DERIVED ENRIQUES

Non Emptiness in the GM case(s).

[KP17] " " "Derived cat of cyclic covers and their branch divisors"

[KP18] Kuz, Perry "Derived category of GM varieties"

[BP22] Bayer, Perry "Kuz conj via K3 cat and group actions"

[BLMS19] Bayer, Lahoz, Macrì, Stellari "Stab conds on Kuz cpt"

[FP23] Feyzbakhsh, Pertusi "Serre-inv stab cond and ulrich bundles on cubic 3-folds"

[PPZ23] Perry, Pertusi, Zhao "Moduli space of stable obj in Enriques cat"

[PR21] Pertusi, Robinett "Stab cond on Ku of GM 3fold and Serre inv"

GM 3-fold : Fano 3-fold of Index 1, deg 10

Def: A GM 3-fold X is a smooth transv. intersection

$n+4 \rightsquigarrow$ GM n-fold for $n=2,\dots,6$

$$X = C_{[\mathbb{C}]}Gr(2,V_5) \cap \mathbb{P}^{7} \cap Q \subseteq \mathbb{P}(\mathbb{C}\oplus\Lambda^2 V_5) = \mathbb{P}^{10}$$

$C_{[\mathbb{C}]}Gr(2,V_5) \subseteq \mathbb{P}(\Lambda^2 V_5)$, $Q \subseteq \mathbb{P}^7$ quad hypersur

X smooth, $[\mathbb{C}] \notin X \Rightarrow \gamma_X : X \longrightarrow Gr(2,V_5)$ projection from $[\mathbb{C}]$

- $[\mathbb{C}] \notin \mathbb{P}^7 \Rightarrow X \simeq Gr(2,V_5) \cap \mathbb{P}^7 \cap Q$ (ordinary)
- $[\mathbb{C}] \in \mathbb{P}^7 \Rightarrow X \xrightarrow{2:1} Gr(2,V_5) \cap \mathbb{P}^6$ (special)

  $\supseteq$ quadric section S K3 surface of degree 10 (BN general)

Rmq : X ordinary GM 3-fold is ramification locus of a special GM 4-fold ie

$$\exists \quad X_4 \xrightarrow{2:1} Gr(2,V_5) \cap \mathbb{P}^7 \supseteq X$$

- $\boxed{D^b(X) = \langle Ku(X), \mathcal{O}_X, \mathcal{U}_X^\vee \rangle}$ where $\mathcal{U}_X = \gamma_X^* \mathcal{U}$ [KP17]

[Kuz-Perry] $Ku(X)$ is an Enriques category Serre funct $Hom_{k}(E, S(F)) \simeq Hom_k(F,E)$

(X special) • $\mathbb{Z}/2\mathbb{Z}$ action given by the involution ι associated to the double cover

[KP18]

- CY2-cover : $D^b(S)$ · $\tau = T_{\mathcal{U}_S} \circ T_{\mathcal{O}_S} \circ (\otimes \mathcal{O}_S(1))$

(X ordinary) CY2-cover : $Ku(X_4) \circlearrowleft \mathbb{Z}/2\mathbb{Z}$ given by involution

In family : $\mathcal{X} \xrightarrow{\pi} S$ fam of GM 3fold, [DK18] $\mathcal{X} \longrightarrow Gr(2, \mathcal{V}_5)$

$$D_{perf}(\mathcal{X}) = \langle Ku(\mathcal{X}), \pi^*(D_{perf}(S)) \otimes \mathcal{O}_{\mathcal{X}}, \pi^*(D_{perf}(S)) \otimes \mathcal{U}_{\mathcal{X}}^\vee \rangle$$

that on the fibers recovers the s.o dec of X

Stability conditions on GM 3-folds [BLMS19]

* $v: K(X) \to K_{num}(X) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \langle k_1, k_2 \rangle = K$ [Kuz09]

* $X \rightsquigarrow$ tilt-procedure to obtain weak stability condition on $D^b(X)$
  $\rightsquigarrow$ the restriction to $Ku(X)$ is a num prop stab cond $\sigma_X = \sigma(\alpha, \beta)$ (*)

  $\exists g \in \widetilde{GL}_2^+ : S \circ \sigma = g \circ \sigma$

- [PR21] These stab. conditions are Serre invariant ($\Rightarrow$ τ invariant)

- Works in family: $\exists \underline{\sigma} = (\sigma_s) \in Stab(\mathcal{X})$ st σ_s one of the stab cond on $\mathcal{X}_s$ of (*)

Thm $v \in K_{num}(X)$, σ Serre-inv stab. condition $\quad M_\sigma(Ku(X), v) \neq \emptyset$

Reductions: 1. All Serre's inv conditions are in the same $\widetilde{GL}_2^+$-orbit [FP23]
We can suppose $\boxed{\sigma = \sigma_X}$

2. $M_\sigma(\mathcal{C}, v) \ni F \Rightarrow F^{\oplus k} \in M_\sigma(\mathcal{C}, kv)$
We can suppose v primitive

Idea: (A) Prove the thm for Y special GM 3-fold $\boxed{\text{st HYP1+2}}$
and show that $\exists E \in M_{\sigma_Y}(Ku(Y), v)$ not $\mathbb{Z}/2\mathbb{Z}$-inv.

(B) deform $X = \mathcal{X}_0$ to $Y = \mathcal{X}_1$ special GM 3-fold (as in (A)) in family $\mathcal{X} \xrightarrow{\pi} S$
with $\underline{\sigma} \in Stab(\mathcal{X})$ st $\sigma_0 = \sigma_X$, $\sigma_1 = \sigma_Y$

$M_{\underline{\sigma}}(Ku(\mathcal{X})/S, v) \longrightarrow S$ universally closed
$+$ $\boxed{\text{smothness criterion}}$ smooth at E $\Longrightarrow$ surjective

(A) Y special GM 3-fold: v is prim: $M_{\sigma_Y}(Ku(Y), v)$ of stable obj
with associated K3 S

- $D^b(S) \xrightarrow{\Phi} Ku(Y)$ CY2-cover: $K(Y)^{\langle \iota \rangle} \simeq D^b(S)$ σ_S induced stab. cond

- $\boxed{\text{If } \exists\, w \in K_{num}(S) \text{ st } \Phi_*(w) = v, \text{ and } w^2 \geq -2}$ $M_{\sigma_S}(S, w)$ smooth of dim $2 + w^2$

[HYP1]

$$\Phi : M_{\sigma_S}(S,w) \longrightarrow M_{\sigma_Y}(Ku(Y),v)$$ hence the latter is non empty

$\searrow$ $M_{\sigma_Y}(Ku(Y),v)^{<\iota>}$ $\nearrow$

Rmq ι_X fixes $K_{num}(Ku(Y))$

$\leftarrow$ $\mathbb{Z}/2\mathbb{Z}$-linearised obj in $M_{\sigma_Y}(Ku(Y),v)$

Fact
$$\bigsqcup_{\phi_X(w)=v} M_{\sigma_S}(S,w) \twoheadrightarrow M_{\sigma_Y}(Ku(Y),v)^{<\iota>} \subseteq M_{\sigma_Y}(Ku(Y),v)$$

$\dim \leq w^2+2$ (for $M_{\sigma_Y}(Ku(Y),v)^{<\iota>}$), $\dim \geq 1-\chi(v,v) = \dim Ext^1 - \dim Ext^2$ (for $M_{\sigma_Y}(Ku(Y),v)$)

[HYP2] $\boxed{w^2+2 < 1-\chi(v,v)}$ $\Rightarrow \exists\, E \in M_{\sigma_Y}(Ku(Y),v)$ not fixed by ι

Key: For S general ($\rho=1$), $\Phi_* : K_{num}(S) \to K_{num}(Ku(Y))$ is not surjective
so we really have to deforme to have [hyp 1]

(B) Given X GM 3-fold, $\exists\, Y \longrightarrow Gr(2,V_5)\cap\mathbb{P}^6$ special GM 3-fold
$\quad\quad$ $\cup$ S, $\quad \Phi : D^b(S) \longrightarrow Ku(Y)$

st $\exists\, w \in K_{num}(S)$ for which
- $\Phi_*(w) = v$, $w^2 \geq -2$
- $w^2+2 < 1-\chi(v,v)$

Idea:
- Period map for K3 surf is surjective but not all deg 10 K3 are branch locus of a special GM 3-fold. Need to avoid some "bad" divisors classified in [GLT15]
- Computations! $v = k_2 \left(= 2 - H + \frac{5}{6}p \in K_{num}(Ku(Y))\right)$

Take (S,H) deg 10 with $\mathrm{Pic}\, S = \langle H, L\rangle = \begin{pmatrix} 10 & 5 \\ 5 & 0 \end{pmatrix}$ and $\exists\, Y \longrightarrow Gr(2,V_5)\cap\mathbb{P}^6 \supseteq S$

$w = +\mathcal{O}_S(L) - 2\mathcal{O}_S$

[PPZ23] also prove that for

Thm2 X ordinary GM 3-fold, v primitive: $M_\sigma(Ku(X),v)$ is smooth projective of dimension $1-\chi(v,v)$.

We need ordinary [Zha21] $\exists\, Y$ special st $M_\sigma(Ku(Y),k_1)$ singular

Pf (Thm 2) . X ordinary GM 3fold, $W \xrightarrow{2:1} Gr(2,V_5) \cap \mathbb{P}^7 \supseteq X$ branch locus

$\mathrm{Sing}\left(M_{\sigma_X}(Ku(X), v)\right) \subseteq \{E \mid \tau(E) \simeq E\} \ni E$

Fact $E \in \mathcal{C} \circlearrowleft \langle \tau \rangle$ τ-inv + stable, $\exists F \in \mathcal{C}^{\langle \tau \rangle}$ st $\mathrm{Forg}\, F = E$

$$\begin{array}{ccccc} \rho(F') & & E \quad \tau\text{-inv} & & \\ Ku(W) & \xrightarrow{\mathrm{Inf}} & Ku(X) & \xrightarrow{\Phi} & \\ s \mid \rho & & \downarrow 2 & & \\ (Ku(W)^{\mathbb{Z}/2\mathbb{Z}}) & \longrightarrow & Ku(W)^{\mathbb{Z}/2\mathbb{Z}} & \longrightarrow & Ku(W) \\ F' & \longmapsto & E' \quad \tau\text{-inv} & & \rho(F') + \tau\rho(F') \\ \text{(using Fact)} & & & & \Uparrow \end{array}$$

$\Rightarrow [\mathrm{Forg}\, E'] = 2[\rho(F')]$

but Φ_* surj and v prim

and $\mathrm{Inf}(\rho(F')) = (\rho(F') + \tau(\rho(F')), \theta)$

- Involutions on special GM 4folds

$W \xrightarrow{2:1} Gr(2,V_5) \cap \mathbb{P}^7 \supseteq X$ GM 3-fold, $\circlearrowleft \tau$

then $Ku(X)$ Enriques
$Ku(W)$ 2CY cover with residual action induced by τ

$\Phi : Ku(X) \simeq Ku(W)^{\mathbb{Z}/2\mathbb{Z}} \to Ku(W)$

- $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \subseteq K_{num}(Ku(W))$

- $\sigma_W \in \mathrm{Stab}^\circ(Ku(W)) \to \mathbb{Z}/2\mathbb{Z}$-inv $\Rightarrow \sigma_X$ Serre inv
proper full num

$\Phi_X(K_{num}(Ku(X))) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \forall\, w \in \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ $\exists\, v \in K_{num}(Ku(X))$ st $\Phi_*(v) = w$

$$\Rightarrow \bigsqcup_{\Phi_X(v) = w} M_{\sigma_X}(Ku(X), v) \twoheadrightarrow M_{\sigma_W}(Ku(W), w)^{\langle \tau \rangle}$$

$\neq \emptyset$ $\qquad\qquad$ $\neq \emptyset$

$2 + (-2)(v^2) = \dfrac{2 - 2v^2}{1 - v^2}$