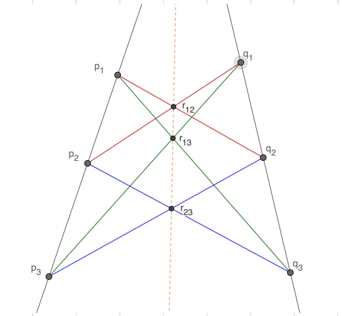
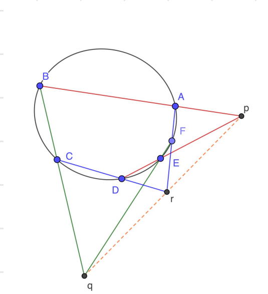
Il teorema di Cayley Bacarach

(Eisenbud, Green, Harris)

Γ, d max $\{|\Gamma'|, \Gamma' \subseteq \Gamma$ ogni curva di deg d per Γ', contiene $\Gamma\}$

Pappo, 400 → Pascal, 1600 → Chasles, 1835





$X_1, X_2 \subseteq \mathbb{P}^2$ cubiche

$\Gamma = X_1 \cap X_2$, $|\Gamma| = 9$

Ogni cubica X che interseca Γ in almeno 8 pt, contiene Γ.

$r_{ij} = p_i q_j \cap p_j q_i$ $j \neq i$

$\Gamma \subseteq \mathbb{P}^n$ $\underset{d}{\wedge}$ $|\Gamma|$ condizioni $\qquad H_\Gamma(d) = \mathrm{codim}_{S_d}(I(\Gamma)_d)$

$S = k[x_0, \dots, x_n]$, $I(\Gamma)$

$|\Gamma'| = 8$, $\Gamma' \subseteq \Gamma$ $\qquad I(\Gamma)_3 = I(\Gamma')_3$

Chasles: $H_\Gamma(3) = H_{\Gamma'}(3)$

CB $\quad X_1, \dots, X_n \subseteq \mathbb{P}^n$ ($F_1, \dots, F_n$; $d_1, \dots, d_n$), $\quad \Gamma = X_1 \cap \dots \cap X_n$, $|\Gamma| = \prod d_i$

$\Gamma = \Gamma' \sqcup \Gamma''$, $d \leq s := \sum d_i - (n+1)$

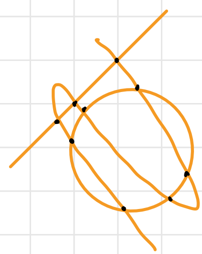
$$H_\Gamma(d) - H_{\Gamma'}(d) = |\Gamma''| - H_{\Gamma''}(s-d) = i_{\Gamma''}(s-d)$$

$$H_\Gamma(d) - H_{\Gamma'}(d) = \dim\left(I(\Gamma')_d / I(\Gamma)_d\right)$$

$n = 2$, $d_1 = d_2 = 3$, $s = 3$

- $d = 3$: $|\Gamma''| = 1$. $\qquad H_\Gamma(3) - H_{\Gamma'}(3) = 1 - 1 = 0$
- $d = 1$, $|\Gamma''| = 3$ $\qquad I(\Gamma)_1 = 0$, $\qquad 3 - H_{\Gamma'}(1) = 6 - H_{\Gamma''}(2)$

$\qquad \dim_k(I(\Gamma'')_1) = \dim_k(I(\Gamma'')_2)$



• SCHEMI

- A , $(X = \mathrm{Spec}\, A, \mathcal{O}_X)$ $X_f = \{p \in X : f \notin p\}$ $\mathcal{O}_X(X_f) \simeq A_f$
$\mathcal{O}_{X,p} \simeq A_p$

$V(I) \leftrightarrow \mathrm{Spec}(A/I)$, $\mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \dots, x_n])$

- A $\mathrm{Proj}\, A = \{p \triangleleft A \text{ omog} \mid A_+ \not\subset p\} = X$
$A_+ = \bigoplus_{i>0} A_i$ $(X, \mathcal{O}_X)$ $(X_f, \mathcal{O}_X|_{X_f}) = (\mathrm{Spec}\, A_{(f)}, \mathcal{O})$
$\mathcal{O}_{X,p} \simeq A_{(p)}$

• $\mathbb{P}^n_k = \mathrm{Proj}(\underbrace{k[x_0, \dots, x_n]}_{S})$ $(\mathbb{P}^n_k)_{x_i} \simeq \mathrm{Spec}(S_{(x_i)})$

$\Gamma = \mathrm{Proj}(S/I)$ $I = I^{sat} = \{F : \exists n : F \cdot S_n \subseteq I\}$

• $\dim X = \sup_p \mathcal{O}_{X,p}$ $\dim \Gamma = \dim(S/I) - 1$

$\underline{\dim \Gamma = 0}$ $\Gamma = \mathrm{Proj}(S/I)$

• $H_{S/I} \rightsquigarrow$ di tipo poly di deg 0 , def $h(1)$ $\deg \Gamma = h(1)$
$P_{S/I}(z) = \dfrac{h(z)}{(1-z)}$

• $\Gamma \to \deg_k \Gamma = \sum_{p \in \Gamma} [k(p) : k] \cdot \dim_k \mathcal{O}_{\Gamma,p} = \deg \Gamma$

$X_1, \dots, X_n \subseteq \mathbb{P}^n_k$ $\Gamma = X_1 \cap \dots \cap X_n$ $\dim \Gamma = 0 \iff \dim\left(S/(F_1, \dots, F_n)\right) = 1$
$F_1 \quad F_n$ $\Gamma = \mathrm{Proj}\left(S/(F_1, \dots, F_n)\right)$

$\iff$ $F_1, \dots, F_n$ sono parte di un sop di S
$\iff$ $\underline{F_1, \dots, F_n \text{ succ. reg}}$ $I = (F_1, \dots, F_n)$, S/I

$I = I^{sat}$: $F \notin I$, $F \cdot S_n \subseteq I$ $(x_0, \dots, x_n) \subseteq Z(S/I)$ <u>an</u>.

RESIDUALE

$\Gamma = \mathrm{Proj}(S/I)$, $\Gamma' \subseteq \Gamma$, $\Gamma' = \mathrm{Proj}(S/I')$, $I' \supseteq I$

$\Rightarrow \mathrm{res}_\Gamma \Gamma' = \Gamma'' = \mathrm{Proj}(S/I'')$, $I'' = I : I'$

I saturato $\Rightarrow$ I'' sat

$F S_n \subseteq I'' = I : I'$
$\Leftrightarrow F S_n \cdot I' \subseteq I \Leftrightarrow F I' \subseteq I$
$\Leftrightarrow F \in I''$

$$\deg \Gamma' + \deg \Gamma'' = \deg \Gamma$$

A artiniano locale: A Gor $\Leftrightarrow \dim(\mathrm{soc}(A)) = 1$

$A = S/J$ di dim 0
$A_0 = k$
ℓ max $A_\ell \neq 0$

$0 \neq A_\ell = \mathrm{soc}(A) \simeq k$

$Q: \quad A_i \times A_{\ell-i} \to A_\ell \simeq k$ non degenere

$\Gamma = \mathrm{Proj}(S/I)$, $\quad R = S/I$,

$\Gamma' = \mathrm{Proj}\Big(\underbrace{R/I'R}_{R'}\Big)$, $\quad \Gamma'' = \mathrm{Proj}\Big(\underbrace{R/I''R}_{R''}\Big)$

$p \in U_i$

$\mathcal{O}_{\Gamma,p} = R_{(p)} = \left(\underline{R_{(x_i)}}\right)_{D(p)}$ Gorenstein, artiniani, locali

$\mathcal{O}_{\Gamma',p} = R'_{(p)}$,

$Q_p: \quad R_{(p)\,i} \times R_{(p)\,\ell-i} \to k$ non degenere

$I''R = \mathrm{Ann}(I'R) \quad \to \quad (I''R)_{(p)} = \mathrm{Ann}_{R_{(p)}}((I'R)_{(p)})$

$\dim((I'R)_{(p)\,i}) = \mathrm{codim}((I''R)_{(p)\,\ell-i})$

$\dim_k R'_{(p)} + \dim_k R''_{(p)} = \dim_k R_{(p)}$

$$\boxed{\deg \Gamma = \deg \Gamma' + \deg \Gamma''}$$

$\underline{CB}$: $\underset{d_1}{F_1}, \dots, \underset{d_n}{F_n} \subseteq S = k[x_0, \dots, x_n]$ succ. omog regolare

$\Gamma = \mathrm{Proj}\,(\underbrace{S/I}_{R})$, $\underset{R'}{\Gamma'} \subseteq \Gamma$, $\underset{R''}{\Gamma''} = \mathrm{res}_\Gamma \Gamma'$ $\quad s = \sum d_i - (n+1)$

$$\underbrace{H_\Gamma(d)}_{H_R(d) = \mathrm{codim}_{S_d}(I_d)} - \underset{H_{R'}(d)}{\underset{\shortparallel}{H_{\Gamma'}(d)}} = \deg \Gamma'' - \underset{H_{R''}(s-d)}{\underset{\shortparallel}{H_{\Gamma''}(s-d)}}$$

- TEO PUREZZA $\exists\, \ell \in S_1$, $\ell \notin Z(S/I) \rightarrow A = R/(\ell)$

$$P_A(z) = P_R(z) \cdot (1-z) = h(z)$$

$\deg \Gamma = h(1) = \dim_k A$

$A' = R'/(\ell) = R/_{I'R}/(\ell) = A/(I'R, \ell)$ $\qquad J' = (I'R, \ell) = I'A$

$A'' = R''/(\ell)$ $\qquad J'' = (I''R, \ell)$

$\dim A = \dim A' + \dim A''$ $\quad (\bigstar)$

- $A = S/(F_1, \dots, F_n, \ell)$ $\qquad P_A(z) = \dfrac{1}{(1-z)^{n+\cancel{1}}} \cdot (1-z^{d_1}) \cdots (1-z^{d_n})\cancel{(1-z)}$

$\underline{\deg P_A} = \sum (d_i - 1) = \sum d_i - n = s + 1$

Q : $A_i \times A_{(s+1)-i} \rightarrow A_{s+1} \simeq k$ $\qquad J', J''$

$I''R = \mathrm{Ann}(I'R)$

$J'' \subseteq \mathrm{Ann}(J') = (J')^\perp$ $\quad (\bigstar)$

$J''_{(s+1)-i} \subseteq (J'_i)^\perp$ $\qquad \dim J'_i = \mathrm{codim}\, J''_{(s+1)-i}$ $\quad (\cdot)$

$\dim A'_i + \dim A''_{(s+1)-i} = \dim A_i$

$(\cdot)$ $\boxed{\dim J'_i = \dim A''_{(s+1)-i}}$

- $A = R/(\ell)$ $\qquad H_A(d) = H_R(d) - H_R(d-1)$

$$H_R(d) = \sum_{i=0}^{d} H_A(i)$$

- $i_{\Gamma''}(k) = \deg \Gamma'' - H_{R''}(k)$ def è 0

$$i_{\Gamma''}(k) - i_{\Gamma''}(k-1) = H_{R''}(k-1) - H_{R''}(k) = -H_{A''}(k)$$

$$\sum_{i=k}^{\infty} H_{A''}(k) = i_{\Gamma''}(k-1)$$

$$H_\Gamma(d) - H_{\Gamma'}(d) = \sum_{i=0}^{d} H_A(i) - H_{A'}(i) \qquad A' = A/J'$$

$$= \sum_{i=0}^{d} \dim_k J'_i$$

$$= \sum_{i=0}^{d} H_{A''}((s+1)-i)$$

$$= \sum_{j=s+1-d}^{s+1} H_{A''}(j) = i_{\Gamma''}(s-d)$$

<u>Congettura</u> $d_1 = \dots = d_n = 2$, $\deg \Gamma = 2^n$ $\qquad s = 2n-(n+1) = n-1$

$$D(d) = \max\{\deg \Gamma' \mid \Gamma' \subseteq \Gamma,\ H_\Gamma(d) > H_{\Gamma'}(d) \leftrightarrow i_{\Gamma''}(s-d) > 0\}$$

$$D(n-1) = 2^n - 2$$

$$D(d) = 2^n - 2^{n-d}$$

$$D(1) = 2^n - 2^{n-1} = 2^{n-1}$$

$$D(0) = 2^n - 2^n$$

Γ intersezione completa di n quadriche, $X \subseteq \mathbb{P}^n$ ipersup di deg d
$\cap$
$\mathbb{P}^n$ che contiene Γ_0, $\Gamma_0 \subset \Gamma$

$\deg \Gamma_0 > 2^n - 2^{n-d} \implies X$ contiene Γ