On the geometry of singular EPW cubes

A **hyper-Kähler manifold** is a simply connected manifold whose space of holomorphic 2-forms is generated by a symplectic form.

We study examples of hyper-Kähler manifolds of **dimension 4** and **6** constructed from A in the Lagrangian Grassmannian $\mathrm{LG}(\bigwedge^3 V_6)$. ($V_6$: $\mathbb{C}$-vector space of dim 6)

Not: We denote by $\Sigma \subset \mathrm{LG}(\bigwedge^3 V_6)$ the divisor of Lagrangians A that contain decomposable vectors. ($v_1 \wedge v_2 \wedge v_3$)

Double EPW sextics

Given $A \in \mathrm{LG}(\bigwedge^3 V_6)$, we can construct the *EPW sextic* stratification

$$Y_A^{\geq i} \subset \mathbf{P}(V_6).$$

Theorem (O'Grady). If $A \notin \Sigma$, then

$$\mathbf{P}(V_6) \supset Y_A^{\geq 1} \supset Y_A^{\geq 2} \supset Y_A^{\geq 3} \supset Y_A^{\geq 4} = \varnothing$$

($Y_A^{\geq 1}$: sextic hypersurface; $Y_A^{\geq 2}$: integral normal surface; $Y_A^{\geq 3} = \varnothing$ for A general (outside the divisor Δ); finite, smooth for $A \in \Delta$)

There exists a double cover ("double EPW sextic")

$$f_A \colon \widetilde{Y}_A \xrightarrow{2:1} Y_A^{\geq 1} \supset Y_A^{\geq 2} \text{ (branched)}$$

such that if $A \notin \Delta$, $\widetilde{Y}_A$ is a hK variety $\sim \mathrm{K3}^{[2]}$.

If $A \in \Delta$, $\widetilde{Y}_A$ is singular along $f_A^{-1}(Y_A^3) = \{y_1, \ldots y_r\}$. We can construct a K3 surface $S_A \subset \mathbf{P}^6$ of genus 6 and a projective resolution

$$\mathrm{Bl}_{\{y_1,\ldots,y_r\}}(\widetilde{Y}_A) \longrightarrow S_A^{[2]} \xrightarrow{\text{small}} \widetilde{Y}_A$$

EPW cubes

Given $A \in \mathrm{LG}(\bigwedge^3 V_6)$, we have another chain of subschemes

$$\mathsf{Z}_A^{\geq i} \subset \mathrm{Gr}(3, V_6).$$

Theorem (IKKR, R)**.** If $A \notin \Sigma$, then

$$\mathrm{Gr}(3, V_6) \supset \mathsf{Z}_A^{\geq 1} \supset \mathsf{Z}_A^{\geq 2} \supset \mathsf{Z}_A^{\geq 3} \supset \mathsf{Z}_A^{\geq 4} \supset \mathsf{Z}_A^{\geq 5} = \varnothing$$

($\mathsf{Z}_A^{\geq 1}$: quartic hypersurf.; $\mathsf{Z}_A^{\geq 2}$: normal integral 6fold; $\mathsf{Z}_A^{\geq 3}$: normal integral 3fold; $\mathsf{Z}_A^{\geq 4} = \varnothing$ for A general (outside the divisor Γ); finite, smooth for $A \in \Gamma$)

where each $\mathsf{Z}_A^{\geq i+1}$ is the singular locus of $\mathsf{Z}_A^{\geq i}$.

Theorem (IKKR, DK)**.** There exists a double cover ("EPW cube")

$$g_A \colon \widetilde{\mathsf{Z}}_A \xrightarrow{2:1} \mathsf{Z}_A^{\geq 2} \supset \mathsf{Z}_A^{\geq 3} \text{ (branched)}$$

such that the singular locus of $\widetilde{\mathsf{Z}}_A$ is $g_A^{-1}(\mathsf{Z}_A^4)$.

If $A \notin \Gamma$, $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler variety $\sim \mathrm{K3}^{[3]}$.

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$A \in \Gamma$

Let $g^{-1}(\mathsf{Z}_A^4) \coloneqq \{z_1, \ldots, z_r\}$.

(a) The blowup $\widetilde{\mathsf{X}}_A \to \widetilde{\mathsf{Z}}_A$ of $\widetilde{\mathsf{Z}}_A$ in $\{z_1, \ldots, z_r\}$ is smooth and the exceptional divisor E_i over the point z_i is the incidence variety $I \subset \mathbf{P}^3 \times (\mathbf{P}^3)^\vee$.

(b) For any choice $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ of contractions of each E_i onto either $\mathbf{P}^3$ or $(\mathbf{P}^3)^\vee$, we obtain an (analytic) small resolution

$$\mathsf{X}_A^\epsilon \to \widetilde{\mathsf{Z}}_A$$

with exceptional locus a disjoint union of r copies of $\mathbf{P}^3$.

For any two analytic resolutions X_A^ϵ and $\mathsf{X}_A^{\epsilon'}$, there is a Mukai flop $\mathsf{X}_A^\epsilon \dashrightarrow \mathsf{X}_A^{\epsilon'}$

(c) There exists a choice of ϵ such that X_A^ϵ is a **projective** smooth quasi-polarized hyper-Kähler sixfold with a projective contraction $\mathsf{X}_A^\epsilon \to \widetilde{\mathsf{Z}}_A$ of r copies of $\mathbf{P}^3$.

Tools for the proof:

[surjectivity of the period map for hyper-Kähler sixfolds + relations between period maps for EPW cubes and double EPW sextics

[Small projective resolutions of $\widetilde{\mathsf{Z}}_A$ (not locally $\mathbb{Q}$-factorial) are the *blow up of Weil divisor classes* that are nontrivial in $\mathrm{Cl}(\widetilde{\mathsf{Z}}_A) / \mathrm{Pic}(\widetilde{\mathsf{Z}}_A)$. ($\mathrm{Cl}(\widetilde{\mathsf{Z}}_A) \cong \mathrm{Pic}(\widetilde{\mathsf{X}}_A) / \oplus \mathbb{Z} E_i$)

$\overset{\mathsf{Y}_A}{\Longrightarrow}$ there exists a line bundle L on $\widetilde{\mathsf{X}}_A$ such that $L_{|E_i} = \mathcal{O}(c_i, 0)$ or $\mathcal{O}(0, c_i)$, with $c_i > 0$, that induces the contraction $\widetilde{\mathsf{X}}_A \to \mathsf{Y}_A$.

$$I \cong \widetilde{\mathsf{X}}_A \to \mathbb{P}^3 \subseteq \mathsf{Y}_A \xrightarrow{\text{small}} \widetilde{\mathsf{Z}}_A \ni z_i$$

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References

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