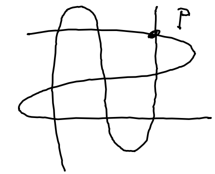
CAYLEY-BACHARACH THEOREM

how to power up geometry using algebra [EGH, '96]

char $k = 0$ → $\mathbb{P}^n$?



<u>Chasles</u> $\mathbb{P}^2 = \mathbb{P}(k^3) \supseteq X_1, X_2$ cubics, $\Gamma = X_1 \cap X_2$ 9 points

cubic: zero locus of a deg 3 poly

↪ (d,e)?

$\Longrightarrow$ Any cubic through 8 pts of Γ will contain Γ

$\Gamma' = \Gamma \setminus \{p\}$

other $\Gamma' \subseteq \Gamma$?

"Pf" $\left\{ \begin{array}{c} \text{cubics} \\ \text{in } \mathbb{P}^2 \end{array} \right\} \overset{(1)}{\supseteq} \left\{ \begin{array}{c} \text{cubics} \\ \text{through } \Gamma' \end{array} \right\} \overset{(2)}{\supseteq} \left\{ \begin{array}{c} \text{cubics} \\ \text{through } \Gamma \end{array} \right\}$

$\underbrace{\text{homo poly of deg 3}}_{\text{vs of dim 10}}$

$F = \sum_{|\alpha|=3} c_\alpha X^\alpha$

each point of Γ is a linear equation for $\{c_\alpha\}$

equations of X_1, X_2

$\Gamma' \rightsquigarrow$ 8 equations for $\{c_\alpha\}$ if those are independent $\Rightarrow$ cod (1) $= 8 \to$ (2) is $=$

Key: <u>Bézout</u> Two plane curves of deg d, e with no common components meet in at most de points

Switch to the algebraic pt of view!

$\mathbb{P}^2 \ni [x_0 : x_1 : x_2]$

$\cup\!\!\mid$

$V(I) = \{ p \in \mathbb{P}^2 \mid f(p) = 0 \ \forall f_d \in I \}$ $\longleftarrow$ $R = S/I$ ↖ gen by homo poly

$S = k[x,y,z]$, $S_d = \langle \text{mon of degree } d \rangle_k$

$V(f)$ curve dy by f ; $R = S/(f)$

$V(f) \cap V(g) = V(f,g)$

$$\boxed{\dim V(I) = \dim_{Krull} R - 1}$$

⚠ (R, V)

$[0:0:1]$ $I = (x,y)$

$[0:0:1]$ $I = (x^2, y)$

$V \longmapsto I(V) = \{ f \in S_d \mid f(p) = 0 \ \forall p \in V \}$

$V' \subseteq V$ $I(V) \subseteq I(V')$

<u>Def</u> $\Gamma' \subseteq \Gamma$ of dim 0

$I' \supseteq I$ ideals of S st S/I , S/I' of dim 1 : Residual scheme of Γ' is $\Gamma'' = V(I'')$

st $I'' = I : I'$

$= \{ f \in S_{hom} \mid f I' \subseteq I \}$

$I''/I = \mathrm{Ann}(I'/I)$

- $\Gamma'' \cup \Gamma' = \Gamma$
- $\deg \Gamma'' + \deg \Gamma' = \deg \Gamma$

<u>CB</u> $\mathbb{P}^2 \supseteq \underset{\deg d,\ e}{X, Y}$, $\Gamma = X \cap Y$ dim 0, $\forall\ \emptyset \neq \Gamma' \subseteq \Gamma$, $\forall a \leq s := d+e-3$

$S \ni F_d, G_e$ st $\dim\left(S/_{I=(F,G)}\right) = 1$, $\forall \underset{(\dim S/I' = 1)}{I' \supseteq I}$, $I'' = I : I'$

$$\dim_k \frac{\{\text{deg } a \text{ curves through } \Gamma'\}}{\{\text{deg } a \text{ curves through } \Gamma\}} = \deg \Gamma'' - \operatorname{codim} \left\{ \begin{array}{c} \text{curves of deg } s-a \\ \text{through } \Gamma'' \end{array} \right\}$$

$$\begin{array}{ccc} \| & & \| \\ \dim_k I'_a / I_a & & \dim_k \left(S / I''_{s-a} \right) \end{array}$$

- Generalize Chasles to $\#\Gamma < 8$ and result is still true if $\text{mult}_\Gamma P = 1$
- $\Gamma = \underset{3\text{pts}}{\Gamma'} \sqcup \underset{6\text{pts}}{\Gamma''}$: $\exists$ conic $\supseteq \Gamma''$ iff $\exists$ line $\supseteq \Gamma'$

Key : $S = k[x,y,z]$ is <u>Gorenstein</u> : $\forall$ J st $A = S/J$ dim 0, $A_i \times A_{m-i} \xrightarrow{\cdot} A_m \simeq k$ is a perfect pairing
$\quad \| $
$A_1 \oplus \cdots \oplus A_m \simeq k$

$\Rightarrow$ • $\left(J'/J\right)_i^{\perp} = \text{Ann}\left(J'/J\right)_{m-i} = \left(J''/J\right)_{m-a}$

• $\dim(\ ) = \operatorname{codim}(\ )^{\perp}$ for a perfect pairing

⚠ Need A of dim 0, so quotient R by a linear form ℓ + dimension count (this add the extra term $\deg \Gamma''$:))