On the period domain of polarized K3 surfaces and hyper-Kähler manifolds of $K3^{[m]}$-type

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Introduction

A *hyper-Kähler manifold* is a simply connected compact Kähler manifold whose space of holomorphic 2-forms is generated by an everywhere non-degenerate form.

Hyper-Kähler manifolds appear naturally in the study of compact Kähler manifolds with trivial real first Chern class (which is the image of the canonical bundle via the first Chern class map). Indeed, the following theorem holds.

Theorem ([Bea83, Théorème 2]). *Each compact Kähler manifold X with trivial real first Chern class admits a finite étale cover*

$$\prod_{i=1}^{k} M_i \longrightarrow X,$$

where each M_i is a complex torus, a Calabi–Yau manifold, or a hyper-Kähler manifold.

In dimension 2, hyper-Kähler manifolds are known as $K3$ *surfaces*, and have been widely studied in the past century. A simple example of K3 surface is the *Fermat quartic*, the smooth quartic in $\mathbb{P}^3_{\mathbb{C}}$ defined by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0. \tag{1}$$

All K3 surfaces are deformation equivalent [Kod64, Theorem 13], hence, in particular, diffeomorphic to the Fermat quartic.

For complex surfaces, the cup-product is a nondegenerate pairing on the second integral cohomology group, and it induces, via the first Chern class map, an intersection form on the Picard group. Moreover, for K3 surfaces, the second integral cohomology group is a torsion-free abelian group of rank 22. The second integral cohomology group of a K3 surface with the intersection form is a unimodular even lattice, isomorphic to the lattice

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U is the hyperbolic plane and $E_8(-1)$ is the standard rank 8 lattice with negative definite scalar product.

A polarized K3 surface of degree $2d$ is a K3 surface S together with an ample invertible sheaf of self-intersection $2d$ and primitive class in $H^2(S, \mathbb{Z})$. There exists a coarse moduli space M_{2d} for polarized K3 surfaces of degree $2d$, namely a space whose closed points parametrize isomorphism classes of polarized K3 surfaces of degree $2d$. This was first shown by Pjateckiĭ–Šapiro and Šafarevič.

Theorem. *There exists a coarse moduli space for polarized K3 surfaces of degree $2d$. It is an irreducible quasi-projective variety of dimension 19.*

The proof is based on the global Torelli theorem that says that K3 surfaces are characterized by their Hodge structure. More precisely, each Hodge isometry between the second cohomology groups of two polarized K3 surfaces is induced by an isomorphism. More precisely, the period morphism

$$\wp_{2d} : M_{2d} \longrightarrow \mathscr{D}_{2d}/O(\Lambda_{K3}, 2d),$$

a regular map between the moduli space of polarized K3 surfaces and their period space (quotient of a Hermitian symmetric domain by an arithmetic group of isometries), is an open embedding.

Some of the theory developed for K3 surfaces can be generalized in higher dimensions. Beauville, Bogomolov, and Fujiki constructed a quadratic form on the second cohomology group of a hyper-Kähler manifold, which is a free abelian group. For a hyper-Kähler manifold X, the lattice $H^2(X,\mathbb{Z})$ only depends on the deformation type of X. A class of examples of hyper-Kähler manifolds is given by the *hyper-Kähler manifolds of* $K3^{[m]}$*-type*, deformations of the m-th Hilbert scheme of points of a K3 surface. The second cohomology group of a hyper-Kähler manifold of K3$^{[m]}$-type with its quadratic form is, for $m \geq 2$, the lattice

$$\Lambda_{K3^{[m]}} = \Lambda_{K3} \oplus \mathbb{Z}(-2(m-1)).$$

Fixing an integer m and the $O(\Lambda_{K3^{[m]}})$-orbit τ of a primitive vector h in $\Lambda_{K3^{[m]}}$ with $h^2 > 0$, there exists a coarse moduli space $\mathcal{M}_\tau^{K3^{[m]}}$ of polarized hyper-Kähler manifolds of K3$^{[m]}$-type with a primitive ample invertible sheaf of orbit τ.

Theorem. *For $m \geq 2$, the moduli space $\mathcal{M}_\tau^{K3^{[m]}}$ is a quasi-projective variety of dimension 20.*

For hyper-Kähler manifolds, the Torelli theorem holds in a weaker form than for K3 surfaces. In particular, the moduli space of polarized hyper-Kähler manifolds of K3$^{[m]}$-type is not irreducible in general. However, the period morphism

$$\wp_\tau : M_\tau^{K3^{[m]}} \longrightarrow \mathscr{D}_\tau/\widehat{O}(\Lambda_{K3^{[m]}}, h)$$

of hyper-Kähler manifolds of K3$^{[m]}$-type (see Section 4.3) and polarisation type $\tau = O(\Lambda_{K3^{[m]}})h$ is an open embedding when restricted to each irreducible component of $\mathcal{M}_\tau^{K3^{[m]}}$.

Fixing an even integral lattice Λ of signature $(2, n_-)$, with $n_- \geq 2$, and a subgroup $\Gamma < O(\Lambda)$ of finite index, we can consider the period space $\mathscr{D}_\Lambda/\Gamma$, where $\mathscr{D}_\Lambda$ is a Hermitian symmetric domain. Borel–Baily studied these quotients and proved that they are normal quasi-projective varieties. This generalizes the period space of polarized K3 surfaces and polarized hyper-Kähler manifolds of K3$^{[m]}$-type.

When Γ is a normal subgroup of $O(\Lambda)$, we obtain a Galois cover

$$q : \mathscr{D}_\Lambda/\Gamma \longrightarrow \mathscr{D}_\Lambda/O(\Lambda).$$

For polarized K3 surfaces of degree $2d$, we obtain the cover

$$q_{2d} : \mathscr{D}_{2d}/O(\Lambda_{K3}, h) \longrightarrow \mathscr{D}_{2d}/O(h^\perp),$$

where h is a primitive vector of square $2d$ in the lattice Λ_{K3} (they are all conjugate).

A reflection with respect to $\beta \in \Lambda$ is an isometry $r_\beta \in O(\Lambda)$ which is the identity on $\beta^\perp$ and acts as $-\mathrm{id}$ on $\mathbb{Z}\beta$. Not all vectors $\beta \in \Lambda$ define a reflection.

Stellari [Ste08, Theorem 3.3] characterized the ramification divisors of q_{2d}, proving that they are in correspondence with the nontrivial classes $[r_\beta]$ in the Galois group $O(h^\perp)/O(\Lambda_{K3}, h)$, for β with negative square. Moreover, he gave a numerical characterization of vectors β such that r_β is nontrivial in $O(h^\perp)/O(\Lambda_{K3}, h)$.

In Theorem 5.5, we generalize the first result of Stellari for even lattices Λ of signature $(2, n_-)$ with $n_- \geq 2$ and all normal subgroups $\Gamma \lhd O(\Lambda)$ of finite index such that $\mathscr{D}_\Lambda/\Gamma$ is irreducible.

The structure of this memoir is the following. In Section 1, we state some preliminary results of complex geometry and we present a brief introduction to lattice theory. Section 2 is devoted to the study of K3 surfaces and of the lattice Λ_{K3}. We present the theory of linear systems on K3 surfaces that leads to the construction of the coarse moduli spaces M_{2d} of polarized K3 surfaces. In Section 3, we focus on the construction of the period morphism for polarized K3 surfaces. Hyper-Kähler manifolds are defined in Section 4, where we present the known examples of deformation types of hyper-Kähler manifolds and the result on the period morphism of polarized hyper-Kähler surfaces proved by Markman. Moreover, following the work of [GHS10], we give a description of the lattice and of various isometry groups in the case of a hyper-Kähler manifold of K3$^{[m]}$-type. Finally, in Section 5, we characterize the ramification divisors of the morphism q, which are the invariant divisors of the period space $\mathscr{D}_\Lambda/\Gamma$ under the action of O/Γ, where $\Gamma \lhd O$ and Γ and O are subgroups of finite index of $O(\Lambda)$. In Section 5.2, we treat the case of polarized K3 surfaces, obtaining again the result proved by Stellari, and in Section 5.3, we study the case of hyper-Kähler manifolds of K3$^{[m]}$-type.

1 Preliminaries

In this section, we will introduce some notation and state some results about complex geometry and lattice theory that we will need in the rest of the memoir.

1.1 Some results of complex geometry

The manifolds considered will always be complex manifolds, of which we recall some useful properties. For more details, we refer to [Huy05].
For each complex manifold X, the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$ defines a long exact sequence in cohomology

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X). \tag{2}$$

There is a natural identification between the group $H^1(X, \mathcal{O}_X^*)$ and the Picard group $\mathrm{Pic}(X)$ of X. Hence, we obtain a morphism $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$, called *first Chern class*.
If the manifold X is compact Kähler, each cohomology group has a Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega_X^p),$$

where $H^q(X, \Omega_X^p) \simeq \overline{H^p(X, \Omega_X^q)}$. We will denote by $H^{p,q}(X)$ the group $H^q(X, \Omega_X^p)$ and by $h^{p,q}$ its dimension.

Moreover, we can define the groups

$$H^{p,q}(X, \mathbb{Z}) = \mathrm{Im}(H^k(X, \mathbb{Z}) \to H^k(X, \mathbb{C})) \cap H^{p,q}(X),$$

where the morphism $H^k(X, \mathbb{Z}) \to H^k(X, \mathbb{C})$ is induced by the sheaf inclusion $\mathbb{Z} \subset \mathbb{C}$.

Theorem 1.1 (Lefschetz theorem on (1,1)-classes)**.** *Let X be a compact Kähler manifold. The image of the first Chern class $c_1 : \mathrm{Pic}(X) \longrightarrow H^2(X, \mathbb{Z})$ is $H^{1,1}(X, \mathbb{Z})$.*

We now consider a compact complex surface S. The cup product is a nondegenerate integral bilinear form on the second integral cohomology group. Thus, via the first Chern map, we can induce an intersection form on $\mathrm{Pic}(S)$ by setting

$$\forall L, M \in \mathrm{Pic}(S) \qquad L \cdot M = c_1(L) \smile c_1(M) \in H^4(S, \mathbb{Z}) \simeq \mathbb{Z}.$$

We will denote by L^2 the number $L \cdot L$.

Theorem 1.2 (Wu's Formula)**.** *Let S be a compact complex surface. For each class $a \in H^2(S, \mathbb{Z})$, we have*

$$a \smile a \equiv a \smile c_1(\omega_S) \pmod 2.$$

More generally, for a compact complex variety X of dimension n, we can define an intersection product of n invertible sheaves on X using the first Chern class. Namely, we set

$$\forall L_1, \ldots, L_n \in \mathrm{Pic}(X) \qquad L_1 \cdot \cdots \cdot L_n = c_1(L_1) \smile \cdots \smile c_1(L_n) \in H^{2n}(X, \mathbb{Z}) \simeq \mathbb{Z}.$$

If S is a surface, the intersection form on $\mathrm{Pic}(S)$ also has a geometric definition (see [Bea96, Chapter 1]). It can be shown that, if C is a smooth curve contained in S and L an invertible sheaf on S, then

$$L \cdot \mathcal{O}_S(C) = \deg(L|_C).$$

Theorem 1.3 (Riemann–Roch Theorem)**.** *Let S be a compact complex surface. For any invertible sheaf L on S, one has*

$$\chi(S,L)=\chi(S,\mathcal{O}_S)+\frac{L^2-L\cdot\omega_S}{2}.$$

Theorem 1.4 (Noether Formula)**.** *For each compact complex surface S,*

$$\chi(S,\mathcal{O}_S)=\frac{\chi_{\mathrm{top}}(S)+\omega_S^2}{2},$$

where $\chi_{\mathrm{top}}(S):=\sum_{i=0}^4(-1)^i b_i(S)$ is the topological Euler characteristic.

If X is a compact Kähler manifold, the first cohomology group $H^1(X,\mathbb{C})$ decomposes as $H^{0,1}(X)\oplus H^{1,0}(X)$, hence the first Betti number $b_1(X)$ is equal to $h^{0,1}(X)+h^{1,0}(X)=2h^{0,1}(X)$. In the case of a general compact complex surface S, the following result holds.

Theorem 1.5 ([BHPV04, IV.Theorem 2.7])**.** *Let S be a compact complex surface. Then*

- $b_1(S)=h^{0,1}(S)+h^{1,0}(S)$;
- *if* $b_1(S)$ *is even, then* $h^{0,1}(S)=h^{1,0}(S)$;
- *if* $b_1(S)$ *is odd, then* $h^{1,0}(S)=h^{0,1}(S)-1$.

Moreover, the first Betti number provides a characterization of compact Kähler surfaces.

Theorem 1.6 ([BHPV04, IV.Theorem 3.1])**.** *A compact complex surface is Kähler if and only if its first Betti number is even.*

1.2 An introduction to lattice theory

A *lattice* Λ is a free $\mathbb{Z}$-module of finite rank with a nondegenerate integral symmetric bilinear form q. The lattice Λ is called *even* if

$$\forall x\in\Lambda \qquad x^2:=q(x)\in 2\mathbb{Z}.$$

For each field K containing $\mathbb{Q}$, we will denote by Λ_K the vector space $\Lambda\otimes_{\mathbb{Z}}K$. The vector space Λ_K is endowed with the K-linear extension of the bilinear form q, which is still nondegenerate.
The signature of Λ is the signature of $\Lambda_{\mathbb{R}}$ and will be denoted by (n_+,n_-).
If n_+ or n_- is zero, the lattice is called *definite*; otherwise, Λ is *indefinite*.

Definition 1.7. Given a lattice Λ, its *dual lattice* is

$$\Lambda^\vee:=\{x\in\Lambda_{\mathbb{Q}}\mid\forall y\in\Lambda\ \ x\cdot y\in\mathbb{Z}\}=\mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z}).$$

Clearly, there is an inclusion $\Lambda\hookrightarrow\Lambda^\vee$. The *discriminant group* of Λ is the quotient

$$A_\Lambda=\Lambda^\vee/\Lambda.$$

It is a finite abelian group. We denote by $\mathrm{disc}(\Lambda)$ the cardinality of A_Λ. The lattice is called *unimodular* if A_Λ is trivial or equivalently if $\Lambda^\vee = \Lambda$.
For each $x \in \Lambda$ nonzero, the *divisibility* of x, denoted by $\mathrm{div}(x)$, is the minimal positive generator of the ideal $x \cdot \Lambda \subset \mathbb{Z}$. Thus, the element

$$x_* = \left[\frac{x}{\mathrm{div}(x)}\right]$$

is an element of A_Λ of order $\mathrm{div}(x)$.

Definition 1.8. The *length* of a lattice Λ, denoted by $\ell(\Lambda)$, is the minimal number of generators of its discriminant group.

The quadratic form q on Λ induces a quadratic form $q_\mathbb{Q}$ on $\Lambda_\mathbb{Q}$. When Λ is even, we obtain a quadratic form q_Λ on the discriminant group A_Λ with values in $\mathbb{Q}/2\mathbb{Z}$, given by the formula

$$\forall x \in \Lambda^\vee \qquad q_\Lambda([x]) \equiv q_\mathbb{Q}(x) \pmod{2\mathbb{Z}}.$$

The quadratic form q_Λ is well defined on the quotient A_Λ. Indeed for all $y \in \Lambda$, the number $q_\mathbb{Q}(x+y) = q_\mathbb{Q}(x) + 2x \cdot y + y^2$ is equal to $q_\mathbb{Q}(x)$ modulo $2\mathbb{Z}$.

Definition 1.9. The *group of isometries* of A_Λ, denoted by $O(A_\Lambda)$, is the group of group automorphisms of A_Λ that preserve q_Λ.

For lattices Λ_1 and Λ_2, the direct sum $\Lambda_1 \oplus \Lambda_2$ will always indicate the orthogonal direct sum. Observe that there exists an isomorphism $A_{\Lambda_1 \oplus \Lambda_2} \simeq A_{\Lambda_1} \times A_{\Lambda_2}$ compatible with the discriminant forms.

An injective morphism of lattices $\Lambda_1 \hookrightarrow \Lambda$ (which respects the quadratic forms) is a *primitive embedding* if the cokernel is torsion free. An element $x \in \Lambda$ is *primitive* if it is not a nontrivial multiple of some element of Λ.

For example, the orthogonal complement $\Lambda_1^\perp$ of a sublattice Λ_1 of Λ is a primitive lattice that intersects Λ_1 trivially. Observe that in general, the inclusion

$$\Lambda_1 \oplus \Lambda_1^\perp \hookrightarrow \Lambda$$

is of finite index but not necessarily surjective.

Remark 1.10. If $\Lambda_1 \hookrightarrow \Lambda$ is an embedding of lattices with Λ_1 unimodular, then $\Lambda = \Lambda_1 \oplus \Lambda_1^\perp$. Indeed, for all $x \in \Lambda$, there exists $x_1 \in \Lambda_1$ such that the linear form $s_x : \Lambda_1 \to \mathbb{Z}$, defined as $s_x(y) = x \cdot y$, is equal to s_{x_1} (because $\Lambda_1 \simeq \Lambda_1^\vee$). Hence, $x - x_1$ is in $\Lambda_1^\perp$ and $x = x + (x - x_1)$.

Theorem 1.11 ([Nik80, Theorem 1.14.4])**.** *Let Λ be a unimodular, even, indefinite lattice of signature (n_+, n_-) and let Λ_1 be an even lattice of signature (m_+, m_-). If $m_\pm < n_\pm$ and $\ell(\Lambda_1) + 2 \leq \mathrm{rk}(\Lambda) - \mathrm{rk}(\Lambda_1)$, there exists a primitive embedding*

$$\Lambda_1 \hookrightarrow \Lambda$$

that is unique up to post composition by isometries of Λ.

As a particular case of Theorem 1.11, we obtain the following corollary.

Corollary 1.12. *Let Λ be a unimodular, even lattice with signature (n_+, n_-), such that $n_\pm > 1$. For each $d \in \mathbb{Z}$, there exists a primitive vector $k \in \Lambda$ with $k^2 = 2d$, and this element is unique up to isometries of Λ.*
In particular, all primitive vectors of fixed degree $2d$ are conjugate by an isometry of Λ.

Example 1.13. To set up some notation, we include a list of standard example of lattices that we will use frequently.

a) We will denote by $\mathbb{Z}$ the lattice of rank 1 with intersection matrix 1. More generally, we will denote by $\mathbb{Z}(n)$ the lattice of rank 1 with intersection matrix n, for all $n \neq 0$. We will write $\mathbb{Z}(n) = \mathbb{Z}k$ if the lattice is generated by the vector k.
The discriminant group of $\mathbb{Z}(n)$ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and is generated by the element $k_* = [k/n]$. If n is even, the quadratic form $q_{A_{\mathbb{Z}(n)}}$ takes the value $1/n$ on k_*.

Lemma 1.14. *Let n be a nonzero integer. Consider the group $G = \mathbb{Z}/2n\mathbb{Z}$ with quadratic from $q : G \to \mathbb{Q}/2\mathbb{Z}$ defined by $q(1) = \dfrac{1}{2n}$. The group of isometries of G with respect to q is*

$$O(G, q) \simeq (\mathbb{Z}/2\mathbb{Z})^{\rho(n)},$$

where $\rho(n)$ is the number of distinct primes dividing n.

Proof. An isometry of G is determined by the image a of the generator 1, where a is invertible modulo $2n$ and satisfies

$$a^2 q(1) = q(a) = q(1) \in \mathbb{Q}/2\mathbb{Z} \iff \frac{a^2}{2n} = \frac{1}{2n} \in \mathbb{Q}/2\mathbb{Z}.$$

Hence $O(G, q) = \{a \in (\mathbb{Z}/2n\mathbb{Z})^\times \mid a^2 \equiv 1 \pmod{4n}\}$. Observe that $O(G, q)$ is an abelian group whose elements are all involutions: in order to prove the theorem, we just have to compute its cardinality.
We write $n = 2^e p_1^{e_1} \cdots p_r^{e_r}$, where $e \geq 0$, $r \geq 0$, $e_i > 0$, and $p_1, \dots, p_r$ are the distinct odd prime factors of n.
The condition $a^2 \equiv 1 \pmod{4n}$ is equivalent to the system

$$\begin{cases} a \equiv \pm 1 & \pmod{p_i^{e_i}} \text{ for all } i \in \{1, \dots, r\}; \\ a \equiv \pm 1 & \pmod{2^{e+1}}, \end{cases}$$

which has 2^r solutions modulo $2n$ if $e = 0$, and 2^{r+1} solutions modulo $2n$ if $e > 0$. Observe that each solution to the system is coprime with $2n$, hence, in both cases, the cardinality of $O(G, q)$ is $2^{\rho(n)}$. □

b) The hyperbolic plane U is the rank-2 lattice with intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Namely, U is generated by two vectors e and f such that $e^2 = f^2 = 0$ and $e \cdot f = 1$. It is a unimodular even lattice.

c) The E_8-*lattice* is the subgroup of $\mathbb{Q}^8$ generated by all vectors of $\mathbb{Z}^8$ whose coordinate sum is even, and by the vector $\frac{1}{2}(1,\ldots,1)$. We can verify that the restriction to E_8 of the canonical quadratic form of $\mathbb{Q}^8$ is even, hence E_8 is an even lattice that is positive definite. Moreover it is the only even, unimodular, positive definite lattice of rank 8.
We will indicate by $E_8(-1)$ the lattice obtained by inverting the sign of the quadratic from on E_8.

The following theorem shows that the last two examples determine all even, unimodular and indefinite lattices.

Theorem 1.15 ([Ser93, Chapter 5]). *Let Λ be an indefinite, even, unimodular lattice of signature (n_+, n_-). The number $\tau := n_+ - n_-$ is a multiple of 8 and*

$$\Lambda \simeq \begin{cases} U^{\oplus n_-} \oplus E_8^{\oplus \frac{\tau}{8}} & \text{if } n_- < n_+; \\ U^{\oplus n_+} \oplus E_8(-1)^{\oplus \frac{-\tau}{8}} & \text{if } n_+ \leq n_-. \end{cases}$$

Remark 1.16. Let Λ be a unimodular indefinite lattice and let $h \in \Lambda$ be a vector of square $2d$. We show that

$$h^\perp = \mathbb{Z}(-2d) \oplus M,$$

where M is unimodular.
Indeed, the lattice Λ is of the form $U \oplus \Lambda'$ and, because of Corollary 1.12, we can assume $h = e + df$, where $\{e, f\}$ is a basis of U as in Example 1.13.b). Therefore, using $h_U^\perp = \mathbb{Z}(e - df) \subset U$, we obtain the description of $h^\perp$.

Observe that each isometry of Λ induces an isometry of the discriminant group A_Λ. Indeed for each $f \in O(\Lambda)$, the morphism

$$\begin{aligned} \bar{f} : A_\Lambda &\longrightarrow A_\Lambda \\ [x] &\longmapsto [f_{\mathbb{Q}}(x)] \end{aligned}$$

is a well defined isometry of A_Λ. We obtain a morphism

$$O(\Lambda) \longrightarrow O(A_\Lambda). \tag{3}$$

We denote by $\widetilde{O}(\Lambda)$ the kernel of this morphism and we call it the *stable orthogonal group*. We also define the group

$$\widehat{O}(\Lambda) = \{f \in O(\Lambda) \mid \bar{f} = \pm \mathrm{id} \in O(A_\Lambda)\}.$$

Clearly, $\widetilde{O}(\Lambda)$ is a subgroup of $\widehat{O}(\Lambda)$ of index at most 2. Both $\widetilde{O}(\Lambda)$ and $\widehat{O}(\Lambda)$ are normal subgroups of $O(\Lambda)$.

Theorem 1.17 ([Nik80, Theorem 1.14.2]). *Let Λ be an even indefinite lattice with $\ell(\Lambda) + 2 \leq \mathrm{rk}\Lambda$. Then the morphism*

$$r : O(\Lambda) \longrightarrow O(A_\Lambda)$$

is surjective.

If Λ satisfies the hypotheses of the theorem, it follows that

$$O(A_\Lambda) \simeq O(\Lambda)/\widetilde{O}(\Lambda).$$

The following result, proved in [Eic74, Satz 10.4], generalizes Corollary 1.12.

Lemma 1.18 (Eichler)**.** *Let Λ be a even lattice containing the direct sum of two hyperbolic planes. The $\widetilde{O}(\Lambda)$-orbit of a primitive vector h is uniquely determined by the integer h^2 and the element $h_* = [h/\operatorname{div}(h)]$ of A_Λ.*

1.2.1 Extension of isometries of a sublattice

Let M be a primitive sublattice of an even lattice L. We want to characterize isometries of $M^\perp$ that extend to isometries of L, following [Nik80, Section 1.5].

We define

$$O(L, M) = \{f \in O(L) \mid f|_M = \mathrm{id}\},$$

the group of isometries of L that are the identity on M. Analogously, we define the groups $\widetilde{O}(L, M) = \widetilde{O}(L) \cap O(L, M)$ and $\widehat{O}(L, M) = \widehat{O}(L) \cap O(L, M)$.

Clearly, each isometry in $O(L, M)$ restricts to an isometry of $M^\perp$; namely, we have a morphism

$$\rho : O(L, M) \longrightarrow O(M^\perp).$$

We say that an isometry $f \in O(M^\perp)$ extends to an isometry of $O(L, M)$ if it is in the image of the restriction ρ.

Consider the chain of sublattices

$$M \oplus M^\perp < L < L^\vee < M^\vee \oplus (M^\perp)^\vee, \tag{4}$$

from which we obtain the subgroup

$$H = L/(M \oplus M^\perp) < (M^\vee \oplus (M^\perp)^\vee)/(M \oplus M^\perp) = A_M \times A_{M^\perp}.$$

Moreover, we consider the projections

$$p : H \hookrightarrow A_M \times A_{M^\perp} \twoheadrightarrow A_{M^\perp} \qquad \text{and} \qquad q : H \hookrightarrow A_M \times A_{M^\perp} \twoheadrightarrow A_M$$

Since M is primitive in L, the morphism p is injective. Indeed, each $l \in L$ can be written as $l = qm + rm'$ with $q, r \in \mathbb{Q}$ and m and m' vectors in M and $M^\perp$ respectively. Since $l \cdot L \subset \mathbb{Z}$, we obtain that qm is an element of $M^\vee$ and rm' is an element of $(M^\perp)^\vee$. Hence,

$$p([l]) = [rm'] = 0 \in A_{M^\perp} \qquad \text{implies} \qquad rm' \in M^\perp.$$

Therefore, the vector $l - rm' = qm$ is in L. Since M is primitive, this implies $qm \in M$, and therefore $l \in M \oplus M^\perp$.

Analogously, we show that the morphism q is injective.

By computing the indices from the chain (4), we obtain

$$\operatorname{disc}(M)\operatorname{disc}(M^\perp) = |H|^2 \operatorname{disc}(L). \tag{5}$$

Moreover, the injectivity of p and q implies $|H| \leq \mathrm{disc}(M^\perp)$ and $|H| \leq \mathrm{disc}(M)$. Hence, if the lattice L is unimodular, then $|H| = \mathrm{disc}(M) = \mathrm{disc}(M^\perp)$, and the morphisms p and q are isomorphisms.

Proposition 1.19 ([Nik80, Corollary 1.5.2])**.** *An isometry $g \in O(M^\perp)$ extends to $O(L, M)$ if and only if $\bar{g}|_{p(H)} = \mathrm{id}$.*

Proof. We prove the proposition in the case of $M = \mathbb{Z}h$ for some primitive vector h of L. In this case, we will denote by $O(L, h)$ the group $O(L, \mathbb{Z}h)$.

Each isometry $g \in O(h^\perp)$ extends uniquely to an isometry $\tilde{g} \in O(L_\mathbb{Q}, h)$, defined by $\tilde{g}(h) = h$ and $\tilde{g}|_{h^\perp_\mathbb{Q}} = g$. The isometry g extends to $O(L, h)$ if and only if $\tilde{g}|_L \in O(L, h)$, namely if and only if $\tilde{g}(l) \in L$ for all $l \in L$.

Observe that each vector $l \in L$ can be written as $l = qh + rv$, with $v \in h^\perp$ and $r, q \in \mathbb{Q}$. Therefore, $\tilde{g}(l) = qh + rg(v)$.

Since $l \cdot h^\perp \subset \mathbb{Z}$, we obtain that $b = r\mathrm{div}(v)$ is an integer. Notice moreover that $\mathrm{div}(g(v)) = \mathrm{div}(v)$ because g in an isometry of $h^\perp$. Hence, we obtain

$$p([l]) = [rv] = b\left[\frac{v}{\mathrm{div}(v)}\right] \in A_{h^\perp} \qquad \text{and} \qquad \bar{g}(p([l])) = [rg(v)] = b\left[\frac{g(v)}{\mathrm{div}(g(v))}\right] \in A_{h^\perp}.$$

Observe that $\bar{g}(p([l])) = p([l])$ if and only if $r(g(v) - v) \in h^\perp$, which is equivalent to

$$\tilde{g}(l) - l \in h^\perp = h^\perp_\mathbb{Q} \cap L. \tag{6}$$

Since $l \in L$ and $\tilde{g}(l) - l \in h^\perp_\mathbb{Q}$, equation (6) is equivalent to $\tilde{g}(l) \in L$. ☐

Therefore,

$$O(L, h) = \{g \in O(h^\perp) \mid \bar{g}|_{p(H)} = \mathrm{id}\}.$$

If L is unimodular, we showed earlier that $p(H) = A_{h^\perp}$.

Corollary 1.20. *If L is a unimodular lattice and $h \in L$ is a primitive vector, then*

$$O(L, h) = \widetilde{O}(h^\perp).$$

For a general lattice L, from the chain (4), it follows that

$$A_L = L^\vee/L \simeq \left(L^\vee/(\mathbb{Z}h \oplus h^\perp)\right) \Big/ \left(L/(\mathbb{Z}h \oplus h^\perp)\right)$$

Therefore A_L is the quotient of $A_{\mathbb{Z}h} \oplus A_{h^\perp}$ by the subgroup H.

Proposition 1.21. *For each lattice L and each primitive vector $h \in L$, there is an inclusion $\widetilde{O}(h^\perp) \hookrightarrow \widetilde{O}(L, h)$.*

Proof. Since $p(H) < A_{h^\perp}$, Proposition 1.19 implies that each isometry $g \in \widetilde{O}(h^\perp)$ extends to an isometry of $O(L, h)$, which we will still denote by g. By definition of $\widetilde{O}(h^\perp)$, the isometry g satisfies $\bar{g}|_{A_{h^\perp}} = \mathrm{id}$. Moreover $g|_{\mathbb{Z}h} = \mathrm{id}$, hence $\bar{g}$ is the identity on $A_{\mathbb{Z}h} \oplus A_{h^\perp}$, and therefore on A_L.

Finally, the morphism $\widetilde{O}(h^\perp) \to \widetilde{O}(L, h)$ is injective because restriction has a left inverse. ☐

We have the following chain of inclusions

$$\widetilde{O}(h^\perp) \xhookrightarrow{i_1} \widetilde{O}(L, h) \xhookrightarrow{i_2} \widehat{O}(L, h) \xhookrightarrow{i_3} O(L, h) \xhookrightarrow{i_4} O(h^\perp), \tag{7}$$

where the index of i_2 divides 2 and the inclusions i_3 and $i_3 i_2$ define normal subgroups of $O(L, h)$.

2 K3 Surfaces

2.1 Definition and first properties

Definition 2.1. A *complex K3 surface* is a compact complex surface S such that the canonical bundle $\omega_S = \Omega_S^2$ is trivial and $H^1(S, \mathcal{O}_S) = 0$.

The triviality of Ω_S^2 implies the existence of a nowhere vanishing holomorphic 2-form. It is in particular nowhere degenerate as an alternating form on the tangent space. Such a 2-form is called a *symplectic form.*

Observe that, if S is a smooth complex projective algebraic variety of dimension 2 such that $\omega_S = \Omega_{S/k}^2 \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$, the GAGA principle implies that $S^{\mathrm{an}} = S(\mathbb{C})$ is a complex K3 surface.

Example 2.2. The following are some examples of projective K3 surfaces.

a) *Complete intersections.* We want to identify smooth complete intersections S of multidegree $2 \leq d_1 \leq \cdots \leq d_n$ in $\mathbb{P}_{\mathbb{C}}^{n+2}$ that are K3 surfaces. Using the exact sequence that defines the complete intersection, it can be shown that $H^1(S, \mathcal{O}_S) = 0$. Moreover, the adjunction formula implies

$$\omega_S = \omega_{\mathbb{P}_{\mathbb{C}}^{n+2}} \otimes \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{n+2}}(d_1 + \cdots + d_n)|_S = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{n+2}}(d_1 + \cdots + d_n - n - 3)|_S.$$

Hence S is a K3 surface if and only if $\sum d_i = n + 3$. This gives only 3 cases: a smooth quartic in $\mathbb{P}_{\mathbb{C}}^3$, a smooth complete intersection of a quadric and a cubic in $\mathbb{P}_{\mathbb{C}}^4$, and a smooth complete intersection of three quadrics in $\mathbb{P}_{\mathbb{C}}^5$.

b) A double covering $\pi : S \to \mathbb{P}_{\mathbb{C}}^2$ branched along a smooth plane sextic curve C is a K3 surface. From the equation $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(C) = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(3)^{\otimes 2}$, we get $\pi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2} \oplus \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(-3)$. Hence $H^1(S, \mathcal{O}_S) = 0$, as π is a finite morphism and $\pi_* \mathcal{O}_S$ has zero first cohomology group. Moreover, the canonical bundle formula for branched covers shows that

$$\omega_S = \pi^*(\omega_{\mathbb{P}_{\mathbb{C}}^2} \otimes \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(-3)) = \mathcal{O}_S.$$

Example 2.3. Let A be a complex torus of dimension 2. The surface A has a natural involution $\iota : x \mapsto -x$ whose set of fixed points has cardinality 16. Let $\tilde{A} \to A$ be the blow up of A in these points. As the blown up points are ι-invariant, ι extends to an involution $\tilde{\iota}$ of $\tilde{A}$. The quotient $S = \tilde{A}/\tilde{\iota}$ is a complex K3 surface [Bea96, Proposition VIII.11], called the "Kummer surface" associated with A. Moreover, S is projective if and only if A is projective, hence we obtain many K3 surfaces that are not projective.

Let S be a K3 surface. By definition, one has $h^0(S, \mathcal{O}_S) = 1$ and $h^1(S, \mathcal{O}_S) = 0$, and Serre duality, implies $h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) = 1$. Hence $\chi(S, \mathcal{O}_S) = 2$.
The Riemann–Roch Theorem 1.3 for K3 surfaces becomes

$$\chi(S, L) = 2 + \frac{L^2}{2}, \tag{8}$$

where $\chi(L) = h^0(S, L) - h^1(S, L) + h^0(S, L^{-1})$ by Serre duality.

The condition $H^1(S, \mathcal{O}_S) = 0$ implies, using Theorem 1.5 and Theorem 1.6, that each K3 surface is in particular Kähler and $H^1(S, \mathbb{Z}) = 0$. From the Noether formula (Theorem 1.4), we can compute

$$2 + b_2(S) = \chi_{\text{top}}(S) = 12\chi(S, \mathcal{O}_S) - c_1^2(S) = 24. \tag{9}$$

Remark 2.4. The Picard group of a K3 surface S is torsion free.
Indeed, first observe that, if L is a nontrivial invertible sheaf that is a torsion element in $\text{Pic}(S)$, then $L^2 = 0$. From the Riemann–Roch formula (8) and Serre duality it follows that

$$2 = \chi(S, L) = h^0(S, L) - h^1(S, L) + h^0(S, L^{-1}) \leq h^0(S, L) + h^0(S, L^{-1}).$$

Hence L or L^{-1} is effective, and in particular has a nonzero global section s. For each nonzero n, the section $s^{\otimes n}$ is a global section of $L^{\otimes \pm n}$ that has the same zero set as s. Therefore, if $L^{\otimes n}$ is trivial, L is also trivial.

For a K3 surface S, the long exact sequence (2) becomes

$$0 = H^1(S, \mathcal{O}_S) \longrightarrow \text{Pic}(S) \longrightarrow H^2(S, \mathbb{Z}) \longrightarrow H^2(S, \mathcal{O}_S).$$

Since $\text{Pic}(S)$ and $H^2(S, \mathcal{O}_S)$ are torsion free, we obtain that the group $H^2(S, \mathbb{Z})$ is also torsion free. Hence $(H^2(S, \mathbb{Z}), \smile)$ is a lattice of dimension 22.

Poincaré Duality implies that the cup product is unimodular, and Theorem 1.2 shows that it is even. Moreover, from the Hodge Index Theorem, we obtain that $H^{1,1}(S)$ has signature $(1, h^{1,1}(S) - 1)$, where $h^{1,1}(S) = 22 - 2h^{2,0}(S) = 20$. In particular, if $H^2(S, \mathbb{C})$ has signature (n_+, n_-), we have $n_+ \leq 3$. Hence, Theorem 1.15 implies that the lattice $(H^2(S, \mathbb{Z}), \smile)$ has signature $(3, 19)$ and is isomorphic to

$$\Lambda_{\text{K3}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

As $H^1(S, \mathcal{O}_S) = 0$, the first Chern map is injective and so $\rho(S)$, the Picard number of S (the rank of $\text{Pic}(S) \otimes \mathbb{R}$) is less than or equal to $b_2(S) - 2h^{0,1}(S) = 20$.
If S is projective, the signature of the intersection form on $\text{Pic}(S)$ is $(1, \rho(S) - 1)$.

Since each K3 surface S is Kähler, there is a Hodge decomposition

$$H^2(S, \mathbb{C}) = H^{0,2}(S) \oplus H^{1,1}(S) \oplus H^{2,0}(S)$$

such that $H^{0,2}(S) \simeq \overline{H^{2,0}(S)}$, $H^{1,1}(S) \perp (H^{2,0}(S) \oplus H^{0,2}(S))$, and $H^{2,0}(S)$ is of dimension one. Therefore the Hodge structure of S is uniquely determined by $H^{2,0}(S) \subset H^2(S, \mathbb{C})$.

Theorem 2.5 (Torelli)**.** *Two K3 surfaces S and S' are isomorphic if and only if there exists an isometry*

$$\varphi \colon H^2(S, \mathbb{Z}) \longrightarrow H^2(S', \mathbb{Z})$$

such that $\varphi_{\mathbb{C}}(H^{2,0}(S)) = H^{2,0}(S')$.
Moreover, there exists an isomorphism $u \colon S' \to S$ such that $u^* = \varphi$ if and only if

$$\varphi_{\mathbb{C}}(\text{Kah}(S)) \cap \text{Kah}(S') \neq \varnothing,$$

where $\text{Kah}(S) \subset H^{1,1}(S, \mathbb{R})$ *is the cone of Kähler classes on S.*

2.2 Linear systems on K3 surfaces and polarized K3 surfaces

Let X be a projective variety. For each invertible sheaf L on X, we will denote by $|L|$ the associated complete linear system, namely $|L| := \mathbb{P}(H^0(X, L))$, or equivalently, if $L = \mathcal{O}_X(D)$,

$$|L| = \{E \in \mathrm{Div}(X) \mid E \sim D \text{ and } E \text{ is effective}\}.$$

The invertible sheaf L induces a rational map

$$\varphi_L : X \dashrightarrow \mathbb{P}(H^0(X, L)^\vee)$$

that is regular on the complement of the base locus $\mathrm{Bs}(L)$ of L.

Definition 2.6. A sheaf L on X is called *very ample* if it induces a closed embedding $X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, and it is called *ample* if there exists $k \in \mathbb{N}$ such that $L^{\otimes k}$ is very ample.

There is a numerical characterization of ampleness ([Laz04, Theorem 1.2.23]).

Theorem 2.7 (Nakai–Moishezon Criterion)**.** *An invertible sheaf L over a proper scheme X is ample if and only if*

$$(L|_Y)^{\dim(Y)} > 0$$

for every irreducible subvariety $Y \subset X$ of positive dimension.

Definition 2.8. An invertible sheaf L on X is called *nef* if $c_1(L|_C) = \deg(L|_C) \geq 0$ for every irreducible curve $C \subset X$.

Theorem 2.9 (Kleiman)**.** *An invertible sheaf L over a projective scheme X is nef if and only if*

$$(L|_Y)^{\dim(Y)} \geq 0$$

for every irreducible subvariety $Y \subset X$ of positive dimension.

We now consider a projective surface S. An invertible sheaf L on S is ample if and only if

$$L^2 > 0 \qquad \text{and} \qquad L \cdot C = \deg(L|_C) > 0 \qquad \text{for all curves } C \subset S,$$

and is nef if and only if

$$L \cdot C = \deg(L|_C) \geq 0 \qquad \text{for all curves } C \subset S.$$

Kleiman's Theorem implies that for each nef invertible sheaf L, we have $L^2 \geq 0$.

Theorem 2.10 (Kodaira vanishing)**.** *Each ample invertible sheaf L on a smooth projective variety X satisfies*

$$\forall i < \dim(X) \qquad H^i(X, L^\vee) = 0.$$

We now restrict ourselves to the case of K3 surfaces. The Riemann–Roch formula (8) and Kodaira vanishing imply that each ample invertible sheaf L on a K3 surface S has a space of global sections of dimension $h^0(S, L) = \chi(S, L) = 2 + \frac{L^2}{2}$.

A main result in the study of projective K3 surfaces is the following.

Theorem 2.11 ([May72, 4.Corollary 6]). *Let L be an ample invertible sheaf on a projective K3 surface. The sheaf $L^{\otimes k}$ is globally generated for all $k \geq 2$ and is very ample for all $k \geq 3$.*

As a consequence, we obtain that if a K3 surface S has an ample invertible sheaf of degree $L^2 = 2d > 0$, then it has an embedding

$$\varphi_{L^{\otimes 3}} : S \hookrightarrow \mathbb{P}(H^0(S, L^{\otimes 3})^\vee) = \mathbb{P}^{9d+1}_{\mathbb{C}}.$$

An invertible sheaf L on a K3 surface S is said to be primitive if its class $c_1(L)$ is a primitive element of the lattice $H^2(S, \mathbb{Z})$. This is equivalent to say that the sheaf L is primitive in $\mathrm{Pic}(S)$.

Definition 2.12. A *polarized K3 surface of degree* $2d$ is a projective K3 surface S together with a primitive ample invertible sheaf L of square $L^2 = 2d$.
A *pseudo-polarized K3 surface of degree* $2d$ is a projective K3 surface S together with a primitive invertible nef sheaf L of square $L^2 = 2d > 0$.

Theorem 2.13. *For each $d > 0$, there exists a polarized K3 surface of degree $2d$.*

Proof. We will only treat the case $d = 3k - 1$; for the other cases, see [Bea96, Proposition VIII.15].
Let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be a quartic K3 surface that contains a line ℓ, for example the Fermat quartic (1), and consider H the hyperplane section: we have that $H^2 = 4$, $H \cdot \ell = 1$ and $\ell^2 = -2$. Hence $H - \ell$ defines a rational map $\varphi : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ which is the projection from the line ℓ. The morphism φ is defined everywhere on S: indeed, it extends to a rational morphism on $\mathbb{P}^3_{\mathbb{C}}$ which is everywhere defined on $\mathrm{Bl}_\ell(\mathbb{P}^3_{\mathbb{C}})$. Hence, it is defined on the strict transform $\widehat{S}$ of S under $\mathrm{Bl}_\ell(\mathbb{P}^3_{\mathbb{C}}) \to \mathbb{P}^3_{\mathbb{C}}$. Since $\widehat{S}$ is the blow up of S with respect to ℓ and ℓ is a smooth divisor of S, we obtain that $\widehat{S}$ is isomorphic to S. Therefore, the sheaf $E = H - \ell$ is globally generated. This implies that the sheaf $L_k = H + (k-1)E$ is very ample. We notice that $L_k^2 = 4 + 6(k-1) = 2d$ and that L_k is also primitive. Indeed $H \cdot L = 1$ hence $L_k \cdot H = 4 + 3(k-1)$ and $L_k \cdot \ell = 1 + 3(k-1)$, with $(4 + 3(k-1), 1 + 3(k-1)) = 1$. □

2.3 Universal local deformation

A *local deformation* of a compact complex manifold X is a smooth and proper morphism $f : \mathscr{X} \to B$, where B is the germ of an analytic space with a distinguished point $0 \in B$ such that the fiber $\mathscr{X}_0$ is isomorphic to X.

Theorem 2.14 (Kuranishi). *Let X be a smooth compact and connected variety with $H^0(X, T_X) = 0$. There exists a pointed analytic space $(B_{\mathrm{univ}}, 0)$ and a deformation $f_{\mathrm{univ}} : \mathscr{X}_{\mathrm{univ}} \to B_{\mathrm{univ}}$ with $\mathscr{X}_0 \simeq X$ that satisfies the following* universal property*: for each local deformation $f : \mathscr{X} \to B$ there exists a unique morphism $B \to B_{\mathrm{univ}}$ such that f is the pullback of f_{univ} via this morphism:*

$$\begin{array}{ccc} \mathscr{X} \simeq \mathscr{X}_{\mathrm{univ}} \times_{B_{\mathrm{univ}}} B & \longrightarrow & \mathscr{X}_{\mathrm{univ}} \\ \downarrow f & & \downarrow f_{\mathrm{univ}} \\ B & \longrightarrow & B_{\mathrm{univ}}. \end{array}$$

The morphism f_{univ} is called the universal local deformation *of X.*
Moreover, the Zariski tangent space of B_{univ} at 0 is isomorphic to $H^1(X,T_X)$ via the Kodaira–Spencer application $T_{B_{\mathrm{univ}},0} \to H^1(X,T_X)$ and B_{univ} is defined around 0 by $h^2(X,T_X)$ equations in a smooth analytic space of dimension $h^1(X,T_X)$.

Let S be a K3 surface. Each nowhere vanishing 2-form on S induces an isomorphism $\Omega_S \simeq T_S$, hence,

$$H^0(S,T_S) \simeq H^0(S,\Omega_S) \simeq H^2(S,T_S)$$

where the last isomorphism follows from Serre duality. Since $H^0(S,\Omega_S) \subset H^1(S,\mathbb{C}) = 0$, all three terms are 0. Moreover, from (9), we have

$$h^1(S,T_S) = h^1(S,\Omega_S^1) = b_2(S) - 2 = 20.$$

Hence we obtain the following corollary.

Corollary 2.15. *Let S be a K3 surface. There exists a universal local deformation*

$$f_{\mathrm{univ}} : \mathscr{S}_{\mathrm{univ}} \longrightarrow B_{\mathrm{univ}}$$

where $(B_{\mathrm{univ}},0)$ is a germ of a smooth analytic space of dimension 20.

We say that a variety X' is a smooth deformation of X if there exists a smooth proper morphism $f : \mathscr{X} \to B$ and points $b, b' \in B$ such that $\mathscr{X}_b \simeq X$ and $\mathscr{X}_{b'} \simeq X'$, where B is smooth and connected.

Lemma 2.16. *All smooth deformations of a K3 surface are K3 surfaces.*

Proof. Let $f : \mathscr{S} \to B$ be a deformation of a K3 surface $S \simeq \mathscr{S}_0$, where 0 is a distinguished point of B. We want to show that for all $b \in B$, the fiber $\mathscr{S}_b$ is a K3 surface.
From Ehresmann's Lemma [Huy05, Proposition 6.2.2], it follows that the Betti numbers are stable under deformation, hence $b_1(\mathscr{S}_b) = 0$. This implies in particular $H^1(\mathscr{S}_b,\mathcal{O}_{\mathscr{S}_b}) = 0$. Moreover, also the topological Euler characteristic is preserved, hence $\chi_{\mathrm{top}}(\mathscr{S}_b) = 24$.

In order to show that $\mathscr{S}_b$ is K3, we only need to prove that $\omega_{\mathscr{S}_b}$ is trivial. From [Tos18, Lemma 5.6] we obtain $\omega_{\mathscr{S}_b} = \omega_{\mathscr{S}}|_{\mathscr{S}_b}$. The number $\omega^2_{\mathscr{S}_b}$ is an integer that depends continuously on b, therefore it is always 0. Using the Noether formula (Theorem 1.4), we obtain

$$h^0(\mathscr{S}_b,\mathcal{O}_{\mathscr{S}_b}) + h^2(\mathscr{S}_b,\mathcal{O}_{\mathscr{S}_b}) \geq \chi(\mathscr{S}_b,\mathcal{O}_{\mathscr{S}_b}) = \frac{\chi_{\mathrm{top}}(\mathscr{S}_b) + \omega^2_{\mathscr{S}_b}}{12} = 2.$$

Hence, we have $h^2(\mathscr{S}_b,\mathcal{O}_{\mathscr{S}_b}) \geq 2 - h^0(\mathscr{S}_b,\mathcal{O}_{\mathscr{S}_b}) = 1$. By Serre duality, we obtain that $\omega_{\mathscr{S}_b}$ has a nonzero global section: the triviality of $\omega_{\mathscr{S}_b}$ is therefore equivalent to showing that

$$h^0(\mathscr{S}_b,\omega^\vee_{\mathscr{S}_b}) \geq 1. \tag{10}$$

We show that this is an open and closed condition on B: since the property is verified for 0 and B is connected, this concludes the proof.
From [Har77, III.Theorem 12.8], for each i, the function

$$b \longmapsto h^i(\mathscr{S}_b,\omega^{\otimes j}_{\mathscr{S}_b})$$

is uppersemicontinuous on B for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. Hence, the condition (10) is a closed condition. Moreover, we have

$$2 = \chi(\mathscr{S}_b,\mathcal{O}_{\mathscr{S}_b}) + (\omega^\vee_{\mathscr{S}_b})^2 \overset{R.R}{=} \chi(\mathscr{S}_b,\omega^\vee_{\mathscr{S}_b}) \leq h^0(\mathscr{S}_b,\omega^\vee_{\mathscr{S}_b}) + h^2(\mathscr{S}_b,\omega^\vee_{\mathscr{S}_b}).$$

The condition $h^2(\mathscr{S}_b,\omega^\vee_{\mathscr{S}_b}) \leq 1$ is open and implies condition (10). Hence the set of points $b \in B$ that satisfy condition (10) is also open. □

2.4 Moduli space of polarized K3 surfaces

Definition 2.17. A *family of polarized K3 surfaces of degree* $2d$ is a pair $(f : \mathscr{S} \to B, \mathcal{L})$ where $f : \mathscr{S} \longrightarrow B$ is a smooth and projective morphism of $\mathbb{C}$-schemes, and $\mathcal{L}$ is an invertible sheaf on $\mathscr{S}$ with the property that, for all $b \in B$,

$$(\mathscr{S}_b = \mathscr{S} \times_B b, \mathcal{L}|_{\mathscr{S}_b})$$

is a polarized K3 surface of degree $2d$.

Two such families $(f : \mathscr{S} \to B, \mathcal{L})$ and $(f' : \mathscr{S}' \to B, \mathcal{L}')$ are isomorphic if there exists a B-isomorphism $u : \mathscr{S} \to \mathscr{S}'$ and an invertible sheaf $L \in \mathrm{Pic}(B)$, such that $u^*\mathcal{L}' = \mathcal{L} \otimes f^*L$.

We fix an integer d and consider the *moduli functor*

$$\mathcal{M}_{2d} : (\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$$

such that, for each $\mathbb{C}$-scheme B,

$$\mathcal{M}_{2d}(B) = \{(f : \mathscr{S} \to B, \mathcal{L}) \text{ family of polarized K3 surfaces of degree } 2d\ \}/\text{isom.}$$

and, for each $\mathbb{C}$-morphism $g : B' \to B$,

$$\begin{aligned} \mathcal{M}_{2d}(g) : \mathcal{M}_{2d}(B) &\longrightarrow \mathcal{M}_{2d}(B') \\ [(f : \mathscr{S} \longrightarrow B, \mathcal{L})] &\longmapsto [(f_{B'} : \mathscr{S} \times_B B' \longrightarrow B', g^*_{\mathscr{S}}\mathcal{L})] \end{aligned}$$

where $g_{\mathscr{S}} : B' \times_B \mathscr{S} \longrightarrow \mathscr{S}$ is the pullback of g via f.

The presence of nontrivial automorphisms on some polarized K3 surfaces prevents the existence of a fine moduli space [Deb19, Remarque 6.11], hence we can only hope for a *coarse moduli space*, namely a $\mathbb{C}$-scheme M_{2d} with a natural transformation $\eta : \mathcal{M}_{2d} \longrightarrow h_{M_{2d}}$ (where $h_{M_{2d}}(S) = \mathrm{Hom}\,(S, M_{2d})$ for all $\mathbb{C}$-schemes S), such that

a) the morphism
$$\eta_{\mathbb{C}} : \mathcal{M}_{2d}(\mathbb{C}) \xrightarrow{\sim} h_{M_{2d}}(\mathbb{C})$$
is an isomorphism between the set of closed points of M_d and the set of classes of polarized K3 surfaces of degree $2d$ modulo isomorphism;

b) for each $\mathbb{C}$-scheme N and natural transformation $\Phi : \mathcal{M}_{2d} \to h_N$, there exists a unique $\mathbb{C}$-morphism $\pi : M_{2d} \to N$ such that $\eta = h_\pi \circ \Phi$, where $h_\pi : h_N \to h_M$ is the morphism induced by π.

The following theorem says that such a scheme exists and it is actually a quasi-projective variety.

Theorem 2.18. *The moduli functor $\mathcal{M}_{2d}$ admits a coarse moduli space M_{2d} that is an irreducible quasi-projective variety of dimension 19.*

We present the idea of a construction of the moduli space, following the path proposed by Viehweg [Vie90].

Let (S, L) be a polarized K3 surface of degree $2d$. From Theorem 2.11, there exists a closed immersion

$$\varphi_{L^{\otimes 3}} : S \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$$

where $N = 9d+1$ and such that $\varphi^*_{L^{\otimes 3}}(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(1)) = L^{\otimes 3}$. Moreover, from the Riemann–Roch formula (8), we obtain that the Hilbert Polynomial of $\varphi_{L^{\otimes 3}}(S)$ is

$$P(t) = \chi(S, L^{\otimes 3t}) = 9dt^2 + 2.$$

Hence we shall consider the Hilbert scheme $\mathrm{Hilb} = \mathrm{Hilb}^{P(t)}_{\mathbb{P}^N_{\mathbb{C}}}$ of all closed subschemes of $\mathbb{P}^N_{\mathbb{C}}$ with Hilbert polynomial P. It is a projective scheme that represents the Hilbert functor

$$\underline{\mathrm{Hilb}} : (\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$$

that associates with each $\mathbb{C}$-scheme B the set of subschemes $Z \subseteq \mathbb{P}^N_B$ flat over B such that all closed fibers $Z_b \subset \mathbb{P}^N_{\mathbb{C}}$ have Hilbert polynomial P. In particular, there exists a universal family $\mathcal{Z} \subset \mathbb{P}^N_{\mathrm{Hilb}}$, flat over Hilb, whose closed fibers have Hilbert polynomial P, see [EH00].

Proposition 2.19 ([Huy16, 5.Proposition 2.1])**.** *Let $\underline{H} : (\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \to (\mathrm{Sets})$ be the functor that maps a $\mathbb{C}$-scheme B to the set of B-flat closed subschemes $Z \subset \mathbb{P}^N_B$ whose closed fibers have Hilbert polynomial $P(t)$ such that*

a) *there exist $\mathcal{L} \in \mathrm{Pic}(Z)$, $L_0 \in \mathrm{Pic}(B)$ such that*

$$p^*(\mathcal{O}(1)) \simeq \mathcal{L}^{\otimes 3} \otimes f^*(L_0)$$

where $p : Z \to \mathbb{P}^N_{\mathbb{C}}$ and $f : Z \to B$ are the natural projections,

b) *for each fiber Z_b of $f : Z \to B$, restriction induces an isomorphism*

$$H^0(\mathbb{P}^N_{k(b)}, \mathcal{O}(1)) \xrightarrow{\sim} H^0(Z_b, L_b^{\otimes 3});$$

c) *$(f : Z \to B, \mathcal{L})$ is a family of polarized K3 surfaces (of degree $2d$).*

The functor $\underline{H}$ is represented by an open subscheme $H \subset \mathrm{Hilb}$, with universal family

$$\mathcal{Z}_H := \mathcal{Z} \times_{\mathrm{Hilb}} H \longrightarrow H.$$

Hence, mapping $Z \in \underline{H}(B)$ to $[(f : Z \to B, \mathcal{L})]$, where $\mathcal{L}$ is defined as in a), defines a functor

$$\underline{H} \longrightarrow \mathcal{M}_{2d}.$$

It can be proven that the sheaf $\mathcal{L}$ is uniquely determined up to tensoring with the pullback of an invertible sheaf of B, hence the morphism is well defined.

The Hilbert scheme comes with a natural $\mathrm{PGL} = \mathrm{PGL}(N+1)$-action that is defined functorially as the functor transformation given by

$$\forall \varphi \in \underline{\mathrm{PGL}}(B) \quad \forall Z \in \underline{\mathrm{Hilb}}(B) \qquad \varphi \cdot Z \longmapsto \varphi_{\mathbb{P}^N_B}(Z),$$

where $\varphi_{\mathbb{P}^N_B} : \mathbb{P}^N_B \to \mathbb{P}^N_B$ is the B-morphism induced by viewing φ as a family of automorphisms of $\mathbb{P}^N_{\mathbb{C}}$ that vary over B. As the conditions that define H are stable under the PGL-action, we obtain an action on H. Moreover, we observe that the functor $\underline{H} \to \mathcal{M}_{2d}$ is equivariant, hence we obtain a functor

$$\theta : \underline{H}/\underline{PGL} \longrightarrow \mathcal{M}_{2d}.$$

It can be shown [Huy16, 5.Theorem 2.2] that θ is injective and locally surjective and that if there exists a categorical quotient Q of H/PGL, then it is a coarse moduli space for $\mathcal{M}_{2d}$, [Huy16, 5.Theorem 2.3].

Viehweg's work shows that there exists a categorical quotient Q which is a quasi-projective scheme, completing the proof of the existence of a quasi-projective coarse moduli space for polarized K3 surfaces.
We will show another construction of the moduli space for polarized K3 surfaces in Section 3.3.

3 Periods

3.1 Local period map

Let $f : \mathscr{X} \to (B, 0)$ be a local deformation of a smooth compact connected Kähler manifold X with smooth base B, with $\mathscr{X}_0 \simeq X$, and such that $\mathscr{X}_b$ is Kähler for each $b \in B$. The higher direct image sheaf $R^2 f_* \mathbb{Z}$ is a locally constant sheaf whose fiber at $b \in B$ is

$$(R^2 f_* \mathbb{Z})_b = H^2(\mathscr{X}_b, \mathbb{Z}).$$

By choosing a path γ in B joining 0 and b, we can identify $H^2(\mathscr{X}_b, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$, and the identification only depends on the homotopy class of the path. Therefore, we can define the *monodromy morphism*

$$\rho : \pi_1(B, 0) \longrightarrow \mathrm{Aut}(H^2(X, \mathbb{Z})).$$

In particular, if B is simply connected, then $R^2 f_* \mathbb{Z}$ is canonically isomorphic to the constant sheaf $H^2(X, \mathbb{Z})$.

We observe that $R^2 f_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{C} \simeq R^2 f_* \mathbb{C}$. These local systems induce a holomorphic vector bundle $R^2 f_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_B \simeq R^2 f_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B$ whose fiber at b is naturally isomorphic to

$$(R^2 f_* \mathbb{C})_b \simeq H^2(\mathscr{X}_b, \mathbb{C}).$$

Proposition 3.1 ([Huy16, 6.Lemma 2.1])**.** *There is a natural injection*

$$f_* \Omega^2_{\mathscr{X}/B} \hookrightarrow R^2 f_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B$$

of holomorphic vector bundles which on each fibre yields the natural inclusion $H^{2,0}(\mathscr{X}_b) \subset H^2(\mathscr{X}_b, \mathbb{C})$.

Remark 3.2. Observe that, since $\mathscr{X}_b$ is compact Kähler for each b, the Hodge Decomposition Theorem implies that the Hodge numbers of the manifolds $\mathscr{X}_b$ are all the same, up to shrinking B. Indeed, [GR84, Theorem 5.10.4] implies that the function

$$b \longmapsto h^q(\mathscr{X}_b, \Omega^p_{\mathscr{X}/B}|_{\mathscr{X}_b})$$

is uppersemicontinuous, where $\Omega^p_{\mathscr{X}/B}|_{\mathscr{X}_b} \simeq \Omega^p_{\mathscr{X}_b}$. Therefore,

$$h^{p,q}(\mathscr{X}_b) \geq h^{p,q}(X)$$

locally around $0 \in B$. Since the Betti numbers are preserved by deformation (see Ehresmann's Lemma [Huy05, Proposition 6.2.2]), using the Hodge Decomposition Theorem we obtain $h^{p,q}(\mathscr{X}_b) = h^{p,q}(X)$ in an open neighbourhood of $0 \in B$.

In particular, we have that $h^{2,0}(\mathscr{X}_b)$ is constant.

Up to shrinking B, we can always suppose that it is simply connected. In this case, the local system is the constant sheaf associated with $H^2(X, \mathbb{C}) \simeq \mathbb{C}^{b_2(X)}$. Therefore the holomorphic bundle $f_* \Omega^2_{\mathscr{X}/B}$ is a subbundle of $R^2 f_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B \simeq \mathcal{O}_B^{h^2(X,\mathbb{C})}$ of rank $h^{2,0}(X)$. Thus, from [GH94, Section 1.5], it induces a holomorphic map

$$B \longrightarrow \mathrm{Gr}(h^{2,0}(X), H^2(X, \mathbb{C})) \tag{11}$$

such that $f_*\Omega^2_{\mathscr{X}/B}$ is obtained as the pullback of the universal subbundle on the Grassmannian $\mathrm{Gr}(h^{2,0}(X), H^2(X,\mathbb{C}))$. Explicitly, the image of $b \in B$ under this morphism is the subspace $(f_*\Omega^2_{\mathscr{X}/B})_b = H^{2,0}(\mathscr{X}_b)$ contained in $H^2(\mathscr{X}_b,\mathbb{C}) \simeq H^2(X,\mathbb{C})$.

We now restrict ourselves to smooth deformations $f : \mathscr{S} \to (B,0)$ of a K3 surface S, where B is smooth and simply connected. In this case, Proposition 3.1 yields a subbundle $f_*\Omega^2_{\mathscr{S}/B} \hookrightarrow R^2 f_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B$ which at each point $b \in B$ is the inclusion of the line $H^{2,0}(\mathscr{S}_b)$ in the vector space $H^2(\mathscr{S}_b,\mathbb{C})$, which we identify with $H^2(S,\mathbb{C})$.

Therefore, the morphism defined in (11) gives the *period map*

$$\wp_B : B \longrightarrow \mathbb{P}(H^2(S,\mathbb{C})),$$

defined by $\wp(b) = [H^{2,0}(\mathscr{S}_b,\mathbb{C})]$.

Let ω_b be a symplectic form on $\mathscr{S}_b$, namely $H^{2,0}(\mathscr{S}_b,\mathbb{C}) \simeq \mathbb{C}\omega_b$. The 2-form ω_b satisfies the Hodge–Riemann relations

$$\int \omega_b \wedge \bar{\omega}_b > 0 \qquad \text{and} \qquad \int \omega_b \wedge \omega_b = 0.$$

Therefore, the image of $\wp$ is contained in the period domain

$$\mathscr{D}_S \coloneqq \{x \in \mathbb{P}(H^2(S,\mathbb{C})) \mid x \cdot x = 0, x \cdot \bar{x} > 0\},$$

a complex manifold.

We have obtained the following result.

Proposition 3.3. *Given a smooth deformation $f : \mathscr{S} \to B$ of a K3 surface S, with B smooth simply connected, the period map defined by*

$$\begin{aligned} \wp_B : B &\longrightarrow \mathbb{P}(H^2(S,\mathbb{C})) \\ b &\longmapsto [H^{2,0}(\mathscr{S}_b)] \end{aligned}$$

is a holomorphic map that takes values in the period domain $\mathscr{D}_S \subset \mathbb{P}(H^2(S,\mathbb{C}))$.

We can apply this theory to the universal local deformation, of which we have proven the existence in Section 2.3. We thus obtain a morphism

$$\wp_{\mathrm{univ}} : B_{\mathrm{univ}} \longrightarrow \mathscr{D}_S,$$

after possibly shrinking of the base B_{univ}.

Moreover, if we fix an isometry $H^2(S,\mathbb{Z}) \simeq \Lambda_{K3}$, we can identify the period domain $\mathscr{D}_S$ with the K3 period domain

$$\mathscr{D}_{K3} = \{x \in \mathbb{P}((\Lambda_{K3})_{\mathbb{C}}) \mid x \cdot x = 0, x \cdot \bar{x} > 0\}.$$

For K3 surfaces, the following local Torelli theorem holds, see [Huy16, 6.Proposition 2.8]

Theorem 3.4 (Local Torelli). *The period map $\wp_{\text{univ}} : B_{\text{univ}} \to \mathscr{D}_S$ is a local isomorphism at 0.*

Remark 3.5. Using this Local Torelli Theorem, we can prove the existence of a universal local deformation of a polarized K3 surface (S, L) of degree $2d > 0$.
Let $f : \mathscr{S} \to (B, 0)$ be the universal local deformation of S, with B smooth simply connected, and let $\wp : B \to \mathscr{D}_S$ be the associated period map. The class $\ell = c_1(L) \in H^2(S, \mathbb{Z})$ is of type $(1, 1)$, therefore it is orthogonal to the subspace $H^{2,0}(S)$ of S. Hence, the period $\wp(0)$ is in $\mathscr{D} \cap \mathbb{P}(\ell_{\mathbb{C}}^{\perp})$.

As B is simply connected, we have natural identifications $H^2(\mathscr{S}_b, \mathbb{Z}) \simeq H^2(S, \mathbb{Z})$ for each $b \in B$. The class $\ell \in H^2(\mathscr{S}_b, \mathbb{Z})$ is of type $(1, 1)$ if and only if it is orthogonal to the subspace $H^{2,0}(\mathscr{S}_b)$, namely if and only if $\wp(b) \in \mathscr{D} \cap \mathbb{P}(\ell_{\mathbb{C}}^{\perp})$. In this case, it corresponds by Theorem 1.1 to an invertible sheaf L_b of $\mathscr{S}_b$ of square $2d$. Since being ample is an open condition verified in 0, after possibly shrinking $(B, 0)$, we can suppose that L_b is ample for each $b \in B$.
Observe that

$$\mathscr{D}_{\ell_{\mathbb{C}}^{\perp}} := \mathscr{D} \cap \mathbb{P}(\ell_{\mathbb{C}}^{\perp}) = \{x \in \mathbb{P}(\ell_{\mathbb{C}}^{\perp}) \mid x \cdot x = 0, x \cdot \bar{x} > 0\} \tag{12}$$

is a smooth variety, because $\ell_{\mathbb{C}}^{\perp}$ is a (nondegenerate) lattice because of the assumption $d > 0$. Therefore, from Theorem 3.4, we obtain that $B^{(\ell)} := \wp^{-1}(\mathscr{D}_{\ell_{\mathbb{C}}^{\perp}}) \subset B$ is a smooth hypersurface. Hence, we get a morphism $f^{(\ell)}$, defined as the restriction of f to $f^{-1}(B^{(\ell)})$, which is still smooth and proper.

$$\begin{array}{ccccc} \mathscr{S} & \xrightarrow{\quad f \quad} & B & \xrightarrow{\quad \wp \quad} & \mathscr{D} \\ \uparrow & & \uparrow & & \uparrow \\ \mathscr{S}^{(\ell)} := f^{-1}(B^{(\ell)}) & \xrightarrow{f^{(\ell)}} & B^{(\ell)} & \longrightarrow & \mathscr{D} \cap \mathbb{P}(\ell_{\mathbb{C}}^{\perp}). \end{array}$$

We show that there exists a line bundle $\mathcal{L}$ on $\mathscr{S}^{(\ell)}$ such that the stalk of $\mathcal{L}$ in $\mathscr{S}_b^{(\ell)}$ is equal to L_b for each $b \in B^{(\ell)}$. Observe that ℓ is a global section of $R^2 f_*^{(\ell)}\mathbb{Z}$ that is zero under the projection to $R^2 f_*^{(\ell)}\mathcal{O}_{\mathscr{S}^{(\ell)}}$. Since B is simply connected, we have $H^2(\mathscr{S}, \mathbb{Z}) \simeq \Gamma(B, R^2 f_*\mathbb{Z})$ and $H^2(\mathscr{S}, \mathcal{O}_{\mathscr{S}}) \simeq \Gamma(B, R^2 f_*\mathcal{O}_{\mathscr{S}})$ and similarly for $B^{(\ell)}$. Thus, we have

$$\begin{array}{ccc} \ell \in R^2 f_*^{(\ell)}\mathbb{Z} & \longrightarrow & R^2 f_*^{(\ell)}\mathcal{O}_{B^{(\ell)}} \ni 0 \\ \wr\downarrow & & \downarrow\wr \\ H^2(\mathscr{S}^{(\ell)}, \mathbb{Z}) & \longrightarrow & H^2(\mathscr{S}^{(\ell)}, \mathcal{O}_{\mathscr{S}^{(\ell)}}) \end{array}$$

and $\ell \in R^2 f_*^{(\ell)}\mathbb{Z}$ gives rise to a unique class in $H^2(\mathscr{S}^{(\ell)}, \mathbb{Z})$ that vanishes under the projection to $H^2(\mathscr{S}^{(\ell)}, \mathcal{O}_{\mathscr{X}^{(\ell)}})$. Hence, it is an element $\mathcal{L}$ of the Picard group of $\mathscr{S}^{(\ell)}$.

Therefore $(f^{(\ell)} : \mathscr{S}^{(\ell)} \to B^{(\ell)}, \mathcal{L})$ is a local deformation of (S, L) and it can be proved that it satisfies the universal property of universal local deformations.

3.2 Period domains of type IV

In this section, we want to generalize the definition of the period domain constructed for local polarized deformations of K3 surfaces and study the properties of these generalized domains.

Let Λ be an even indefinite lattice of signature $(2, n_-)$ such that $\ell(A_\Lambda)$ is less than n_- and $n_- \geq 2$.

Example 3.6. Key examples of such a lattice are lattices of the form

$$\Lambda = \mathbb{Z}k \oplus M,$$

where $k^2 = -2d$ and M is a unimodular, even, indefinite lattice of signature $(2, n_- - 1)$. Given a primitive vector h of square $2d$ in the unimodular K3-lattice Λ_{K3}, its orthogonal complement $h^\perp$ is of this type, as showed in Remark 1.16.

The zero locus of the quadratic form induced on $\Lambda_\mathbb{C}$ is a smooth quadric in $\mathbb{P}(\Lambda_\mathbb{C})$. The open analytic subset

$$\mathscr{D} = \{[x] \in \mathbb{P}(\Lambda_\mathbb{C}) \mid x \cdot x = 0, x \cdot \bar{x} > 0\}$$

of this quadric is a complex manifold, called the *period domain*.
Notice that for $\Lambda = \ell^\perp$, where $\ell \in \Lambda_{K3}$ primitive vector of degree $2d$, it is the same domain we have constructed in (12) in Remark 3.5.

The period domain $\mathscr{D}$ is diffeomorphic to $\mathrm{Gr}^{\mathrm{po}}(2, \Lambda_\mathbb{R}) \subset \mathrm{Gr}(2, \Lambda_\mathbb{R})$, the open set of oriented planes $P \subset \Lambda_\mathbb{R}$ on which the quadratic form is positive definite. The diffeomorphism associates with each $[x] \in \mathscr{D}$ the oriented plane $P = \mathrm{Re}(x)\mathbb{R} \oplus \mathrm{Im}(x)\mathbb{R}$. Indeed, as $x^2 = 0$ and $x \cdot \bar{x} > 0$, the plane P is positive definite and only depends on $[x] \in \mathbb{P}(\Lambda_\mathbb{C})$. Conversely, each oriented and positive definite plane P with direct orthonormal basis $\{v_1, v_2\}$ defines an element $[v_1 + iv_2] \in \mathscr{D}$ independent of the choice of the basis.

Having fixed an orthonormal basis $\{e_1, e_2, f_1, \dots, f_n\}$ of $\Lambda_\mathbb{R}$, where $e_1\mathbb{R} \oplus e_2\mathbb{R}$ is positive definite, $\mathrm{Gr}^{\mathrm{po}}(2, \Lambda_\mathbb{R})$ has two connected components determined by the sign of the determinant of the matrix given by the projection of each 2-oriented plane P onto $e_1\mathbb{R} \oplus e_2\mathbb{R}$. So we get a decomposition

$$\mathscr{D} = \mathscr{D}^+ \sqcup \mathscr{D}^-,$$

where the two connected components are diffeomorphic, exchanged by complex conjugation.

Observe that the condition $n_- > \ell(A_\Lambda)$ implies that the lattice Λ is of the form $U \oplus \Lambda'$. Indeed, from [Nik80, Corollary 1.10.2], since $l(A_\Lambda) < n_- = 1 + (n_- - 1)$, it follows that there exists a lattice Λ' of signature $(1, n_- - 1)$ and such that $A_{\Lambda'} = A_\Lambda$. Therefore, the lattice $U \oplus \Lambda'$ has the same signature and the same discriminant quadratic form as the lattice Λ, and from [Nik80, Corollary 1.13.3] we obtain $\Lambda \simeq U \oplus \Lambda'$.

For such a lattice $\Lambda = U \oplus \Lambda'$, we show that the isometry $g = -\mathrm{id}_U \oplus \mathrm{id}_{\Lambda'} \in O(\Lambda)$ exchanges the two connected components of $\mathrm{Gr}^{\mathrm{po}}(2, \Lambda_\mathbb{R})$. Indeed, if $\{e_1, f_1\}$ is a basis of U and e_2 is a positive vector of $\Lambda'_\mathbb{R}$, then $g(v) \cdot e_1 = -v \cdot e_1$ and $g(v) \cdot e_2 = v \cdot e_2$ for each vector $v \in \Lambda$. Observe that g induces the identity on the discriminant group, hence $g \in \widetilde{O}(\Lambda)$.

Since each isometry of Λ acts on $\mathbb{P}(\Lambda_\mathbb{C})$ and preserves $\mathscr{D}$, we get an action of $O(\Lambda)$ on $\mathscr{D}$. We show that this action is properly discontinuous.

Proposition 3.7 ([Huy16, 6.Proposition 5]). *There exists a diffeomorphism*

$$D \xrightarrow{\sim} \mathrm{Gr}^{\mathrm{po}}(2, \Lambda_{\mathbb{R}}) \xrightarrow{\sim} O(2, n_-)/SO(2) \times O(n_-)$$

Proof. By identifying $\Lambda_{\mathbb{R}}$ with the vector space $\mathbb{R}^{2+n_-}$ endowed with the quadratic form of signature $(2, n_-)$, we obtain a transitive action of $O(2, n_-)$ on $\mathrm{Gr}^{\mathrm{po}}(2, \Lambda_{\mathbb{R}})$. Let P be the plane spanned by the two positive vectors of the canonical basis of $\mathbb{R}^{2+n_-}$. The stabilizer of $P \in \mathrm{Gr}^{\mathrm{po}}(2, \Lambda_{\mathbb{R}})$ is $SO(2) \times O(P^{\perp}) = SO(2) \times O(n_-)$. □

The group $SO(2) \times O(n_-)$ is compact and $O(\Lambda)$ is a discrete subgroup of the locally compact group $O(2, n_-)$. Hence, from [Wol84, Lemma 3.1.1], it follows that $O(\Lambda)$ acts on D properly discontinuously.

Recall the following theorem by Borel–Baily.

Theorem 3.8 ([Huy16, Theorem 6.1.13]). *For all subgroups Γ of $O(\Lambda)$ of finite index, the quotient $\mathscr{D}/\Gamma$ is a normal quasi-projective variety.*

Moreover, if there exists $g \in \Gamma$ that exchanges the two components of $\mathscr{D}$ (for example if $\Gamma \supset \widetilde{O}(\Lambda)$), the variety $\mathscr{D}/\Gamma$ is irreducible.

3.3 Period map for polarized K3 surfaces

In Section 3.1, we have constructed a period morphism for each local deformation with smooth and simply connected base B. In this section, we generalize that construction and we build a global period map for polarized K3 surfaces.

Consider a smooth deformation $f : \mathscr{S} \to B$ of K3 surfaces. There exists an étale covering $\widetilde{B} \to B$ such that the pullback $\tilde{f} : \widetilde{\mathscr{S}} \to \widetilde{B}$ admits a *marking*, that is an isomorphism

$$R^2 \tilde{f}_* \mathbb{Z} \xrightarrow{\sim} \widetilde{B} \times \Lambda_{K3}.$$

Indeed, we consider $\widetilde{B} := \mathrm{Isom}(R^2 f_* \mathbb{Z}, \underline{\Lambda_{K3}}) \to B$, the natural $O(\Lambda_{K3})$-principal bundle associated to $R^2 f_* \mathbb{Z}$. The fiber at $b \in B$ is the set of isometries $H^2(\mathscr{S}_b, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3}$. Therefore, the pullback

$$\begin{array}{ccc} \widetilde{\mathscr{S}} & \longrightarrow & \mathscr{S} \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ \widetilde{B} & \longrightarrow & B \end{array}$$

has a marking $R^2 \tilde{f}_* \mathbb{Z} \xrightarrow{\sim} \widetilde{B} \times \Lambda_{K3}$. Indeed, at the point $(b, \varphi) \in \widetilde{B}$, with $b \in B$ and $\varphi : H^2(\mathscr{S}_b, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3}$, we have a canonical isomorphism of the fiber of $R^2 \tilde{f}_* \mathbb{Z}$ with Λ_{K3}, and this isomorphisms glue together.

The trivialization of the local system $R^2 \tilde{f}_* \mathbb{Z}$ yields, as in the case of simply connected local deformations, a period morphism

$$\wp_{\widetilde{B}} : \widetilde{B} \longrightarrow \mathscr{D} \subset \mathbb{P}((\Lambda_{K3})_{\mathbb{C}}).$$

We now restrict ourselves to families of polarized K3 surfaces of degree $2d$. We fix a primitive vector h of Λ_{K3} with $h^2 = 2d$ and denote by Λ_{2d} the lattice $h^{\perp}$, as described in

Remark 1.16.
Consider a family $(f : \mathscr{S} \to B, \mathcal{L})$ of polarized K3 surfaces of degree $2d$. The image of $\mathcal{L}$ via its first Chern class is a global section ℓ of $R^2 f_* \mathbb{Z}$. The orthogonal system $\ell^\perp \subset R^2 f_* \mathbb{Z}$ is defined fiberwise as the orthogonal complement of $\ell_b = \ell|_{\mathscr{S}_b}$ in $H^2(\mathscr{S}_b, \mathbb{Z})$.
Therefore, the fibers of $\ell^\perp$ are lattices that are isomorphic to Λ_{2d}.
Similarly to the construction above, we have an étale cover $\widetilde{B}' \to B$ such that

$$\widetilde{B}'_b = \{f : H^2(\mathscr{S}_b, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3} \mid f(\ell) = h\},$$

and we obtain a period morphism

$$\wp_{\widetilde{B}'} : \widetilde{B}' \longrightarrow \mathscr{D}_{\Lambda_{2d}} \subset \mathbb{P}((\Lambda_{2d})_{\mathbb{C}}).$$

Observe that $\widetilde{B}'$ embeds into the space $\widetilde{B}$ constructed above and $\wp_{\widetilde{B}'}$ is the restriction of the period map $\wp_{\widetilde{B}}$.
The group $O(\Lambda, h) \simeq \widetilde{O}(\Lambda_{2d})$ (Corollary 1.20) acts on $\widetilde{B}'$ and on $\mathscr{D}_{\Lambda_{2d}}$ and the morphism $\wp$ is equivariant. Passing to the quotient, we obtain a holomorphic morphism

$$\wp_B : B \longrightarrow \mathscr{D}_{\Lambda_{2d}} / \widetilde{O}(\Lambda_{2d}).$$

After the discussion of Section 3.2, we see that $\mathscr{D}_{\Lambda_{2d}} / \widetilde{O}(\Lambda_{2d})$ is an irreducible quasi-projective variety.

Recall the following result from Borel [Bor72, Theorem3.10] (see also [Huy16, Remark 6.4.2]).

Theorem 3.9 (Borel)**.** *If Y is a nonsingular complex algebraic variety and $\varphi : Y \to \mathscr{D}_{\Lambda_{2d}} / \widetilde{O}(\Lambda_{2d})$ is a holomorphic map, then φ is algebraic.*

In particular, if B is algebraic the period morphism $\wp_B$ is an algebraic morphism.

Fix a degree $2d$. In Section 2.4, we constructed a subscheme H of the Hilbert scheme, whose closed points parametrize polarized K3 surfaces (S, L) of degree $2d$ with an embedding in $\mathbb{P}^N_{\mathbb{C}}$, for $N = 9d + 1$, such that $L^{\otimes 3} = \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(1)|_S$. It comes with a universal family of polarized K3 surfaces $(\mathscr{Z}_H \to H, \mathcal{L})$, thus we can apply the previous construction to the underlying complex manifolds.

We obtain a period morphism

$$\wp : H \longrightarrow \mathscr{D}_{\Lambda_{2d}} / \widetilde{O}(\Lambda_{2d}).$$

The morphism $\wp$ is clearly invariant under the action of PGL. Moreover, the Torelli theorem implies that the set of orbits of H/PGL injects into $\mathscr{D}_{\Lambda_{2d}} / \widetilde{O}(\Lambda_{2d})$. Indeed, if $(S \subset \mathbb{P}^N_{\mathbb{C}}, L)$ and $(S' \subset \mathbb{P}^N_{\mathbb{C}}, L')$ are two polarized K3 surfaces of degree $2d$ with the same image $\bar{x} \in \mathscr{D}_{\Lambda_{2d}} / \widetilde{O}(\Lambda_{2d})$ under $\wp$, we have a commutative diagram

$$\begin{array}{ccc} H^2(S, \mathbb{Z}) & \xrightarrow[\sim]{\varphi} & \Lambda_{K3} \\ \Phi \big\downarrow \wr & & \wr \big\downarrow g \\ H^2(S', \mathbb{Z}) & \xrightarrow[\varphi']{\sim} & \Lambda_{K3} \end{array}$$

where φ (similarly φ') is an isometry between $H^2(S,\mathbb{Z})$ (respectively $H^2(S',\mathbb{Z})$) and Λ_{K3}, that sends $c_1(L)$ (respectively $c_1(L')$) to h, and $g \in O(\Lambda_{K3}, h) = \widetilde{O}(\Lambda_{2d})$ is such that

$$(g \circ \varphi)(\omega_S) = \varphi'(\omega_{S'}).$$

Therefore, the obtained isomorphism $\Phi = (\varphi')^{-1} \circ g \circ \varphi : H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ is a Hodge isomorphism that sends the ample class L of S to the ample class L' of S'. Since each ample class is Kähler, because of a theorem of Kodaira [Huy05, Proposition 5.3.1], the Torelli Theorem 2.5 implies the existence of an isomorphism $u : S' \xrightarrow{\sim} S$ such that $u^* = \Phi$ on the second cohomology groups. We observe that the morphism $S' \xrightarrow{u} S \xhookrightarrow{i} \mathbb{P}^N_{\mathbb{C}}$ is such that

$$(i \circ u)^*(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(1)) = u^*(L^{\otimes 3}) = L'^{\otimes 3}.$$

Thus, it is obtained by composing the inclusion $S' \subset \mathbb{P}^N_{\mathbb{C}}$ with an automorphism of $\mathbb{P}^N_{\mathbb{C}}$. Hence, the surfaces (S, L) and (S, L') are in the same PGL-orbit.
Moreover, using the local Torelli theorem, it can be proven that the set of PGL-orbits is an analytic open subset of the algebraic variety $\mathscr{D}_{\Lambda_{2d}}/\widetilde{O}(\Lambda_{2d})$.

In particular, we obtain again the existence of a coarse moduli space for polarized K3 surfaces of degree $2d$.

Theorem 3.10. *One can put a structure of quasi-projective variety M_{2d} on the orbit space H/PGL such that it is a coarse moduli space for the functor $\mathcal{M}_{2d}$. Moreover, M_{2d} can be realized as a Zariski open subscheme of the quasi-projective variety $\mathscr{D}_{\Lambda_{2d}}/\widetilde{O}(\Lambda_{2d})$.*

Proof. Since $\wp : H \to \mathscr{D}_{\Lambda_{2d}}/\widetilde{O}(\Lambda_{2d})$ is algebraic, its image is a constructible set which is also an analytic open subset. Therefore, by [Gro05, Exp. XII, Corollary 2.3], it is open in the Zariski topology. Thus $M_{2d} := \wp(H)$ has a natural algebraic structure whose close points parametrize polarized K3 surfaces of degree $2d$.
Additionally, for each family of polarized K3 surfaces $(f : \mathscr{S} \to B, \mathcal{L}) \in \mathcal{M}_{2d}(B)$, with the above construction, we obtain a period morphism

$$\wp_b : B \longrightarrow \mathscr{D}_{\Lambda_{2d}}/\widetilde{O}(\Lambda_{2d})$$

with image in M_{2d}. This provides a holomorphic map $B \to M_{2d}$ that is algebraic by Theorem 3.9. Therefore, we have a natural transformation $\mathcal{M}_{2d} \to h_{M_{2d}}$ that induces an isomorphism on closed points. □

Corollary 3.11. *The moduli space M_{2d} is an irreducible quasi-projective variety of dimension* 19.

Proof. In Section 3.2, we proved that $\mathscr{D}_{\Lambda_{2d}}/\widetilde{O}(\Lambda_{2d})$ is an irreducible quasi-projective algebraic variety. Moreover, it is of dimension 19, because it is the quotient of a Zariski open set of an hypersurface of $\mathbb{P}((\Lambda_{2d})_{\mathbb{C}}) \simeq \mathbb{P}^{20}_{\mathbb{C}}$ by a properly discontinuous action. As M_{2d} is a Zariski open subset of $\mathscr{D}_{\Lambda_{2d}}/\widetilde{O}(\Lambda_{2d})$, we obtain the wanted result. □

4 Hyper-Kähler manifolds

4.1 Definition and first properties

Definition 4.1. A *hyper-Kähler manifold* is a simply connected compact Kähler manifold X such that $H^0(X, \Omega_X^2) = \mathbb{C}\omega$, where ω is a holomorphic 2-form on X which is nowhere degenerate.

Results from Section 2.1 imply that hyper-Kähler manifolds of dimension 2 are K3 surfaces. From the definition, it follows that each hyper-Kähler manifold X has trivial canonical bundle ω_X and even dimension. In the same way as we did for K3 surfaces, it can be shown that $H^2(X, \mathbb{Z})$ is torsion free.

Let X be a hyper-Kähler manifold of dimension $2m$. There exists an integral quadratic form q_X on the free abelian group $H^2(X, \mathbb{Z})$ of signature $(3, b_2(X)-3)$, called the *Beauville–Fujiki form*. It satisfies

$$\forall x \in H^2(X, \mathbb{Z}) \qquad x^{2m} = c_X q(x)^m, \tag{13}$$

where c_X is a positive rational number, called the *Fujiki constant*. It is a topological invariant of X. Moreover, it satisfies $q_X(x) > 0$ for all Kähler classes $x \in H^2(X, \mathbb{Z})$, thus in particular for all ample classes. For more details, see [GHJ03, Chapter 23].

Example 4.2. We present here two families of hyper-Kähler manifolds.
These, and two other examples in dimensions 6 and 10 found by O'Grady, are the only known deformations types of hyper-Kähler manifolds in even dimension greater than or equal to 4.

a) *Hilbert powers of K3 surfaces* [Bea83, Section 6]. Let S be a K3 surface and let $S^{[m]}$ be the Douady–Hilbert space that parametrizes analytic subspaces of S of length $m \geq 2$. It is a smooth compact Kähler manifold of dimension $2m$. Beauville proved that it is a hyper-Kähler manifold and that its second cohomology group is

$$H^2(S^{[m]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

where 2δ is the class of the exceptional divisor of the natural projection $S^{[m]} \to S^{(m)}$.

This decomposition is orthogonal for the Beauville–Fujiki quadratic form. This form restricts to the intersection form on $H^2(S, \mathbb{Z})$ and is such that $q_{S^{[m]}}(\delta) = -2(m-1)$. Hence, the lattice $(H^2(S^{[m]}, \mathbb{Z}), q_{S^{[m]}})$ is isomorphic to

$$\Lambda_{K3^{[m]}} := \Lambda_{K3} \oplus \mathbb{Z}(-2(m-1)).$$

Definition 4.3. For $m \geq 2$, a hyper-Kähler manifold is *of type K3*$^{[m]}$ if it is a smooth deformation of $S^{[m]}$ for some K3 surface S.

Since the second cohomology group is invariant under deformation, the lattice associated with a hyper-Kähler manifold of type K3$^{[m]}$ is always isomorphic to $\Lambda_{K3^{[m]}}$.

b) *Generalized Kummer varieties* [Bea83, Section 7]. Let A be a complex torus of dimension 2 and let m be an integer greater than or equal to 2. The Douady–Hilbert space $A^{[m+1]}$ is a complex manifold with a nowhere degenerate holomorphic 2-form, but it is not simply connected. Given the application

$$\begin{aligned} A^{[m+1]} &\longrightarrow A \\ (p_0, \dots, p_m) &\longmapsto \sum_{i=0}^{m} p_i, \end{aligned}$$

we define $K_m(A)$ as the inverse image of 0. For $m = 1$, it is isomorphic to the Kummer surface associated with A, constructed in Example 2.3. For $m \geq 2$, Beauville proved that it is a hyper-Kähler manifold of dimension $2m$ and that there is a decomposition

$$H^2(K_m(A), \mathbb{Z}) \simeq H^2(A, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

orthogonal for the quadratic form and with $q_{K_m(A)}(\delta) = -2(m+1)$. Therefore,

$$(H^2(K_m(A), \mathbb{Z}), q_{K_m(A)}) \simeq U^{\oplus 3} \oplus \mathbb{Z}(-2(m+1)).$$

Definition 4.4. For $m \geq 2$, a hyper-Kähler manifold is *of Kummer type* K_m if it is a smooth deformation of $K_m(A)$ for some complex torus A and some integer m.

4.2 Moduli space of polarized hyper-Kähler manifolds

Similarly to the case of polarized K3 surfaces, using Viehweg's results [Vie90], we can construct a quasi-projective coarse moduli space of polarized hyper-Kähler manifolds (X, H) of fixed dimension $2m$ and fixed degree H^{2m}. Indeed, Matsusaka's Big Theorem implies the existence of a constant $k(m)$ such that, for any hyper-Kähler manifold X of dimension $2m$ and any ample invertible sheaf H on X, the sheaf $H^{\otimes k}$ is very ample for each $k \geq k(m)$.

Let (Λ, q) be a lattice. If we fix a hyper-Kähler manifold X_0 with an isometry $(H^2(X_0, \mathbb{Z}), q_{X_0}) \simeq (\Lambda, q)$, and the $O(\Lambda)$-orbit τ of a primitive element of Λ of positive square, this construction gives a coarse moduli space of polarized hyper-Kähler manifolds of type (X_0, τ): its closed points parametrize equivalence classes (X, H) where X is a hyper-Kähler manifold that is a smooth deformation of X_0, and H is an ample line bundle on X whose class in Λ is in the orbit τ.

We restrict ourselves to hyper-Kähler manifolds of $K3^{[m]}$-type, whose second cohomology group is isomorphic to the lattice

$$\Lambda_{K3^{[m]}} := \Lambda_{K3} \oplus \mathbb{Z}(-2(m-1)).$$

Let τ be the $O(\Lambda_{K3^{[m]}})$-orbit of a primitive element $h \in \Lambda_{K3^{[m]}}$ of positive square $2d$. We denote by $M_\tau^{K3^{[m]}}$ the coarse moduli space of polarized hyper-Kähler manifolds of type $K3^{[m]}$ and polarization type τ. This moduli space is not irreducible, but the number of irreducible components is known, see [Apo14] and [Son22].

For each primitive vector h in $\Lambda_{K3^{[m]}}$, the numbers h^2 and $\mathrm{div}(h)$ are invariant under the action of $O(\Lambda_{K3^{[m]}})$. In general, they do not determine the orbit of $O(\Lambda_{K3^{[m]}})h$, but they do in some special cases.

Proposition 4.5 ([GHS10, Corollary 3.7]). *Let d, t, and γ be integers such that $\gamma \mid (2t, 2d)$ and let*

$$L_{2t} = \Lambda_{K3} \oplus \mathbb{Z}(-2t)$$

be the lattice associated with a hyper-Kähler manifold of type $K3^{[t+1]}$. We set $\omega = (\frac{2t}{\gamma}, \frac{2d}{\gamma}, \gamma)$.
If $\omega = 1$ and there exists a primitive $h \in L_{2t}$ such that $h^2 = 2d$ and $\operatorname{div}(h) = \gamma$, then all such vectors belong to the same $O(L_{2t})$-orbit.
Moreover, such a vector always exists if $\gamma = 1$, or $\gamma = 2$ and $d + t \equiv 0 \pmod 4$.

In these cases, we denote by $M^{[m]}_{\gamma,2d}$ the moduli space $M^{K3^{[m]}}_{\tau}$, where τ is the unique orbit of a vector of square $2d$ and divisibility γ. We have the following result ([Deb17]).

Theorem 4.6. *(Gritsenko–Hulek–Sankaran, Apostolov) If $\gamma = 1$, or $\gamma = 2$ and $d+m \equiv 1 \pmod 4$, the quasi-projective moduli space $M^{[m]}_{\gamma,2d}$ of polarized hyper-Kähler manifolds of type K3$^{[m]}$ with a polarization of square $2d$ and divisibility γ is irreducible of dimension 20.*

4.3 Period map for polarized hyper-Kähler manifolds of K3$^{[m]}$-type

Extending the construction of Section 3.3, a period morphism for polarized hyper-Kähler manifolds of K3$^{[m]}$-type with fixed polarization type has been constructed. Having fixed a dimension $2m$ and a polarization type $\tau = O(\Lambda)h$, for some primitive vector $h \in \Lambda_{K3^{[m]}}$, there is a *period morphism*

$$\wp_{K3^{[m]}} : M^{K3^{[m]}}_{\tau} \longrightarrow \mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[m]}}, h),$$

with the property that its restriction to any irreducible component of $M^{K3^{[m]}}_{\tau}$ is an open embedding.

The key property of the period morphism for K3 surfaces is the Torelli Theorem 2.5. For hyper-Kähler manifolds, a weaker version of the Torelli theorem holds. Before enunciating the theorem, we give the definition of *parallel transport operators.*

Definition 4.7. Let X and Y be deformation equivalent hyper-Kähler manifolds. An isomorphism $\varphi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(Y,\mathbb{Z})$ is a *parallel transport operator* if there exists a smooth and proper family $f : \mathscr{X} \to B$ of hyper-Kähler manifolds and a path $\gamma : [0,1] \to B$ with $\mathscr{X}_{\gamma(0)} = X$ and $\mathscr{X}_{\gamma(1)} = Y$, such that φ is induced by the parallel transport of the local system $R^2 f_* \mathbb{Z}$ along γ.

Theorem 4.8 ([Mar11, Theorem 1.3]). *Let X and Y be hyper-Kähler manifolds that are deformation equivalent. The manifolds X and Y are isomorphic if and only if there exists a parallel transport operator $\varphi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(Y,\mathbb{Z})$ which is an isomorphism of integral Hodge structures.*
Moreover, let $\varphi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(Y,\mathbb{Z})$ be a parallel transport operator which is an isomorphism of integral Hodge structures. There exists an isomorphism $u : X \xrightarrow{\sim} Y$ such that $\varphi = u_*$ on the second cohomology group, if and only if u maps some Kähler class on X to a Kähler class on Y.

Therefore, instead of considering the group $O(\Lambda_{K3}, h)$ of isometries that preserve the class h, in this case we should consider the group of Hodge isometries that preserve h and that are parallel transport operators. In the case of polarized hyper-Kähler manifolds of $\mathrm{K3}^{[m]}$-type, this group is $\widehat{O}(\Lambda_{K3^{[m]}}, h)$, see [Mar11, Lemma 9.2].

Recall that, from Proposition 1.19, we have

$$O(\Lambda_{K3^{[m]}}, h) = \{g \in O(h^\perp) \mid g|_{p(H)} = \mathrm{id}\},$$

where $p(H)$ is the image of $\Lambda_{K3^{[m]}}/(\mathbb{Z}h \oplus (h^\perp))$ via the projection $A_{\mathbb{Z}h} \oplus A_{h^\perp} \to A_{h^\perp}$. If g is an isometry of $h^\perp$ such that $g|_{p(H)} = \mathrm{id}$, we will still denote by g its extension to $\Lambda_{K3^{[m]}}$.

Let l be the generator of the $\mathbb{Z}(-2(m-1))$ factor in $\Lambda_{K3^{[m]}}$. We saw in Section 1.2 that the discriminant group of $\Lambda_{K3^{[m]}}$ is a cyclic group of order $2(m-1)$ generated by the element $l_* = [l/2(m-1)]$. Therefore, we have the identification

$$\widehat{O}(\Lambda_{K3^{[m]}}, h) = \{g \in O(h^\perp) \mid \bar{g}|_{p(H)} = \mathrm{id} \quad \text{and} \quad \bar{g}(l_*) = \pm l_* \in A_{\Lambda_{K3^{[m]}}}\}. \tag{14}$$

We would like to give a more explicit description of the group $\widehat{O}(\Lambda_{K3^{[m]}}, h)$. In order to do so, we first describe the lattice $h^\perp$ for a primitive vector $h \in \Lambda_{K3^{[m]}}$.

4.4 Groups of isometries of some lattices

We consider a slightly more general situation. Given a positive integer t, let L_{2t} be the lattice

$$L_{2t} = M \oplus U \oplus \mathbb{Z}(-2t), \tag{15}$$

where M is an even unimodular lattice and the factor $\mathbb{Z}(-2t)$ is generated by a vector l. The discriminant group $A_{L_{2t}}$ is a cyclic group of order $2t$ generated by l_*. The lattice $\Lambda_{K3^{[m]}}$ is a lattice of type $L_{2(m-1)}$.

Let h be a primitive vector of L_{2t} of square $2d > 0$ and divisibility γ. Recall that from (7) we have a chain of subgroups

$$\widetilde{O}(h^\perp) \xhookrightarrow{i_1} \widetilde{O}(L_{2t}, h) \xhookrightarrow{i_2} \widehat{O}(L_{2t}, h) \xhookrightarrow{i_3} O(L_{2t}, h) \xhookrightarrow{i_4} O(h^\perp), \tag{16}$$

where we can describe the group $\widehat{O}(L_{2t}, h)$ as in (14).

In this section we study the lattice $h^\perp$ and the group $A_{h^\perp}$, following [GHS10]. Moreover we describe the image of $\widehat{O}(L_{2t}, h)$ in $O(A_{h^\perp})$ in some cases. In particular:

- in Proposition 4.10 we describe the lattice $h^\perp$.
- When $(\frac{2t}{\gamma}, \gamma) = 1$:
  - we compute the discriminant group $A_{h^\perp}$ and show that $\widetilde{O}(L_{2t}, h) = \widetilde{O}(h^\perp)$ (Proposition 4.14);
  - we describe the image of the group $\widehat{O}(L_{2t}, h)$ in $O(A_{h^\perp})$ under the morphism $r : O(h^\perp) \to O(A_{h^\perp})$ introduced in Theorem 1.17 (Remark 4.15 for $t = 1$ or $\gamma > 2$ and Proposition 4.16 for $\gamma \in \{1, 2\}$).

– we discuss the normality of $\widehat{O}(L_{2t}, h)$ in $O(h^\perp)$, and show that, if $(t, d) = 1$, then $\widehat{O}(L_{2t}, h) \lhd O(h^\perp)$ (Corollary 4.19).

4.4.1 The lattice $h^\perp$

The vector h in the lattice L_{2t} (15) can be written as

$$h = ax + cl$$

where $x \in M \oplus U$ is primitive and a, c are coprime integers. The divisibility of h is $\gamma = (a, 2tc) = (a, 2t)$. In particular $\gamma \mid 2t$ and we can write $a = \gamma a_1$ for some $a_1 \in \mathbb{Z}$. Observe moreover that, since $\gamma \mid a$, we have $(c, \gamma) = 1$. Finally,

$$h_* = \left[\frac{h}{\mathrm{div}(h)}\right] = c\frac{2t}{\gamma} l_* \in A_{L_{2t}}. \tag{17}$$

By computing the square of h, we obtain

$$2d = h^2 = \gamma^2 a_1^2 x^2 - 2tc^2,$$

where x^2 is an even integer. Thus, the quotient $\frac{d+tc^2}{\gamma^2}$ is an integer that we denote by b.

Given $\{e, f\}$ a standard basis of U, we consider the vector

$$\tilde{h} = \gamma(e + bf) + cl. \tag{18}$$

It has divisibility equal to $(\gamma, 2tc)$, which is γ since $\gamma \mid 2t$, square equal to $2\gamma^2 b - 2tc^2 = 2d$, and

$$\tilde{h}_* = \left[\frac{\tilde{h}}{\mathrm{div}(\tilde{h})}\right] = \frac{c}{\gamma} 2tl_* = h_* \in A_{L_{2t}}.$$

Since by Eichler's Lemma, the $\widetilde{O}(L_{2t})$-orbit of h is determined by h^2 and h_*, and we are only interested in the $O(L_{2t})$-orbit of h, we can suppose that the vector h is of the form (18).

Note that the element h_* of $A_{L_{2t}}$ (see (17)) is determined by $c \pmod{\gamma}$. If $c = \gamma n + c'$, then, for $b' = b + tn^2 - \frac{2t}{\gamma} nc$, the vector

$$h' = \gamma(e + b'f) + c'l$$

has square

$$2\gamma^2 b' - 2tc'^2 = 2\gamma^2 b + 2t\gamma^2 n^2 - 4t\gamma nc - 2t(\gamma^2 n^2 - 2\gamma nc + c^2) = 2\gamma^2 b - 2tc^2 = 2d,$$

and $h'_* = h_*$ in $A_{L_{2t}}$.

Therefore, given a primitive vector $h \in L_{2t}$ of square $2d$, divisibility γ, and such that the class $h_* \in A_{L_{2t}}$ corresponds to the element $c\frac{2t}{\gamma}$ of order γ of $\mathbb{Z}/2t\mathbb{Z}$ for some integer c modulo γ and prime to γ, we may assume that h is of the form

$$h = \gamma(e + bf) + cl \tag{19}$$

up to isometries of L_{2t}.

Remark 4.9. We notice that if h is a primitive vector of L_{2t} of divisibility γ and square $2d$, then

$$\gamma^2 \mid d + tc^2. \tag{20}$$

Therefore, in general not all pairs $(2d, \gamma)$ can be realized as $(h^2, \mathrm{div}(h))$ for some primitve vector $h \in L_{2t}$.
For instance, if $\gamma = 1$ the condition (20) is always verified. If $\gamma = 2$, then c is necessary 1 and we obtain that d must verify $d + t \equiv 0 \pmod 4$.

Proposition 4.10 ([GHS10, Proposition 3.6])**.** *Let h be a primitive vector of L_{2t} of square $2d$ and divisibility γ, and let c be the integer modulo γ and prime to γ defined in the above discussion. Then,*

$$h^\perp = M \oplus \begin{pmatrix} -\dfrac{2d+2c^2t}{\gamma^2} & c\dfrac{2t}{\gamma} \\ c\dfrac{2t}{\gamma} & -2t \end{pmatrix}.$$

Proof. Since the orthogonal $h^\perp$ only depends on the $O(L_{2t})$-orbit of h, we can suppose

$$h = \gamma(e + bf) + cl,$$

where $\{e, f\}$ is a standard basis of U and $\gamma^2 b = d + tc^2$. Hence $h^\perp = M \oplus B$, where B is the orthogonal complement of h in the lattice $U \oplus \mathbb{Z}l$. The lattice B has rank two and

$$B = \left\langle h_1 = e - bf, h_2 = c\frac{2t}{\gamma} f + l \right\rangle.$$

Indeed, h_1 and h_2 are orthogonal to h, each vector $v \in U \oplus \mathbb{Z}l$ can be written as $v = x_1 h_1 + x_2 h_2 + yf$, and such a vector v is orthogonal to h if and only if $y = 0$. Direct computations show that the intersection matrix on B is the one written in the statement. □

Remark 4.11. If $\gamma = 1$, then $h_* = 0 \in A_{L_{2t}}$. Therefore $c = 0$, the lattice B is diagonal (or"split"), and

$$h^\perp \simeq M \oplus \mathbb{Z}(-2d) \oplus \mathbb{Z}(-2t).$$

Conversely, if the lattice B is split, we have $c = 0$. Hence, from Equation (19) we obtain $\gamma = 1$, since h is primitive.

Definition 4.12. A polarization type $O(L_{2t})h$ is said to be *split* if $\mathrm{div}(h) = 1$, *non-split* otherwise.

4.4.2 The groups $\widetilde{O}(L_{2t}, h)$ and $\widehat{O}(L_{2t}, h)$

As already observe for the lattice $\Lambda_{K3^{[m]}}$, from Proposition (1.19) we have

$$O(L_{2t}, h) = \{g \in O(h^\perp) \mid g|_{p(H)} = \mathrm{id}\},$$

where H is the group $L_{2t}/(\mathbb{Z}h \oplus h^\perp)$ and p is the projection $A_{\mathbb{Z}h} \times A_{H^\perp} \to A_{h^\perp}$.

In the proof of Proposition 4.10, we have also proved that

$$H = L_{2t}/(\mathbb{Z}h \oplus h^\perp) = (U \oplus \mathbb{Z}l)/(\mathbb{Z}h \oplus h^\perp) = \langle [f] \rangle.$$

We describe the image $p(H)$, that is generated by $p([f])$. The vector

$$k_1 = \frac{\gamma}{2d}h - f$$

is in $(h^\perp)^\vee$. Indeed, we can compute $k_1 \cdot h_1 = -1$ and $k_1 \cdot h_2 = 0$. Notice, moreover, that $p(\bar{f}) = -\bar{k}_1$. Hence, the group $p(H)$ is generated by $\bar{k}_1 \in A_{h^\perp}$.

Therefore, the groups $\widetilde{O}(L_{2t}, h) = \widetilde{O}(L_{2t}) \cap O(L_{2t}, h)$ and $\widehat{O}(L_{2t}, h) = \widehat{O}(L_{2t}) \cap O(L_{2t}, h)$ can be described as

$$\widetilde{O}(L_{2t}, h) = \{g \in O(h^\perp) \mid \bar{g}(\bar{k}_1) = \bar{k}_1 \in A_{h^\perp} \text{ and } \bar{g}(l_*) = l_* \in A_{L_{2t}}\} \tag{21}$$

and

$$\widehat{O}(L_{2t}, h) = \{g \in O(h^\perp) \mid \bar{g}(\bar{k}_1) = \bar{k}_1 \in A_{h^\perp} \text{ and } \bar{g}(l_*) = \pm l_* \in A_{L_{2t}}\}. \tag{22}$$

4.4.3 The discriminant group $A_{h^\perp}$

We study the discriminant group $A_{h^\perp}$: from Equation (5) follows that

$$\operatorname{disc}(\mathbb{Z}h)\operatorname{disc}(h^\perp) = |H|^2 \operatorname{disc}(L_{2t}), \tag{23}$$

where $\operatorname{disc}(\mathbb{Z}h) = 2d, \operatorname{disc}(L_{2t}) = 2t$ and $\operatorname{disc}(h^\perp) = |A_{h^\perp}|$.

Observe that the element $\bar{k}_1$ has order $\frac{2d}{\gamma}$ in $A_{h^\perp}$. Indeed, given an integer $n \in \mathbb{Z}$, the vector nk_1 is in $h^\perp = h^\perp{}_{\mathbb{Q}} \cap L_{2t}$ if and only if $n\frac{\gamma}{2d}h \in L_{2t}$, hence if and only if $n\frac{\gamma}{2d} \in \mathbb{Z}$.

We have showed in Section 1.2.1 that the morphism p is injective, hence we obtain

$$|H| = |p(H)| = \frac{2d}{\gamma}.$$

From Equation (23) we get

$$2d \cdot |A_{h^\perp}| = 2t \left(\frac{2d}{\gamma}\right)^2,$$

from which we obtain that $A_{h^\perp}$ is an abelian group of cardinality $\frac{2d}{\gamma}\frac{2t}{\gamma}$.

Remark 4.13. If $h \in L_{2t}$ is a primitive vector of square $2d$ and divisibility γ, then

$$\omega := \left(\frac{2t}{\gamma}, \frac{2d}{\gamma}, \gamma\right) = \left(\frac{2t}{\gamma}, \gamma\right).$$

Indeed, from (19) we can suppose that $h = \gamma(e + bf) + cl$, and therefore

$$2d = h^2 = 2b\gamma^2 - 2tc^2 = \gamma\left(2b\gamma - \frac{2t}{\gamma}c^2\right). \tag{24}$$

Hence $(\frac{2t}{\gamma}, \gamma) \mid \frac{2d}{\gamma}$.

Proposition 4.14 ([GHS10, Proposition 3.12]). *Let $h \in L_{2t}$ be a primitive vector with $h^2 = 2d$ and $\operatorname{div}(h) = \gamma$. We suppose $\omega = (\frac{2t}{\gamma}, \gamma) = 1$. Then, we have an isometry*

$$A_{h^\perp} \simeq \mathbb{Z}/\tfrac{2d}{\gamma}\mathbb{Z} \times \mathbb{Z}/\tfrac{2t}{\gamma}\mathbb{Z}$$

such that the subgroup $p(H) < A_{h^\perp}$ corresponds to the factor $\mathbb{Z}/\frac{2d}{\gamma}\mathbb{Z}$ and the intersection form on $A_{h^\perp}$ is defined by $q(1,0) = -\frac{\gamma^2}{2d}$ and $q(0,1) = -\frac{\gamma^2}{2t}$.

In this case, we have

$$\widetilde{O}(L_{2t}, h) = \widetilde{O}(h^\perp).$$

Proof. Keeping the notation of the previous proposition, we may assume $h = \gamma(e+bf)+cl$, where $\gamma^2 b = d + c^2 t$ and $\{e, f\}$ is the standard basis of the hyperbolic plane U contained in L_{2t}.

We have proven that the lattice $h^\perp$ is of the form $h^\perp = M \oplus B$, where B is the rank 2 sublattice of $U \oplus \mathbb{Z}l$ generated by the vectors $h_1 = e - bf$ and $h_2 = c\frac{2t}{\gamma}f + l$. Therefore, the group $A_{h^\perp}$ is isomorphic to A_B.

The vectors

$$k_1 = \frac{\gamma}{2d}h - f \qquad \text{and} \qquad \tilde{k}_2 = \frac{c}{2d}h + \frac{1}{2t}l$$

form a basis of $B^\vee$ that is (up to sign) dual to the basis $\{h_1, h_2\}$ of B. Consider the vector

$$k_2 := ck_1 - \gamma\tilde{k}_2 = cf + \frac{\gamma}{2t}l = \frac{\gamma}{2t}h_2.$$

This vector k_2 is orthogonal to k_1, and $\langle \bar{k}_1 \rangle$ and $\langle \bar{k}_2 \rangle$ are two subgroups of A_B. We have already shown that the order of $\bar{k}_1$ in A_B is $\frac{2d}{\gamma}$. Analogously, we can show that the order of $\bar{k}_2$ is $\frac{2t}{\gamma}$. We show that, for $\omega = 1$, the subgroups $\langle \bar{k}_1 \rangle$ and $\langle \bar{k}_2 \rangle$ intersect trivially.

We fix some notation: let γ', t', and d' be integers such that $(\gamma', t', d') = 1$ and

$$\gamma = \omega\gamma', \qquad \frac{2d}{\gamma} = \omega d', \qquad \text{and} \qquad \frac{2t}{\gamma} = \omega t'.$$

Suppose $n\bar{k}_1 \in \langle \bar{k}_2 \rangle$, namely that there exists some integer $m \in \mathbb{Z}$ such that

$$n\bar{k}_1 = m\bar{k}_2 \in A_B.$$

Then $nk_1 - mk_3$ is a vector in B, and its component along e is $\frac{n\gamma^2}{2d}$. Since B is contained in L_{2t} and e is primitive in L_{2t}, the coefficient of $nk_1 - mk_2$ along e must be an integer. We obtain

$$\frac{n\gamma^2}{2d} = \frac{n\gamma'}{d'} \in \mathbb{Z}. \tag{25}$$

Moreover, by computing the order of $n\bar{k}_1 = m\bar{k}_3$ in A_B, we obtain $d' \mid n$. Indeed,

$$\frac{\omega d'}{(n, \omega d')} = \operatorname{ord}_{A_B}\left(n\bar{k}_1\right) = \operatorname{ord}_{A_B}\left(m\bar{k}_3\right) = \frac{\omega t'}{(m, \omega t')}.$$

Hence, we obtain

$$d'(m, \omega t') = t'(n, \omega d') = t'\left(\frac{n}{(n, d')}, \omega\frac{d'}{(n, d')}\right)(n, d'). \tag{26}$$

Let δ be the integer $\frac{d'}{(n,d')}$. By definition, δ is coprime with $\frac{n}{(n,d')}$, and $d' = \delta(n,d')$. Equation (26) implies $\delta \mid t'$. Moreover, since $d' \mid n\gamma'$ by (25), we obtain $\delta \mid \gamma'$. Since $(\gamma', d', t') = 1$, it follows that $\delta = 1$. Therefore d' divides n and we can write $n = d'n'$, so that

$$\begin{aligned} n\bar{k}_1 &= \frac{n\gamma}{2d}h - nf = \frac{n'd'}{\omega d'}h - nf \\ &= \frac{n'}{\omega}(\gamma(e+bf)+cl) - nf \\ &\equiv \frac{n'c}{\omega}l + (2n'\gamma'b - n)f \pmod{B}. \end{aligned}$$

We now use our assumption $\omega = 1$. Then nk_1 is an element of $L_{2t} \cap B^\vee = B$. Therefore, the intersection $\langle \bar{k}_1 \rangle \cap \langle \bar{k}_2 \rangle$ is trivial. Since the order of $A_{h^\perp}$ is equal to the order of $\langle \bar{k}_1 \rangle \times \langle \bar{k}_2 \rangle$, we obtain

$$A_{h^\perp} \simeq \langle \bar{k}_1 \rangle \times \langle \bar{k}_2 \rangle \simeq \mathbb{Z}/\tfrac{2d}{\gamma}\mathbb{Z} \times \mathbb{Z}/\tfrac{2t}{\gamma}\mathbb{Z}. \tag{27}$$

This proves the first part of the proposition.

By Proposition 1.21 we have an inclusion $\widetilde{O}(h^\perp) \hookrightarrow \widetilde{O}(L_{2t}, h)$. We show that, for $\omega = 1$, this morphism is surjective, namely that for each isometry $g \in \widetilde{O}(L_{2t}, h)$, its restriction to $h^\perp$ satisfies $\bar{g}|_{A_{h^\perp}} = \mathrm{id}$.

Let g be an isometry of $\widetilde{O}(L_{2t}, h)$. From (21), it follows that $g(k_1) \equiv k_1 \pmod{h^\perp}$, hence

$$\frac{\gamma}{2d}h - f = k_1 \equiv g(k_1) = g\left(\frac{\gamma}{2d}h - g(f)\right) = \frac{\gamma}{2d}h - g(f) \pmod{h^\perp}.$$

Hence, we obtain $g(f) \equiv f \pmod{h^\perp}$. Moreover, there exists, still by (21), a vector $m \in L_{2t}$ such that $g\left(\frac{l}{2t}\right) = \frac{l}{2t} + m$. Since g is an isometry, then

$$h \cdot \frac{l}{2t} = g(h) \cdot g\left(\frac{l}{2t}\right) = h \cdot g\left(\frac{l}{2t}\right) = h \cdot \frac{l}{2t} + h \cdot m,$$

which implies that the vector m is in $h^\perp$. Therefore $g(k_3) = g\left(cf + \frac{\gamma}{2t}l\right) \equiv k_3 \pmod{h^\perp}$. Since, for $\omega = 1$, the group $A_{h^\perp}$ is generated by k_1 and k_3, we obtain $\bar{g}|_{A_{h^\perp}} = \mathrm{id}$, which is what we wanted. □

4.4.4 Normality of $\widehat{O}(L_{2t}, h)$ in $O(h^\perp)$

We consider the chain of subgroups

$$\widetilde{O}(h^\perp) \xhookrightarrow{i_1} \widetilde{O}(L_{2t}, h) \xhookrightarrow{i_2} \widehat{O}(L_{2t}, h) \xhookrightarrow{i_3} O(L_{2t}, h) \xhookrightarrow{i_4} O(h^\perp). \tag{28}$$

We want to understand if $\widehat{O}(L_{2t}, h)$ is a normal subgroup of $O(h^\perp)$, which will be used in Section 5.3. A summary of the results that follow can be found in Remark 4.20.

Proposition 4.14 implies that, if $\omega = 1$, the inclusion i_1 is an equality.

For $t = 1$, the group $A_{L_{2t}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, therefore $\widetilde{O}(L_{2t}) = \widehat{O}(L_{2t}) = O(L_{2t})$, hence we also have equalities

$$\widetilde{O}(\Lambda_{K3^{[2]}}, h) = \widehat{O}(\Lambda_{K3^{[2]}}, h) = O(\Lambda_{K3^{[2]}}, h).$$

Moreover, in this case, the condition $\omega = 1$ is always verified, hence $\widehat{O}(\Lambda_{K3^{[2]}}, h) = \widetilde{O}(h^\perp)$ is a normal subgroup of $O(h^\perp)$ (the inclusion i_1, i_2, and i_3 are equalities).

We restrict ourselves to the case $\omega = 1$. In this case, the chain (16) takes the form

$$\widetilde{O}(h^\perp) = \widetilde{O}(L_{2t}, h) \xhookrightarrow{i_2} \widehat{O}(L_{2t}, h) \xhookrightarrow{i_3} O(L_{2t}, h) \xhookrightarrow{i_4} O(h^\perp). \tag{29}$$

Remark 4.15. Beri et al. [BBBF] showed that, in the case $\omega = 1$, the inclusion

$$i_2 : \widetilde{O}(L_{2t}, h) \longhookrightarrow \widehat{O}(L_{2t}, h)$$

is trivial if and only if $t = 1$ or $\gamma > 2$. Their proof goes as follows.

Suppose that there exists $g \in \widehat{O}(L_{2t}, h) \setminus \widetilde{O}(L_{2t}, h)$. The isometry g satisfies (see (21) and (22))

$$g\left(\frac{l}{2t}\right) = -\frac{l}{2t} + m$$

for some vector $m \in L_{2t}$. Therefore,

$$\gamma(e + bf) + cl = h = g(h) = \gamma g(e + bf) - cl + 2tcm, \tag{30}$$

from which we obtain

$$\frac{2t}{\gamma}cm - \frac{2c}{\gamma}l = (e + bf) - g(e + bf) \in L_{2t}.$$

Since $\gamma \mid 2t$, this implies that $\frac{2c}{\gamma}$ is an integer. Since γ and c are coprime, it follows that γ divides 2.

Conversely, if γ is either 1 or 2, the integer c, being prime to γ and determined modulo γ, is 0 or 1 respectively. In this cases, the vector $y = ctf + l$ defines a reflection r_y on L_{2t} such that $r_y(l_*) = -l_* \in A_{L_{2t}}$. Indeed, the vector y has square $-2t$ and divisibility equal to $(ct, 2t)$. Hence, it defines a reflection on L_{2t}, because $-2t \mid 2(ct, 2t)$ (see Section 5) and

$$r_y\left(\frac{l}{2t}\right) = \frac{l}{2t} - \frac{2}{2t}\frac{y \cdot l}{y^2}y = \frac{l}{2t} - \frac{2}{2t}(ctf + l) \equiv -\frac{l}{2t} \pmod{L_{2t}}.$$

We have proven in Proposition 4.14 that, for $\omega = 1$, the discriminant group $A_{h^\perp}$ is of the form

$$A_{h^\perp} = \mathbb{Z}/\tfrac{2d}{\gamma}\mathbb{Z} \times \mathbb{Z}/\tfrac{2t}{\gamma}\mathbb{Z},$$

where the first factor is equal to the subgroup $p(H)$. Observe that, if $\gamma = 1$ or $\gamma = 2$, the reflection r_y induces the isometry

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on the discriminant group. Indeed, keeping the notation of Proposition 4.14, the vector y is equal to $\frac{2t}{\gamma}k_2$, therefore r_y acts as $-\mathrm{id}$ on k_2 and as id on k_1 (which is orthogonal to k_2). Hence, if $\frac{2t}{\gamma} \neq 2$, the isometry r_y is an element of $\widehat{O}(L_{2t}, h) \setminus \widetilde{O}(L_{2t}, h)$. Since $\gamma \mid 2$, $t \neq 1$ and $(\frac{2t}{\gamma}, \gamma) = 1$ the condition $\frac{2t}{\gamma} \neq 2$ is always satisfied, therefore the inclusion i_2 has index 2 in these cases.

Hence, if $\omega = 1$ and $\gamma > 2$, the group $\widehat{O}(L_{2t}, h)$ is equal to $\widetilde{O}(h^\perp)$, and thus it is a normal subgroup of $O(h^\perp)$.

Proposition 4.16. *If $\omega = 1$ and γ is 1 or 2, the image of $\widehat{O}(L_{2t}, h)$ under the morphism $O(h^\perp) \xrightarrow{r} O(A_{h^\perp})$ defined in Theorem 1.17 is*

$$r(\widehat{O}(L_{2t}, h)) = \{\mathrm{id}, s\}.$$

Proof. We already showed in Remark 4.15 that the isometry $s \in O(A_{h^\perp})$ is in the image $r(\widehat{O}(L_{2t}, h))$. Moreover, in Proposition 4.14, we showed that if $g \in \widetilde{O}(L_{2t}, h)$, then $r(g) = \mathrm{id}$.

Let g be an isometry of $\widehat{O}(L_{2t}, h) \setminus \widetilde{O}(L_{2t}, h)$, namely g satisfies $r(g)(\bar{k}_1) = \bar{k}_1$ and $g(\frac{l}{2t}) = -\frac{l}{2t} + m$ for some $m \in L_{2t}$ (see (22)). We show that $r(g)(\bar{k}_2) = -\bar{k}_2$; that implies $r(g) = s$.

For $\gamma = 1$, the integer c is 0, and the vector $\frac{l}{2t}$ is orthogonal to $h = e + bf$. Hence, since g is an isometry of L_{2t} that fixes h, the vector $g(\frac{l}{2t})$ is orthogonal to $g(h) = h$, and that implies $m \in h^\perp$. Moreover, in this case, $k_2 = \frac{l}{2t}$, thus we have $r(g)(\bar{k}_2) = -\bar{k}_2$.

For $\gamma = 2$, the integer c is 1, and we have $h = 2(e + bf) + l$. Recall that in the proof of Proposition 4.14, we showed that if $r(g)(\bar{k}_1) = \bar{k}_1$, then $g(f) \equiv f \pmod{h^\perp}$. Therefore, from equation (30), we obtain

$$2(e + bf) + l = h = g(h) \equiv 2(g(e) + bf) - l + 2tm \pmod{h^\perp}.$$

Hence, we have $2tm \equiv 2e + 2l - 2g(e) \pmod{h^\perp}$. Observe that the vector $h_1 = e - bf$ is in the lattice $h^\perp$, therefore $g(h_1)$ is in $h^\perp$ too, and that implies $g(e) \equiv bg(f) \equiv bf \pmod{h^\perp}$. Thus, we obtain

$$2tm \equiv 2e + 2l - 2g(e) \equiv 2e + 2l - 2bf \equiv 2l \equiv 2(l + tf) - 2tf \equiv -2tf \pmod{h^\perp},$$

where we used that $y = l + tf$ is orthogonal to h. Therefore, the vector $m + f$ is an integral vector that belong to the lattice $h^\perp$, thus we have $m \equiv -f \pmod{h^\perp}$.

Finally, by computing the image of $k_2 = f + \frac{l}{t}$, we have

$$g(k_2) \equiv f - \frac{l}{t} + 2m \equiv f - \frac{l}{t} - 2f \equiv -k_2 \pmod{h^\perp}.$$

As explained above, this proves $r(g) = s$.

□

Theorem 1.17 implies that, if the unimodular part M of $h^\perp$ has rank at least 2 (which is the case for $L_{2t} = \Lambda_{K3^{[t+1]}}$), the morphism $O(h^\perp) \xrightarrow{r} O(A_{h^\perp})$ is surjective. In particular, in this case and under the hypotheses of the previous proposition, the group $\widehat{O}(L_{2t}, h)$ is normal in $O(h^\perp)$ if and only if the group

$$K = \left\{ \mathrm{id}, s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

is a normal subgroup of $O(A_{h^\perp})$.

Example 4.17. The group K is not always a normal subgroup of $O(A_{h^\perp})$. For example, for $t=9$, $\gamma=2$ and $d=15$, the group $A_{h^\perp}$ is of the form

$$A_{h^\perp} = \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$$

with quadratic form defined by $q(1,0) \equiv -\frac{2}{15} \pmod{2\mathbb{Z}}$ and $q(0,1) \equiv -\frac{2}{9} \pmod{2\mathbb{Z}}$. The morphism g defined by the matrix $\begin{pmatrix} 1 & 10 \\ 6 & 2 \end{pmatrix}$ is an isometry of $A_{h^\perp}$: indeed it is an involution and for each $(x,y) \in A_{h^\perp}$, we can compute

$$q(g(x,y)) = q((x+10y, 6x+2y)) \equiv -\frac{2}{15}x^2 - \frac{2}{9}y^2 = q(x,y) \pmod{2\mathbb{Z}}.$$

Moreover

$$\begin{pmatrix} 1 & 10 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 10 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}.$$

Therefore, in this case, K is not a normal subgroup of $O(A_{h^\perp})$.

Lemma 4.18. *Let A be the group*

$$A = \mathbb{Z}/\tfrac{2d}{\gamma}\mathbb{Z} \times \mathbb{Z}/\tfrac{2t}{\gamma}\mathbb{Z}.$$

If t and d are coprime integers and γ is either 1 or 2, then $K = \{\mathrm{id}, s\}$ is a normal subgroup of $\mathrm{Aut}(A)$.

Proof. If $\gamma = 2$, then $A = \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/t\mathbb{Z}$ with $(t,d)=1$. Hence $A \simeq \mathbb{Z}/td\mathbb{Z}$ and from Lemma 1.14 we obtain that $\mathrm{Aut}(A)$ is abelian, and in particular it follows $K \lhd \mathrm{Aut}(A)$. We now consider the case $\gamma = 1$, hence $A = \mathbb{Z}/2d\mathbb{Z} \times \mathbb{Z}/2t\mathbb{Z}$. Let $g = \begin{pmatrix} a & e \\ b & f \end{pmatrix}$ be an automorphism of A. The order of (a,b) in A is equal to the order $2d$ of $(1,0)$ in A, hence we obtain

$$\mathrm{lcm}\left(\frac{2d}{(a,2d)}, \frac{2t}{(b,2t)}\right) = \mathrm{ord}_A\,(a,b) = 2d.$$

Since t and d are coprime, it follows that $t \mid b$ and we can write $b = tb'$. Analogously, we can write $e = de'$. Therefore, we can compute

$$gs = \begin{pmatrix} a & de' \\ tb' & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -de' \\ tb' & -f \end{pmatrix}$$

and

$$sg = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & de' \\ tb' & f \end{pmatrix} = \begin{pmatrix} a & de' \\ -tb' & -f \end{pmatrix}.$$

Since $tb' \equiv -tb' \pmod{2t}$ and $de' \equiv -de' \pmod{2d}$, we obtain $sg = gs$ and hence $g^{-1}sg = s$.

In both cases, we proved $K \lhd \mathrm{Aut}(A)$. □

Since $O(A_{h^\perp})$ is a subgroup of $\mathrm{Aut}(A_{h^\perp})$ that contains K, we obtain the following corollary.

Corollary 4.19. *Let $h \in L_{2t}$ be a primitive vector of square $2d$ such that $(t,d)=1$. The group $\widehat{O}(L_{2t}, h)$ is a normal subgroup of $O(h^\perp)$.*

Proof. Since the divisiblity of h divides $(2t, 2d)$, if t and d are coprime, it follows that γ is either 1 or 2. Moreover, from $(t, d) = 1$ we obtain $\omega = (\frac{2t}{\gamma}, \frac{2d}{\gamma}, \gamma) = 1$. Proposition 4.14 provides an isomorphism

$$A_{h^\perp} \simeq \mathbb{Z}/\tfrac{2d}{\gamma}\mathbb{Z} \times \mathbb{Z}/\tfrac{2t}{\gamma}\mathbb{Z}$$

where t, d and γ satisfy the hypotheses of Lemma 4.18. Therefore, the group K is normal in $\mathrm{Aut}(A_{h^\perp})$ and hence in $O(A_{h^\perp})$. Since $K = r^{-1}(\widehat{O}(L_{2t}, h))$, we obtain that $\widehat{O}(L_{2t}, h)$ is a normal subgroup of $O(h^\perp)$. □

Remark 4.20. To sum up, we have proved the following.

- If $t = 1$, then $\widehat{O}(L_{2t}, h) = \widetilde{O}(h^\perp)$;
- if $t > 1$ and $\omega = 1$, then
  - for $\gamma > 2$, then $\widehat{O}(L_{2t}, h) = \widetilde{O}(h^\perp)$,
  - for $\gamma \in \{1, 2\}$, then $\widehat{O}(L_{2t}, h) = r^{-1}\left(\left\{\mathrm{id}, s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}\right)$.

Moreover, we proved that in the following cases

a) $\omega = 1$ and $\gamma > 2$;

b) $(t, d) = 1$,

the group $\widehat{O}(L_{2t}, h)$ is a normal subgroup of $O(h^\perp)$.

5 Invariant divisors

5.1 Ramification divisors of quotients of period domains

For each even indefinite lattice Λ of signature $(2, n_-)$, with $n_- \geq 2$, we have defined in Section 3.2 a period domain

$$\mathscr{D} = \{[x] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid x \cdot x = 0, x \cdot \bar{x} > 0\}$$

with a natural action of the group of isometries $O(\Lambda)$. For each subgroup $\Gamma < O(\Lambda)$ of finite index, we have shown that the quotient

$$\mathcal{K}_\Gamma = \mathscr{D}/\Gamma$$

is a quasi-projective variety, which is irreducible if there exists an element of Γ that exchanges the two connected components of $\mathscr{D}$. In the following, we will suppose that there exists such an element in Γ. Observe that $-\mathrm{id}$ acts trivially on $\mathscr{D}$, hence on $\mathcal{K}_\Gamma$. Let $\widehat{\Gamma}$ be the group generated by Γ and $-\mathrm{id}$: then $\mathcal{K}_{\widehat{\Gamma}} = \mathcal{K}_\Gamma$.

Consider a subgroup $O < O(\Lambda)$ of finite index such that

$$\Gamma \lhd O < O(\Lambda).$$

Since $-\mathrm{id}$ is in the center of $O(\Lambda)$, the group $\widehat{\Gamma}$ is a normal subgroup of $\widehat{O}$. The group

$$G = \widehat{O}/\widehat{\Gamma}$$

is a finite group that acts on $\mathcal{K}_\Gamma$.

Remark 5.1. An element $[x] \in \mathcal{K}_\Gamma$ is fixed by $g \in G$ if and only if there exists an isometry $f \in O$ such that $[f] = g$ and x is an eigenvector of $f_{\mathbb{C}}$.
Indeed, if $[x] \in \mathcal{K}_\Gamma$ is fixed by $g = [f]$, then $[f(x)] = [x] \in \mathcal{K}_\Gamma$. This means that there exists $\tilde{f} \in \widehat{\Gamma}$ such that the lines $f_{\mathbb{C}}(x)\mathbb{C}$ and $\tilde{f}_{\mathbb{C}}(x)\mathbb{C}$ are equal. Replacing f by $\tilde{f}^{-1}f$ we obtain that x is an eigenvector of $f_{\mathbb{C}}$. Conversely, by definition, each eigenvector of $f_{\mathbb{C}}$ defines a line that is fixed by $[f]$.

Therefore, the action of G on $\mathcal{K}_\Gamma$ is (very)-generally faithful: let X be the subset

$$X = \bigcup_{g \in G \setminus \{\mathrm{id}\}} \bigcup_{\substack{g=[f] \\ \lambda \in \mathrm{Sp}(f)}} V_\lambda(f) \subset \mathscr{D},$$

which is a countable union of closed subvarieties of $\mathscr{D}$ of codimension greater than or equal to 1. Then, for x not contained in X, the stabilizer of $[x]$ in $\mathcal{K}_\Gamma$ is trivial.

Hence, the action of G on $\mathcal{K}_\Gamma$ yields a Galois cover

$$q : \mathcal{K}_\Gamma \longrightarrow \mathcal{K}_O \tag{31}$$

with Galois group G.

The varieties $\mathcal{K}_\Gamma$ and $\mathcal{K}_O$ are normal varieties. By restricting the morphism q to the preimage of the smooth locus of $\mathcal{K}_O$, the purity of the branch locus implies that the branch locus has codimension 1. We want to characterize the ramification divisors of the cover q, namely the irreducible algebraic divisors of $\mathcal{K}_\Gamma$ contained in the fixed locus of a nontrivial element of G.

5.1.1 Heegner divisors and reflections

For all $\beta \in \Lambda$ primitive with $\beta^2 < 0$, we consider

$$\mathcal{H}_{\beta^\perp} = \mathrm{Im}\left(\mathscr{D} \cap \mathbb{P}(\beta^\perp \otimes \mathbb{C}) \longrightarrow \mathcal{K}_\Gamma\right).$$

Since the lattice $\beta^\perp$ has signature $(2, n_- - 1)$, we observe that

$$\mathscr{D}_{\beta^\perp} := \mathscr{D} \cap \mathbb{P}(\beta^\perp \otimes \mathbb{C}) = \{[x] \in \mathbb{P}(\beta^\perp \otimes \mathbb{C}) \mid x \cdot x = 0, x \cdot \bar{x} > 0\} = \mathrm{Gr}^{\mathrm{po}}(2, \beta^\perp \otimes \mathbb{R})$$

is not empty, and it is a hypersurface of $\mathscr{D}$.

Moreover, $\mathcal{H}_{\beta^\perp}$ is always an irreducible algebraic divisor of $\mathcal{K}_\Gamma$. Indeed $\mathcal{H}_{\beta^\perp}$ is the image of the morphism i defined by

$$\begin{array}{ccc} \mathscr{D}_{\beta^\perp} & \hookrightarrow & \mathscr{D} \\ \downarrow & & \downarrow \pi \\ \mathscr{D}_{\beta^\perp}/\{g \in \Gamma \mid g(\beta) = \pm\beta\} & \overset{i}{\hookrightarrow} & \mathscr{D}/\Gamma = \mathcal{K}_\Gamma. \end{array}$$

Observe that the restriction induces a morphism

$$\{g \in \Gamma \mid g(\beta) = \pm\beta\} \longrightarrow O(\beta^\perp)$$

whose image is a subgroup of finite index of $O(\beta^\perp)$. As Γ is of finite index in $O(\Lambda)$, and for each $g \in \Gamma$ with $g(\beta) = \pm\beta$, the isometry g^2 is in $O(\Lambda, \beta)$, it is enough to show that $O(\Lambda, \beta)$ is of finite index in $O(\beta^\perp)$. Proposition 1.19 implies

$$O(\Lambda, \beta) = \{f \in O(\beta^\perp) \mid \bar{f}|_{p(H)} = \mathrm{id}\},$$

where $p(H) < A_{\beta^\perp}$. Since $A_{\beta^\perp}$ is a finite group, its automorphism group is finite, hence there exists an integer n such that for each $f \in O(\Lambda)$, the isometry f^n restricts to the identity on the discriminant group. In particular, it is the identity on $p(H)$ and hence extends to $O(\Lambda, \beta)$.

Therefore, from Borel–Baily's theorem, it follows that

$$\mathscr{D}_{\beta^\perp}/\{g \in \Gamma \mid g(\beta) = \pm\beta\}$$

is algebraic. Another result of Borel [Bor72, Theorem 3.10] (with [Huy16, Remark 6.4.2]), implies that the morphism i is algebraic, hence $\mathcal{H}_\beta^\perp$ is an algebraic divisor of $\mathcal{K}_\Gamma$. Finally we observe that $\mathcal{H}_\beta^\perp$ is also irreducible. Indeed, $\mathscr{D} \cap \mathbb{P}(\beta^\perp \otimes \mathbb{C})$ has 2 connected components exchanged by complex conjugation, hence they are contained in two different components of $\mathscr{D}$. This implies that they are identified in the quotient.

Definition 5.2. A *Heegner divisor* of $\mathcal{K}_\Gamma$ is a divisor of the form $\mathcal{H}_{\beta^\perp} \subset \mathcal{K}_\Gamma$ for some primitive $\beta \in \Lambda$ with $\beta^2 < 0$.

Lemma 5.3. *Let β and γ be primitive vectors of Λ with negative squares. The divisors $\mathcal{H}_{\beta^\perp}$ and $\mathcal{H}_{\gamma^\perp}$ of $\mathcal{K}_\Gamma$ are equal if and only if β and γ are in the same $\widehat{\Gamma}$-orbit.*

Proof. Let $\pi : \mathscr{D} \to \mathcal{K}_\Gamma$ be the canonical projection. For each vector β that defines a Heegner divisor, the divisor $\mathcal{H}_{\beta^\perp}$ is the image via π of the period domain $\mathscr{D}_{\beta^\perp} \subset \mathscr{D}$. The connected components of $\mathscr{D}_{\beta^\perp}$ are $\mathscr{D}^+_{\beta^\perp} = \mathscr{D}^+ \cap \mathscr{D}_{\beta^\perp}$ and $\mathscr{D}^-_{\beta^\perp} = \mathscr{D}^- \cap \mathscr{D}_{\beta^\perp}$.

Observe that

$$\pi^{-1}(\mathcal{H}_{\beta^\perp}) = \bigcup_{g \in \widehat{\Gamma}} \mathscr{D}_{g(\beta)^\perp}.$$

Clearly, if β and γ are in the same $\widehat{\Gamma}$-orbit, they define the same Heegner divisor.

Conversely, if $\mathcal{H}_{\gamma^\perp} = \mathcal{H}_{\beta^\perp}$, then $\mathscr{D}_{\gamma^\perp}$ is contained in $\pi^{-1}(\mathcal{H}_{\beta^\perp})$ and in particular

$$\mathscr{D}^+_{\gamma^\perp} \subset \pi^{-1}(\mathcal{H}_{\beta^\perp}) \cap \mathscr{D}^+ = \bigcup_{g \in \widehat{\Gamma}} \mathscr{D}^+_{g(\beta)^\perp}.$$

Since $\mathscr{D}^+_{\gamma^\perp}$ is irreducible, there exists $g \in \widehat{\Gamma}$ such that $\mathscr{D}^+_{\gamma^\perp} = \mathscr{D}^+_{g(\beta)^\perp}$. As complex conjugation exchanges $\mathscr{D}^+$ and $\mathscr{D}^-$, we obtain $\mathscr{D}_{\gamma^\perp} = \mathscr{D}_{g(\beta)^\perp}$.

We show that this implies $g(\beta) = n\gamma$ for some integer $n \in \mathbb{Z}$, or equivalently that $g(\beta)^\perp = \gamma^\perp$. If not, the closed subvariety $\mathbb{P}(g(\beta)^\perp_{\mathbb{C}}) \cap \mathbb{P}(\gamma^\perp)$ is a hypersurface of $\mathbb{P}(\gamma^\perp)$ that contains $\mathscr{D}_{\gamma^\perp}$. Therefore, it contains its closure $\{[x] \in \mathbb{P}(\gamma^\perp) \mid x^2 = 0\}$, which is an irreducible quadric, hence not contained in any hypersurface.

With the same reasoning, from $\mathscr{D}_{\beta^\perp} \subset \pi^{-1}(\mathcal{H}_{\gamma^\perp})$, we obtain that there exists $g' \in \widehat{\Gamma}$ such that $g'(\gamma) = n'\beta$ for some $n' \in \mathbb{Z}$. Hence, we obtain $n = \pm 1$, and therefore γ and β are in the same $\widehat{\Gamma}$-orbit.

□

For each primitive vector $\beta \in \Lambda$, the reflection with respect to $\beta^\perp$ in $\Lambda_{\mathbb{Q}}$ is given by the formula

$$\forall x \in \Lambda \qquad r_\beta(x) = x - \frac{2x \cdot \beta}{\beta^2}\beta.$$

Observe that since β is primitive, r_β is in $O(\Lambda)$ if and only if $\beta^2 \mid 2\mathrm{div}(\beta)$.

Definition 5.4. A primitive vector $\beta \in \Lambda$ with $\beta^2 < 0$ *defines a nontrivial reflection in* G if $\beta^2 \mid 2\mathrm{div}(\beta)$, the reflection r_β is in the group $\widehat{O}$, and $[r_\beta] \in G$ is nontrivial.

If β defines a nontrivial reflection in G, then the Heegner divisor $\mathcal{H}_{\beta^\perp}$ is contained in the fixed locus of r_β.

5.1.2 The ramification divisors of $q : \mathcal{K}_\Gamma \longrightarrow \mathcal{K}_O$

The following theorem generalizes [Ste08, Proposition 3.8]: in situation (31), we show that the divisorial components of the ramification of q are Heegner divisors associated with reflections in G.

Theorem 5.5 ([GHS07, Corollary 2.13])**.** *Let Λ be an even lattice of signature $(2, n_-)$ with $n_- \geq 2$, and let Γ and O be subgroups of finite index of $O(\Lambda)$ such that $\Gamma \lhd O$, and Γ contains an isometry g that exchanges the two connected components of $\mathscr{D}$.*
An irreducible divisor $D \subset \mathcal{K}_\Gamma$ is fixed by a nontrivial element g of G if and only if it is a Heegner divisor $\mathcal{H}_{\beta^\perp}$, where β defines a nontrivial reflection in G and $g = [r_\beta]$.
Moreover, each irreducible divisor is contained in the fixed locus of at most one nontrivial element $g \in G$.

Proof. Remark 5.1 implies that the set of points of $\mathcal{K}_\Gamma$ fixed by $g \in G$ is

$$\mathrm{Fix}(g) = \pi\left(\bigcup_{[f]=g} \bigsqcup_{\lambda\in \mathrm{Sp}(f_{\mathbb{C}})} \mathbb{P}(V_\lambda(f_{\mathbb{C}})) \cap \mathscr{D}\right),$$

where $\pi : \mathscr{D} \to \mathcal{K}_\Gamma$ is the canonical projection and $V_\lambda(f_{\mathbb{C}})$ is the eigenspace of $f_{\mathbb{C}}$ relative to the eigenvalue λ.

Observe that if $\mathrm{Fix}(g)$ contains an irreducible divisor D, there exists an isometry $f \in \widehat{O}$ with $[f] = g$ and an eigenvalue λ of $f_{\mathbb{C}}$ such that $V_\lambda(f_{\mathbb{C}})$ has codimension 1. Indeed, D has codimension 1 in $\mathscr{D}$ and

$$\begin{aligned} D &= \pi\left(\pi^{-1}(D) \cap \bigcup_{[f]=g} \bigsqcup_{\lambda\in \mathrm{Sp}(f_{\mathbb{C}})} \mathbb{P}(V_\lambda(f_{\mathbb{C}})) \cap \mathscr{D}\right) \\ &= \bigcup_{\substack{[f]=g \\ \lambda\in \mathrm{Sp}(f_{\mathbb{C}})}} \pi\left(\pi^{-1}(D) \cap \mathbb{P}(V_\lambda(f_{\mathbb{C}}))\right), \end{aligned}$$

where the union is over a countable set, as Γ is countable. Hence at least one of the pieces $\pi(\pi^{-1}(D) \cap \mathbb{P}(V_\lambda(f_{\mathbb{C}})))$ has codimension 1, and so has $\pi^{-1}(D) \cap \mathbb{P}(V_\lambda(f_{\mathbb{C}}))$ and this implies the claim.

Moreover, since D is irreducible, we obtain

$$D = \pi(\mathbb{P}(V_\lambda(f_{\mathbb{C}})) \cap \mathscr{D}).$$

For each real operator, the eigenspace relative to an eigenvalue λ has the same dimension as the eigenspace relative to $\bar{\lambda}$. Since $f_{\mathbb{C}}$ is a real operator and an isometry, and the codimension of $V_\lambda(f_{\mathbb{C}})$ is 1, it follows that $\lambda = \pm 1$. Up to changing f into $-f$, we can suppose $\lambda = 1$.

Since $\mathrm{codim}(V_1(f_{\mathbb{Q}})) = \mathrm{codim}(\ker(\mathrm{id} - f_{\mathbb{Q}})) = \mathrm{codim}(V_1(f_{\mathbb{C}})) = 1$, there exists $\beta \in \Lambda$ primitive such that

$$V_1(f_{\mathbb{Q}}) = \beta^\perp \text{ and } f_{\mathbb{Q}|_{\beta^\perp}} = \mathrm{id}.$$

Observe moreover that $\beta^2 < 0$. Indeed if $x \in \mathscr{D} \cap \mathbb{P}(V_1(f_{\mathbb{C}}))$, then $f_{\mathbb{C}}(\bar{x}) = \bar{x}$, so $P = \mathrm{Re}(x)\mathbb{R} \oplus \mathrm{Im}(x)\mathbb{R} \subset V_1(f_{\mathbb{C}})$. As P is positive definite, it follows that $n_+(V_1(f_{\mathbb{C}})) = 2$, hence $\beta^2 < 0$. Hence f satisfies $f|_{\beta^\perp} = \mathrm{id}$ and $f(\beta) = -\beta$, namely $f_{\mathbb{Q}}$ is the reflection with respect to β and $[r_\beta] = [f] = g \in G$ is nontrivial. Therefore D is a Heegner divisor and β defines a nontrivial reflection in G.

Suppose there exists g and g' in G such that $D \subset \mathrm{Fix}(g) \cap \mathrm{Fix}(g')$. We have proved that there exist vectors β and γ that define nontrivial reflections such that $D = [\mathcal{H}_{\beta^\perp}] = [\mathcal{H}_{\gamma^\perp}]$ with $g = [r_\beta]$ and $g' = [r_\gamma]$. Lemma 5.3 implies that $\gamma = g\beta$ for some $g \in \widehat{\Gamma}$. Thus, since $r_{g\beta} = g r_\beta g^{-1}$, it follows that $g' = [r_{g\beta}] = [r_\beta] = g$. □

Hence, the ramification divisors of the morphism $q : \mathcal{K}_\Gamma \to \mathcal{K}_O$ are parametrized by the $\widehat{\Gamma}$-orbits of vectors $\beta \in \Lambda$ that define a nontrivial reflection in $\widehat{O}/\widehat{\Gamma}$.

We notice that given $g \in G$, the fixed locus $\mathrm{Fix}(g)$ may contain several divisorial components, namely we could have $g = [r_\beta]$ for several vectors β that are not in the same $\widehat{O}$-orbit.

5.2 Polarized K3 surfaces

We apply the results of Section 5.1 to the moduli space M_{2d} of polarized K3 surfaces of degree $2d$. We have constructed in Section 3.3 the period morphism

$$\wp : M_{2d} \hookrightarrow \mathscr{D}_\Lambda / \widetilde{O}(\Lambda),$$

where Λ is the orthogonal of a primitive vector $h \in \Lambda_{K3}$ of square $2d$, described in Example 3.6. In this case, Λ is a lattice of type L_{2d} (see (15)), hence the discriminant group is $A_\Lambda = \mathbb{Z}/2d\mathbb{Z}$ with quadratic form q_Λ defined by $q_\Lambda(\bar{1}) = \left[-\frac{1}{2d}\right]$.

The lattice Λ satisfies the hypothesis of Theorem 1.17, so $\widetilde{O}(\Lambda)$ is a normal subgroup of $O(\Lambda)$ with quotient

$$O(A_\Lambda) \simeq O(\Lambda)/\widetilde{O}(\Lambda).$$

Therefore, the lattice Λ and the group $\Gamma = \widetilde{O}(\Lambda) \lhd O(\Lambda)$ satisfy the hypotheses of Theorem 5.5. In this case, the group $\widehat{\Gamma}$ is equal to $\widehat{O}(\Lambda)$, and

$$G = O(\Lambda)/\widehat{O}(\Lambda) \simeq O(A_\Lambda)/\pm \mathrm{id}.$$

Hence, the ramification divisors of

$$q : \mathcal{K}_{\widehat{O}(\Lambda)} \longrightarrow \mathcal{K}_{O(\Lambda)}$$

are parametrized by the $\widehat{O}(\Lambda)$-orbits of vectors $\beta \in \Lambda$ that define a nontrivial reflection in G. We want to give numerical conditions for a vector $\beta \in \Lambda$ to define nontrivial reflections in G.

We will consider the (slightly) more general situation of a lattice Λ of the form $M \oplus \mathbb{Z}k$ with M even unimodular of signature $(2, n_-)$, with $n_- \geq 1$, $k^2 = -2d$, and $\Gamma = \widetilde{O}(\Lambda)$.

Theorem 5.6 ([GHS07, Corollary 3.4]). *Let $\beta \in \Lambda$ be a primitive vector with $\beta^2 < 0$. The vector β defines a nontrivial reflection in G if and only if β satisfies the conditions*

a) $\beta^2 \mid (2\mathrm{div}(\beta), 2d)$;

b) $\beta^2 \notin \{-2, -2d\}$.

Proof. Each $\beta \in \Lambda$ can be written in the form

$$\beta = am + bk,$$

where $a, b \in \mathbb{Z}$ and $m \in M$ is a primitive vector. The condition β primitive is equivalent to $(a, b) = 1$. If β is a primitive vector that defines a reflection, we have by Definition 5.4 $\beta^2 \mid 2\mathrm{div}(\beta) = 2(a, 2db)$. We can compute

$$\beta^2 = a^2 m^2 - 2db^2 \tag{32}$$

where m^2 is even.
From $\beta^2 \mid 2a \mid am^2$, we obtain $\beta^2 \mid 2db^2$. As β is primitive we have $(a,b)=1$. From $\beta^2 \mid (2a, 2db^2)$, we obtain $\beta^2 \mid 2d$. Thus condition a) is verified.

We observe that $r_\beta \in O(A_\Lambda)$ is the automorphism

$$1 = \left[\frac{k}{2d}\right] \longmapsto \left[r_\beta\left(\frac{k}{2d}\right)\right] = \left[\left(1 + 2\cdot\frac{2d}{\beta^2}b^2\right)\frac{k}{2d}\right] = 1 + 2\cdot\frac{2d}{\beta^2}b^2$$

of $\mathbb{Z}/2d\mathbb{Z}$. Hence $[r_\beta] \in G$ is nontrivial if and only if $1 + 2\cdot\frac{2d}{\beta^2}b^2 \not\equiv \pm 1 \pmod{2d}$, namely if and only if

$$\frac{2d}{\beta^2}b^2 \not\equiv 0 \pmod{d} \quad \text{and} \quad 1 + \frac{2d}{\beta^2}b^2 \not\equiv 0 \pmod{d}.$$

We show that if β is primitive with $\beta^2 < 0$ and satisfies a), this two conditions are equivalent to $\beta^2 \neq -2$ and $\beta^2 \neq -2d$.

- $\frac{2d}{\beta^2}b^2 \equiv 0 \pmod{d}$ implies $\beta^2 \mid 2b^2$. Since $\beta^2 \mid 2a$ and $(a,b)=1$, this implies $\beta^2 \mid 2$. As $\beta^2 < 0$, the only possibility is $\beta^2 = -2$. Conversely, for $\beta^2 = -2$, we have $d \mid \frac{2d}{\beta^2}b^2$.

- $1 + \frac{2d}{\beta^2}b^2 \equiv 0 \pmod{d}$ implies $d\beta^2 \mid \beta^2 + 2db^2$ and in particular β^2 is divisible by d. From a), it also satisfies $\beta^2 \mid 2d$. Hence $\beta^2 \in \{-d, -2d\}$. Observe that if $\beta^2 = -d$, the integer d is even, and, since $\beta^2 d \mid \beta^2 + 2db^2$, we get that d divides the odd number $1 + 2b^2$, which is absurd. The only possibility is $\beta^2 = -2d$. In this case, since $\beta^2 \mid 2a$ and m^2 is even, we obtain

$$\beta^2 d = -2d^2 \mid a^2m^2 = \beta^2 + 2db^2 \implies 1 + \frac{2d}{\beta^2}b^2 \equiv 0 \pmod{d}.$$

□

Corollary 5.7 ([Ste08, Theorem 3.3]). *The divisorial components of the ramification locus of $q : \mathcal{K}_{\widehat{O}(\Lambda)} \longrightarrow \mathcal{K}_{O(\Lambda)}$ are the Heegner divisors $\mathcal{H}_{\beta^\perp}$ such that β is primitive and satisfies the conditions*

a) $\beta^2 \mid (2\mathrm{div}(\beta), 2d)$*;*

b) $\beta^2 < 0$ *and* $\beta^2 \notin \{-2, -2d\}$*.*

Proof. We observed in Section 5.1 that the ramification components of the ramification locus of $q : \mathcal{K}_{\widehat{O}(\Lambda)} \longrightarrow \mathcal{K}_{O(\Lambda)}$ are the irreducible divisors of $\mathcal{K}_{\widehat{O}(\Lambda)}$ contained in the fixed locus of some nontrivial element of G, that we have characterized in Theorem 5.5.
Let $D \subset \mathcal{K}_{\widehat{O}(\Lambda)}$ be an irreducible divisor contained in the fixed locus of a nontrivial $g \in G$. From Theorem 5.5, we have $D = \mathcal{H}_{\beta^\perp}$ where β defines a nontrivial reflection in G, so by Theorem 5.6 it satisfies a) and b).

Conversely, if β satisfies a) and b), it defines a nontrivial reflection $[r_\beta] \in G$ and the divisor $\mathcal{H}_{\beta^\perp}$ is contained in its fixed locus. □

In particular the components of the ramification divisor of q meets the image of the period morphism $\wp : M_{2d} \hookrightarrow \mathscr{D}_{\widetilde{O}(\Lambda)}$ and hence define divisors on the moduli space M_{2d}. Indeed by [Huy16, 6.Ramark 3.7] we have that

$$\wp(M_{2d}) = \mathscr{D}_{\widetilde{O}(\Lambda)} \setminus \bigcup_{\beta^2=-2} \mathcal{H}_{\beta^2}.$$

5.3 Polarized hyper-Kähler manifolds of K3$^{[m]}$-type

We now apply the results of Section 5.1 to moduli spaces of polarized hyper-Kähler manifolds of K3$^{[m]}$-type. In Section 4.3 we constructed, for each positive integer m and each polarisation type $\tau = O(\Lambda_{K3^{[m]}})h$, a period morphism

$$\wp : M_\tau^{K3^{[m]}} \longrightarrow \mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[m]}}, h),$$

where $h^\perp$ is the lattice described in Proposition 4.10. In order to apply Theorem 5.5 to the cover

$$q : \mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[m]}}, h^\perp) \longrightarrow \mathscr{D}_{h^\perp}/O(h^\perp), \tag{33}$$

we need to study the normality of the subgroup $\widehat{O}(\Lambda_{K3^{[m]}}, h)$ of $O(h^\perp)$.

Let $2d$ be the square of h and let γ be the divisibility of h. We set

$$\omega = \left(\frac{2(m-1)}{\gamma}, \frac{2d}{\gamma}, \gamma\right).$$

In Remark 4.20, we proved that in the following cases, the group $\widehat{O}(\Lambda_{K3^{[m]}}, h)$ is a normal subgroup of $O(h^\perp)$.

a) $(m-1, d) = 1$;

b) $\omega = 1$ and $\gamma > 2$.

If $\widehat{O}(\Lambda_{K3^{[m]}}, h) \lhd O(h^\perp)$, the cover q described in (33) is a ramified Galois cover of group

$$G \simeq O(h^\perp)/\langle \widehat{O}(\Lambda_{K3^{[m]}}, h), -\mathrm{id}\rangle,$$

therefore, again by Remark 4.20, we obtain that

- If $t = 1$, or $\omega = 1$ and $\gamma > 2$, then $G \simeq O(A_{h^\perp})/\{\pm\mathrm{id}\}$;
- if $\omega = 1$ $\gamma \in \{1, 2\}$, then $G \simeq O(A_{h^\perp})/\langle s, -\mathrm{id}\rangle$.

In Theorem 5.5, we showed that the ramification divisors of q are parametrized by vectors $\beta \in h^\perp$ that define nontrivial reflections in G.

5.3.1 Vectors β that define nontrivial reflections

As in the case of polarizedK3 surfaces studied in Section 5.2, we would like to characterize vectors $\beta \in h^\perp$ that define nontrivial reflections in G, at least in some cases. We do that in the case $\gamma = 1$.

As in Section 4.4, we consider a lattice $L_{2t} = M \oplus U \oplus \mathbb{Z}l$ and a vector $h \in L_{2t}$ of square $2d$ and divisibility 1. We have shown in Section 4.3 that, up to isometries of L_{2t}, we can suppose that $h = e + df$, and that $\Lambda = h^\perp$ is a lattice

$$\Lambda = M \oplus \mathbb{Z}k \oplus \mathbb{Z}l,$$

where $k = e - df$, and with $k^2 = -2d$ and $l^2 = -2t$. Proposition 4.14 shows that the discriminant group A_Λ is isomorphic to

$$A_\Lambda \simeq \langle \bar{k}_1 \rangle \times \langle \bar{k}_2 \rangle \simeq \mathbb{Z}/2d\mathbb{Z} \times \mathbb{Z}/2t\mathbb{Z},$$

where $k_1 = \frac{e+df}{2d} - f = \frac{e-df}{2d} = k_*$ and $k_2 = \frac{l}{2t} = l_*$.

Each vector $\beta \in \Lambda$ can be written as

$$\beta = am + bk + cl$$

where a, b, c are relatively prime integers and $m \in M$ is a primitive vector. Such a vector β has divisibility $\mathrm{div}(\beta) = (a, 2db, 2tc)$ and square

$$\beta^2 = a^2m^2 - 2db^2 - 2tc^2. \tag{34}$$

If β defines a reflection, then, since $\beta \cdot k_* = -b$, we obtain

$$\begin{aligned} [r_\beta(k_*)] &= \left[k_* - 2\frac{\beta \cdot k_*}{\beta^2}\beta \right] \\ &= \left[k_* + 2\frac{b}{\beta^2}(2dbk_* + 2tcl_*) \right] \\ &= \left[\left(1 + \frac{4db^2}{\beta^2}\right) k_* + \frac{4tcb}{\beta^2} l_* \right] \end{aligned}$$

in A_Λ, and an analogous computation gives $[r_\beta(l_*)]$. Hence $[r_\beta] \in O(A_\Lambda)$ is the matrix

$$\begin{pmatrix} \left[1 + \frac{4db^2}{\beta^2}\right]_{2d} & \left[\frac{4dbc}{\beta^2}\right]_{2d} \\ \left[\frac{4tcb}{\beta^2}\right]_{2t} & \left[1 + \frac{4tc^2}{\beta^2}\right]_{2t} \end{pmatrix} \in O(\mathbb{Z}/2d\mathbb{Z} \times \mathbb{Z}/2t\mathbb{Z}). \tag{35}$$

The vector β defines a reflection if and only if $\beta^2 \mid 2\mathrm{div}(\beta)$. Observe that this implies

$$\beta^2 \mid 4db \qquad \text{and} \qquad \beta^2 \mid 4tc,$$

and therefore the entries of the matrix (35) are integers.

Recall that as $\gamma = 1$, if $\widehat{O}(L_{2t}, h) \lhd O(h^\perp)$, the group G is isomorphic to $O(A_\Lambda)/\{\pm s, \pm \mathrm{id}\}$, where $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The next theorem characterize vectors β that define a reflection r_β trivial in G.

Theorem 5.8. *Let $\beta \in M \oplus \mathbb{Z}(-2d) \oplus \mathbb{Z}(-2t)$ be a primitive vector with $\beta^2 < 0$. Let k be a generator of the factor $\mathbb{Z}(-2d)$ and let l be a generator of the factor $\mathbb{Z}(-2t)$.*
The vector β defines a reflection r_β such that $[r_\beta]$ is contained in the group $\{\pm s, \pm \mathrm{id}\}$ if and only if β satisfies the conditions:

a) $\beta^2 \mid 2\mathrm{div}(\beta)$;

b) *one has*

- *either $\beta^2 = -2$;*
- *or $\beta^2 = -2t$ and $2td \mid \beta \cdot k$;*
- *or $\beta^2 = -2d$ and $2td \mid \beta \cdot l$;*
- *or $\beta^2 = -2td$, $(t, d) = 1$, and $2td \mid (\beta \cdot k, \beta \cdot l)$.*

Proof. The vector β defines a reflection if and only if $\beta^2 \mid 2\mathrm{div}(\beta)$, and we have

$$\mathrm{div}(\beta) = (a, 2db, 2tc) \mid 2\,\mathrm{lcm}(t, d)(a, b, c) = 2\,\mathrm{lcm}(t, d), \tag{36}$$

where the last equality holds because β is primitive. Therefore,

$$\beta^2 \mid 4\,\mathrm{lcm}(t, d). \tag{37}$$

We want to characterize the reflections r_β such that

$$[r_\beta] \in \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm \mathrm{id} \right\}, \tag{38}$$

where the matrix $[r_\beta]$ is given in equation (35).

Assume (38) holds. The off-diagonal terms are then zero, namely $2t\beta^2 \mid 4tbc$ and $2d\beta^2 \mid 4dbc$, or equivalently

$$\beta^2 \mid 2bc. \tag{39}$$

As for the diagonal terms, we want to understand when they are equal to ± 1. For the first entry, we have

(a) $1 + \dfrac{4db^2}{\beta^2} \equiv 1 \pmod{2d}$ if and only if $2d\beta^2 \mid 4db^2$, or equivalently $\beta^2 \mid 2b^2$.

(b) $1 + \dfrac{4db^2}{\beta^2} \equiv -1 \pmod{2d}$ if and only if $2d\beta^2 \mid 2(\beta^2 + 2db^2)$, so exactly when

$$d\beta^2 \mid \beta^2 + 2db^2, \tag{40}$$

which yields $d \mid \beta^2$ and $\beta^2 \mid 2db^2$. We show that necessarily $2d \mid \beta^2$. Indeed, if not, β^2 and d has the same valuation at 2, hence d is even and, from $\beta^2 \mid 2db^2$, we obtain $\beta^2 \mid db^2$. Therefore, by (40), the even number d divides the odd number $1 + 2\frac{db^2}{\beta^2}$, and clearly it is not possible.
In conclusion, $1 + \dfrac{4db^2}{\beta^2} \equiv -1 \pmod{2d}$ implies $2d \mid \beta^2$ and $\beta^2 \mid 2db^2$.

The same argument applied to the second diagonal term yields analogous results with t and c in place of d and b respectively. Namely, we have

(a') $1+\dfrac{4tc^2}{\beta^2} \equiv 1 \pmod{2t}$ if and only if $\beta^2 \mid 2c^2$.

(b') $1+\dfrac{4db^2}{\beta^2} \equiv -1 \pmod{2d}$ if and only if

$$t\beta^2 \mid \beta^2 + 2tc^2. \tag{41}$$

Moreover, the last condition implies $2t \mid \beta^2$ and $\beta^2 \mid 2tc^2$.

Putting all together, we can characterize primitive vectors β such that $[r_\beta] \in \{\pm\mathrm{id}, \pm s\}$.

- $[r_\beta] = [\mathrm{id}]$ if and only if $\beta^2 = -2$.
  Clearly, for $\beta^2 = -2$, the matrix (35) is the identity. Conversely, if $[r_\beta] = [\mathrm{id}]$, the two diagonal terms are equal to 1, and (a) and (a') give
  $$\beta^2 \mid 2b^2 \qquad \text{and} \qquad \beta^2 \mid 2c^2.$$
  Moreover $\beta^2 \mid 2\mathrm{div}(\beta) \mid 2a$. Since $(a,b,c) = 1$, we obtain $\beta^2 \mid 2$ hence the only possibility is $\beta^2 = -2$, as we assumed $\beta^2 < 0$.

- $[r_\beta] = s$ if and only if $\beta^2 = -2t$ and $t \mid \beta \cdot (k/2d)$.
  Indeed, in this case, the second diagonal term must be equal to -1, hence from (b') we obtain
  $$\beta^2 = 2ti \text{ for some negative integer } i, \text{ and} \qquad 2ti = \beta^2 \mid 2tc^2.$$
  Moreover, since the first diagonal term is 1, (a) implies $2ti = \beta^2 \mid 2b^2$, and therefore $i \mid b^2$. Finally, we also have $2ti = \beta^2 \mid 2\mathrm{div}(\beta) \mid 2a$, hence
  $$i \mid (a, b^2, c^2) = 1.$$
  Thus we obtain $\beta^2 = -2t$. The number β^2 divides both $2b^2$ and $2a$. Since by (39) β^2 also divides $2bc$, we obtain $-2t = \beta^2 \mid 2(a, b^2, bc) \mid 2b$, which implies
  $$t \mid -b = \beta \cdot (k/2d).$$
  Conversely, for $\beta^2 = -2t$ with $\beta^2 \mid 2\mathrm{div}(\beta)$ and $t \mid b = \beta \cdot (k/2d)$, we show that $[r_\beta]$ is equal to s. Indeed, the only nontrivial check is to show that $\left[1+\dfrac{4tc^2}{\beta^2}\right]_{2t}$ is equal to $[-1]_{2t}$. By equation (41), this condition can be rewritten as $t\beta^2 \mid \beta^2 + 2tc^2$. By equation (34) we have
  $$\beta^2 + 2tc^2 = a^2m^2 - 2db^2,$$
  where $2t = \beta^2 \mid 2\mathrm{div}(\beta) \mid 2a$, $t \mid b$ by hypothesis, and m^2 is even. Hence, we obtain $t\beta^2 = 2t^2 \mid a^2m^2 - 2db^2 = \beta^2 + 2tc^2$, which is the condition we needed.

- Analogously, $[r_\beta] = -s$ if and only if $\beta^2 = -2d$ and $d \mid \beta \cdot (l/2t)$.

- $[r_\beta] = -\mathrm{id}$ if and only if $\beta^2 = -2td$ and $\beta^2 \mid (\beta\cdot k, \beta\cdot l)$.
  Observe that, if β is a primitive vector that defines a reflection, the conditions $\beta^2 = -2td$ and $\beta^2 \mid (\beta\cdot k, \beta\cdot l)$ imply $(t,d) = 1$.
  Indeed, we have $\beta\cdot k = -2db$ and $\beta\cdot l = -2dc$. Hence $\beta^2 \mid (\beta\cdot k, \beta\cdot l)$ is equivalent to $2td \mid 2db$ and $2td \mid 2tc$, and namely to $t \mid b$ and $d \mid c$.
  Therefore, since $\beta^2 \mid 2\mathrm{div}(\beta) \mid 2a$, it follows that (t,d) divides a, and also divides b and c, from the previous observation. Since β is primitive, this implies $(t,d) = 1$.

If $[r_\beta] = -\mathrm{id}$, the diagonal terms must be equal to -1. From (b) and (b') we obtain

$$2d \mid \beta^2 \qquad \text{and} \qquad \beta^2 \mid 2db^2,$$

$$2t \mid \beta^2 \qquad \text{and} \qquad \beta^2 \mid 2tc^2.$$

Therefore, we get $2\,\mathrm{lcm}(t,d) \mid \beta^2$, which in turns divides $4\,\mathrm{lcm}(t,d)$, by (37), so β^2 is either $-2\,\mathrm{lcm}(t,d)$ or $-4\,\mathrm{lcm}(t,d)$.
We exclude the case $\beta^2 = -4\,\mathrm{lcm}(t,d)$. Indeed, in this case, from $\beta^2 \mid 2db^2$ we obtain $\frac{4\,\mathrm{lcm}(t,d)}{2d} \mid b^2$, hence $2 \mid b^2$, and analogously, from $\beta^2 \mid 2tc^2$, we get $2 \mid c^2$. Since we also have $2 \mid a$, because $\beta^2 \mid 2\mathrm{div}(\beta) \mid 2a$, we get a contradiction, β being primitive.

Therefore $\beta^2 = -2\,\mathrm{lcm}(t,d)$. If we denote by s the number (t,d), and we write $t = s\tau$ and $d = s\delta$, then $(\tau,\delta) = 1$ and $\mathrm{lcm}(t,d) = s\tau\delta$. We show that, still under the hypothesis $[r_\beta] = -\mathrm{id}$, we have $s = 1$.

The condition (40) is equivalent to $1 + \dfrac{2db^2}{\beta^2} \equiv 0 \pmod d$. Thus,

$$1 + \frac{2s\delta b^2}{2s\tau\delta} \equiv 0 \pmod s, \qquad \text{hence} \qquad \frac{b^2}{\tau} \equiv -1 \pmod s.$$

In particular, we can write $b^2 = \tau b_1$, where $(b_1, s) = 1$. Analogously, using (41), we show that there exists c_1 coprime with s such that $c^2 = \delta c_1$. The vanishing of the off-diagonal terms condition (see equation (39)) gives

$$2s\tau\delta = -\beta^2 \mid 2bc.$$

Hence, for each prime q that divides s, we have

$$v_q(s) + v_q(\tau\delta) = v_q(s\tau\delta) \le v_q(bc) = \frac{v_q(b^2c^2)}{2} = \frac{v_q(\tau b_1\delta c_1)}{2} = \frac{v_q(\tau\delta)}{2}, \tag{42}$$

where in the last equality we used that $v_q(b_1) = v_q(c_1) = 0$ because $q \mid s$ and s is coprime to both b_1 and c_1. Equation (42) implies $v_q(s) \le 0$, which is absurd since $q \mid s$. Hence $s = 1$.

Therefore, we are left to consider the case $(t,d) = 1$, and $\beta^2 = -2td$. The divisibility relations $\beta^2 \mid 2\mathrm{div}(\beta) \mid 4db$ and $\beta^2 \mid 2db^2$ imply

$$-2td = \beta^2 \mid (4db, 2db^2) = 2d(2b, b^2),$$

and thus $t \mid (2b, b^2)$. Moreover, from equation (39) we have $-2td = \beta^2 \mid 2bc$, therefore we obtain

$$t \mid (bc, 2b, b^2) = b(c, 2, b).$$

We prove that this implies $t \mid b$. If $(c, 2, b) = 1$, the statement is clear. Otherwise, since $2 \mid (b, c)$ and β is primitive, then $2 \nmid a$. Therefore, since $-2td = \beta^2 \mid 2\mathrm{div}(\beta) \mid 2a$, we obtain that t is odd and hence if t divides $2b$, then it also divides b.

Analogously, we obtain $d \mid c$. Since $\beta \cdot k = -2db$ and $\beta \cdot l = -2tc$, we obtain the condition $\beta^2 \mid (\beta \cdot k, \beta \cdot l)$.

Conversely, as in the case $[r_\beta] = s$, direct computations show that if β is a primitive vector that defines a reflection of square $\beta^2 = -2td$ and such that $t \mid b$ and $d \mid c$, then $[r_\beta] = -\mathrm{id}$. Equation (39) is easily verified and the computation for equations (40) and (41) is the same as in the case $[r_\beta] = s$.

□

We observe that the conditions found on β are invariant under the action of $\widehat{O}(\Lambda_{K3^{[m]}}, h)$. Indeed, if $g \in \widehat{O}(\Lambda_{K3^{[m]}}, h)$, we have $g(l) = \pm l + 2tn$ and $g(k) = k + 2dn'$ for some $n, n' \in \Lambda$.

Therefore, if $\beta^2 = -2d$, then $2td \mid \beta \cdot l$ if and only if $2td \mid g(\beta) \cdot l$. Indeed,

$$\beta \cdot l = g(\beta) \cdot g(l) = g(\beta) \cdot (\pm l) + 2tg(\beta) \cdot n,$$

and, since $\beta^2 \mid 2\mathrm{div}(\beta) = \mathrm{div}(g(\beta))$, we have that $2td \mid 2tg(\beta) \cdot n$. The invariance of the other conditions can be shown in a similar way using $g(k) = k + 2dn'$.

This is our main result: we determine the ramification divisors of the Galois cover (33), in the case of polarised hyper-Kähler manifolds of polarisation type of square $2d$ and divisibility 1. It applies in particular when $(m-1, d) = 1$.

Corollary 5.9. *Let $h \in \Lambda_{K3^{[m]}}$ be a primitive vector of square $2d$ and divisibility 1 such that $\widehat{O}(\Lambda_{K3^{[m]}}, h)$ is a normal subgroup of $O(h^\perp)$. The divisorial components of the ramification locus of $q : \mathcal{K}_{\widehat{O}(\Lambda_{K3^{[m]}}, h^\perp)} \longrightarrow \mathcal{K}_{O(h^\perp)}$ are the Heegner divisors $\mathcal{H}_{\beta^\perp}$ such that β is primitive and satisfies the conditions*

a) $\beta^2 \mid 2\mathrm{div}(\beta)$;

b) β^2 *is such that:*

- $\beta^2 \neq -2$;
- *if* $\beta^2 = -2(m-1)$, *then* $2(m-1)d \nmid \beta \cdot k$;
- *if* $\beta^2 = -2d$, *then* $2(m-1)d \nmid \beta \cdot l$;
- *if* $\beta^2 = -2(m-1)d$, *then* $2(m-1)d \nmid (\beta \cdot k, \beta \cdot l)$.

5.3.2 Hyper-Kähler fourfolds

We now restrict to the case $m = 2$ of hyper-Kähler fourfolds of polarization type $\tau = O(\Lambda_{K3^{[2]}})h$, where h is a primitive vector of square $2d$ and divisibility γ. Since $\gamma \mid 2(m-1)$, we obtain that γ is either 1 or 2.

We consider the case $\gamma = 1$: since $m-1=1$, the group $\widehat{O}(\Lambda_{K3^{[2]}}$ is a normal subgroup of $O(h^\perp)$ and defines the Galois cover

$$q : \mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[2]}}, h^\perp) \longrightarrow \mathscr{D}_{h^\perp}/O(h^\perp).$$

Corollary 5.9 implies that the divisorial components of the ramification locus of q, are the Heegner divisors $\mathcal{H}_{\beta^\perp}$ such that β is primitive and satisfies the conditions

a) $\beta^2 \mid 2\mathrm{div}(\beta)$;

b) $\beta^2 \neq -2$ and if $\beta^2 = -2d$, then $2d \nmid \beta \cdot l$.

Observe that, from Equation (37), if β defines a reflection, then $\beta^2 \mid 4d$.

In [DM19], Debarre and Macrì characterized the image of the period morphism of polarized hyper-Kähler fourfolds of square $2d$ and divisibility γ. We would like to characterize those ramification divisors that meet the this image.

For each primitive rank-2 sublattice K of $\Lambda_{K3^{[2]}}$ of signature $(1,1)$ that contains the vector h, the authors denote by $\mathscr{D}^{(1)}_{2d,K}$ the divisor of $\mathscr{D}_{h^\perp}/\widetilde{O}(h^\perp)$ cut out by the codimension-2 subspace $\mathbb{P}(K^\perp_{\mathbb{C}}) \subset \mathbb{P}((\Lambda_{K3^{[2]}})_{\mathbb{C}})$. Namely, if $K \cap h^\perp = \mathbb{Z}\beta$ for some primitive vector $\beta \in h^\perp$, the divisor $\mathscr{D}^{(1)}_{2d,K}$ is the Heegner divisor $\mathcal{H}_{\beta^\perp}$. Moreover, for each positive integer D, the authors set

$$\mathscr{D}^{(1)}_{2d,D} := \bigcup_{\mathrm{disc}(K^\perp)=D} \mathscr{D}^{(1)}_{2d,K} \subset \mathscr{D}_{h^\perp}/\widetilde{O}(h^\perp).$$

The image of the period morphism

$$\wp_{K3^{[2]}} : M^{K3^{[2]}}_{2d,1} \longrightarrow \mathscr{D}_{h^\perp}/\widetilde{O}(h^\perp),$$

for polarized hyper-Kähler fourfolds of K3$^{[2]}$-type and polarization type defined by a vector h of square $2d$ and divisibility 1 is described in [DM19, Theorem 6.1]. In particular, they show that the following holds.

Proposition 5.10 ([DM19, Theorem 6.1]). *The image of the period morphism $\wp_{K3^{[2]}}$ is the complement of certain irreducible Heegner divisors contained in the hypersurfaces $\mathscr{D}^{(1)}_{2d,2d}$, $\mathscr{D}^{(1)}_{2d,8d}$, $\mathscr{D}^{(1)}_{2d,10d}$ and $\mathscr{D}^{(1)}_{2d,\frac{2d}{5}}$, where the last case occurs only for $d \equiv \pm 5 \pmod{25}$.*

We now determine when a Heegner divisor $\mathcal{H}_{\beta^\perp}$ is contained in one of these hypersurfaces, for $\beta \in h^\perp$ primitive vector of negative square that defines a reflection.

Proposition 5.11. *The Heegner divisors $\mathcal{H}_{\beta^\perp}$ defined by primitive vectors β that define a reflection, such that*

- $\beta^2 \neq -2, \beta^2 \neq -8$;
- *and if $d \equiv \pm 5 \pmod{25}$, $\beta^2 \neq -10$ and $\beta^2 \neq -45$,*

define divisors in the moduli space $M^{K3^{[2]}}_{2d,1}$.

Proof. Observe that, if $K \cap h^\perp = \mathbb{Z}\beta$ for some vector β of negative square, then the lattices $K^\perp$ and $\langle h, \beta\rangle^\perp$ are equal. In particular, using [GHS13, Lemma 7.2], we can compute

$$\operatorname{disc}(K^\perp) = \operatorname{disc}(\langle h, \beta\rangle^\perp) = \frac{-\beta^2 \operatorname{disc}(h^\perp)}{\operatorname{div}(\beta)^2} = \frac{-4d\beta^2}{\operatorname{div}(\beta)^2}, \tag{43}$$

where we used that $\operatorname{disc}(h^\perp) = |A_{h^\perp}| = 2d \cdot 2$.
Therefore, the Heegner divisor $\mathcal{H}_{\beta^\perp}$ is contained in the locus $\mathscr{D}^{(1)}_{2d, \frac{-4d\beta^2}{\operatorname{div}(\beta)^2}}$.

If β is a primitive vector of negative square that defines a reflection, then $\beta^2 \mid 2\operatorname{div}(\beta)$ and, since $\operatorname{div}(\beta)$ always divides β^2, we have that β^2 is equal to either $-\operatorname{div}(\beta)$ or $-2\operatorname{div}(\beta)$. Hence,

- for $\beta^2 = -\operatorname{div}(\beta)$, the formula in (43) yields

  $$\operatorname{disc}(\langle h, \beta\rangle^\perp) = -\frac{4d}{\beta^2},$$

  where $\beta^2 \mid 2d$. Hence the Heegner divisor $\mathcal{H}_{\beta^\perp}$ is contained in the locus $\mathscr{D}^{(1)}_{2d, -2\frac{2d}{\beta^2}}$. Proposition 5.10 implies that, if

  $$-2\frac{2d}{\beta^2} \notin \left\{2d, 8d, 10d, \frac{2d}{5}\right\},$$

  where the last case only occurs for $d \equiv \pm \pmod{25}$, the Heegner divisor $\mathcal{H}_{\beta^\perp}$ meets the image of $\wp_{K3^{[2]}}$. Namely,

  $$\text{if } \beta^2 \neq -2 \text{ and, for } d \equiv \pm 5 \pmod{25},\ \beta^2 \neq -10,$$

  the divisor $\mathcal{H}_{\beta^\perp}$ defines a divisor of the moduli space $M^{K3^{[2]}}_{2d,1}$.

- for $\beta^2 = -2\operatorname{div}(\beta)$, the formula in (43) yields

  $$\operatorname{disc}(\langle h, \beta\rangle^\perp) = -\frac{16d}{\beta^2},$$

  where $\beta^2 \mid 4d$. Hence the Heegner divisor $\mathcal{H}_{\beta^\perp}$ is contained in the locus $\mathscr{D}^{(1)}_{2d, -2\frac{8d}{\beta^2}}$. Proposition 5.10 implies that, if

  $$-2\frac{8d}{\beta^2} \notin \left\{2d, 8d, 10d, \frac{2d}{5}\right\},$$

  where the last case only occurs for $d \equiv \pm 5 \pmod{25}$, the Heegner divisor $\mathcal{H}_{\beta^\perp}$ meets the image of $\wp_{K3^{[2]}}$. Namely,

  $$\text{if } \beta^2 \neq -2,\ \beta^2 \neq -8, \text{ and, for } d \equiv \pm \pmod{25}, \text{ if } \beta^2 \neq -45,$$

  the divisor $\mathcal{H}_{\beta^\perp}$ defines a divisor of the moduli space $M^{K3^{[2]}}_{2d,1}$.

□

5.3.3 Hyper-Kähler fourfolds of polarisation of square 2.

We consider the case of a polarisation type defined by a vector h of square 2 ($d = 1$). In this case γ is 1 (see Remark 4.9),the group of isometries of $A_{h^\perp}$ is

$$O(A_{h^\perp}) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{Z}/2\mathbb{Z},$$

and, since $-\mathrm{id}$ and id define the same isometry of $A_{h^\perp}$, we have $G \simeq Z/2\mathbb{Z}$.

If β is a primitive vector of $\Lambda_{K3^{[2]}}$ that defines a reflection, then $\beta^2 \in \{-2, -4\}$ (see (37)). For $\beta^2 = -2$, the reflection $[r_\beta]$ is the identity on the discriminant group, while for $\beta^2 = -4$, the action of $[r_\beta]$ on the discriminant group is nontrivial, and $\mathrm{Fix}([r_\beta])$ contains the Heegner divisor $\mathcal{H}_{\beta^\perp}$. Proposition 5.11 implies that $\mathcal{H}_{\beta^\perp}$ meets the image of the period morphism

$$\wp_{K3^{[2]}} : M_{2,1}^{K3^{[2]}} \longrightarrow \mathscr{D}_{h^\perp}/\widetilde{O}(h^\perp),$$

and therefore induces a divisor in the moduli space $M_{2,1}^{K3^{[2]}}$ of polarised hyper-Kähler fourfolds of square 2.

Observe that, if we write $h^\perp = M \oplus \mathbb{Z}k \oplus \mathbb{Z}l$, where $k^2 = -2$ and $l^2 = -2$, the vector $\gamma = k + l$ is a vector of square -4 that defines a nontrivial reflection in G.

Remark 5.12. All primitive vectors $\beta \in \Lambda$ of square -4 that define a reflection, are conjugated by an element of $\widetilde{O}(\Lambda)$, hence $\mathcal{H}_{\beta^\perp} = \mathcal{H}_{\gamma^\perp}$.
From Eichler's Lemma 1.18, we know that the $\widetilde{O}(\Lambda)$-orbit of a vector β is uniquely determined by β^2 and $\beta_* \in A_{h^\perp}$. Notice that, for each primitive vector β of square -4 that defines a reflection, $\mathrm{div}(\beta) = 2$. Indeed, $\mathrm{div}(\beta) \mid 2$ from Equation (36), and $-4 = \beta^2 \mid 2\mathrm{div}(\beta)$ because β defines a reflection.
We write $\beta = am + bk + cl$, where a, b, c are integers and $m \in M$ is a primitive vector. Recall that $A_{h^\perp} = \langle k_* \rangle \times \langle l_* \rangle$ where $k_* = \left[\frac{k}{2}\right]$ and $l_* = \left[\frac{l}{2}\right]$. Therefore,

$$\beta_* = \left[\frac{\beta}{\mathrm{div}(\beta)}\right] = \bar{b}k_* + \bar{c}l_* \in A_{h^\perp}.$$

We show that $\bar{b} = \bar{c} = 1 \in \mathbb{Z}/2\mathbb{Z}$, hence $\beta_* = k_* + l_*$. This is enough to finish the proof.

Since $\mathrm{div}(\beta) = (a, 2b, 2c) = 2$, we can write $a = 2a_1$ for some integer a_1. By computing the square of β

$$-4 = (2a_1)^2 m^2 - 2b^2 - 2c^2$$

we obtain that $2 \mid b^2 + c^2$, from which we obtain that b and c have the same parity. Since a is even and a, b, c are coprime, it follows that b and c are both odd.

Corollary 5.13. *Let $h \in \Lambda_{K3^{[2]}}$ be a primitive vector of square 2. The ramification divisor of the cover*

$$q : \mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[2]}}, h^\perp) \longrightarrow \mathscr{D}_{h^\perp}/O(h^\perp).$$

is irreducible and meets the image of the period morphism

$$\wp_{K3^{[2]}} : M_{2,1}^{K3^{[2]}} \longrightarrow \mathscr{D}_{h^\perp}/\widetilde{O}(h^\perp).$$

Proof. The components of the ramification divisor of q are the irreducible divisors D of $\mathscr{D}_{h^\perp}/\widetilde{O}(h^\perp)$ contained in the fixed locus of some nontrivial element of G.
Keeping the notation of the previous remark, we write $h^\perp = M \oplus \mathbb{Z}k \oplus \mathbb{Z}l$. The only nontrivial element of G is $[r_\gamma]$, where $\gamma = k + l$. Theorem 5.5 shows that if D is an irreducible divisor contained in $\mathrm{Fix}([r_\gamma])$, then there exists a vector β that defines a nontrivial reflection in G, such that $D = \mathcal{H}_{\beta^\perp}$ and $[r_\gamma] = [r_\beta]$. Now, since $[r_\beta]$ is nontrivial, then $\beta^2 \neq -2$, and hence $\beta^2 = -4$. From the previous remark, we have $\mathcal{H}_{\beta^\perp} = \mathcal{H}_{\gamma^\perp}$. Therefore the ramification divisor of q is irreducible.
Finally, Proposition 5.11 show that $\mathcal{H}_{\gamma^\perp}$ meets in the image of $\wp_{K3^{[2]}}$. □

The moduli space $M_{2,1}^{K3^{[2]}}$ contains a dense open subset $U_{2,1}$ that is the moduli space of double EPW sextics (see [Deb17, Example 3.5]). The involution $[r_\gamma] \in G$ defines an involution on $U_{2,1}$ which is the duality involution of double EPW sextics studied by O'Grady in [O'G08, Theorem 1.1]. Moreover the Heegner divisors $\mathcal{H}_{\gamma^\perp}$ induces a divisor on $U_{2,1}$, that is exactly the divisors of autodual double EPW sextics.

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