreover we have isomorphisms $\alpha : L = \mathcal{L}_1 \simeq \mathcal{L}_{-1}$ defined by a path from $\{1\}$ to $\{-1\}$ and $\delta : \mathcal{L}_{-1} \simeq \mathcal{L}_1 = L$ defined by a path from $\{-1\}$ to $\{1\}$ s.t. $\delta \circ \alpha = T$. We introduce the complex C_2

$$(C_2) \qquad L \oplus \mathcal{L}_{-1} \xrightarrow{\partial_2} \mathcal{L}_{-1} \oplus L : \partial_2(b_1, b_{-1}) = (b_{-1} - \alpha(b_1), \delta(b_{-1}) - b_1)$$

The morphisms $\varphi_1, \psi_1, \psi_{-1}$ and φ_{-1} above can be combined to define a quasi-isomorphism of complexes $\varphi : C_1 \to C_2$. Then we introduce the complex C_3

$$(C_3) \qquad L \xrightarrow{\partial_3} L : \partial_3(b) = (T - Id)(b)$$

and the morphism $D : C_3 \to C_2$ defined by $D^0(b) = (b, \alpha(b))$ in degree 0 and $D^1(a) = (0, a)$ in degree 1. Finally we can check that D is a quasi-isomorphism, since for example in C_2, ker $\partial_2 \simeq \{(b_1, b_{-1}) | \alpha(b_1) = b_{-1}$ and $\delta(b_{-1}) = b_1\}$, hence ker $\partial_2 \simeq \{b \in L | T(b) = b\}$.

1.4. System of $n-$linear first order differential equations and local systems

We consider here holomorphic equations, however the theory can be developed for differentiable equations.

Definition 1.10. A first order holomorphic system of $n-$linear differential equations on $\mathbb{C}^n$ is written as

$$\frac{du}{dz} = A(z)u$$

where $u \in \mathbb{C}^n$, z is a coordinate on an open subset U of $\mathbb{C}$ and $A : U \to End(\mathbb{C}^n)$ is an holomorphic map in the vector space of endomorphisms of $\mathbb{C}^n$.
When A is independent of z, the system is said to have constant coefficients.

Classically the system is referred to as homogeneous in $n-$unknowns $u_i(z) \in \mathbb{C}, i \in [1, n]$, with holomorphic coefficients as entries of A, $a_{ij} : U \to \mathbb{C}, i, j \in [1, n]$, and it is written in the following form:

$$\frac{du_i}{dz} = \sum_{j=1}^{n} a_{ij}(z)u_j(z), \; i = 1, 2, \dots, n$$

this is one equation and it is not true that we have n distinct equations with independent variables.
If we consider a differentiable map $A : I \to End(\mathbf{R}^n)$ on an interval I of $\mathbf{R}$, then Cauchy's theorem confirms the existence of global solutions, defined on the whole interval, which form a real vector space of dimension n, the isomorphism with $\mathbf{R}^n$ being determined by the initial condition given at a fixed time $t \in I$ with varying position $u \in \mathbf{R}^n$.
The extension of Cauchy's theorem to the holomorphic case can be found in the book of Cartan [1]. This proves the existence of unique local solutions with fixed initial conditions. A subtle point to study is the existence and behavior of a global solution on $U \subset \mathbb{C}$, namely to decide whether a local solution can be extended to all U. This behavior is a central point in our subject here.
Let $\gamma : I \to U$ denotes a path in U defined on a real interval. Identifying $\mathbb{C}^n$ with $\mathbf{R}^{2n}$ we deduce by composition a map $A \circ \gamma : I \to \mathbf{R}^{2n}$ defining a real differential system. Applying the result on the existence of global solutions on I, it is not difficult to check that the local solutions can be extended along each path.
The problem arise when we extend a solution along a non trivial loop. Since a solution defined near the origin and extended along a path does not necessarily coincide, upon first return to the origin, with itself, that is we don't obtain necessarily the original solution. In conclusion holomorphic solutions cannot be extended necessarily to the whole open set. However the extensions along two paths with the same origin and the same end point, coincide at the same end point if the two paths are homotopic.

Corollary 1.11. *Global solutions are defined on a simply connected open subset V in U and form a complex vector space of dimension n.*

The notion of local system is the abstract concept which takes care of this behavior and of the basic properties of the space of solutions.

Proposition 1.12. *Given an homogeneous system of $n-$linear first order differential equations with holomorphic coefficients on an open subset U of $\mathbb{C}$, the sheaf defined* *by holomorphic global solutions on each open subset $V \subset U$ form a local system $\mathcal{L}$ on U.*

Proof. The restriction of $\mathcal{L}$ to a simply connected open subset V is isomorphic to the constant sheaf $\mathbb{C}^n_V$. □

1.5. Connections and Local Systems

We introduce the concept of connections directly on analytic manifolds. The generalization of the concept of system of $n-$linear first order differential equations is in two directions. First, since the coordinate space may be of dimension higher than 1, we are concerned with partial differential equations in many variables and second the definition is compatible with transition transformations on the manifold.

Definition 1.13. Let F be a locally free holomorphic $\mathcal{O}_X-$module on a complex analytic manifold X. A connection on F is a $\mathbb{C}_X-$linear map

$$\nabla : F \to \Omega^1_X \otimes_{\mathcal{O}_X} F$$

satisfying the following condition for all sections f of F and φ of $\mathcal{O}_X$:

$$\nabla(\varphi f) = d\varphi \otimes f + \varphi \nabla f$$

known as Leibnitz condition.

1.5.1. Properties. We define a morphism of connections as a morphism of $\mathcal{O}_X-$modules which commutes with ∇.

The definition of ∇ extends to differential forms in degree p as a $\mathbb{C}-$linear map

$$\nabla^p : \Omega^p_X \otimes_{\mathcal{O}_X} F \to \Omega^{p+1}_X \otimes_{\mathcal{O}_X} F \text{ s.t. } \nabla^p(\omega \otimes f) = d\omega \otimes f + (-1)^p \omega \wedge \nabla f$$

The connection is said to be integrable if its curvature $\nabla^1 \circ \nabla : F \to \Omega^2_X \otimes_{\mathcal{O}_X} F$ *vanishes* (the curvature is a linear morphism).

Then it follows that the composition of maps $\nabla^{i+1} \circ \nabla^i = 0$ vanishes for all $i \in \mathbb{N}$ for an integrable connection.

In this case a DeRham complex is associated to ∇

$$(\Omega^*_X \otimes_{\mathcal{O}_X} F, \nabla) \colon = F \to \Omega^1_X \otimes_{\mathcal{O}_X} F \cdots \Omega^p_X \otimes_{\mathcal{O}_X} F \xrightarrow{\nabla^p} \cdots \Omega^n_X \otimes_{\mathcal{O}_X} F$$

The contraction of ∇ with a vector field X is denoted by ∇_X. For two vector fields X, Y, let $[X, Y]$ denotes the vector field defined as the bracket of X and Y, then the connection is integrable if and only if $\nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X$ for all X, Y.

Proposition 1.14. *The horizontal sections F^∇ of a connection ∇ on a module F on an analytic smooth variety X, are defined as the solutions of the differential equation*

$$F^\nabla = \{f : \nabla(f) = 0\}.$$

When the connection is integrable, F^∇ is a local system of dimension equals to $\dim F$.

Proof. Based on the relation between differential equations and connections, locally we can find a small open subset $U \subset X$ isomorphic to an open set of $\mathbb{C}^n$ s.t. $F_{|U}$ is isomorphic to $\mathcal{O}^m_U$. This isomorphism corresponds to the choice of a frame $\{e_i\}_{i\in[1,m]}$ of F on U and extends to the tensor product of F with the module of differential forms: $\Omega^1_U \otimes F \simeq (\Omega^1_U)^m$. The connection matrix Ω_U is a matrix of differential forms $\{\omega_{ij}\}_{i,j\in[1,m]}$, sections of Ω^1_U, defined as follows: its $i-$th column is the transpose of the line image of $\nabla(e_i)$ in $(\Omega^1_U)^m$. Then the restriction of ∇ to U corresponds to a connection on $\mathcal{O}^m_U$ denoted ∇_U and defined on sections $y = (y_1, \cdots, y_m)$ of $\mathcal{O}^m_U$ on U, written in column $\nabla_U {}^t y = d({}^t y) + \Omega_U \, {}^t y$ or

$$\nabla_U \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} dy_1 \\ \vdots \\ dy_m \end{pmatrix} + \Omega_U \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

the equation is in $End(T,F)_{|U} \simeq (\Omega^1 \otimes F)_{|U}$ where T is the tangent bundle to X. Let $(x_1, \cdots, x_n)$ denotes the coordinates of $\mathbb{C}^n$, then ω_{ij} decompose as

$$\omega_{ij} = \textstyle\sum_{k\in[1,n]} \Gamma^k_{ij}(x) dx_k$$

so that the equation of the coordinates of horizontal sections is given by linear partial differential equations for $i \in [1,m]$ and $k \in [1,n]$

$$\frac{\partial y_i}{\partial x_k} + \sum_{j\in[1,m]} \Gamma^k_{ij}(x) y_j = 0$$

The solutions form a local system of dimension m since the Frobenius condition is satisfied by the integrability hypothesis on ∇. □

Remark 1.15. In terms of the basis $e = (e_1, \cdots, e_m)$ of $F_{|U}$, a section s is written as

$s = \sum_{i\in[1,m]} y_i e_i$ and $\nabla s = \sum_{i\in[1,m]} dy_i \otimes e_i + \sum_{i\in[1,m]} y_i \nabla e_i$ where
$\nabla e_i = \sum_{j\in[1,m]} \omega_{ij} \otimes e_j$.

The connection appears as a global version of linear differential equations, independent of the choice of local coordinates on X.

Remark 1.16. The natural morphism $\mathcal{L} \to (\Omega^*_X \otimes_{\mathbb{C}} \mathcal{L}, \nabla)$ defines a resolution of $\mathcal{L}$ by coherent modules, hence induces isomorphisms on cohomology

$$H^i(X, \mathcal{L}) \simeq H^i(R\Gamma(X, (\Omega^*_X \otimes_{\mathbb{C}} \mathcal{L}, \nabla)))$$

where we take hypercohomology on the right. On a smooth differentiable manifold X, the natural morphism $\mathcal{L} \to (\Omega^*_X \otimes_{\mathbb{C}} \mathcal{L}, \nabla)$ defines a soft resolution of $\mathcal{L}$ and induces isomorphisms on cohomology

$$H^i(X, \mathcal{L}) \simeq H^i(\Gamma(X, (\Omega^*_X \otimes_{\mathbb{C}} \mathcal{L}, \nabla))$$

1.5.2. Connections defined by local systems. We associate to a local system $\mathcal{L}$ on X, a vector bundle $\mathcal{L}_X := \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{L}$ on X.

The transition transformations are deduced from the corresponding transformations of $\mathcal{L}$. Then a connection is defined on $\mathcal{L}_X$ as follows

$$\forall g \in \Gamma(U, \mathcal{O}), \forall s \in \Gamma(U, \mathcal{L}), \quad \nabla(g \otimes s) = dg \otimes s$$

The connection is well defined since the transition transformations are defined by matrices with locally constant coefficients.

Example. i) Let D_r be an open complex disc centered at 0 of radius r, then the connection on the trivial line bundle $\mathcal{O}_{D_r^*}$ defined as $\nabla u = du - \frac{\alpha}{z} udz$ admits for flat sections the vector space of solutions of the equation $du = \frac{\alpha}{z} udz$ generated by all determinations z^α on open subsets of D_r^*. Since the extension of z^α along a loop around the origin produces the new determination $e^{\alpha(log z + 2i\pi)} = e^{2i\pi\alpha} z^\alpha$, the flat sections define a local system on D_r^* with fiber $\mathbb{C}$ and monodromy $T = e^{2i\pi\alpha} Id$ on $\mathbb{C}$.
ii) More generally for $M = D_r - \{b_1, \ldots, b_n\}$ a disc with n points deleted, the connection on the trivial line bundle $\mathcal{O}_M$ defined as $\nabla u = du - (\Sigma_{i\in[1,n]} \frac{\alpha_i}{z-b_i}) udz$ admits for flat sections the vector space of solutions of the equation
$du = (\Sigma_{i\in[1,n]} \frac{\alpha_i}{z-b_i}) udz$ generated by all determinations $\Pi_{i\in[1,n]}(z-b_i)^{\alpha_i}$ on open subsets of M. They define a local system with fiber $\mathbb{C}$ and monodromy $T_k = e^{2i\pi\alpha_k} Id$ around each point b_k. The bundle associated to the local system is isomorphic to $\mathcal{O}_M$, hence trivial.

Theorem 1.17 (Deligne). *The functor $(F, \nabla) \mapsto F^\nabla$ is an equivalence between the category of integrable connections on X and the category of complex local systems on X with quasi-inverse defined by $\mathcal{L} \mapsto \mathcal{L}_X$.*

Proof. i) The correspondence giving horizontal sections is functorial. The canonical morphism, compatible with the connections, $\mathcal{O}_X \otimes_{\mathbb{C}} F^\nabla \to F : g \otimes s \mapsto gs$ is an isomorphism. In fact, since the connection is integrable, there exists locally a basis consisting of horizontal sections, hence locally every section s of (F, ∇) is written as a sum $s = \sum_{i\in[1,n]} g_i h_i$ where h_i is horizontal and g_i is an analytic function, then $\nabla(s) = \sum_{i\in[1,n]} g_i h_i$.
ii) Let $\mathcal{L}$ denotes a local system on X, then the canonical morphism $\mathcal{L} \to \mathcal{L}_X$: $s \mapsto 1 \otimes s$ is an isomorphism onto $\mathcal{L}_X^\nabla$. □

1.6. Fibrations and local systems (Gauss-Manin connection)

We prove that the $i-$th group of rational cohomology of the fibers of a proper submersion of manifolds $f : M \to N$ form a local system $R^i f_* \mathbb{Q}$ on N for all integers i. First we recall notions on the higher direct image of a sheaf.

1.6.1. Cohomology via sheaf theory techniques. In this section we recall notions on cohomology constructed via sheaf theory. Basic references are Godement [5] and Warner [14]. Cohomology attach to a topological space, a group or a vector space in degree i known as $i-$th cohomology group and to a continuous map a linear function on the groups (for this reason the cohomology is an invariant said to be linear depending on topology only in contrast to other structure, differentiable for example. The first technical constructions were simplicial, based on a triangulation of the space, but later cohomology with various coefficients constructed via sheaf theory proved to be of more flexible use in various domains of mathematics.
- *Real coefficients.* The real field $\mathbf{R}$ defines on a topological space M, a constant sheaf denoted $\mathbf{R}_M$ or simply $\mathbf{R}$.
On a differentiable manifold M, the DeRham complex of differential forms $\mathcal{E}_M^*$ is

a resolution of the constant sheaf $\mathbf{R}_M$ (Poincaré's lemma).
DeRham theorem asserts that the $i-th$ *cohomology of the complex of global sections* $\Gamma(M, \mathcal{E}^*_M)$ *is isomorphic to the* $i-th$ *cohomology vector space:*

$$H^i(M, \mathbf{R}) \simeq H^i(\Gamma(M, \mathcal{E}^*_M))$$

This construction of cohomology is fit to analysis. It is so important, that it can be considered as a definition, for a first approach to cohomology.

- *The pushforward functor* f_* *and its derived functors.* We may think of a map of topological spaces $f : M \to N$ as a family of spaces consisting of the fibers $M_v = f^{-1}(v)$ at various points v of N, whose cohomologies can be viewed as a family of groups. However the fact that the union M of these fibers has itself the structure of a topological space, as well the presence of a topology on the parameter space N and the continuity of the morphism f can be used to obtain a more rich structure on the family of cohomologies $\{H^i(M_v, \mathbb{Z})\}_v, v \in N$, namely the structure of a sheaf $R^i f_* \mathbb{Z}$ defined by the "presheaf " associating to an open subset U in N the group $H^i(f^{-1}(U), \mathbb{Z})$.

Definition 1.18. Let $f : M \to N$ be a continuous map and $\mathcal{F}$ be a sheaf of abelian groups on M.
i) The direct image sheaf $f_*\mathcal{F}$ is associated to the presheaf on N defined by the global sections on inverse of open sets: $U \to \Gamma(f^{-1}(U, \mathcal{F})$.
ii) For any f_*-acyclic resolution $\mathcal{K}^*$ of $\mathcal{F}$ on M, the complex of sheaves $Rf_*\mathcal{F}$ on N is defined as $Rf_*\mathcal{F}\colon = f_*\mathcal{K}^*$ on N and called the higher direct image of the sheaf $\mathcal{F}$. Its $i-th$ cohomology sheaves is defined as $R^if_*\colon = \mathcal{H}^i(f_*\mathcal{K}^*)$ and called the $i-th$ derivative of the direct image functor (flabby or fine resolutions are examples of acyclic resolutions).

If we view f as giving rise to the family of fibers $f^{-1}(v)$ and if f is proper, the sheaf $R^i f_* \mathbb{Z}_M$ gives rise to the family of cohomology of the fibers $H^i(f^{-1}(v), \mathbb{Z}) \simeq (R^i f_* \mathbb{Z}_M)_v$. The sheaf structure contains more information than merely the family of cohomology of the fibers (for example the monodromy invariant recalled below). Even if f is not proper the direct image is still interesting, for example in the case of the embedding j of the punctured disk in $\mathbb{C}$ the fiber at 0 of $R^1 j_* \mathbb{Z}$ is isomorphic to $\mathbb{Z}$ generated by the Poincaré dual of the homology class of a loop around 0.

Example. Let X be an analytic variety, Y a normal crossing divisor (NCD) in X and $j : X - Y \to X$ the open embedding. The local information on the topology near Y as a product of discs punctured or not is reflected in the higher direct image $Rj_*\mathbb{C}$ of the constant sheaf $\mathbb{C}$ on $X - Y$. Let $y \in Y$ and U_y a neighbourhood of y isomorphic to a product of complex discs D^n s.t. $D^n - D^n \cap Y \simeq D^{*p} \times D^q$, then $(R^i j_* \mathbb{C})_y \simeq H^i(D^{*p}, \mathbb{C}) \simeq \wedge^i(\mathbb{C}^p)$.
The result follows from a general relation known as Kunneth formula for a product of spaces and a general statement:
$(R^i j_* \mathbb{C})_y$ *is isomorphic to the inductive limit of* $H^i(U_y, \mathbb{C})$ *for small open sets* U_y.

We see on this example that in practice we don't need to go back to the definition and construct a flabby resolution of $\mathbb{C}$ to compute the cohomology groups.

However, it happens that one need to work directly on the complex level, like in Hodge theory where Deligne needed filtered complexes to define the weight and the Hodge filtrations on the cohomology of algebraic varieties. In this case, although the analytic DeRham complex Ω^*_{X-Y} is not fine, the direct image complex $j_*\Omega^*_{X-Y}$ is quasi-isomorphic to $Rj_*\mathbb{C}$ since we can find enough small Stein open sets U_y with acyclic cohomology for coherent analytic coefficients sheaves. Moreover it was necessary to introduce a subcomplex of forms having logarithmic singularities along Y in order to define the correct Hodge filtration (see(2.2.1.)).

Theorem 1.19 (differentiable fibrations). *Let $f : M \to N$ be a proper differentiable submersive morphism of manifolds. For each point $v \in N$ there exists an open neighbourhood U_v of v such that the differentiable structure of the inverse image $M_{U_v} = f^{-1}(U_v)$ decomposes as a product of a fibre at v with U_v:*

$$f^{-1}(U_v) \xrightarrow[\simeq]{\varphi} U_v \times M_v \quad s.t. \quad pr_1 \circ \varphi = f_{|U_v}$$

The proof follows from the existence of a tubular neighbourhood of the submanifold M_v. Let $V \subset M$ containing M_v be isomorphic to an open neighbourhood of the zero section in the normal bundle $N_{M_v/M}$ and endowed with a retraction $P : V \to M_v$. Since the differential of the map $P \times f_{|V} : V \to M_v \times N$ is invertible on the compact manifold M_v and the restriction of $P \times f_{|V}$ is injective on M_v, there exists an open neighbourhood $V' \subset V$ such that the restriction of $P \times f_{|V}$ to V' is an open embedding. Since f is proper we find U_v such that $f^{-1}(U_v)$ is included in V' and satisfy the statement of the theorem.

- We will retain that if M, N are smooth complex algebraic varieties and f is a smooth algebraic proper morphism, the theorem will apply only to the underlying differentiable structure. The algebraic structure or the underlying analytic structure do not decompose into a product, since two smooth nearby fibers are not necessarily isomorphic as analytic or algebraic varieties but only as differentiable varieties.

Remark. i) The morphism obtained by composition $P = pr_2 \circ \varphi : M_{U_v} \to M_v$ induces for each point $w \in U_v$ a diffeomorphism of the fibers $M_w \simeq M_v$, equal to the identity on M_v for $w = v$. It defines a retraction by deformation from M_{U_v} onto M_v.

ii) Let $f : (M, \partial M) \to N$ be a differentiable morphism of manifolds with boundary. Suppose f proper and submersive, as well as its restriction to the boundary ∂M of M. Then f is locally differentially trivial on N, that is at each point v in N there exists a commutative diagram

$$\begin{array}{ccc} U \times (f^{-1}(v),\ \partial f^{-1}(v)) & \simeq & (f^{-1}(U),\ \partial f^{-1}(U)) \\ pr_U \searrow & & \swarrow f_{|U} \\ & U & \end{array}$$

where U is an open neighbourhood of v in N and the isomorphism is a differentiable morphism of manifolds with boundary.

1.6.2. Locally constant cohomology. The differentiable result above for a proper submersive morphism $f : M \to N$ has a linear version on the cohomology of the fibers

Proposition 1.20 (locally constant cohomology). *In each degree i, the cohomology sheaf of the fibers $R^i f_* \mathbb{Z}$ is constant on a small neighbourhood U_v of any point v of fiber $H^i(M_v, \mathbb{Z})$ i.e there exists an isomorphism between the restriction $(R^i f_* \mathbb{Z})_{|U_v}$ with the constant sheaf $H^i_{U_v}$ defined on U_v by the vector space $H^i = H^i(M_v, \mathbb{Z})$.*

Proof. Let U_v be isomorphic to a ball in $\mathbf{R}^n$ over which f is trivial, then for any small ball B_ρ included in U_v, the restriction $H^i(M_{U_v}, \mathbb{Z}) \to H^i(M_{B_\rho}, \mathbb{Z})$ is an isomorphism since M_{B_ρ} is a deformation retract of M_{U_v}. □

1.6.3. Complex algebraic case. Let $f : X \to V$ be an algebraic, proper and smooth morphism of complex algebraic varieties (analytically f^a is a submersion), then f defines a differentiable locally trivial fiber bundle on V (that is the trivialisations are differentiable but not necessarily analytic). The problem of discovering properties to distinguish such class of local systems (called geometric) is a fundamental problem in geometry.
We still denote by f the differentiable morphism $X^{dif} \to V^{dif}$ associated to f, then the complex of real differential forms $\mathcal{E}^*_X$ is a fine resolution of the constant sheaf $\mathbf{R}$ and $R^i f_* \mathbf{R} \simeq \mathcal{H}^i(f_* \mathcal{E}^*_X)$.

Example. Let $f : X \to S^1$ be a locally trivial fibration with typical fiber F at some point t. The direct image sheaves $R^i f_* \mathbb{Q}$ are local systems on S^1. In this case the fibration is defined by a monodromy homeomorphism $T : F \to F$ which induces on cohomology isomorphisms $T_i : H^i(F, \mathbb{Q}) \to H^i(F, \mathbb{Q})$, the monodromy of $R^i f_* \mathbb{Q}$ on S^1 where the fiber $(R^i f_* \mathbb{Q})_t$ at t is identified with $H^i(F, \mathbb{Q})$. It follows that: $H^1(S^1, R^i f_* \mathbb{Q}) \simeq Coker(T_i - Id)$ and $H^0(S^1, R^i f_* \mathbb{Q}) \simeq Ker(T_i - Id)$.

Example (Geometric local system). Let $f \colon X \to V$ be a smooth and proper morphism of smooth analytic varieties. It follows that f is a differentiable bundle on V (since f is a submersion) i.e every point y in V has a neighbourhood U_y such that $f^{-1}(U_y)$ is diffeomorphic to a product $U_y \times X_y$ of U_y with the fiber of X at y. Namely let $\gamma \colon [0, 1] \to V$ be a differentiable path in V between two points y_0 and y_1, then it defines a diffeomorphism $\gamma_* \colon X_{y_0} \to X_{y_1}$ inducing an isomorphism γ^* on cohomology.
This isomorphism on cohomology depends on the path up to homotopy and hence defines a representation of the fundamental group $\pi_1(V, y_0)$ on the cohomology $H^i(X_{y_0}, \mathbb{Z})$ or equivalently, the family $H^i(X_y, \mathbb{Z})$ forms a local system on V. In this example the structure of the sheaf on the higher direct cohomology is defined by the cohomology of the fibers and the monodromy.
The monodromy in this case is induced by the diffeomorphism defined on the fiber $f^{-1}(v)$ by a trivialization of $f_{|\gamma}$, in particular it is compatible with the cup-product on cohomology.
Suppose now that V is a punctured disc D^*, then $\pi_1(D^*, t)$ is isomorphic to $\mathbb{Z}$. The

action of a generator (a circle through t) is the monodromy operator on $H^i(X_t, \mathbb{Z})$ and denoted by T.
Suppose again V is a disc but f smooth only over D^*. Then the monodromy is related to the singularities of the fiber X_0 at the origin of D. For example, it is a necessary condition that the monodromy defined by f over D^* is trivial, for f to be smooth over D . A morphism f over D^* giving rise to a non trivial monodromy cannot be extended to a smooth morphism over D. The local system is called geometric to recall that it is constructed as direct image of cohomology of algebraic varieties. Such local systems reflect special topological properties of algebraic varieties. The concept of Variation of Hodge structures is introduced in order to take care of such additional properties. At the expense of hard technical constructions, such structure leads to results subsequent to the geometry.

1.6.4. Relative DeRham complex. Let $f : X \to V$ be a smooth morphism of analytic manifolds, the bundle of relative differential forms is defined as $\Omega^1_{X/V} := \Omega^1_X / f^*(\Omega^1_V)$ and $\Omega^p_{X/V} := \wedge^p \Omega^1_{X/V}$ so that the differential d on Ω^p_X induces a differential on the relative forms and a relative DeRham complex $(\Omega^*_{X/V}, d)$ is defined and can be extended for any local system $\mathcal{L}$ to a complex $\Omega^*_{X/V}(\mathcal{L}) : = (\Omega^*_{X/V} \otimes_{\mathbb{C}} \mathcal{L}, \nabla)$; at a point $v \in V$, there exists an isomorphism $\Omega^*_{X(v)} \simeq \Omega^*_{X/V} \otimes \mathbb{C}(v)$ where $\mathbb{C}(v) = \mathcal{O}_{V,v}/\mathcal{M}_{V,v} \simeq \mathbb{C}$. The complex of holomorphic forms is not fine to compute cohomology but Grothendieck showed the interest in the notion of Hypercohomology (see later 3.6) which is used in the next result, the best that we can hope for and which is indeed proved in [4]

Theorem 1.21 (Deligne). *There exists natural isomorphisms of holomorphic bundles on V*

$$R^p f_*(\mathcal{L}) \otimes \mathcal{O}_V \simeq R^p f_* \Omega^*_{X/V}(\mathcal{L})$$

This result generalizes the classical DeRham theorem in the case where V is reduced to a point but with coefficient in a local system in remark 1.16. It leads to a description of the Gauss-Manin connection on $R^p f_* \Omega^*_{X/V}(\mathcal{L}$ with $R^p f_*(\mathcal{L})$ isomorphic to the horizontal sections.

Definition 1.22. Suppose f a locally trivial topological fibration, then the connection defined by the local system $R^p f_* \mathcal{L}$ on the bundle $R^p f_* \Omega^*_{X/V}(\mathcal{L})$ is the Gauss-Manin connection.

2. Singularities of Local Systems and Systems of differential equations with meromorphic coefficients: Regularity

In the previous paragraph we obtained a general result on the equivalence of the two categories defined by Local Systems on one side and flat Connections on a manifold on the other side; moreover Gauss-Manin connections are associated to the cohomology of the fibers of a smooth morphism of smooth varieties.
A morphism, in general acquire singular fibers at some critical values in the space

of parameters, then the Gauss-Manin connection is defined on the complement of the set of critical values (known also as the discriminant of the morphism). Hironaka's result on desingularisation, suggest that the case of normal crossing divisor as discriminant is the most important to study.
Historically, connections which are meromorphic on analytic varieties and holomorphic on the complement of a divisor were studied first. As usual it is natural to start the study on a small disc, where an asymptotic property near a singularity known as regular singularity has been described first.
The corresponding singularities of local systems appears in the DeRham complex defined by the connection. The discovery of perverse sheaves later, will be presented in the third section. The references to this section are [3] and [4].

2.1. System with meromorphic coefficients on the complex disc

Let $K = \mathcal{M}_0$ denotes the field of germs of meromorphic functions at 0. We consider the above homogeneous system of $m-$linear first order differential equations with coefficients $a_{ij}(z)$ meromorphic at 0

$$\frac{du}{dz} = A(z)u, \quad A(z) \text{ matrix with entries } a_{ij}(z) \text{ meromorphic at } 0$$

Taking the coefficients of the system in K is a convenient way to make sure that 0 is the only singular point of the coefficients, equivalently the system is meromorphic on a disc D_r where we shrink the radius enough to have a unique singular point at the origin (we may suppose $r = 1$ for convenience). The solutions are vectors $u(z) = (u_1(z), \ldots, u_m(z))$ of holomorphic functions on any sector in D^* (simply connected region of the disc defined by the rays of angle θ satisfying:$\theta_1 < \theta < \theta_2$); there exist always solutions since the restriction of the system to such sector is holomorphic. The main point of study here is that the solutions cannot be extended in general to univalent solutions on the whole disc.

Definition 2.1. A fundamental matrix of solutions consists of a basis of solutions of m vectors where each vector is written as a column of m holomorphic functions defined on a sector.

In fact it is convenient to write the equation in matrix form as follows

$$\frac{dU}{dz}(z) = A(z)U(z) \tag{2.1}$$

where $A(z)$ is a matrix with entries $a_{ij}(z) \in K$ and $U(z)$ represents a matrix of m independent solutions. This equation can be treated as a first order equation.

Example. Let Γ be a constant $(m, m)-$ complex matrix. It is easy to check that a solution of the matrix equation: $z\frac{d}{dz}U(z) - \Gamma\, U(z) = 0$ on the complex disc D^* is given on any sector by $U(z) = exp((log\, z)\, \Gamma)$ where $log\, z$ is a determination of the logarithm. In particular the equation defined by the matrix

$$A(z) = \frac{1}{z}\Gamma \tag{2.2}$$

where Γ is a Jordan matrix with eigenvalue α admits the solutions $u_i(z) = (z^\alpha(logz)^i, z^\alpha(logz)^{i-1}, \cdots, z^\alpha, 0, \cdots)$ for $0 \leq i < m$ forming the columns of the exponential matrix $exp(logz\,\Gamma)$.

2.1.1. Multivalued solutions. For certain canonical constructions it is convenient to avoid the choice of a sector, then there is an advantage to introduce the space of multivalued solutions [4].
By definition, *multivalued functions* on D_r^* are holomorphic functions on the universal covering $\widetilde{D}_r^*$ of D_r^* (called Poincaré half plane H for $r = 1$).

$$\widetilde{D}_r = \{t \in \mathbb{C}, |z = e^{2i\pi t}| < r\},\ \pi : \widetilde{D}_r \to D_r^* : t \mapsto z = e^{2i\pi t} \tag{2.3}$$

Definition 2.2. Let $\mathcal{F}$ be a sheaf of complex vector spaces on D_r^*. A multivalued section of $\mathcal{F}$ is a section of $\pi^{-1}\mathcal{F}$.

The covering π admits sections s on sectors of D_r^*. Given a multivalued section $\tilde{f}$ of $\mathcal{F}$, a section s of π defines a section $\tilde{f} \circ s$ of $\mathcal{F}$ on a sector of D_r^*.

Definition 2.3. The multivalued solutions of an homogeneous system of $m-$linear first order differential equations on D_r^* form a vector space of finite dimension, solutions of the differential operator obtained on $\widetilde{D}_r$ by the change of variable $z = e^{2i\pi t}$.

Example. The general theory below is similar to the above example (2.2) that we discuss again.
i) The matrix $\tilde{U}(t) = exp((2i\pi t)\Gamma)$ consists of multivalued solutions on the Poincaré half plane H of the equation: $\frac{d}{dt}\tilde{U}(t) = 2i\pi\Gamma\,\tilde{U}(t)$.
ii) *Monodromy.* If we view the solutions in z near a point $z_0 \neq 0$ as sections of a *local system* $\mathcal{L}$ and we follow a solution $U(z)$ along a circle, we obtain upon the first return to z_0 a new basis of sections of $\mathcal{L}$: $exp(2i\pi\Gamma)U(z) = TU(z)$, where T is the monodromy and $logT = 2i\pi\Gamma$.
On H, $\tilde{U}(t)$ consists of sections of $\pi^{-1}\mathcal{L}$ and satisfy:
$\tilde{U}(t+1) = exp(2i\pi\Gamma)\tilde{U}(t)$ that is $exp(2i\pi(t+1)\Gamma) = exp(2i\pi\Gamma)exp(2i\pi t\Gamma)$.
iii) If we introduce the fibre bundle $\mathcal{L}_{D^*} = \mathcal{O}_{D^*} \otimes \mathcal{L}$, the *holomorphic sections* defined by $\tilde{U}'(t) = exp(-2i\pi t\Gamma)\tilde{U}(t)$ have period 1 on H, that is
$\tilde{U}'(t+1) = exp((-2i\pi t - 2i\pi)\Gamma)\tilde{U}(t+1) = exp(-2i\pi t\Gamma)(exp{-2i\pi\Gamma})exp(2i\pi\Gamma)\tilde{U}(t) = \tilde{U}'(t)$
Hence if $(u_1'(t), \dots, u_m'(t))$ is a multivalued solution then the product of the matrix $exp(-tlogT)$ with the vector $(u_1'(t), \dots, u_m'(t))$ is the inverse image of a global holomorphic section of $\mathcal{L}_{D^*}$.

2.1.2. Canonical form of the solutions of the meromorphic system.
We consider again the above general system (2.1) defined by $A(z)$ lifted to H. Let $\widetilde{S}(t) = (\widetilde{U}_1(t), \cdots, \widetilde{U}_m(t))$ be a set of m independent multivalued solutions; each vector $\widetilde{U}_k(t)$ has m holomorphic functions on $H = \widetilde{D}^*$ as components.
The *Monodromy* is induced on the solutions by the action of the translation : $t \mapsto t+1$ on $\widetilde{D}^*$. As the coefficients of the system defined by $A(2i\pi t)$ are of period

1, the substitution of t by $t+1$ transform the basis $\widetilde{S}(t)$ of the vector space of solutions into another basis, hence there is a matrix $C \in GL(m, \mathbb{C})$ s.t.

$$\widetilde{S}(t+1) = \widetilde{S}(t)C$$

The matrix C defines a linear transformation of the space of solutions, that is the monodromy transformation.
The logarithm of the monodromy. Let Γ be a matrix s.t. $e^{2i\pi\Gamma} = C$ with eigenvalues λ satisfying the condition $0 \leq Re(\lambda) < 1$, then the the matrix $\widetilde{S}(t)e^{-2i\pi t\Gamma}$ has period 1 in t. Considering the change of variable $z = e^{2i\pi t}$, we get a *matrix* $\Sigma(z)$ *with coefficients holomorphic on* D_r^* s.t.

$$\Sigma(z) \in Gl(m, \mathcal{O}_{D_r^*}) : \Sigma(e^{2i\pi t}) = \widetilde{S}(t)e^{-2i\pi t\Gamma}$$

then for each determination of $logz$, the columns of $\Sigma(z)e^{\Gamma logz}$ form a basis of the vector space of solutions (called also a fundamental system of solutions) since if we put formally $z = e^{2i\pi t}$ and $2i\pi t = logz$, we recover $\widetilde{S}(t)$.
For example for $m = 1$ and $\Gamma = \alpha$, the solution $z^\alpha = e^{2i\pi\alpha t}$ satisfy $e^{2i\pi\alpha(t+1)} = e^{2i\pi\alpha t}e^{2i\pi\alpha}$, hence the matrix C has one entry $e^{2i\pi\alpha}$.

Remark 2.4. The condition $0 \leq Re(\lambda) < 1$ on the eigenvalues is arbitrary and we could add to λ an integer.

In summary we have

Proposition 2.5. *Let $A(z)$ be a matrix (m,m) with coefficients holomorphic on D_r^*, meromorphic at 0, and consider the equation*

$$\frac{dU}{dz} = A(z)U.$$

There exists a matrix Γ with constant complex coefficients and an (m,m) matrix $\sum(z)$ with coefficients holomorphic on D_r^* such that a fundamental system of multi-valued solutions of the equation is of the form $\widetilde{S}(t) = \Sigma(e^{2i\pi}t)e^{2i\pi t\Gamma}$ or equivalently $S(z) = \Sigma(z)e^{logz\Gamma}$ is a solution on any sector of D_r^* with a fixed determination of $logz$. The monodromy matrix is then defined as $C = e^{2i\pi\Gamma}$.

Remark 2.6. i) To construct the matrix C we need to choose a basis of the solutions and then study the action of T, that is the transformation of the solutions by the change of variable $\theta + 2i\pi$ (one turn around zero). Then C and Γ are defined up to conjugation by the matrix of change of the basis. Hence we can reduce Γ to the Jordan canonical form in the example (2.2).
ii) If the matrix $A(z)$ is of the form $\frac{B(z)}{z}$ where $B(z)$ is holomorphic s.t. the difference of two eigenvalues of $B(0)$ is never a non zero integer, the matrix C is conjugate to $exp(2i\pi B(0))$ [3,1), p 137].

2.1.3. Equivalent Systems. Consider a matrix $M(z) \in GL(n, K)$ and let

$$G(z) = M^{-1}(AM - \frac{dM}{dz})$$

then $V = M^{-1}U$ is a solution of the equation $\frac{dV}{dz} = G(z)V$ if and only if U is a solution of the equation $\frac{dU}{dz} = A(z)U$.

Definition 2.7. Let $A(z)$ and $G(z)$ be two matrices with meromorphic coefficients at 0, the two systems $\frac{dU}{dz} = A(z)U$ and $\frac{dV}{dz} = G(z)V$ are said to be equivalent if there exists an invertible matrix $M(z)$ such that $G(z)$ is related to $A(z)$ by the above formula.

2.1.4. Regular singularity.

Theorem 2.8 (regular singular point). *Given a system $\frac{dU}{dz} = A(z)U$, the following conditions are equivalent:*
i) the system is equivalent to a system of the form $\frac{dV(z)}{dz} = \frac{B(z)}{z}V(z)$ where $B(z)$ is a matrix with holomorphic coefficients
ii) the system is equivalent to a system of the form $\frac{dV(z)}{dz} = \frac{\Gamma}{z}V(z)$ where Γ is a matrix with constant coefficients
iii) there exists a matrix $\sum(z)$ with meromorphic coefficients at 0 s.t. a fundamental system of solutions is given as $S(z) = \Sigma(z)e^{log z \Gamma}$.

Counterexample. The singularity of the equation $z^2 \frac{df}{dz} + f = 0$ is not regular since it has $e^{\frac{1}{z}}$ as solution.

Definition 2.9. A system $\frac{dU}{dz} = A(z)U$ has regular singular point at 0 if the equivalent conditions of the theorem are satisfied.

2.1.5. Linear differential equations on a punctured disc. Let $D_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$ denotes the complex open disc of radius ε centered at 0. A set of $n+1$ meromorphic functions $a_i(z)$ for $i \in [0, n]$ on the complex disc D_r with a unique isolated singular point at 0 defines a differential operator

$$P = \sum_{i \in [0,n]} a_i(z)(\frac{d}{dz})^{n-i}$$

of degree n if $a_0(z) \not\equiv 0$, acting on the holomorphic functions on D_ε for $\varepsilon < r$. The study of the solutions u on D_ε^* of the equation:

$$(\tfrac{d}{dz})^n u(z) = -\tfrac{1}{a_0(z)}(\textstyle\sum_{i \in [1,n]} a_i(z)(\tfrac{d}{dz})^{n-i} u(z))$$

can be reduced to the case of a linear system if we introduce the new variables

$$u_0 = u, u_1 = (\tfrac{d}{dz})u, \cdots, u_i = (\tfrac{d}{dz})^i u, \ldots, u_{n-1} = (\tfrac{d}{dz})^{n-1} u,$$

then the system is written as

$$\tfrac{d}{dz}u_0 = u_1, \tfrac{d}{dz}u_i = u_{i+1}, \ldots, \tfrac{d}{dz}u_{n-1} = -\tfrac{1}{a_0(z)}(\textstyle\sum_{i \in [1,n]} a_i(z)u_{n-i}).$$

Corollary 2.10. *i) The holomorphic solutions of the differential equation $P(u) = 0$ on a simply connected open subset $U \subset D_r^* = D_r - \{0\}$ form a complex vector space of dimension n.*
ii) The sheaf of solutions on D_r^* is a local system.

The corollary follows from the existence and uniqueness of local holomorphic solutions of the equation. Holomorphic version of Cauchy conditions for the existence of solutions of the equation $P(u) = 0$ apply near any point $z_0 \neq 0 \in D_r^*$ and show the existence of a unique holomorphic solution u_1 with given initial values for its derivatives to the order $n-1$ at z_0. The solutions near z_0, being in correspondence with these initial values in $\mathbb{C}^n$, form a complex vector space of dimension n on a neighbourhood of z_0 and extend on a simply connected open subset.

Remark 2.11. i) The regularity condition has been introduced by Fuchs as follows: the order of the pole of $\frac{a_i}{a_0}$ at 0 is at most i. Later we will extend the notion of regularity to meromorphic connections always associated to a differential equation [3,1), p 143].
ii) Although the solutions are sections of $\mathcal{O}_{D_r^*}$ which is a trivial fibre bundle, the local system $\mathcal{L}$ is not necessarily trivial ($\mathcal{L}$ defines an analytic bundle denoted by $\mathcal{L} \otimes \mathcal{O}_{D_r^*}$ which is trivial).

Example. A determination of the function $z^\alpha = e^{\alpha log z}$ is the solution of the equation $z\frac{du}{dz} - \alpha u = 0$ on any sector. The value of one determination on the complement of a ray in D_r is multiplied by $e^{2i\pi\alpha}$ after extension across the ray in the positive orientation.

2.1.6. Monodromy. Let P be an holomorphic differential operator on a punctured disc D_r^* and consider the vector space E of solutions near a point v in D_r^*. The extension of a solution along a circle S^1 through v defines the *invertible linear monodromy operator* $T : E \to E$.

2.2. Connections with Logarithmic singularities

The object of regularity is to study the behavior of a connection near singularities along a divisor Y on X. Since resolution of singularities leads naturally to a normal crossing divisor (NCD), special techniques have been developed in this case by Deligne, based on the logarithmic complex.

2.2.1. Logarithmic Complex.
i) Let X be a smooth complex algebraic variety. A normal crossing divisor Y in X is defined by a system of local parameters of the regular local ring $\mathcal{O}_{X,y}$, then Y is the union of its irreducible components Y_i and it is written as $Y = \cup_{i\in I} Y_i$; for all subset M of I, let $Y_M = \cap_{i\in M} Y_i$, $Y_M^* = Y_M - \cup_{i\notin M} Y_i$, then given a general point $y \in Y_M^*$ there exist analytic local coordinates $z_j, j \in [1, n]$ at y such that Y_M is defined by $z_i = 0$ for $i \leq p$. A neighbourhood $U(y)$ of y is isomorphic to D^{p+k} and $U(y)^* \simeq (D^*)^p \times D^k$ where D is a complex disc, denoted with a star when the origin is deleted, with fundamental group $\Pi_1(U(y)^*)$ a free abelian group generated by p elements representing classes of closed paths around the origin, one for each D^* in the various one dimensional axis with coordinate z_i.
ii) Let $j : X - Y \to X$ denotes the open embedding, the sheaf $\Omega_X^m(LogY)$ of differential forms of degree m with logarithmic poles on Y is the subsheaf of $j_*\Omega_{X-Y}^m$ defined locally near y by

$$\Omega^m_X(LogY)_y :=\{\omega \in (j_*\Omega^m_X)_y : f\omega \in \Omega^m_{X,y} \text{ and } fd(\omega) \in \Omega^{m+1}_{X,y}\}$$

where f is a reduced equation of Y near y.

2.2.2. Properties. The above definition is independent of the choice of coordinate equations of the components $Y_i, i \leq p$ of Y near y.

From now on we will suppose the component Y_i of Y smooth.

i) In terms of the equations z_i, the forms $\frac{dz_i}{z_i}, i \leq p, dz_i, p < i < n$ form an $\mathcal{O}_{X,y}-$basis of $\Omega^1_X(LogY)_y$.

ii)$\Omega^m_X(LogY) \simeq \wedge^m \Omega^1_X(LogY)$.

iii) There exists a global residue morphism $Res_i : \Omega^1_X(LogY) \to \mathcal{O}_{Y_i}$ with value, the restriction to Y_i of the locally defined coefficient of $\frac{dz_i}{z_i}$.

Definition 2.12. Let F be a vector bundle on X. A connection with logarithmic poles along Y, $\nabla : F \to \Omega^1_X(LogY) \otimes_{\mathcal{O}_X} F$ has a matrix with logarithmic poles.

It extends to $\nabla^i : \Omega^i_X(LogY) \otimes_{\mathcal{O}_X} F \to \Omega^{i+1}_X(LogY) \otimes_{\mathcal{O}_X} F$; it is integrable if $\nabla^1 \circ \nabla = 0$, so that a logarithmic complex $\Omega^*_X(LogY)(F) := (\Omega^*_X(LogY) \otimes_{\mathcal{O}_X} F, \nabla)$ is defined in this case.

The composition map: $R_i \otimes Id \circ \nabla : F \to \Omega^1_X(LogY) \otimes_{\mathcal{O}_X} F \to \mathcal{O}_{Y_i} \otimes F$ vanishes on the product $\mathcal{I}_{Y_i}F$ of F with the ideal $\mathcal{I}_{Y_i}$ defining Y_i. It induces a linear map called the residue endomorphism of the connection

$$Res_i(\nabla) : F \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_i} \to F \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_i}.$$

At the point $y \in Y$, the residue Res_i induces a linear endomorphism $Res_i(y)$ on the fibre $F(y)$ of the vector bundle defined by F.

Let $U(y)$ denotes a polydisc D^n with center y. The universal covering $\widetilde{U}(y)$ of $D^{*,p} \times D^{n-p}$ is defined by

$\{t = (t_1, \cdots, t_n) \in \mathbb{C}^n : \forall i \leq p, \mathrm{Im} t_i > 0 \,\text{and}\, \forall i > p, | t_i |< \varepsilon\}$,

with $t \to (e^{2i\pi t_1}, \cdots, e^{2i\pi t_p}, t_{p+1}, \cdots, t_n) \in D^{*,n}$ as covering map. We denote by $\mathcal{L}$ the local system defined by the horizontal sections, by L its fibre and by $\widetilde{L}$ the global sections of the inverse image of $\mathcal{L}$ on $\widetilde{U}(y)$. The monodromy action T_j for $j \leq p$ is defined on $\widetilde{U}(y)$ by the formula: $T_j v(t) = v(t_1, \cdots, t_j + 1, \cdots, t_n)$ for any section v. Moreover there is an isomorphism between the fibre $F(y)$ and the vector space $\widetilde{L}$ (see the proof of next theorem); via this isomorphism, we have the following relation proved in ([4],II,5)

Lemma 2.13. $T_i = exp(-2i\pi Res_i(\nabla))$.

In what follows we need to choose a section τ of the projection $\mathbb{C}$ on $\mathbb{C}/\mathbb{Z}$, given for example by a region of $\mathbb{C}$ defined by the conditions on real z, $\Re z \in [0, 1[$; such section will appear, when we fix a determination of the logarithm, in the proof in [4](see also[3,2)]) of the following result due to Manin in dimension one.

Theorem 2.14 (Logarithmic extension). *Let Y be a NCD in X, F_{X^*} an holomorphic vector bundle on $X - Y$ and ∇ a connection on F_{X^*}. There exists a locally free module F_X on X which extends F_{X^*}, moreover the extension is unique if the*

following conditions are satisfied
i) ∇ *has logarithmic poles with respect to* F_X*.*

$$\nabla\colon F_X \to \Omega^1_X(logY)\otimes F_X$$

ii) The eigenvalues of the residues of ∇ *with respect to* F_X *belongs to the image of* τ*.*

Proof. a) The local system $\mathcal{L}$ is said *locally unipotent along* Y *if at any point* $y\in Y$ *all* T_j *are unipotent*, in which case the extension we describe is called *canonical*. First we work locally on a neighbourhood of a point y of Y, which amounts to suppose from now on X^* *isomorphic to a product of discs punctured or not.* Let $\mathcal{L}=F^{\nabla}_{X^*}$ and let $\widetilde{L}$ denotes the vector space of multivalued sections of $\mathcal{L}$, that is the subspace of horizontal sections of the analytic sheaf $\widetilde{F}_{X^*}$ on $\widetilde{X^*}$. The bundle F_{X^*} is isomorphic to $\mathcal{O}_{X^*}\otimes_{\mathbb{C}}\widetilde{L}$. In such case of local unipotent monodromy actions, the endomorphisms $N_i = Res_i\nabla$ are nilpotent and related to the logarithm of the monodromy by the formula: $N_i = -\frac{1}{2i\pi}LogT_i = \frac{1}{2i\pi}\sum_{k>0}(I-T_i)^k/k$. The connection on $\mathcal{O}_X\otimes_{\mathbb{C}}\widetilde{L}$ defined by the matrix

$$\nabla \quad = \quad \Sigma_{i\le p}N_i\, dz_i/z_i$$

is isomorphic to the local extension we are looking for. For $v\in\widetilde{L}$, we define a section $\tilde{v}\in\widetilde{F}_{X^*}$ via the action of the monodromy, explicitly described by the formula

$$\tilde{v}=(exp(2i\pi\Sigma_{i\le p}t_iN_i)).v$$

Notice that the exponential is a linear sum of multiples of $Id-T_j$ with analytic coefficients, hence its action defines an analytic section.
We have, for all $t\in\widetilde{X^*}$, $\tilde{v}(t+e_j)=\tilde{v}(t)$, since for $e_j=(0,\cdots,1_j,\cdots,0)$,$\tilde{v}(t+e_j)=[exp(2i\pi N_j)exp(2i\pi\Sigma_{i\le p}t_iN_i)].v(t+e_j)=[exp(2i\pi\Sigma_{i\le p}t_iN_i]exp(2i\pi N_j).v(t+e_j)$ where $exp(2i\pi N_j).v(t+e_j)=T_j^{-1}.v(t+e_j)=T_j^{-1}T_j.v(t)=v(t)$;
hence $\tilde{v}$ is the inverse image of a section of F_{X^*} denoted by

$$\tilde{\text{v}}=(exp(\Sigma_{i\le p}(logz_i)N_i)).v$$

where $2i\pi t_i=logz_i$, moreover $\nabla\tilde{v}=\Sigma_{i\le p}\widetilde{N_i.v}\otimes\frac{dz_i}{z_i}$.
Let $j: X^*\to X$ be the inclusion, then we can describe F_X as a subsheaf of $j_*F_{X^*}$ by the condition that a basis of $\widetilde{L}$ is sent onto a basis of $F_{X,y}$.

b) In general the local system is defined by a representation of $\Pi_1(X^*)$ on the vector space L, i.e the action of commuting automorphisms T_i for $i\in[1,p]$ indexed by the local components Y_i of Y at y. The automorphisms T_i decomposes as a product of commuting automorphisms, semi-simple and unipotent $T_i=T_i^sT_i^u$. Since L is a $\mathbb{C}$-vector space, T_i^s can be represented by the diagonal matrix of its eigenvalues. If we consider families of eigenvalues λ_i for each T_i we have the spectral decomposition of L

$$L=\oplus_{\lambda.}L^{\lambda.} \qquad , \qquad L^{\lambda.}=\cap_{i\in[1,n]}(\cup_{j>0}\ker\,(T_i-\lambda_iI)^j)$$

where the direct sum is over all families $(\lambda.) \in \mathbb{C}^p$. The logarithm of T_i is defined as the sum

$$LogT_i = LogT_i^s + LogT_i^u$$

$LogT_i^s$ is the diagonal matrix formed by $log\,\lambda_i$ for all eigenvalues λ_i of T_i^s and for a fixed determination of log, while $LogT_i^u = -\Sigma_{k\geq 1}(1/k)(I - T_i^u)^k$ is defined as a finite sum of nilpotent endomorphisms. Let $N_i = -\frac{1}{2i\pi}LogT_i^u$ and $D_i = -\frac{1}{2i\pi}LogT_i^s$, then the i-th residue is $D_i + N_i$ with eigenvalues $\alpha_i \in [0,1[$ such that $\lambda_i = e^{-2i\pi\alpha_i}$.

Let $\mathcal{U}_X^{\lambda.}$ denotes the vector bundle defined by $\mathcal{O}_X$ with the connection $\nabla^{\lambda.}$ defined by the matrix

$$\Sigma_{i\leq p}\alpha_i dz_i/z_i \quad , \quad \alpha_i = -\frac{1}{2i\pi}log\lambda_i$$

where the determination of $log\lambda_i$ is such that α_i is in the image of the section τ. Let $\mathcal{U}^{\lambda.}$ denotes the local system of horizontal sections. The local system $\mathcal{L}$ on X^* decomposes into

$$\mathcal{L} \quad = \quad \oplus_{\lambda.}(\mathcal{U}^{\lambda.}) \otimes \mathcal{L}^{\lambda.}$$

where $\mathcal{L}^{\lambda.}$ is unipotent, then we put $F_X = \oplus_{\lambda.}\mathcal{U}_X^{\lambda.} \otimes \mathcal{L}_X^{\lambda.}$ where $\mathcal{L}_X^{\lambda.}$ is the extension of $\mathcal{O}_{X^*} \otimes \mathcal{L}^{\lambda.}$ defined above in (a).

c) The crucial step is in the uniqueness since the patching process of the local system extends uniquely to a patching process of the bundle F_X. This result is explained with details in [3,2)].
A basic ingredient in the proof,is that the eigenvalues of the residue are constants along Y_i and F_X is unique up to isomorphisms if we suppose the eigenvalues of the residues in the image of the section τ.

Local description of F_X as a subsheaf of $j_*F_{X^*}$.

Let t_i be the coordinates of the product of $p-$upper half planes and $n-p$ discs, then the universal covering map of a neighbourhood of y is given by $z_j = exp(2i\pi t_j), j = 1,\ldots,p$ and $z_j = t_j$ for $j > p$. We associate to an element v of $\widetilde{L}^{\lambda.}, \lambda. = e^{-2i\pi\alpha.}$, the section $\tilde{v}$ of F_X defined by the formula

$$\tilde{\mathrm{v}} = exp(2i\pi\Sigma_{j\leq p}t_j(\alpha_j I + N_j)).v = \Pi_{j\leq p}z_j^{\alpha_j}\exp(\Sigma_{j\leq p}logz_j N_j).v$$

It can be checked that this section descends to a section near $y \in Y_M$ in X^*. A basis of L is sent on a basis of $(F_X)_y$ and we have

$$\nabla\tilde{v} = \Sigma_{j\leq p}[\widetilde{\alpha_j v} + \widetilde{N_j.v}] \otimes \frac{dz_j}{z_j}.$$

Notice that a different choice for the section τ would add an integer k to α_j, hence would multiply the basis $\tilde{\mathrm{v}}$ by z_j^k and modify the extension, but a different choice of the determination of $logz_j$ would add an integer $2i\pi k$ and hence change v by $T^{-k}v$ in the expression of $\tilde{\mathrm{v}}$ which is only a linear transformation of the basis and does not modify the extension. □

The main application of the above construction is proved in ([4],II,6)

Theorem 2.15 (Logarithmic DeRham cohomology). *The integrable connection* ∇ *defines a DeRham complex with coefficients in the canonical extension* F_X *of a flat bundle on* X^*, *quasi-isomorphic to* $Rj_*F^{\nabla}_{X^*}$

$$Rj_*F^{\nabla}_X \quad \cong \quad \Omega^*_X(LogY)\otimes F_X$$

In particular $H^i(X-Y,F^{\nabla}_{X^*})\simeq H^i(R\Gamma(X,\Omega^*_X(LogY)\otimes F_X))$.

2.3. Meromorphic Connections on the disc

To study the behavior of a connection near singularities along any divisor Y (not necessarily a NCD) on X it is useful to introduce meromorphic connections. We start with the disc case.

Let $\mathcal{O}_D[0]$ denotes the sheaf of holomorphic functions on D^*, meromorphic at 0. We define now a meromorphic connection on $\mathcal{O}_D[0]-$ modules

Definition 2.16. Let D be a complex disc and M a locally free $\mathcal{O}_D[0]-$ module of finite rank. A meromorphic connection on M is a $\mathbb{C}-$linear operator $\nabla : M\to M$ satisfying

$$\forall h\in\mathcal{O}_D[0],\forall u\in M,\quad \nabla_z(hu)=\frac{dh}{dz}u+h\nabla_z u$$

The definition is for modules on a disc and will be extended to varieties. In this definition we did contract the differential form dz with ∂_z.

Let K be the germ of $\mathcal{O}_D[0]$ at 0, then the above definition apply for the $K-$vector space E, germ of M at 0.

If $\mathbf{e}=(e_1,\cdots,e_n)$ is a basis of E over K, we can write $\nabla_z e_i=\sum_j a_{ji}(z)e_j$, then ∇_z is defined by the matrix

$$A=(a_{ji}(z))\in\ End(n,K)$$

with respect to this basis, then we have in matrix form

$\nabla_z\mathbf{e}=\mathbf{e}A$ and for $u=\sum u_i(z)e_i$, $\nabla_z u=\sum_i(\frac{du_i}{dz}+\sum_j a_{ji}(z)u_j(z))e_i$.

If $\mathbf{f}=(f_1,\cdots,f_n)$ is a new basis defined on $\mathbf{e}$ by a matrix B ($\mathbf{f}=\mathbf{e}B$), then $\nabla_z\mathbf{f}=\mathbf{e}(\frac{dB}{dz}+AB)=\mathbf{f}B^{-1}(\frac{dB}{dz}+AB)$

Example. Let v a vector in E with matrix Q on $\mathbf{f}$, $v=\mathbf{f}Q=\mathbf{e}BQ$ and let $P=BQ$ s.t. $v=\mathbf{e}P$, then $dQ/dz=-(B^{-1}(\frac{dB}{dz}+AB))Q$ is equivalent to $dP/dz=-AP$ (the horizontality of v is independent of the basis).

2.3.1. Connections and Systems of linear differential equations. The equation $\nabla_z u=0$ is equivalent for $u=\mathbf{e}U$ to the system $\frac{dU(z)}{dz}=-AU$.

Hence a system defines a connection with respect to the canonical basis of K^n.

We deduce from the expression of the matrix of a connection with respect to a basis that equivalent classes of systems of linear differential equations give isomorphic meromorphic connections.

Corollary 2.17. *There is a correspondence between equivalent classes of systems of linear differential equations and isomorphisms classes of meromorphic connections on the disc.*

Example. i) The differential operator $P = z(d/dz) - \alpha$, defines on the sheaf $\mathcal{O}_D[0]$ the connection $\nabla_z(f) = P(f)$.

ii) Let $M = Ke^{\frac{1}{z}}$, this notation means that we consider M as a K subspace of the inductive limit of $\Gamma(D^*_\varepsilon, \mathcal{O})$ so to induce the natural connection. Hence we consider $\nabla_z(fe^{\frac{1}{z}}) := \frac{df}{dz}e^{\frac{1}{z}} - f\frac{1}{z^2}e^{\frac{1}{z}}$, then (M, ∇_z) is isomorphic to (K, ∇'_z) s.t. $\nabla'_z(f) = \frac{df}{dz} - f\frac{1}{z^2}$.

2.4. Regular meromorphic connections

For any divisor Y on X we denote by $\mathcal{O}_X[Y]$ the sheaf of rings of holomorphic functions on $X^* = X - Y$, meromorphic along Y. It is a coherent sheaf of rings since $\mathcal{O}_X[Y]$ is locally isomorphic to $\mathcal{O}_X[h^{-1}]$ where h is a local equation of Y.

Definition 2.18. i) Let Y be a divisor on X and F a vector bundle on $X - Y$. An $\mathcal{O}_X[Y]$-coherent module $\widetilde{F}$ with an isomorphism $\widetilde{F}_{|X-Y} \simeq F$ is called a meromorphic extension of F.
ii) A connection on $\widetilde{F}$ is defined as a $\mathbb{C}-$linear map $\widetilde{F} \to \Omega^1_X \otimes_{\mathcal{O}_X[Y]} \widetilde{F}$ satisfying the usual (Leibnitz) condition.
iii)A connection ∇ on F is said to be meromorphic with respect to $\widetilde{F}$ if it extends to a connection on $\widetilde{F}$.
iv) A coherent module $\widetilde{F}$ is effective if there exists an $\mathcal{O}_X$-coherent module G s.t. $\widetilde{F} \simeq \mathcal{O}_X[Y] \otimes_{\mathcal{O}_X} G$.

We recover the definition on the disc D since Ω^1_D is free of rank one generated by dz ($\Omega^1_D \otimes_{\mathcal{O}_D} F \simeq F$).
Let $u : Z \to X$ be a morphism on a connected analytic manifold Z s.t. $u^{-1}(Y)$ is a divisor on Z. The inverse image $u^*\nabla$ of the meromorphic connection is defined on the vector bundle u^*F and its meromorphic extension $u^*\widetilde{F} = \mathcal{O}_D \otimes_{u^{-1}\mathcal{O}_X} u^{-1}\widetilde{F}$ as follows. Near a point $a \in Z$ with local coordinates $(z_1, \dots, z_r)$ and $b = u(a)$ with local coordinates $(x_1, \dots, x_n)$, u is defined by $x_i = u_i(z_1, \dots, z_p)$. For a section $f \in \widetilde{F}_b$ s.t. $\nabla f = \sum dx_i \otimes f_i$ where $f_i \in \widetilde{F}_b$, we define $(u^*\nabla)(u^*f) = \sum du_i \otimes u^*f_i$. It can be checked that the definition of $u^*\nabla$ is independent of all choices and that the inverse image of the local system of flat sections of ∇ consists of the flat sections of $u^*\nabla$ (write u as a composition of a projection and an immersion).

Definition 2.19 (regularity in dim 1). A connection (F, ∇) meromorphic with respect to $\widetilde{F}$ on a disc D is said to be regular at 0 if the system defining the flat sections is regular, that is if we can choose a basis $e = (e_1, \dots, e_m)$ near 0 of $\widetilde{F}$ over K s.t. the matrix of the connection has a simple pole

$$z\nabla_z e_i = -\textstyle\sum_j b_{ij}(z)e_j, \quad b_{ij}(z) \in \mathcal{O}_{D,0}$$

Example. We deduce in the one dimensional case, by (2.8,ii) and the remark (2.6,i) on the reduction of the logarithm of the monodromy via Jordan form that a regular connection on a disc, meromorphic at 0, is isomorphic to a direct sum of meromorphic connections of the form $M_{\alpha,\ell}$, where $M_{\alpha,\ell}$ is the meromorphic connection with a basis $e_1, \dots, e_\ell$ such that

$$z\nabla e_i = \alpha\, e_i + e_{i+1} \text{ for } i < l \quad \text{and } z\nabla e_\ell = \alpha\, e_\ell$$

$M_{\alpha,\ell}$ can be realized as the vector subspace of $\widetilde{\mathcal{O}}$ over K with basis :

$$e_i, i \in [1,l] : e_i = e_{l,l+1-i} \text{ where } e_{\ell,j} = z^\alpha \frac{(log z)^{j-1}}{(j-1)!}$$

Definition 2.20 (regularity). i) A connection (F,∇) meromorphic with respect to $\widetilde{F}$ is said to be regular if for any morphism $u : D \to X$ s.t. $u^{-1}(Y) = \{0\}$ its inverse $u^*(F,\nabla)$ is regular on D with respect to $u^*\widetilde{F}$.
ii) A connection (F,∇) meromorphic with respect to $\widetilde{F}$ has logarithmic poles along a normal crossing divisor Y in X if there exists a bundle G on X s.t. $\widetilde{F} \simeq G \otimes_{\mathcal{O}_X} \mathcal{O}_X[Y]$ and ∇ restricts to $G \to \Omega^1_X(LogY) \otimes_{\mathcal{O}_X} G$.

2.4.1. Riemann-Hilbert correspondence.

Theorem 2.21. *The functor $(\widetilde{F},\nabla) \to (\widetilde{F},\nabla)_{|X-Y}$ on a complex analytic manifold X with a divisor Y induces an equivalence of the following categories*
i) the category of flat meromorphic and regular connections along Y
ii) the category of analytic flat connections on $X - Y$.
iii) the category of finite rank complex local system on $X - Y$.

The proof in [3,2), prop. 5.1] is based on the canonical logarithmic extension across a NCD obtained by the repeated blowing up process until we transform Y into a NCD.

2.4.2. Algebraic results. Let X be a smooth complex algebraic variety and F an algebraic vector bundle on X with its Zariski topology. A connection $\nabla : F \to \Omega^1_X \otimes F$ is said to be algebraic if its image is in the algebraic tensor product with *algebraic differential forms* on X, then an analytic connection denoted also $\nabla : F^{an} \to \Omega^{1,an}_X \otimes F^{an}$ extends ∇. The algebraic connection ∇ is integrable if and only if the associated analytic ∇ is, and in this case the analytic flat sections $F^{an,\nabla}$ form a local system $\mathcal{L}$ for the transcendental topology on $X - Y$.

Regularity. The main difference between algebraic and analytic varieties consists in the fact that an algebraic variety X can be always embedded into a proper algebraic variety $\overline{X}$ s.t. $Y = \overline{X} - X$ is a divisor in $\overline{X}$. We can moreover suppose $\overline{X}$ smooth if X is. Let $j : X \to \overline{X}$ be the inclusion, then we consider the algebraic extension j_*F , its analytic extension $(j_*F)^{an}$ and the extension $j^{an}_*(F^{an})$ (for example in the case of $\mathbb{C}^*$ and $F = \mathcal{O}_{\mathbb{C}^*}$ we get the meromorphic functions at 0 and in the second case the essential singularities of functions at 0). The sheaf $(j_*F)^{an}$ is a vector bundle on $\overline{X} - Y$ meromorphic on $\overline{X}$ along Y with a connection ∇.

Definition 2.22. (regularity at ∞) The algebraic connection (F,∇) is regular at ∞ if $((j_*F)^{an},\nabla)$ is regular as an analytic meromorphic connection along Y.

This definition is independent from the choice of $\overline{X}$.
The following version of the Riemann-Hilbert correspondence is proved in [3,2)]

Theorem 2.23. *The functor* $(F, \nabla) \to (F^{an}, \nabla)$ *is an equivalence of the following categories*
i) the category of algebraic flat connections on X *regular at* ∞
ii) the category of analytic flat connections on X^{an}*.*
Hence with the category of finite rank complex local system on X^{an}*.*

3. Singularities of local systems: Constructible Sheaves

In the previous sections local systems were attached to proper differentiable fibrations. We explore here the structure of higher direct image of constant sheaves by algebraic morphisms, subsequent to the special properties of algebraic morphisms (for example Bertini's theorem on the general fiber of a morphism of algebraic varieties, the counterpart of Sard's theorem in differential geometry).
More precisely, Thom-Whitney stratifications for proper morphisms, are introduced to describe the relative behavior of singular fibers varying on a general base space of dimension higher than one. As a consequence, we are lead to introduce constructible sheaves to describe the derived image by an algebraic morphism.
In parallel local systems were attached to differential equations with holomorphic coefficients. The corresponding subject which is not treated in this section, would be the structure differential modules on analytic varieties; the correspondence being via the DeRham complex attached to such modules.
The references to these subsections are the papers of Lê-Teissier [10],[2] and the book of Goresky-MacPherson [7].

3.1. Stratification theory

3.1.1. Bertini's result. Let $f\colon X \to V$ be a morphism on a smooth variety X of dimension m into a variety V of dimension n, *then there exists a Zariski open subset* S_n *in* V *such that the restriction of* f *to* S_n *is smooth.*

The fibers of f on S_n are smooth.
An algebraic morphism f can be completed, that is f can be factorized as an open immersion followed by a proper morphism. This important property explains the fact that we don't need f to be proper.

- One can try again to describe the restriction $f_n : f^{-1}(V - S_n) \to (V - S_n)$ of f to $V - S_n$. However $f^{-1}(V - S_n)$ may be a singular variety now, nevertheless it follows from Thom's work:

for a proper morphism f*, there exists an open set* S_{n-1} *in* $V - S_n$ *such that any point* $y \in S_{n-1}$ *has an open neighbourhood* U_y *of* y *in* S_{n-1} *satisfying the following property: the topological structure of* $f^{-1}(U_y)$ *decomposes into a product:* $f^{-1}(U_y) \simeq U_y \times f^{-1}(y)$.

Hence a sequence of subspaces S_j, locally closed in V and adapted to f, can be constructed in this way (and will be called a stratification of V adapted to f when it satisfy additional topological properties). This is the subject of the next result.

Example. 1- Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a morphism defined by a polynomial. Bertini's result shows that there exists a subset $A \subset \mathbb{C}$ containing a finite number of critical values such that the restriction of P to $\mathbb{C}^n - P^{-1}(A)$ is smooth.
However P may not define a locally trivial fibration on $\mathbb{C} - A$ if it is not proper. The morphism $P\colon \mathbb{C}^2 \to \mathbb{C}$ defined by the polynomial $X^2Y + X$ has no critical value. Let $X^* = \mathbb{C}^2 - P^{-1}(0)$, for any $a \neq 0$ the morphism $X^*_a \times \mathbb{C}^* \to X^*$: $((x,y),b) \mapsto (\frac{xb}{a}, \frac{ya}{b})$ is an algebraic trivialization over $\mathbb{C}^*$. In particular the fiber over $\mathbb{C}^*$ is smooth and irreducible while the fiber at 0 is the union of $X = 0$ and $XY + 1 = 0$ hence reducible. Hence, the morphism is not topologically locally trivial on $\mathbb{C}$ [S.A. Broughton On the topology of polynomial hypersurfaces, AMS Proceedings of Symposia in Pure Math. volume 40 (1983) Part 1]

2- In the previous example , the morphism is not proper but can be compactified by considering the family of polynomials $P^h_t = X^2Y + XZ^2 - tZ^3$ homogeneous in X, Y, Z defining a variety $V(P^h_t) \subset \mathbb{P}^2 \times \mathbb{C}$. The points $((x,y,z),t) = ((0,1,0),t)$ are singular on the fibers of the projection $V(P^h_t) \to \mathbb{C}$ to the coordinate t s.t. $(0,1,0)$ is contained and singular in all compactified fibers of P.
In general algebraic varieties have the property that they can be compactified; and the morphisms can be factorized by an embedding followed by a proper projection. For example a polynomial $P(x_1,\dots,x_n)$ of degree m in n variables defines a quasi-projective variety $X \subset \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$ such its closure is defined by $P_h(x_0,x_1,\dots,x_n) = x_0^m P(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0})$. To compactify the morphism we introduce the hypersurface $\tilde{X} \subset \mathbb{P}^n \times \mathbb{C}$ defined as $\{x,t) \in \mathbb{P}^n \times \mathbb{C} : P_h(x_0,x_1,\dots,x_n) - tx_0^m = 0\}$.
The open subset $\tilde{X} \cap (\mathbb{C}^n \times \mathbb{C})$, complement of the hyperplane $H_\infty : x_0 = 0$, is the graph of the morphism defined by $t = P(x_1,\dots,x_n)$. The study of the fibration on X is related to the fibration on $\tilde{X}$.

3) Let $p : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ denotes the blowing -up of 0 in $\mathbb{C}^2$; the fiber over $\mathbb{C}^2 - \{0\}$ is reduced to a point and the restriction of p is an isomorphism over $\mathbb{C}^2 - \{0\}$; at the origin the fiber is a projective line.
Let D be a line in $\mathbb{C}^2$ and $\bar{D}$ its strict transform, then the image of $p' : (\widetilde{\mathbb{C}}^2 - \bar{D}) \to \mathbb{C}^2$ is $(\mathbb{C}^2 - D) \cup \{0\}$.

3.1.2. Stratification. Let V be a complex analytic space (resp. algebraic variety) endowed with a decreasing sequence of sub-analytic spaces (resp. algebraic subvarieties) $V = V_d \supseteq V_{d-1} \cdots \supseteq V_0 \supseteq V_{-1} = \emptyset$. The various subspaces $S_l := V_l \setminus V_{l-1}$ (called strata) form a partition of V by locally closed subspaces ($V = \cup_l S_l$). A partition of V is called a (Whitney) stratification when it is subject to the following properties.

(1)- Smoothness. $S_l := V_l \setminus V_{l-1}$ is either empty or a locally closed analytic (resp. algebraic) subset of pure dimension l and *the connected components of S_l are a finite number of non singular varieties.*

(2)- (Local normal topological triviality). Given a point v in a strata S_l in V (S_l is smooth but V is not necessarily smooth along S_l), we consider a local embedding in a complex space $\mathbb{C}^n$ of a neighbourhood U_v of v in V and a transversal section

through v, that is an analytic (resp. algebraic) smooth subspace $\mathcal{N}_{n-l}$ of $\mathbb{C}^n$ intersecting transversally in $\mathbb{C}^n$ each strata $S_j \cap U_y, j \geq l$ adjacent to S_l such that $S_l \cap \mathcal{N}_{n-l}$ is a zero dimensional subspace containing v.

The intersection with a small ball of center v and small radius r: $N(v) = \mathcal{N}_{n-l} \cap V \cap B_r^{2n}$ is called a normal slice. The boundary $L(v) = \partial N(v)$ of $N(v)$ is called the link at v.

The normal slice and the link are canonically partitioned (or stratified) as transverse intersections of partitioned spaces. For r small enough, the homeomorphic type of the pair $(N(v), L(v))$ is a topological local invariant (independent of the embedding, the choice of $\mathcal{N}_{n-l}$ and the point v varying in the (connected) strata ([7] p 41). The partitioned normal slice is homeomorphic to a cone on the link with its canonical partition with respect to the vertex (identified with v) and the product partition on $L(v)\times]0,1]$. Moreover there exist standard (transcendental) neighbourhoods W_v of v in V satisfying:

$$W_v \simeq (N \times (W_v \cap S_l) \simeq (N \times (\mathbb{C}^l)$$

this being a homeomorphism respecting the partitions.

Historically, Whitney introduced two conditions on the stratifications that were enough to obtain local topological triviality as it has been shown by the following *Thom - Mather isotopy lemma.*

Lemma 3.1. *Every stratum Y of a Whitney stratified algebraic set has a neighbourhood which is the total space of a locally trivial topological fibre bundle with base space the stratum.*

Hence we may consider this lemma as an existence theorem for the above stratifications.

Example (Whitney umbrella). Consider the surface $W : x^2 - zy = 0$ in : $\mathbb{C}^3$. Let A_z be the $z-$axis defined by $Z = 0, y = 0$. The singular subset is A_z, but the link at a point $z \neq 0$ has a different topological type than the link at 0.

3.1.3. Topological structure of algebraic morphisms.

Theorem 3.2 (Thom - Whitney). *Let $f : X \to V$ be a proper algebraic map of algebraic varieties.*
There exist finite algebraic Whitney stratifications $\mathcal{X}$ of X and $\mathcal{S}$ of V such that, given any connected component S of a stratum S_l of $\mathcal{S}$ on V :
1) $f^{-1}(S)$ is a union of connected components of strata of $\mathcal{X}$ each of which is mapped submersively to S; in particular, every fiber $f^{-1}(y)$ is stratified by its intersection with the strata of $\mathcal{X}$.
2) for all points $y \in S$ there exists a transcendental open neighbourhood U of y in S and a stratum-preserving homeomorphism $h : U \times f^{-1}(y) \simeq f^{-1}(U)$ such that $f \circ h$ is the projection to U.

Definition 3.3 (Stratification of f). A pair of stratifications $\mathcal{X}$ and $\mathcal{S}$ as above is called a (Thom-Whitney) stratification of f.

The proof is based on Thom Isotopy Lemmas, adapted to the algebraic setting [10].

Definition 3.4 (Constructible sheaves). Let X be an analytic (resp. algebraic) variety and A be a ring. A sheaf $\mathcal{F}$ in the category of A_X-modules is constructible if there exists a stratification (resp. with algebraic strata) $\mathcal{X}$ of X such that its restriction to each stratum is a local system (for the transcendental topology).

A linear version of the result of Thom is :

Proposition 3.5. *The $i-$th higher direct image sheaf $R^i f_* \mathbb{Z}_X$ by an algebraic morphism is constructible.*

Precisely, the restriction of $R^i f_* \mathbb{Z}_X$ in degree i, the cohomology of the fibers, is locally constant on each stratum of a Thom-Whitney stratification.

3.1.4. Basic properties.
i)-*The category of constructible sheaves is abelian.*
Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of constructible sheaves then *ker* f and *coker* f are constructible sheaves.
The proof is based on the following result. If $\mathcal{F}$ (resp. $\mathcal{G}$) is constructible with respect to a stratification $\mathcal{S}_1$ (resp. $\mathcal{S}_2$), then there exists a finer stratification for which $\mathcal{F}$ and $\mathcal{G}$ are constructible.
ii) - Let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be a short exact sequence of sheaves, *if $\mathcal{F}_1$ and $\mathcal{F}_2$ are constructible then $\mathcal{F}$ is also constructible.*
iii) - Let $f : X \to V$ be an algebraic morphism then *the inverse image of a constructible sheaf on V is constructible on X and the direct image of a constructible sheaf on X is constructible on V.*

3.2. Cohomologically Constructible sheaves

In general not only the cohomology of complexes is interesting but also the complexes themselves, however since there is no preference between resolutions of complexes there is a need to define a category which identify all resolutions in some sense. Verdier did find the correct definition of the category by considering morphisms up to homotopy and inverting quasi-isomorphisms (morphisms inducing isomorphisms on cohomology), constructing in this way the derived category of abelian sheaves. Inside this category of abelian sheaves on a variety, we are interested in the subcategory of complexes whose cohomology are constructible sheaves. The correspondence between differential modules and their associated DeRham complexes is a good example where we need derived categories.

3.2.1. The derived category of abelian sheaves $D^+(M, \mathbb{Z})$. In the previous cohomological constructions we needed to choose an acyclic resolution to define the direct image functors. The existence of various acyclic resolutions gives the necessary flexibility for computation; however we need to justify such construction, that is to prove that the various resolutions give isomorphic objects.
At first sight the resolution itself may appears to be of no interest, while only

its cohomology is of interest. However, DeRham resolution has already its own interest in analysis, and the increasing use of cohomology in mathematics showed that one might need to work with the complex itself. For example, considering a diagram of continuous maps $N \xrightarrow{f} V \xrightarrow{g} M$, the higher direct images $R^k(g\circ f)_*\mathcal{F}$ of a sheaf $\mathcal{F}$ on N by $g\circ f$ is linked to the the higher direct images $R^i g_*(R^j f_*\mathcal{F})$ only via a spectral sequence, hence they can never be recovered completely unless we keep some knowledge of the complex $Rf_*\mathcal{F}$ itself instead of its cohomology.
How should we formulate this knowledge without loosing the flexibility in the choice of acyclic resolutions is the problem solved by Verdier [13]. The basic idea is to consider a category where the complex remains the object but to modify the morphisms of complexes to signal that our interest is in fact in its cohomology. In the first step of the construction the morphisms of complexes are considered up to homotopy. This step already transforms significantly the category. For example, the inverse of a morphism of complexes $f : K \to K'$ is a morphism $g : K' \to K$ such that $g\circ f$ (resp. $f\circ g$) is only homotopic to the identity. This is already an important modification of the category, since Deligne,for example, notices that any morphism of complexes is isomorphic to an injective morphism [2] in such category.
In the second step a quasi-isomorphism, that is a morphism of complexes inducing an isomorphism on cohomology, is set to be an isomorphism by declaring invertible all quasi- isomorphisms.
Rigorous proofs and elaborate constructions are needed to develop this concept of Grothendieck-Verdier derived category (see Illusie's article which gives motivations behind these constructions in [8] (see also the Springer book of Iversen in the series universitext (1986) and earlier work in a book by Cartan and Eilenberg published by Princeton University Press (1956)).
The category obtained from the category of complexes of abelian sheaves on a topological space M by the above two step construction is called the derived category of abelian sheaves $D^+(M,\mathbb{Z})$.

3.2.2. Hypercohomology. That is *cohomology with coefficients in a complex of sheaves.* Let

$$\mathcal{L}^* = (\cdots \to \mathcal{L}_j \to \mathcal{L}_{j+1} \to \cdots)$$

be a complex of abelian sheaves on M. The hypercohomology $R^i\Gamma(M,\mathcal{L}^*)$, is the cohomology of the derived functor defined by global sections Γ on M with value in the complex $\mathcal{L}^*$. The definition is in two steps. In the first step one construct a quasi-isomorphism of $\mathcal{L}^*$ with a complex of Γ-acyclic sheaves $\mathcal{A}^*$ (for example fine or flabby or injective sheaves) that is a morphism of complexes $g : \mathcal{L}^* \to \mathcal{A}^*$ inducing isomorphisms on cohomology. In the second step one take the cohomology of global sections as definition of hypercohomology

$$R^i\Gamma(M,\mathcal{L}^*) := H^i(\Gamma(M,\mathcal{A}^*))$$

In general, for each left exact (resp. right exact) functor F from the category of complexes of sheaves to an abelian category, *derived functors denoted RF ((resp. LF) are defined in a similar way.*

Remark 3.6. The classical cohomology theory with coefficients in the group $\mathbb{Z}$ can be viewed as the special case of the cohomolgy with coefficients in the constant sheaf $\mathbb{Z}$.
However the original topological construction of homology (dual to cohomology, see Spanier, Massey, and Dieudonné for historical remarks) remains basic for the intuition in topological problems that motivates the interest in cohomology and still gives powerful methods of computation as in the case of the first construction of Intersection cohomology by MacPherson with Goresky.

3.2.3. The derived category of c-constructible sheaves $D^b_c(X,\mathbb{Q})$.
The main result states that the higher direct image sheaves of constructible sheaves by algebraic morphisms are constructible.
This is the main reason to study cohomology with coefficients in constructible sheaves and for more flexibility in deriving functors. We need to work in the sub-category of the Grothendieck-Verdier derived category of sheaves of $\mathbb{Q}-$modules consisting of complexes of sheaves whose cohomology sheaves are constructible.

Definition 3.7. i) A complex of sheaves of $\mathbb{Q}_X-$modules is c-constructible (for cohomologically constructible) if its cohomology sheaves are constructible.
ii) Let $D(X)$ denotes the derived category of $\mathbb{Q}_X-$modules. The full sub-category of $D(X)$ whose objects are c-constructible sheaves (resp. bounded, bounded at left, bounded at right) is denoted by $D_c(X,\mathbb{Q})$ (resp.$D^b_c(X,\mathbb{Q})$, $D^{+\infty}_c(X,\mathbb{Q})$, $D^{-\infty}_c(X,\mathbb{Q})$).

Proposition 3.8. *The higher direct image of a c-constructible complex by a proper algebraic morphism $f : X \to V$ is c-constructible, hence the derived functor $Rf_* : D^b_c(X,\mathbb{Q}) \to D^b_c(Y,\mathbb{Q})$ is well defined.*

The result follows from Thom-Whitney stratification theory for an algebraic morphism.

4. From Lefschetz theorems to the decomposition theorem

To illustrate the power of the various objects introduced in the last three sections, we give a statement of the decomposition theorem in [2]. However it is not possible to give a proof, since either we deduce the result from the proof in positive characteristic, or we need to develop Hodge theory and in both cases there is still a long way left to the interested reader.

The classical Hard Lefschetz theorem on a non singular complex projective variety $X \hookrightarrow \mathbb{P}^m$ of dimension n with a class $\eta \in H^2(X,\mathbb{Q})$ of an hyperplane section $X_t = H_t \cap X$, states that the iterated cup-product

$$H^{n-i}(X,\mathbb{Q}) \xrightarrow{\eta^i} H^{n+i}(X,\mathbb{Q}) \tag{4.1}$$

is an isomorphism for $i \in [0, n]$.

The relative case. For a smooth projective morphism $f : X \to V$ where $\dim X = n$ and $\dim V = s$, the class of an hyperplane section defines a section $\eta \in R^2 f_* \mathbb{Q}_X$ induces by cup-product a map

$$R^{n-s-i} f_* \mathbb{Q}_X \xrightarrow{\eta^i} R^{n-s+i} f_* \mathbb{Q}_X \tag{4.2}$$

which is an isomorphism.
If we imagine to study the cohomology of X via the fibration on V, we see that we need to consider the cohomology of V with coefficients in the geometric local systems $R^i f_* \mathbb{Q}_X$. Such local systems underly various geometric invariants that have been abstracted into the theory of polarized variations of Hodge structures, and many of the important theorems on *geometric local systems* follow uniquely from this underlying Hodge theory.
For example, in the abelian category of finite local systems which is noetherian and artinian, *the sheaves $R^i f_* \mathbb{Q}_X$ are semisimple*, which means that they split into a finite direct sum of irreducible local subsystems (with no nontrivial local subsystems).
The study of such geometric local systems in Hodge theory has become a central object in the study of cohomology of algebraic varieties.

The isomorphisms η^i in the relative case are compatible with the Hodge structure on the cohomology of the fibres X_t of f. Deligne, using Hodge theory, deduced the degeneration of Leray's spectral sequence, from the relative version of Lefschetz result. By definition, such spectral sequence is associated to the canonical filtration τ on $Rf_* \mathbb{Q}_X$ defined by truncation, that is to say that if $\mathcal{I}_X$ is an injective resolution of $\mathbb{Q}_X$, then the filtered complex $(f_* \mathcal{I}_X, \tau)$ is defined up to a filtered quasi-isomorphism in the derived category of filtered complexes on the base V; then the associated spectral sequence corresponding to the filtration τ is defined up to isomorphism. Such filtration τ defines a filtration L on the cohomology of X

$$H^j(V, R^i f_* \mathbb{Q}_X) \Longrightarrow Gr^i_L H^{i+j}(X, \mathbb{Q})$$

and the degeneration statement asserts that there exists natural isomorphisms:

$$Gr^i_L H^{i+j}(X, \mathbb{Q}) \simeq H^j(V, R^i f_* \mathbb{Q}_X)$$

In particular there exists non canonical isomorphisms of rational vector spaces

$$H^n(X, \mathbb{Q}) \simeq \oplus_{i+j=n} H^j(V, R^i f_* \mathbb{Q}_X)$$

The degeneration translates in the derived category of sheaves on V, into a non-canonical decomposition of the derived direct image complex as a direct sum of its cohomology

$$Rf_* \mathbb{Q}_X \simeq \oplus R^i f_* \mathbb{Q}_X[-i]. \tag{4.3}$$

4.1. The decomposition theorem for projective morphisms

A natural question is to find how far we can relax the hypothesis and keep in the same time the result. In fact the theorems as stated are false for a non necessarily smooth projective morphism. In order to formulate similar results in the presence of singularities of spaces as well of the morphism, various objects and tools introduced in the last two decades proved to be fundamental objects in the study of the topology as well of the geometry of varieties and lead to spectacular extensions of the results [2]. In particular the following are basic subjects in the theory
- Thom-Whitney stratification
- Perverse sheaves and Intermediate extension of a local system.
- General Intersection theory on cohomology.

Since the restriction to a strata f/S is a locally trivial topological bundle, the higher direct cohomology sheaf $(R^i f_* \mathbb{Q}_X)/S$ is locally constant on S. Then we say that $R^i f_* \mathbb{Q}_X$ is constructible on V and $Rf_* \mathbb{Q}_X$ is cohomologically constructible on V.

The category of perverse sheaves. A subcategory of the derived category $D^+(V, \mathbb{Q})$ of $\mathbb{Q}-$sheaves on a variety V, which is abelian, called the category of perverse sheaves, has been introduced in [2] following earlier work in ([7], 1). It appeared to be a fundamental object in the study of topological and geometrical properties of the morphism f.

A complex of sheaves K in $D^+(V, \mathbb{Q})$ is defined to be perverse if the following property is satisfied: there exists a stratification $\mathcal{S}$ of V such that for each strata S, $i_S : S \to V$, the restriction $H^n(i_S^* K) = 0$ for $n > -dimS$ and $H^n(Ri_S^! K) = 0$ for $n < -dimS$.

When K is constructible, these conditions show that the restriction of K to the open strata, is reduced to a local system $\mathcal{L}$ in degree $-dimX$.

The perverse truncation. The main interest in the subcategory of perverse sheaves follows from the construction of a cohomological functor defined on the derived category $D^+(V, \mathbb{Q})$ with value in the category of perverse sheaves, constructed inductively with respect to a stratification of V. Namely, the notion of perverse truncation ${}^p\tau^i$ of a complex K is constructed in [2] and then the notion of $i-$th perverse cohomology ${}^p\mathcal{H}^i(K)$ is defined as the cone of the morphism ${}^p\tau^{i-1}(K) \to {}^p\tau^i(K)$ so to fit in a triangle ${}^p\tau^{i-1}(K) \to {}^p\tau^i(K) \to {}^p\mathcal{H}^i(K)$. Perverse cohomology sheaves in various degrees fit together in a long exact sequence in the abelian category and in fact such exact sequence is the best way to compute these objects, as in any cohomology theory.

The Intermediate extension. Research in the above field has been motivated first by the discovery by Goresky and MacPherson ([7], 1) of special objects called Intersection complexes. They are uniquely defined by local systems on locally closed subsets of V. Their construction use the above Whitney stratification on a singular variety V and in an essential way the local topological triviality of the various strata S_l.

In the abelian category of perverse sheaves which is noetherian and artinian, the irreducible sheaves are Intersection complexes defined by irreducible local systems.The following is Deligne's construction of Intersection complexes.

Let $\mathcal{S} = \{S_l\}_{l\leq d}$ of V (dimension $S_l = l$) and let $j_0 : V - \overline{S}_0 \to V$, $j_l : V - \overline{S}_l \to V - \overline{S}_{l-1}$ for $0 < l < d$ denotes the embedding. The Intermediate extension compatible with $\mathcal{S}$ of a local system $\mathcal{L}$ on the big open strata S_d is defined as:

$$j_{!*}\mathcal{L}[d] = \tau_{\leq -1} Rj_{0*} \cdots \tau_{\leq -l-1} Rj_{l*} \cdots \tau_{\leq -d} Rj_{d-1*}\mathcal{L}[d]$$

where for all sheaves $\mathcal{F}$ constructible with respect to $\mathcal{S}$, we have

$$(Rj_{l*}\mathcal{F})_v \simeq R\Gamma(L_{S_{l,v}}, \mathcal{F})$$

where $L_{S_{l,v}}$ is the link of S_l at v, then $\tau_{\leq -l-1}$ truncates the cohomology up to degree $\leq -l-1$.

The Decomposition theorem. Now we are in a position where we can state the version of the decomposition for a projective morphism and a geometric local system ([2], 6.2.4) so that we don't need to invoke explicitly the Hodge theoretical properties which underly such local system and can refer to ([2], 6.2.10)

Let $f : X \to V$ be a projective morphism and $\mathcal{L}$ be a geometric local system on a smooth open subset U of X, $j : U \to X$, dim. $X = n$. The map defined by iterated cup-product with the class of an hyperplane section $\eta \in R^2 f_*\mathbb{Q}_X$

$$^p\mathcal{H}^{-i}(Rf_*(j_{!*}\mathcal{L}[n])) \xrightarrow{\eta^i} {}^p\mathcal{H}^{i}(Rf_*(j_{!*}\mathcal{L}[n])) \tag{4.4}$$

is an isomorphism.

Then, the degeneration of the perverse Leray's spectral sequence defined by the perverse filtration on $Rf_*(j_{!*}\mathcal{L}[n])$ follows and leads to the decomposition:

There exists a non canonical isomorphism in the derived category

$$Rf_*(j_{!*}\mathcal{L}[n])) \xrightarrow{\simeq} \oplus_i {}^p\mathcal{H}^{i}(Rf_*(j_{!*}\mathcal{L}[n]))[-i] \tag{4.5}$$

Remark 4.1. The proof given in [2] is deduced from the theory in positive characteristic. The interested reader can find a proof via differential modules in the following paper by
Saito M.: Modules de Hodge polarisables. *Publ. RIMS, Kyoto univ.*, 24 (1988), 849-995.
An interesting paper, for constant coefficients by
De Cataldo M.A.A., Migliorini L.: The Hodge Theory of algebraic maps. *Ann. scient. Ec. Norm. Sup.* (2005)
is easier to read. Finally, a paper by
El Zein F.: Topology of Algebraic Morphisms, Contemporary Mathematics 474; http://arxiv.org/abs/math/0702083, states the theorem in the form of a geometrical decomposition formula and treats the isolated singularity case. Such treatment apply also in the general case (to appear later).

References

[1] Cartan H.: Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes *Hermann, Paris* (1961)

[2] Beilinson A.A., Bernstein J., Deligne P.: Analyse et Topologie sur les espaces singuliers Vol.I, *Astérisque, France 100*

[3] Borel A. et al.: Algebraic D-modules,*Perspective in Math. 2*,Academic Press, Boston (1987)
1) Haefliger A.: Local theory of meromorphic connections. 2) Malgrange B.: Regular Connections, after Deligne.

[4] Deligne P.: Equations différentielles à points singuliers réguliers. *Lecture Notes in Math. 163* Springer (1970).

[5] Godement R.: Topologie algébrique et théorie des faisceaux, *Hermann, Paris* (1971)

[6] [SGA]. Séminaire de Géométrie algébrique. par Deligne P. SGA $4\frac{1}{2}$ *Lecture Notes in Math. 569* Springer, Berlin (1977)

[7] Goresky M., Macpherson R. : 1) Intersection homology II. *Inv.Math.* 72 (1983) p 77-129. 2) Stratified Morse theory, *Ergebnisse der Mathematik, 3.folge. Band 14,* Springer-Verlag, Berlin Heidelberg (1988)

[8] Illusie L.: Catégories dérivées et dualité, travaux de J.L. Verdier, *Enseign. Maths. 36*, 369-391 (1990)

[9] Ince E.L.: Ordinary differnial equations, 1926. Dover, New-york (1956)

[10] Lê D. T.- Teissier B.: Cycles évanescents, sections planes et conditions de Whitney II. *Proceedings of Symp. in pure math. 40*, (1983).

[11] MacPherson R. : Chern classes for singular varieties, *Annals of Math. 100*, 423-432 (1974)

[12] Pham F.: Singularités des systèmes différentiels de Gauss-Manin, *Progress in Math.,Birkhauser 2* ,Basel,(1979)

[13] Verdier J.L.: Des Catégories dérivées des catégories abéliennes, *Astérisque 239*,Soc.Math. France, (1996)

[14] Warner. Foundations of differentiable manifolds and Lie groups. *graduate texts in Math. 94* Springer

Literature

In spite of the title, the article does not cover recent developments in the theory. In fact we were heading directly to Variation of Hodge structures underlying geometric local systems, but even this aim has not been attained. Instead an appendix with a basic example with Hodge theoretical oriented references has been added. For a wide view open to other applications in mathematics we suggest the following books covering developments in the theory with an extensive list of research articles in the field.

- Dimca A. Sheaves in topology, *Universitext, Springer Verlag,* (2004).

- Kashiwara M., Schapira P. Sheaves on manifolds, *Grund.math.Wissench. 292, Springer Verlag* (1990). (2003).

- Schurman J. Topology of Singular Spaces and Constructible Sheaves *Monografie Mathematyczene, New series, Polish Academy, Birkhauser, Basel* (2003).

Appendix A. Example: Family of Elliptic Curves

To illustrate the various features of the subject we go back to its origin and compute the local system and Gauss-Manin connection defined by the family of elliptic curves.

A.1. Riemann Surfaces

The classical theory of Riemann surfaces originated in the study of the 'algebraic functions' $w = w(z)$ satisfying the equation with analytic coefficients

$$a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0, \quad a_0(z) \not\equiv 0$$

The main problem consists in the fact that there is no such continuous function $w(z)$ since there exist in general n values of the function for each z. However it is possible to define on each simply connected open subset U in $\mathbb{C}$, an holomorphic function $w(z)$ solution of the equation, called a branch of the function. The behavior of such branches is useful to understand the integrals of rational functions R of z and w

$$F(z) = \int_{z_0}^{z} R(z, w(z))dz$$

The beautiful idea of Riemann, to interpret such branch as a section of a covering space of $\mathbb{C}$, is at the origin of the introduction of the notion of manifolds in modern geometry. This rich subject is treated here as an example, but it is also an historical subject in the field basic in Mathematic.

A.1.1. Elliptic Curves. We consider the equation parameterized by a variable t

$$w^2 = z(z-1)(z-t) \tag{A.1}$$

The point of view of manifolds consists in the introduction of the complex curve S_t defined by the equation $P_t(z, w) = w^2 - z(z-1)(z-t) = 0$ in the variables z, w in $\mathbb{C}^2$. The implicit function theorem shows that for $t \neq 0$ and $t \neq 1$, the curve is smooth. Moreover since the degree in z is 3, this curve which is not compact, is homeomorphic to a torus minus one point(the proof uses Weirstrass periodic meromorphic function $\mathbb{P}$ and its derivative).
This suggest strongly to study the torus itself, that is to compactify the curve. This operation known as the projective completion appeared to be so rich in Mathematic that it became common to view the non compact varieties as compact varieties minus a locus "at infinity" (or open varieties).
Such Riemann surface S_t can be represented as a cover of the Riemann sphere $\mathbb{P}^1$ via the projection onto the variable z that extends to the projective curve onto the projective space. The historical technique to understand such curve via the cuts from 0 to 1 and from t to ∞ in the $z-$plane may be confusing, unless it is coupled with this covering point of view (see [2] for a complete description of the theory).

A.1.2. The local system. The theory of Weirstrass function $\mathcal{P}(z)$ constructed as the sum of the series with indices in $\mathbb{Z}^2$ and its relation to its derivative $\mathcal{P}'(z)$ define an isomorphism between the torus with its analytic structure as a quotient of $\mathbb{C}$ by a lattice isomorphic to $\mathbb{Z}^2$ and the elliptic curve (see the book of Cartan on analytic functions). It follows that the cohomology space $H^1(S_t, \mathbb{Z})$ of the smooth elliptic curve, for $t \neq 0$, $t \neq 1$ is generated by two cycles γ and δ with intersection matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Now if the parameter t vary in $\mathbb{C} - \{0,1\} = P^1 - \{0,1,\infty\}$, we consider the family $S = \cup_t S_t$ and the projection $f : S \to \mathbb{C} - \{0,1\}$ defined by t. The family of cohomology spaces $H^1(S_t, \mathbb{Z})$ form a local system on $\mathbb{C} - \{0,1\}$. Since the fundamental group of $\mathbb{C} - \{0,1\}$ is a free group generated by two loops around the two punctures $\{0,1\}$, the local system is completely determined by the two associated monodromy linear operators around 0 and 1.

The monodromy is related to the presence of a critical point $(0,0)$ (resp. (1,0)) for $P(z,w,t)$ for $t = 0$ (resp. $t = 1$). It is possible to check that the Hessian matrix at these critical points is invertible. Such property of points, known as non degenerate critical points, has been the center of continuous attention by mathematicians and an hypothesis of fundamental results known as Morse theory. We give the basic results in this theory in the next subsection.

We apply these results in our case to the variety $S = \cup_{t \in \mathbb{C}} S_t$. Locally S is defined in $\mathbb{C}^3$ by $P(z,w,t) = w^2 - z(z-1)(z-t) = 0$ and the projection to $\mathbb{C}$ is defined by the projection on the parameter t space.

We conclude from Picard-Lefschetz transformation below that the monodromy linear operators around 0 and 1 are defined resp. by the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

as computed for example in ([1], thm 1.1.20).

A.2. Non degenerate critical points

Let $z = (z_1, \cdots, z_n) \in U \subset \mathbb{C}^n$ and let $f \colon U \to \mathbb{C}$ be a Morse function defined for $n > 1$ by $f(z) = \Sigma_{i=1}^n z_i^2$. The Hessian of f at zero is the matrix $(\partial^2 f / \partial z_i \partial z_j(0))_{i,j}$. The point 0 is a non degenerate critical point if the differential $df(0) = 0$ and the determinant of the Hessian matrix is non zero.
These properties together are independent of the coordinates.
The following result show that the local study of the fibration defined by f near a non degenerate critical point, is isomorphic to the case defined by a Morse function.

Lemma A.1 (Morse). *Let $f : U \to \mathbb{C}$ be an holomorphic map on an open set U in an analytic manifold M with a non degenerate critical point $a \in U$. Locally near a, f can be written in a suitable set of coordinates $z_1, \ldots, z_n$ as $f(z) = f(a) + \sum_{i=1}^n z_j^2$.*

See for example ([3], II, 1.1).

Morse lemma has the following real form

Lemma A.2. *Let $f : X \to \mathbf{R}$ be a differentiable map with a non degenerate critical point $a \in X$. Locally near a, f can be written in a suitable set of coordinates $x_1, \ldots, x_n$ as $f(x) = f(a) - \sum_{j=1}^{r} x_j^2 + \sum_{j=r+1}^{n} x_j^2$.*

The number r is called the index of f at a and it is independent of the choice of coordinates.

A.2.1. Invariants attached to a non degenerate critical point. Let $z = (z_1, \cdots, z_n) \in U \subset \mathbb{C}^n$ and let $f \colon U \to \mathbb{C}$ be defined for $n > 1$ by $f(z) = \Sigma_{i=1}^{n} z_i^2$. There exists a unique critical point at the origin since $\partial f/\partial z_i = 2z_i$ vanish for all i only at such point. The inverse image $Y = f^{-1}(0)$ of zero has an isolated singular point at the origin.
Moreover, this singular point is non degenerate, since the determinant of the Hessian matrix $(\partial^2 f/\partial z_i \partial z_j)(0)_{i,j}$ of f at zero doesn't vanish. The map f induces a morphism $\overline{B_\varepsilon} \to D_{\varepsilon^2}$ from the closed $2n-$ball of radius ε to the disc of radius ε^2. Let $z_j = x_j + iy_j$, then $f(z.) = \Sigma_{j=1}^{n}(x_j^2 - y_j^2) + 2i(\Sigma_{j=1}^{n} x_j y_j)$.

Lemma A.3. *For $|t|$ small enough, the fiber of f at t meets transversally the boundary S^{2n-1}.*

Corollary A.4. *Let $f : B_\varepsilon - f^{-1}(0) \to D^*_{\varepsilon^2}$ be the restriction of the fibration to the punctured disc. There exists δ small enough s.t. $f : f^{-1}D^*_\delta \cap (B_\varepsilon - f^{-1}(0)) \to D^*_\delta$ is a differentiable bundle.*

This corollary follows from the theorem on differentiable fibration. It is valid for any morphism f as proved by Milnor.

A.2.2. *Milnor fiber* is defined for t small enough by

$$F_t = \{z. \in B_\varepsilon \colon f(z.) = t\} \ .$$

Since it is a differentiable invariant, it is denoted by F instead of F_t. For each t, the fiber F_t has an holomorphic structure depending on t in general.

A.2.3. Vanishing cycle. With the usual norm and scalar product on $\mathbf{R}^n$, let

$$Q = \{(u., v.) \in \mathbf{R}^n \times \mathbf{R}^n \colon \parallel u. \parallel = 1, \parallel v. \parallel \leq 1, (u., v.) = 0\}.$$

Q is homeomorphic to the space of tangent vectors to the sphere S^{n-1} of length less or equal to 1. Since the sphere S^{n-1} is a deformation retract of Q, $H_{n-1}(Q, \mathbb{Z}) \simeq H_{n-1}(S^{n-1}, \mathbb{Z}) \simeq \mathbb{Z}$.
We define an isomorphism from Q onto the Milnor fiber F_ρ for real numbers ρ as follows.
Let $z_j = x_j + iy_j$, then F_ρ can be considered as a subset of $\mathbf{R}^n \times \mathbf{R}^n$ defined by three real conditions

$$F_\rho = \{(x., y.) \in \mathbf{R}^n \times \mathbf{R}^n \colon \parallel x. \parallel^2 + \parallel y. \parallel^2 \leq \varepsilon^2, \parallel x. \parallel^2 - \parallel y. \parallel^2 = \rho, \ (x., y.) = 0\}.$$

Let $\sigma \colon = (1/2(\varepsilon^2 - \rho))^{1/2}$, then the following change of variables

$$u. = x./\parallel x.\parallel, v. = y./\sigma, \quad x. = (\sigma^2\| v.\|^2 + \rho)^{1/2}(u.), y. = \sigma(v.)$$

maps F_ρ to Q isomorphically. The image of S^{n-1} is the set $z.$ in F_ρ with real coordinates $z_j = \rho^{1/2}u_j$ for all j. When ρ vary in a small interval $[0, \rho_0]$, the family of embedded spheres S^{n-1}, each in a fiber at ρ, form a ball B of radius ρ_0 in the total space $\mathbb{C}^n$ s.t. the spheres collapse to 0 for $\rho = 0$. In this sense the sphere is a non vanishing cycle on the fiber which vanish in the total space.

Definition A.5. A generator of $H_{n-1}(F, \mathbb{Z})$ is called a vanishing cycle. It is defined by the homology class of an orientation on the embedded sphere S^{n-1} in F.

The homology of F vanish in degrees different from 0 and $n-1$.
For example, if $n = 2$, by a change of variables, we are reduced to the case of $f = z_1^2 - z_2^2$, then Milnor fiber F_ρ is an hyperbola rotating along the circle defined by the vanishing cycle. The singular fiber is given by two complex lines intersecting in one point.
This local situation apply in general near an isolated singularity as proved by Milnor (only the number of vanishing cycles vary).

A.2.4. Picard-Lefschetz transformation. Let $f : X \to \Delta$ be a projective morphism defined on an analytic manifold with value in a complex disc Δ. Let $a \in X$ be a unique non degenerate critical point and let γ be a simple loop in the disc with origin a general point p around the critical value $c = f(a)$ (one positive turn). By the above local theory, a vanishing cycle v_a near a has been constructed on the fiber of a point y_c on the loop γ near c; this cycle v_a can be carried, by a trivialization of f restricted to the induced path γ_{p,y_c} from the loop γ, into a cycle δ (depending on γ), called also vanishing cycle.

Proposition A.6. *The monodromy action is trivial on $H^i(X_p, \mathbb{Q})$ for $i \neq n-1$ and for $i = n-1$*

$$\gamma x = x + \varepsilon_n(x, \delta)\delta$$

where $\varepsilon_n = \pm 1$ is a sign depending on n and (x, δ) is the intersection number $Tr(x \smile \delta)$

See ([3], II, Thm 3.16).

A.3. Picard-Fuchs Equations

Considering again the equation $A1$ in $(z, w) \in \mathbb{C}^2$, the meromorphic differential form $\omega = \frac{dz}{dw}$ on $\mathbb{C}^2$, induces a form on the curve S_t classically written as $\omega_t = \frac{dz}{\sqrt{z(z-1)(z-t)}}$. It can be checked that this form extends for $t \neq 0, 1$ to an holomorphic form on the compact curve, that is ω_t can be written locally as $\omega_t = f(u)du$ where u is a local coordinate and $f(u)$ is holomorphic. For each t, there exists a small ball B_t s.t. the family is topologically trivial over B_t so that we can choose constant homology vectors δ, γ generating the homology of S_t. The form ω_t is closed and its class $[\omega_t]$ decomposes on the dual basis of the cohomology δ^*, γ^* as follows

$$[\omega_t] = (\int_\delta \omega_t)\,\delta^* + (\int_\gamma \omega_t)\,\gamma^*$$

The coefficients are called the periods $A(t) = \int_\delta \omega_t$ and $B(t) = \int_\gamma \omega_t$ of ω_t. By derivation under the integral sign, we can check that the coefficients are in fact holomorphic in t.

Since the basis (δ^*, γ^*) is locally constant, hence horizontal for the Gauss-Manin connection, the derivation of ω_t by the connection is $\omega'_t = A'(t)\delta^* + B'(t)\gamma^*$. It can be checked that $[\omega_t]$ and $[\omega'_t]$ form a basis of the cohomology, so that by derivation we obtain $[\omega''_t]$ which decomposes on such basis, hence we obtain an equation in cohomology classes

$$a(t)\,\omega'' + b(t)\,\omega' + c(t)\,\omega = 0$$

If we consider a cycle ξ in S_t and the function $h(t) = \int_\xi \omega$ then we deduce a differential equation in the function $h(t)$; it is possible to determine the coefficients $a(t), b(t)$ and $c(t)$ and obtain the equation with regular singular points ([1], 1.1.17)

$$t(t-1)\,h''(t) + (2t-1)\,h'(t) + \frac{1}{4}h(t) \;=\; 0$$

References

[1] Carlson J., Muller S., Peters C., Period Mappings and Period Domains, *Cambridge studies in advanced mathematics* 85, (2003).

[2] Springer G. Introduction to Riemann surfaces, *Addison-Wesley,* (1957).

[3] Voisin C. Hodge theory and Complex Algebraic Geometry, *Cambridge studies in advanced mathematics* 76,77,(2002).

Fouad El Zein
Institute of Mathematics of Jussieu, Geometry and Dynamics,
Case 7012, 2 place Jussieu, 75251 Paris Cedex 05
Associate member of CAMS, AUB, Beirut.
e-mail: `elzein@math.jussieu.fr`

Jawad Snoussi
Instituto de Mathemáticas, Unidad Cuernavaca, UNAM, México
e-mail: `jsnoussi@matcuer.unam.mx`