Groups acting on moduli spaces of hyper-Kähler manifolds

§. Motivation

Setting : Algebraic varieties $X \subseteq \mathbb{P}^N_{\mathbb{C}}$ defined by homo poly
smooth

locally $X \simeq \mathbb{C}^n$ where $n = \dim X$

$K_X := \Omega^n_X$ line bundle of n-holomorphic forms $\rightsquigarrow$ $c_1(X) = [K_X] \in H^2(X, \mathbb{Z})$

$\boxed{\text{Curves}}$ $H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$
$c_1(X) = 2g-2$

$c_1 < 0$; $c_1 = 0$; $c_1 > 0$
$\mathbb{P}^1$; elliptic curves ; curves of $g \geq 2$

$\dim X > 1$ $\rightarrow$ we restrict to study (up to deformation) the case $\boxed{c_1 = 0}$

Thm [Beau] X cpx alg var with $c_1(X) = 0$, $\exists$ finite covering

$$A \times \prod \underbrace{Y_i} \times \prod \underbrace{X_j} \longrightarrow X$$

abelian variety
$\mathbb{C}^n/\text{lattice}$

Calabi-Yau var.
- $\pi_1(Y) = 1$
- $\Gamma(\Omega^i_Y) = \begin{cases} 0 \\ \mathbb{C} & i = 0, n \end{cases}$

hyper-Kähler variety
- $\pi_1(X) = 1$
- $\Gamma(\Omega^2_X) = \mathbb{C}\cdot\sigma_X$

everywhere non deg

Cor $\pi_1(X)$ ab
and X simply connected $\Rightarrow$ $\pi_1(X) \simeq \prod CY \times \prod hK$

ex: - dim 2, hK man = CY man, K3 surfaces
simply conn + $K_S = 0$: $S \subseteq \mathbb{P}^3_{\mathbb{C}}$ quartic, $X \xrightarrow{2:1} \mathbb{P}^2_{\mathbb{C}} \supseteq$ sextic

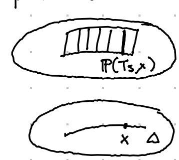
All K3s are diffeo.

- dim $2n$, some example
  - $K3^{[n]}$-type : S K3 $\rightsquigarrow$ $\mathrm{Hilb}^n S$ Hilb scheme of n-points on S

$$S^{[2]} = \mathrm{Bl}_\Delta S^{(2)} \longrightarrow S^{(2)} = \mathrm{Sym}^2 S \supseteq \Delta$$

2-pts $\bullet\ \bullet$

1-pt $\bullet\!\!\rightarrow$ tg vect



- Kum_n
- two other examples in dim 6, 10

§ Properties of hK

[Topological] $H^2(X,\mathbb{Z})$ torsion free $\Rightarrow$ free fin gen $\mathbb{Z}$-mod

Thm [BBF] $\exists\, q_X$ quad form on $H^2(X,\mathbb{Z})$ and c_X constant st

$$\forall \alpha \in H^2(X,\mathbb{Z}) \qquad \alpha^{2n} = c_X \cdot q_X(\alpha)^n$$

$\Lambda_X = (H^2(X,\mathbb{Z}), q_X)$ lattice (topological inv)
st $\operatorname{sign}(\Lambda_X \otimes \mathbb{R}) = (3, b_2(X)-3)$

For $X \sim K3^{[n]}$, $\Lambda_X = \Lambda_{K3^{[n]}}$ is even $(2 \mid q_X(\alpha) \ \forall \alpha)$
deform. equiv

[Complex structure] X projective $\Rightarrow$ Hdg decomposition on

conjugated — wrt q_X

$$H^2(X,\mathbb{C}) = \left(\underbrace{H^0(X,\Omega^2_X)}_{\substack{\| \\ H^{2,0}(X) = \mathbb{C}\sigma_X}} \oplus H^2(X,\mathcal{O}_X)\right) \perp H^1(X,\mathcal{O}_X)$$

where $q_X(\sigma_X)=0,\ q_X(\sigma_X+\bar{\sigma}_X)>0$

Rmk: $H^{2,0}(X)$ is a line and determines the Hdg decomposition of $H^2(X,\mathbb{C})$

- Moduli space: $X \subseteq \mathbb{P}^N_{\mathbb{C}} \leftrightarrow L = X \cap H$ defines $\ell = [L] \in H^2(X,\mathbb{Z})$
  of $K3^{[n]}$-type — divisor on X

$$M_{(X,\ell)} = \left\{ (Y,m) \ \middle|\ \begin{array}{c} Y \text{ def equiv to } X \\ \varphi: \underset{m}{H^2(Y,\mathbb{Z})} \xrightarrow[\text{isometry}]{\sim} \underset{\ell}{H^2(X,\mathbb{Z})} = \Lambda_{K3^{[n]}} \end{array} \right\} \Big/ \sim \text{iso}$$

moduli space of pol. hK

Notice that $\varphi_{\mathbb{C}}: H^2(Y,\mathbb{C}) \to H^2(X,\mathbb{C})$

$H^{2,0} \perp m$ hence $\varphi_{\mathbb{C}}(H^{2,0}) \perp \ell$

"period of Y"

$$\varphi_{\mathbb{C}}(H^{2,0}(Y)) \in \Omega_{\ell^\perp} = \{ x \in \mathbb{P}(\ell^\perp_{\mathbb{C}}) \mid q_X(x)=0,\ q_X(x+\bar{x})>0 \}$$

Thm [Torelli] $\exists\, \Gamma < O(\ell^\perp)$ st $\forall\ M \subseteq M_{(X,\ell)}$ conn. cpt
the application sending (Y,m) to $[\varphi_{\mathbb{C}}(H^{2,0}(Y))]$

$$\text{" } G = O(\ell^\perp)/\Gamma \text{ "} \curvearrowright M \longrightarrow \Omega_{\ell^\perp}/\Gamma \qquad \text{is an embedding}$$

$. Period spaces

Λ lattice, $\mathrm{sign}(\Lambda \otimes \mathbb{R}) = (2, n-)$

$\mathcal{D}_\Lambda = \{ x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid x^2 = 0, \quad x \cdot \bar{x} > 0 \} \subseteq \mathbb{P}(\Lambda_{\mathbb{C}})$
open in a proj. quadric

[Borel-Baily] $\forall\ \Gamma < O(\Lambda)$ of fin index, $\mathcal{D}_\Lambda / \Gamma$ quasi-projective variety

$\forall\ \overset{-id}{\overset{\wedge}{\Gamma}} \trianglelefteq O < O(\Lambda)$, $G = O/\Gamma \curvearrowright \mathcal{D}_\Lambda/\Gamma$ (very)-generically free

Aim: study divisors that are contained in $\mathrm{Fix}(g)$ for some $g \in G \setminus \{id\}$

Thm (-): An irred divisor D is contained in $\mathrm{Fix}(g)$ if and only if

$g = [r_\beta]$ (reflection wrt $\beta^\perp \subseteq \Lambda$) for some $\beta \in \Lambda$, $\beta^2 < 0$ and $D = \mathrm{Im}\left(\mathcal{D}_\Lambda \cap \mathbb{P}(\beta^\perp_{\mathbb{C}}) \to \mathcal{D}_\Lambda/\Gamma \right)$

Characterization of the "ramification divisor" of $\mathcal{D}_\Lambda/\Gamma \longrightarrow \mathcal{D}_\Lambda/O$

in terms of some reflections of G

→ In the geometric case, we expect these reflections to rely hK manifolds that "share some properties"

(ex) K3 surfaces (Stellari, Orlov) $M_{K3,2d} \overset{p}{\hookrightarrow} \mathcal{D}_{2d} \circlearrowleft O(\Lambda_{K3})/\Gamma \simeq \mathbb{Z}/2\mathbb{Z}^{n(d)} \supseteq R$

$S, S' \in M_{K3,2d}$: $p(S) = r(p(S'))$ for some $r \in R$ iff $D^b(S) \simeq D^b(S')$

Higher dimension?

- Pf of the theorem
- What happens in higher dimension hK

Thm (-): An irred divisor D is contained in $\mathrm{Fix}(g)$ if and only if

$$g = [r_\beta] \quad \text{for some } \beta \in \Lambda \quad \text{and} \quad D = \mathrm{Im}\left(\mathcal{D}_\Lambda \cap \mathbb{P}(\beta^\perp_{\mathbb{C}}) \to \mathcal{D}_\Lambda/\Gamma\right)$$

reflection wrt $\beta^\perp \subseteq \Lambda$ $\qquad$ $\beta^2 < 0$

Pf $\quad \mathcal{D}_\Lambda \xrightarrow{\pi} \mathcal{D}_\Lambda/\Gamma$ projection, $g \in G$, $g = [f]$ for $f \in O$

Given $[x] \in \mathcal{D}_\Lambda/\Gamma$, $g \cdot [x] = [f_{\mathbb{C}}(x)] = [x] \in \mathcal{D}_\Lambda/\Gamma$
iff $\exists\, \tilde{f} \in \Gamma$ st $\tilde{f}_{\mathbb{C}} f_{\mathbb{C}}(x) = \lambda x$ ie x eigenvalue of f' st $[f'] = g$

$$\mathrm{Fix}(g) = \pi\left(\bigcup_{[f]=g} \bigsqcup_{\lambda \in \mathrm{Spec}(f)} \mathbb{P}(V_\lambda(f_{\mathbb{C}})) \cap \mathcal{D}_\Lambda\right)$$

$\mathrm{Fix}(g) \supseteq D$ irred. divisor

countable set $\qquad$ one of these has codimension 1

$\Rightarrow \exists\, f$ st $g = [f]$ and $\mathrm{codim}_{\Lambda_{\mathbb{C}}}(V_\lambda(\underbrace{f_{\mathbb{C}}}_{\text{real operator isometry}})) = 1 \to \lambda = \pm 1$

$\to \exists\, \beta \in \Lambda$ st $V_1(f_{\mathbb{Q}}) = \beta^\perp$ so $f = r_\beta$ + D irred, $D = \pi(\mathbb{P}(\beta^\perp_{\mathbb{C}}) \cap \mathcal{D}_\Lambda)$
and $g \neq id$

Rmk: Each irreducible divisor is contained in the fixed locus of at most one element $g \in G$.
but $\mathrm{Fix}([r_\beta])$ can have multiple components

$\rightsquigarrow \mathcal{R} = \{[r_\beta] \in O/\Gamma \text{ st } \beta^2 < 0\}$ reflections that act on $\mathcal{D}_\Lambda/\Gamma \supset M_\tau$ in the hK case

Questions:

- Determine $\mathcal{R}$ (ex: numerical char of β st $r_\beta \in \mathcal{R}$)
- Study the relationship between X and $r(X)$ for $X \in M_\tau$

hK case : $K3^{[m]}$-type, $h \in \Lambda_{K3^{[m]}}$

$h^\perp = \underset{\text{unimodular}}{M} \oplus$ rank 2 lattice

Mon $= \Gamma_{h,m} < O(h^\perp)$ has an explicit description

Thm (–) • Numerical characterization of m, $h \in \Lambda_{K3^{[m]}}$ st $\Gamma_{h,m} \triangleleft O(h^\perp)$

In those cases, $\mathcal{M}_\gamma \hookrightarrow \Omega_{h^\perp}/\Gamma_{h,m}$

$O(h^\perp)/\Gamma_{h,m}$ acts on $\mathcal{M}_\gamma$

$\Rightarrow \mathcal{R} = \{ [r_\beta] \neq id \text{ st } \beta^2 < 0 \}$ acting on $\mathcal{M}_\gamma$

• Numerical char for $\mathcal{R}$ in some $K3^{[m]}$ - cases $(\mathrm{div}(h) = 1)$

ex <u>hK 4folds of $K3^{[2]}$-type</u> : for each $h \in \Lambda_{K3^{[n]}}$ <u>$\mathrm{div}(h) = 1$</u> or 2, $\Gamma_{h,m} \triangleleft O(h^\perp)$

Consider

$${}^{[2]}\mathcal{M}_2^{(1)} \longrightarrow \Omega_{h^\perp}/{}^{[2]}\Gamma_2^{(1)} \circlearrowleft O(h^\perp)/{}^{[2]}\Gamma_2^{(1)} = [r_\gamma]$$

moduli space of polarized hK man of square 2, div 1

Cor : • $\mathrm{Fix}([r_\gamma])$ is an <u>irreducible</u> divisor that meets ${}^{[2]}\mathcal{M}_2^{(1)}$

• We have explicit construction of elements in ${}^{[2]}\mathcal{M}_2^{(2)}$, and

$r_\gamma([X]) = [X']$ iff X and X' are obtained with "dual" constructions