Groups acting on moduli spaces of hyper-Kähler manifolds

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Def A hyper-Kähler manifold is a simply connected compact Kähler smooth manifold X with $H^0(X, \Omega^2_X) = \mathbb{C} \cdot \omega$, where ω is a holomorphic 2-form everywhere non degenerate

[BBF] There exists an integral quadratic form q_X on $H^2(X, \mathbb{Z})$, signature $(3, b_2(X) - 3)$

free abelian group

* X hK manifold is of $K3^{[m]}$-type if it is a deformation of $\mathrm{Hilb}^m S$

S K3 surface

In this case $H^2(X, \mathbb{Z}) = \Lambda_{K3^{[m]}} = \Lambda_{K3} \oplus \mathbb{Z}(-2(m-1))$ even lattice

[Markman] $\tau = O(\Lambda_{K3^{[m]}})\, h$, there is a period morphism

$$\mathcal{P}_\tau : {}^{[m]}\mathcal{M}^\circ_\tau \hookrightarrow \Omega_{h^\perp} / \mathrm{Mon}(K3^{[m]}, h)$$

coarse moduli space of polarized hK manifolds of $K3^{[m]}$-type and polarization type τ

PERIOD SPACE

quotient of a Hermitian symmetric domain by $\mathrm{Mon}(K3^{[m]}, h) < O(h^\perp)$ of finite index

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Λ lattice
signature $(2, n_-)$
where $n_- \geq 1$

$\mathcal{D}_\Lambda$ period domain
$O(\Lambda) \curvearrowright \mathcal{D}_\Lambda$

$\rightsquigarrow$ [Borel-Baily] $\Gamma < O(\Lambda)$ of finite index
$\mathcal{D}_\Lambda / \Gamma$ quasi-projective variety

$\Gamma \triangleleft O < O(\Lambda)$: $q : \mathcal{D}_\Lambda/\Gamma \longrightarrow \mathcal{D}_\Lambda / O$ is a Galois cover of group $G = O/\Gamma$

Thm The divisorial components of the ramification of q are the Heegner divisors $\mathcal{H}_{\beta^\perp}$ where $\beta \in \Lambda$ primitive, $\beta^2 < 0$ defines a nontrivial reflection in O/Γ

$\beta \in \Lambda$, $\beta^2 < 0$ then
$\mathcal{H}_{\beta^\perp} := \mathrm{Im}\left(\mathcal{D}_\Lambda \cap \mathbb{P}(\beta^\perp) \to \mathcal{D}_\Lambda / \Gamma \right)$
is an irreducible divisor of $\mathcal{D}_\Lambda / \Gamma$

β defines a reflection in O
if $\exists\, r_\beta \in O$ such that
$r_\beta(\beta) = -\beta$, $r_\beta|_{\beta^\perp} = \mathrm{id}$

hK manifolds of $K3^{[m]}$-type $\quad {}^{[m]}\mathcal{M}_\gamma^\circ \hookrightarrow \mathcal{D}_{h^\perp} / \mathrm{Mon}(K3^{[m]}, h) \circlearrowleft O/\mathrm{Mon}(K3^{[m]}, h)$

if $\mathrm{Mon}(K3^{[m]}, h) \triangleleft O$

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$h \in \Lambda_{K3^{[m]}}$ of $\overset{h\cdot k=1}{\boxed{\operatorname{div} h = 1}}$: [GHS] $h^\perp = M \oplus \mathbb{Z}k \oplus \mathbb{Z}l$, with $k^2 = -2h^2$, $l^2 = -2(m-1)$

$\rightsquigarrow$ numerical characterizations of $\beta \in h^\perp$ st $\beta^2 < 0$, $r_\beta \in O(h^\perp)$ and $r_\beta \notin \mathrm{Mon}(K3^{[m]}, h)$

Thm $h \in \Lambda_{K3^{[2]}}$ of $h^2 = 2d$, $\operatorname{div}(h) = 1$. [$h \in \Lambda_{K3^{[2]}} \longrightarrow \operatorname{div}(h) \in \{1, 2\}$]
The divisorial components of the ramification of

$$q_{2,h}: \mathcal{D}_{h^\perp} / \mathrm{Mon}(K3^{[2]}, h) \longrightarrow \mathcal{D}_{h^\perp} / O(h^\perp)$$

are the Heegner divisors $\mathcal{H}_{\beta^\perp}$ such that $\beta \in h^\perp$ primitive, $\beta^2 < 0$ and

- $\beta^2 \mid 2\operatorname{div}(\beta)$ [β defines a reflection]
- $\beta^2 \neq -2$ and if $\beta^2 = -2d$, then $2d \nmid \beta \cdot l$ [$r_\beta \notin \mathrm{Mon}(K3^{[2]}, h)$]

ex $h^2 = 2$: ${}^{[2]}\mathcal{M}_2 \hookrightarrow \mathcal{D}_{h^\perp} / \mathrm{Mon}(K3^{[2]}, 2) \xrightarrow{q_{2,2}} \mathcal{D}_{h^\perp} / O(\Lambda_{K3^{[m]}})$ ramification divisor is irreducible

$\cup\!\!\mathrm{I}$ $\mathcal{H}_{\beta^\perp}$ where r_β duality involution of double EPW sextics in ${}^{[2]}\mathcal{M}_2^{(1)}$.