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FIXED LOCUS of the involution of EPW cubes

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A **hyper-Kähler manifold** is a simply connected complex compact Kähler manifold X whose space of holomorphic 2-forms is generated by a symplectic form.

- $H^2(X,\mathbb{Z})$ carries a nondegenerate quadratic form q_X (topological invariant)
- X of $K3^{[n]}$-type if X is a deformation of $\mathrm{Hilb}^n(S)$, for some K3 surface S.

Def. An **EPW cube** is a hyper-Kähler sixfold $\widetilde{\mathsf{Z}}$ of $K3^{[3]}$-type constructed by [IKKR] as the double cover

$$g\colon \underset{\circlearrowleft\,\iota}{\widetilde{\mathsf{Z}}} \xrightarrow{2:1} \mathsf{Z}, \qquad (*)$$
$$\mathsf{Z} \supseteq \mathsf{Z}_3$$

of some integral sixfold Z contained in the Grassmannian $\mathrm{Gr}(3,6)$, branched over the smooth threefold $\mathsf{Z}_3 = \mathrm{Sing}(\mathsf{Z})$.*

* $Z_3 \subseteq Z \subseteq \mathrm{Gr}(3,6)$ are Lagrangian degeneracy loci.

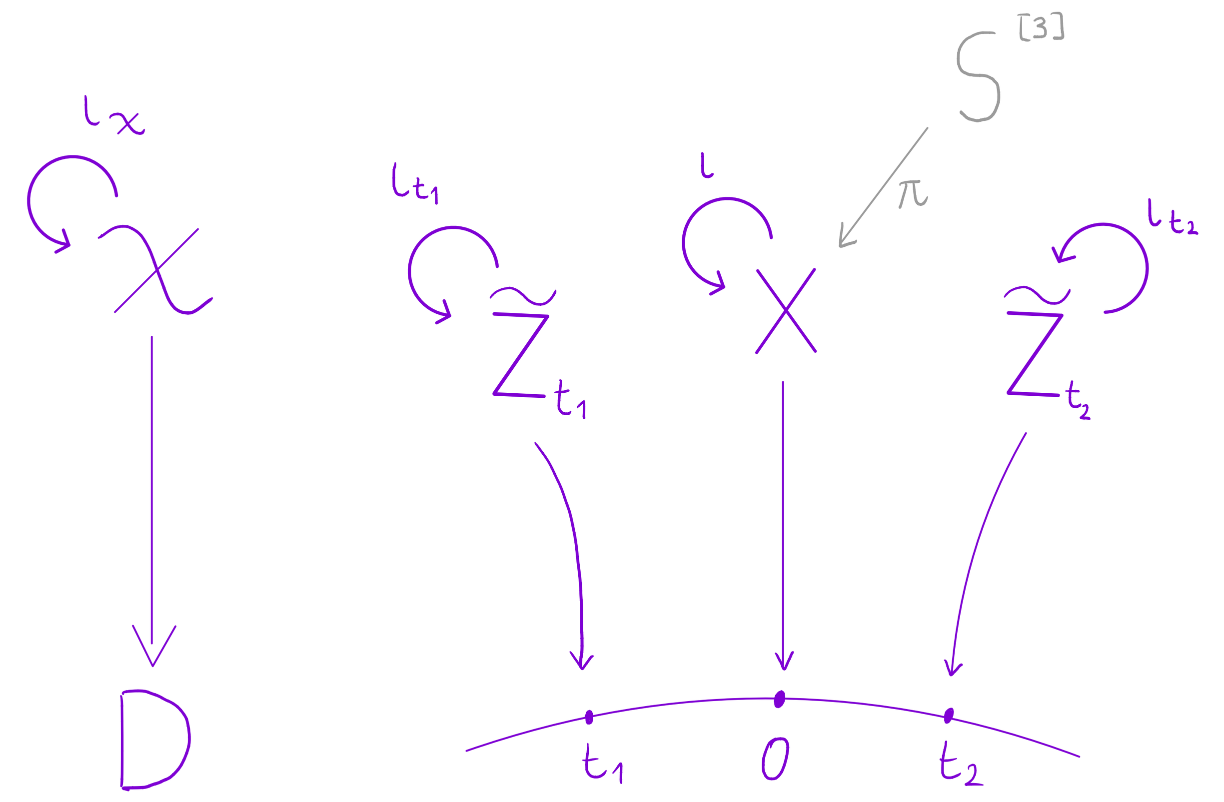
- The pullback of the hyperplane section of Z defines an ample class h on $\widetilde{\mathsf{Z}}$ of square $q_{\widetilde{\mathsf{Z}}}(h) = 4$ and divisibility $\mathrm{div}_q(h) = 2$.
- The construction of $\widetilde{\mathsf{Z}}$ is explicit, and yields a family of EPW cubes of dimension 21.

Let ι be the involution $\widetilde{Z}$ associated with the double cover g .

- Since ι is **antisymplectic**, the fixed locus $\mathsf{W} := g^{-1}(\mathsf{Z}_3)$ is a **smooth Lagrangian threefold**.
- The deformation space of W inside $\widetilde{\mathsf{Z}}$ has dimension $h^0(\mathsf{W}, N_{\mathsf{W}/\widetilde{\mathsf{Z}}}) = h^0(\mathsf{W}, \Omega^1_{\mathsf{W}}) = h^{0,1}(\mathsf{W})$.

Thm.[R] Let $\widetilde{\mathsf{Z}}$ be a smooth EPW cube with associated involution ι. The fixed locus W of ι is a rigid Lagrangian submanifold, namely its first Betti number is 0.

Pf. The proof uses the degeneration methods of [FMOS 1, 2]



① A singular degeneration of EPW cubes

Let S be a degree 2 K3 surface, with $S \xrightarrow{2:1} \mathbf{P}^2$. There exists a divisorial contraction

$$\pi\colon S^{[3]} \longrightarrow X.$$

invariant wrt the involution $j^{[3]}$ of $S^{[3]}$
$\Longrightarrow$ induces an involution $\iota \curvearrowright X$

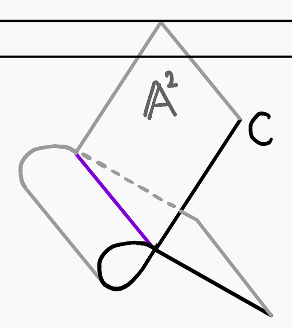
The singular variety X is the special fiber of a degeneration $\mathscr{X} \to (D,0)$, whose generic fiber $\mathscr{X}_t$ is a smooth EPW cube $\widetilde{\mathsf{Z}}_t$.
smooth curve

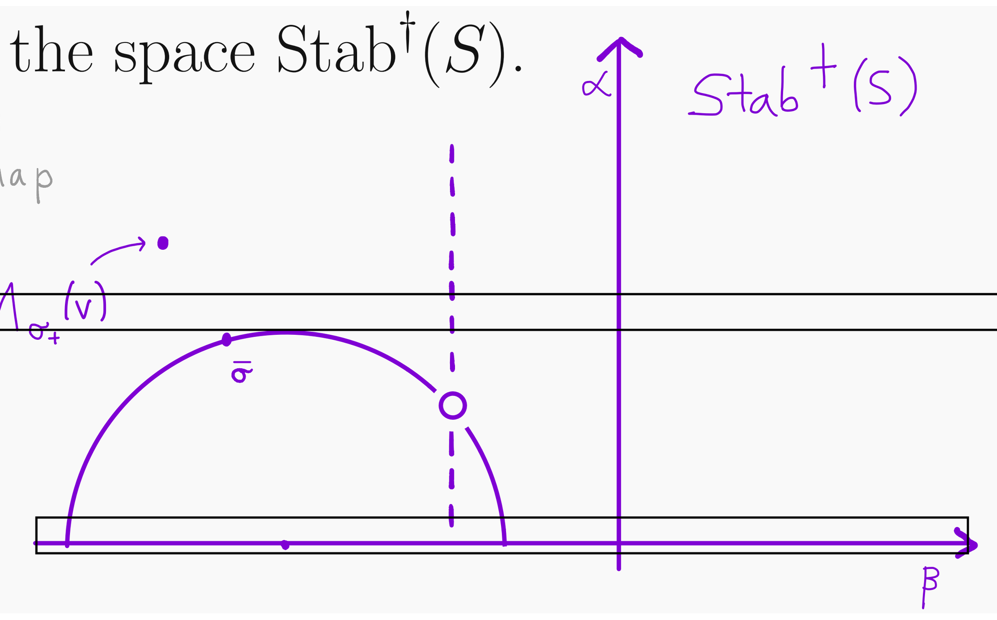
[BM] Contraction of moduli spaces of sheaves on a K3 surface S are determined by walls in the space $\mathrm{Stab}^\dagger(S)$.

uses the normality of $M_{\bar\sigma}(v)$ showed using the Kuranishi map

- Explicit description of the contraction $\pi\colon S^{[3]} \longrightarrow X \simeq M_{\bar\sigma}(v)$
- Fixed locus of $\iota \curvearrowright X = \underset{\substack{\text{dim 2}\\\text{smooth}}}{F_2} \sqcup \underset{\substack{\text{dim 3}\\\text{sing in codim 1}}}{F_3}$

Locally $F_3 \simeq$

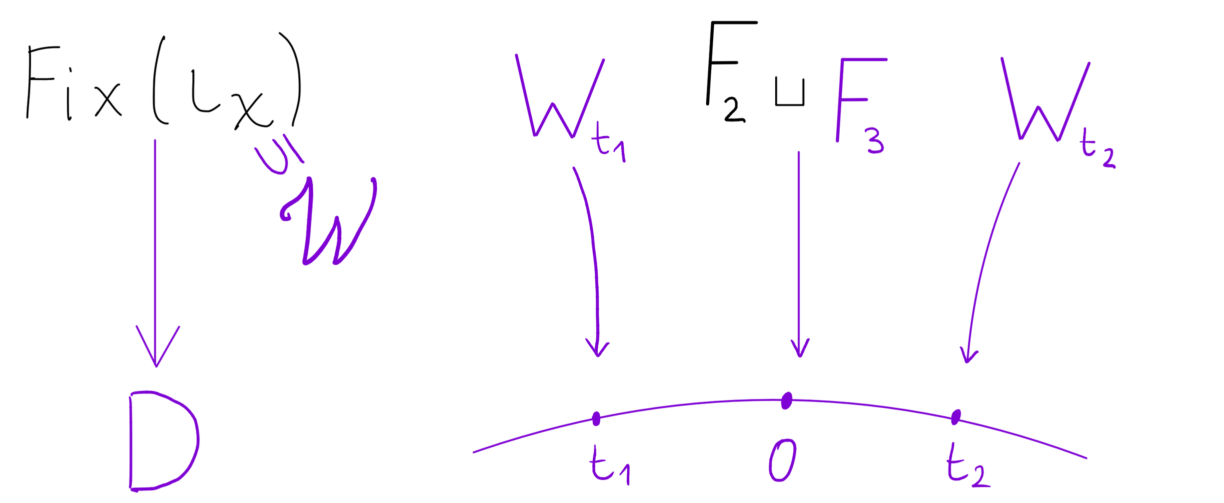




② Degeneration of W

The schematic fixed locus $\mathrm{Fix}(\iota_{\mathscr{X}}) \to (D,0)$ has reduced fibers.

The connected component $\mathscr{W}$ that dominates D has generic fiber equal to W_t and special fiber equal to F_3.



The family $\mathscr{W} \to D$ has slc fibers $\underset{[K]}{\Longrightarrow}$ $h^i(\mathscr{W}_t, \mathscr{O}_{\mathscr{W}_t})$ is constant. Hence $h^1(\mathsf{W}, \mathscr{O}_{\mathsf{W}}) = h^1(F_3, \mathscr{O}) = 0$.

There exists a family $\widetilde{\mathscr{W}} \to B$ whose fibers $\widetilde{\mathscr{W}}_b$ are Lagrangians sumbanifolds of class $2[\mathsf{W}]$ that cover $\widetilde{\mathsf{Z}}$.
The class $2\cdot[W]$ deforms in $\widetilde{Z}$!

References

[IKKR] Iliev, Kapustka, Kapustka, Ranestad, EPW cubes
[M] Markmann, Stable vector bundles on a hyper-Kähler manifold with a rank 1 obstruction map are modulars
[BM] Bayer, Macrì, Projectivity and birational geometry of Bridgeland moduli spaces
[FMOS1] Flapan, Macrì, O'Grady, Saccà, The geometry of antisymplectic involutions, I
[FMOS2] Flapan, Macrì, O'Grady, Saccà, The geometry of antisymplectic involutions, II
[K] Kollár, Families of varieties of general type