ON HIGHER REGULATORS OF SIEGEL VARIETIES

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Abstract. We construct classes in the middle degree motivic cohomology of the Siegel variety of almost any dimension. We compute their image by Beilinson’s higher regulator in terms of Rankin-Selberg type automorphic integrals. In the case of $\text{GSp}(6)$, using results of Pollack and Shah, we relate the integral to a non-critical special value of a degree 8 spin $L$-function.

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1. Introduction

The purpose of the present article is to prove a formula relating classes in the motivic cohomology of Shimura varieties of symplectic groups to certain adelic integrals of Rankin-Selberg type. In some particular cases, these integrals are known to compute non-critical special values of $L$-functions of automorphic forms for these groups, and our results are hence framed in the spirit of Beilinson-Deligne conjectures.
Let \( n \geq 1 \) and let \( G = \text{GSp}_{2n} \) be the symplectic similitude group. Denote by \( \text{Sh}_G \) the Shimura variety associated to \( G \). These Shimura varieties and their cohomology play a prominent role in the study of arithmetic aspects of cuspidal automorphic representations of \( G(A) \) and their associated Galois representations.

Fundamental objects used in most of the approaches to Beilinson-Deligne conjectures and to the theory of Euler systems are modular units. These are elements of the motivic cohomology groups \( H^1_M(\text{Sh}_{\text{GL}_2}, \overline{Q}(1)) \cong \mathcal{O}(\text{Sh}_{\text{GL}_2})^\times \otimes \mathbb{Z} \overline{Q} \), which can be seen as motivic realizations of Eisenstein series. Indeed, by the second Kronecker limit formula, their logarithm is related to limiting values of some real analytic Eisenstein series.

Under some natural restrictions on the minimal \( K_\infty \)-type of \( \pi_\infty \), there is a differential form \( \omega_\Psi \) on \( \text{Sh}_G(U) \) of some well chosen Hodge type with cohomology class \( [\omega_\Psi] \in H^d_{DR}(\text{Sh}_G(U), C) \). Under some natural restrictions on the minimal \( K_\infty \)-type of the discrete series \( \pi_\infty \), this cohomology class induces, via Poincaré duality, a morphism

\[
\langle -, [\omega_\Psi] \rangle : H^{d+1}_D(\text{Sh}_G(U)/R, R(t)) \to \mathbb{C} \otimes Q K,
\]

where \( K \) is a number field containing all the eigenvalues of the spherical Hecke algebra acting on \( \pi \), and where \( H^{d+1}_D(\text{Sh}_G(U)/R, R(t)) \) denotes the real Deligne cohomology group of \( \text{Sh}_G(U) \).

Finally recall the existence of Beilinson’s regulator map

\[
r_D : H^{d+1}_M(\text{Sh}_G(U), \mathbb{Q}(t)) \to H^{d+1}_D(\text{Sh}_G(U)/R, R(t)).
\]

According to Beilinson-Deligne conjectures, if the latter Deligne cohomology group is non-zero, one expects to be able to construct non-zero motivic cohomology elements which are related to special values of \( L \)-functions.

We now state our first main theorem.

**Theorem 1.1** (Theorem 4.12). There exists an explicit differential form \( \xi \), such that

\[
\langle r_D(\text{Eis}_{M,n}), [\omega_\Psi] \rangle = \int_{\text{Sh}_H(U \cap H)} \xi \wedge \omega_\Psi.
\]

The form \( \xi \) is related to either the logarithm of modular units or cup-products of them. Thus, using second Kronecker limit formula, the theorem can be restated as follows. Let \( K_H \) denote a maximal compact subgroup of \( H(R) \) and fix a generator \( X_0 \) of the highest exterior power of \( \text{Lie}(H(R))/\text{Lie}(K_H) \). Moreover, denote by \( E_n(h, s) \) be certain Eisenstein series on \( \text{GL}_2 \) (resp. \( \text{GL}_2 \times_{\text{det}} \text{GL}_2 \)) if \( n \equiv 0, 3 \) modulo 4 (resp. \( n \equiv 1, 2 \) modulo 4) defined in §4.4.
Theorem 1.2 (Theorem 4.17). We have
\[ \langle r_D(E_{\text{M}}), [\omega \Psi] \rangle = C_{\text{U} \cap \text{H}} \int_{\text{H}(\mathbb{Q}) \text{ZG}(\text{A}) \backslash \text{H}(\text{A})} E_n(h, 0) \omega \Psi(X_0)(h) \cdot dh, \]
where \( C_{\text{U} \cap \text{H}} \) is a volume factor depending on \( \text{U} \cap \text{H} \).

We expect that, for certain automorphic representations \( \pi \) of \( \text{G}(\text{A}) \), these Rankin-Selberg type integrals might give integral expression for the Spin \( L \)-function of \( \pi \), hence giving more evidence towards Beilinson’s conjectures. Indeed, this method was successfully used to study Beilinson’s conjectures for \( \text{GL}_2 \) ([14]) and \( \text{GSp}_4 \) ([23]).

We assume that \( \pi \) has trivial central character and that the cusp form \( \Psi \) supports a not identically zero Fourier coefficient of type (42), which takes values in \( \overline{\mathbb{Q}} \). Using the main theorem of [32], we get the following result for \( \text{GSp}_6 \).

Theorem 1.3 (Theorem 5.13). Let \( \Sigma \) be a finite set of primes containing \( \infty \) and the bad primes for \( \pi \) and let \( \Psi = \Psi_{\infty} \otimes \Psi_f \) be as above and unramified outside \( \Sigma \). Then, there exists a cusp form \( \Psi' = \Psi_{\infty} \otimes \Psi_f \in \pi \), with \( \Psi \) and \( \Psi' \) coinciding outside \( \Sigma \), and a section \( f \) for the Eisenstein series \( E_3(h, s) \), such that
\[ \langle r_D(E_{\text{M},3}), [\omega \Psi'] \rangle \in \overline{\mathbb{Q}}^\times I_\infty(f, A \cdot \Psi, 0) \cdot L^\Sigma(\pi, \text{Spin}, 0), \]
where \( L^\Sigma(\pi, \text{Spin}, s) = \prod_{\mathfrak{p} \in \Sigma} L(\pi_{\mathfrak{p}}, \text{Spin}, s) \), \( I_\infty(f, A \cdot \Psi, s) \) is an explicit archimedean integral defined in ([5.3.3]) and \( A \) is the operator of Lemma 5.1.

Some remarks are in order.

Remark 1.4. The motivic cohomology classes \( E_{\text{M},2} \) were first considered in [23] and Theorem 1.3 recovers, up to the calculation of the archimedean integral, the main result of loc. cit.

Remark 1.5. The classes \( E_{\text{M},3} \) were used in [6], where the first and third authors showed that their étale realisations could be assembled into a norm-compatible tower of cohomology classes for some \( p \)-level subgroups, giving rise to an element of the Iwasawa cohomology of the local \( p \)-adic Galois representation associated to the automorphic representation of \( \text{GSp}_6 \). The above theorem shows that the \( p \)-adic \( L \)-functions constructed in [6] are indeed related to special values of complex \( L \)-functions in the spirit of Perrin-Riou conjectures.

Remark 1.6. We expect to express the archimedean factor of the above theorem in terms of certain Gamma factors. This should be achieved by giving explicit models of discrete series representations as in [27], [29] using the differential equations that arise from Schmid’s realisation of discrete series in [35]. This procedure is quite involved in our case (yet still reasonable) and we expect to come back to it in a not so distant future.

To conclude this introduction, according to the taste of the reader, one could also be interested in the non-vanishing of the motivic cohomology group in which our class lives. For such claim, we need to assume (for the moment) the non-vanishing of the archimedean integral and, certainly, the non-vanishing at \( s = 0 \) of the Spin \( L \)-function of the automorphic representation of \( \pi \).

Corollary 1.7 (Corollary 5.15). Assume that there exists an automorphic representation \( \pi \) of \( \text{G}(\text{A}) \) and a cusp form \( \Psi \) in \( \pi \), which satisfy all the running assumptions and such that
\[ I_\infty(f, A \cdot \Psi, 0) \cdot L^\Sigma(\pi, \text{Spin}, 0) \neq 0. \]
Then, the class \( E_{\text{M},3} \) is non-trivial and thus \( H^7_M(\text{Sh}_\text{G}(U), \mathbb{Q}(4)) \) is non-zero.
Remark 1.8. If $\pi$ has a cuspidal spin lift to $\text{GL}_8$, one can weaken the assumption of the above Corollary to the non-vanishing of the archimedean integral (Corollary 5.17).

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2. Preliminaries

In this section, we introduce some notation and define the basic objects. We state a lemma concerning the existence of certain subgroups of the symplectic groups and we use them to define the motivic cohomology classes we will study later.

2.1. Groups. Let $GSp_{2n}$ be the group scheme over $\mathbb{Z}$ whose $R$-points, for any commutative ring $R$ with identity, are described by

$$GSp_{2n}(R) = \{ A \in GL_{2n}(R) : \ t^t A J A = \nu(A) J, \ \nu(A) \in G_m(R) \},$$

where $J$ is the matrix $\left( \begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix} \right)$, for $I_n$ denoting the $n \times n$ identity matrix.

2.1.1. Subgroups. Let $F$ be a totally real $\mathbb{Q}$-algebra of dimension $\delta$; denote by $GSp^*_{2m,F}/\mathbb{Q}$ the subgroup scheme of $\text{Res}_{F/\mathbb{Q}} GSp_{2m,F}$ sitting in the Cartesian diagram

$$GSp^*_{2m,F} \hookrightarrow \text{Res}_{F/\mathbb{Q}} GSp_{2m,F} \xrightarrow{\nu} GSp_{2m,F} \xrightarrow{\text{Res}_{F/\mathbb{Q}}} GSp_{m,F}.$$  

For instance, when $F = \mathbb{Q}^\delta$,

$$GSp^*_{2m,F} = GSp_{2m}^\delta = \{ (g_i) \in GSp_{2m}^\delta : \nu(g_1) = \cdots = \nu(g_\delta) \}.$$  

Consider $F^{2m}$ with its standard $F$-alternating form $\langle , \rangle_F$. We fix the standard symplectic $F$-basis $\{ e_1, \ldots, e_m, f_1, \ldots, f_m \}$ and define $\langle , \rangle_Q$ to be $\text{Tr}_{F/\mathbb{Q}} \circ \langle , \rangle_F$. Then, by definition $GSp^*_{2m,F} \subset GSp(\langle , \rangle_Q)$. Notice that, after opportuneuly fixing a $\mathbb{Q}$-basis of $F$, $GSp(\langle , \rangle_Q)$ becomes isomorphic to $GSp_{2m}^{\nu}$, thus we have an embedding

$$GSp^*_{2m,F} \hookrightarrow GSp(\langle , \rangle_Q) \simeq GSp_{2m}^{\nu}. \quad (1)$$

Example 2.1. Let $F$ be a real étale quadratic extension over $\mathbb{Q}$. Such extensions are parametrised by $a \in Q^{x}_{>0}/(Q^{x}_{>0})^2$, and we identify $F = \mathbb{Q} \oplus \mathbb{Q}\sqrt{a}$, for a representative $a$ of the corresponding class in $Q^{x}_{>0}/(Q^{x}_{>0})^2$. Let $m = 1$; we realise the isomorphism $GSp(\langle , \rangle_Q) \simeq GSp_4$, by choosing the $\mathbb{Q}$-basis of $F^2$ given by

$$\left\{ \frac{1}{\sqrt{a}}e_1, \frac{1}{2}e_1, \sqrt{a}f_1, f_1 \right\}.$$  

Indeed, such a basis represents the alternating form $\langle , \rangle_Q$ as given by $J$.

Let $V_n$ be the standard representation of $GSp_{2n}$ with symplectic basis $\{ e_i, f_i \}$. Given a partition $(n_i)_{1 \leq i \leq t}$ of $n$, we will consider the embedding

$$GSp_{2n_1} \boxtimes \cdots \boxtimes GSp_{2n_t} \hookrightarrow GSp_{2n} \quad (2)$$

induced by the decomposition $V_n = \bigoplus_{i=1}^t V_{n_i}$, where each $V_{n_i}$ is a vector space of dimension $2n_i$ endowed with symplectic basis $\{ e_{s_i-1+1}, \ldots, e_{s_i}, f_{s_i-1+1}, \ldots, f_{s_i} \}$, with $s_k := \sum_{j=1}^k n_j$. 


2.1.2. Modular embeddings. We start with the following combinatorial result.

**Lemma 2.2.** For almost all integers $n$, there exist $n_1 \leq \ldots \leq n_{k(n)} \in \mathbb{N}$ such that $\sum_i n_i = n$ and such that the following is true:

- If $n \equiv 0$ or 3 modulo 4, then $n_1 = 1$ and $\sum_i \left( \frac{n_i+1}{2} \right) = \frac{1}{2} \left( \frac{n+1}{2} \right)$.
- If $n \equiv 1$ or 2 modulo 4, then $n_1 = n_2 = 1$ and $\sum_i \left( \frac{n_i+1}{2} \right) = \frac{1}{2} \left( \frac{n+1}{2} \right) + 1$.

**Proof.** Let $n \equiv 0$ or 3 modulo 4, then we need to find $n_i$’s such that $1 + \sum_i n_i = n$ and such that $1 + \sum_i \left( \frac{n_i+1}{2} \right) = \frac{1}{2} \left( \frac{n+1}{2} \right)$. Using the first condition, the second one boils down to

$$\sum_i n_i^2 = \frac{n^2 + n}{2} - \sum_i n_i - 2 = \frac{(n-1)^2 + (n-1) - 2}{2}.$$ 

Analogously, if $n \equiv 1$ or 2 modulo 4, then we want to find $n_i$’s such that $1 + 1 + \sum_i n_i = n$ and the second condition in the lemma becomes

$$\sum_i n_i^2 = \frac{(n-2)^2 + 3(n-2)}{2}.$$ 

The existence of $n_i$’s satisfying these conditions follows from [33] for $n$ large enough. \qed

Let $n$ be an integer as in the lemma above and let $p_n = (n_i)_i$ be a partition of $n$ given by the lemma. Note that $p_n$ is not necessarily unique. We agree that $n_1 \leq n_2 \leq \ldots \leq n_{k(n)}$.

We set $\varepsilon_n = 1$ (resp. 2) if $n \equiv 0$ or 3 (resp. 1 or 2) modulo 4. Let us consider the sequence of subsets of $p_n$ defined inductively by

$$X_1 = \begin{cases} \{n_1\} & \text{if } \varepsilon_n = 1 \\ \{n_1, n_2\} & \text{if } \varepsilon_n = 2 \end{cases}$$

$$X_s = \left\{ n_i \in p_n \mid n_i = \min \left\{ p_n - \bigcup_{j=1}^{s-1} X_j \right\} \right\}, \text{for } s \geq 2.$$ 

For any $i \geq 2$, let $\delta_i = |X_i|$, $F_s$ denote a totally real étale $\mathbb{Q}$-algebra of dimension $\delta_s$ and let $m_i$ denote the common value of the elements of $X_i$. By composing the maps of (1) and (2), we construct the embedding

$$\iota_n : H_n := \text{GSp}_2 \boxtimes \text{GSp}^*_2 \boxtimes \text{GSp}^*_2 \boxtimes \text{GSp}^*_{2m_2, F_2} \cdots \boxtimes \text{GSp}^*_{2m_1, F_1} \hookrightarrow \text{GSp}_{2n} \equiv: G_n$$

if $\varepsilon_n = 1$ and the embedding

$$\iota_n : H_n := \text{GSp}_2 \boxtimes \text{GSp}_2 \boxtimes \text{GSp}^*_2 \boxtimes \text{GSp}^*_2 \boxtimes \text{GSp}^*_{2m_2, F_2} \cdots \boxtimes \text{GSp}^*_{2m_1, F_1} \hookrightarrow \text{G}_n$$

if $\varepsilon_n = 2$.

**Remark 2.3.** Some examples of embeddings $\iota_n$ for small values of $n$ are:

- $\text{GSp}_2 \boxtimes \text{GSp}_2 \hookrightarrow \text{GSp}_4$,
- $\text{GSp}_2 \boxtimes \text{GSp}^*_{2, F} \hookrightarrow \text{GSp}_6$,
- $\text{GSp}_2 \boxtimes \text{GSp}_2 \boxtimes \text{GSp}_4 \hookrightarrow \text{GSp}_8$,
- $\text{GSp}_2 \boxtimes \text{GSp}_2 \boxtimes \text{GSp}_6 \hookrightarrow \text{GSp}_{10}$,
- $\text{GSp}_2 \boxtimes \text{GSp}_4 \boxtimes \text{GSp}_8 \hookrightarrow \text{GSp}_{14}$.

The first one was the one used by [23] and [25] and the second one, the example of most interest to this article, was considered in [32] and [6]. We also remark that there is no partition satisfying the condition of Lemma 2.2 for $n = 6, 9, 10, 13, 16, 17, 26, 33$ (and these are most probably all the exceptions of Lemma 2.2).
2.2. Shimura varieties. We keep the notations of the previous section. In particular, we let $p_n$ be a partition of $n$ as in Lemma 2.2. Let $S = \text{Res}_C/RG_m/C$ be the Deligne torus.

After identifying $\text{GSp}_{2m_1, F_1}/R$ with $\text{GSp}_{2m_2, R}$, denote by $X_{H_n}$ the $H_n(R)$-conjugacy class of
\[
H_n = h \mapsto H_n^x \mapsto R, \quad x + iy \mapsto \left( \begin{array}{cc}
x & y \\
y & -x \end{array} \right),
\]
The pair $(H_n, X_{H_n})$ defines a Shimura datum of reflex field $Q$. Denote by $\text{Sh}_{H_n}$ the corresponding Shimura variety of dimension $\sum_i \binom{n_i + 1}{2}$.

Remark 2.4. If $U \subseteq H_n(A_f)$ is a fibre product (over the similitude characters) $U_1 \times A^\vee \cdots \times A^\vee U_t$ of sufficiently small subgroups, we have
\[
\text{Sh}_{H_n}(U) = \text{Sh}_{\text{GL}_2}(U_1) \times G_m \times \cdots \times G_m \times \text{Sh}_{\text{GSp}_{2m_1, F_1}}(U_t),
\]
where $\times G_m$ denotes the fibre product over the zero dimensional Shimura variety of level $D = \text{det}(U_t)$
\[
\pi_0(\text{Sh}_{\text{GL}_2})(D) = \hat{Z}^*/D.
\]
Notice that the embedding $\iota_n : H_n \rightarrow G_n$ induces another Shimura datum $(G_n, X_{G_n})$ of reflex field $Q$. For any neat open compact subgroup $U$ of $G_n(A_f)$, denote by $\text{Sh}_{G_n}(U)$ the associated Shimura variety of dimension $d_n := n(n+1)/2$. We also write $\iota_n : \text{Sh}_{H_n} \hookrightarrow \text{Sh}_{G_n}$ the embedding of codimension $c_n = d_n - \sum_i \binom{n_i + 1}{2} = \frac{1}{2}(d_n + 1 - \varepsilon_n)$ induced by the group homomorphism $\iota_n : H_n \hookrightarrow G_n$.

2.3. Motivic cohomology classes for $\text{GSp}_{2n}$. We now define the cohomology classes we want to study in this article.

2.3.1. Modular units. The inputs of our construction are the modular units already considered by Beilinson. Let us introduce them following the description given in [20, §5.3].

Let $T_2$ denote the diagonal maximal torus of $\text{GSp}_2 = \text{GL}_2$ and let $B_2$ denote the standard Borel. Define the algebraic character $\lambda : \text{Sh}_{\text{GL}_2} \rightarrow \text{G}_m$ by $\lambda(\text{diag}(t_1, t_2)) = t_1/t_2$. For any $s \in C$, let
\[
\mathcal{B}_s = \bigoplus_{\text{type } \eta_\infty = \lambda} \text{Ind}_{B_2(A_f)}^{\text{GL}_2(A_f)} \eta_f|_{\lambda_f}^s
\]
be the direct sum over characters $\eta = \eta_\infty \otimes \eta_f : T_2(Q)\backslash T_2(A) \rightarrow C^\times$ such that the restriction of $\eta_\infty$ to the identity component $T_2(R)^+$ of $T_2(R)$ coincides with the character induced by $\lambda$ on $R$-points, where $\lambda_f$ denotes the character induced by $\lambda$ on $A_f$-points, and where we omit the characters $\eta$ with
\[
\eta \left( \begin{array}{cc} t \\ t^{-1} \end{array} \right) = |t|^2, t \in A^\times.
\]
Note that $\mathcal{B}_s$ is denoted by $\mathcal{B}_{s,0}$ in [20, 5.3]. If $s$ is an integer, we have an action of $\text{Aut}(C)$ on $\mathcal{B}_s$ (see [20, (3.5.5)]) and we let $\mathcal{B}_s^0$ be the subspace of $\text{Aut}(C)$-invariant functions. Beilinson defines a $\text{GL}_2(A_f)$-equivariant morphism (see [20, (3.5.4)])
\[
\text{Eis}_{\lambda} : \mathcal{B}_s^0 \rightarrow H^1_{\lambda}(\text{Sh}_{\text{GL}_2}, Q(1)) \simeq O(\text{Sh}_{\text{GL}_2})^\times \otimes Z \bar{Q}, \phi_f \mapsto u(\phi_f),
\]
where $H^1_{\lambda}(\text{Sh}_{\text{GL}_2}, Q(1))$ denotes $\lim_{\leftarrow} H^1_{\lambda}(\text{Sh}_{\text{GL}_2}(V), Q(1))$ and $O(\text{Sh}_{\text{GL}_2})^\times \otimes Z \bar{Q}$ denotes $\lim_{\leftarrow} (O(\text{Sh}_{\text{GL}_2}(V))^\times \otimes Z \bar{Q})$, the limits being taken over all neat compact open subgroups $V \subset \text{GL}_2(A_f)$. 
2.3.2. The construction. In the case $\varepsilon_n = 1$, let

$$V_1 \subset \text{GSp}_2(A_f), V_2 \subset \text{GSp}_{2m_2,F_2}(A_f), \ldots, V_t \subset \text{GSp}_{2m_t,F_t}(A_f)$$

denote compact open subgroups. If $\varepsilon_n = 2$ we make a similar choice, adapting the notation in an obvious way. We assume that the images of the $V_s$ by the similitude characters are the same. Taking the fiber products over the similitude character, we obtain a compact open subgroup needed to describe the component at infinity of the automorphic representations under consideration.

Remark 2.7. To be the composite of these morphisms.

When $\varepsilon_n = 2$, the projection on the second factor of $\text{Sh}_{H_n}(V)$ is also a morphism $p_2 : \text{Sh}_{H_n}(V) \to \text{Sh}_{\text{GL}_2}(V_1)$. Hence when $n$ is such that $\varepsilon_n = 1$, we have the sequence of morphisms

$$\mathcal{B}^0_{V_1} \xrightarrow{\text{Eis}_M} H^1_M(\text{Sh}_{\text{GL}_2}(V_1), \overline{Q}(1)) \xrightarrow{p_1^*} H^1_M(\text{Sh}_{H_n}(V), \overline{Q}(1)) \xrightarrow{\Delta_n^1} H^{d_n+1}_M(\text{Sh}_{G_n}(U), \overline{Q}(t_n)).$$

Definition 2.5. We define $\text{Eis}_{M,n} : \mathcal{B}^0_{V_1} \to H^{d_n+1}_M(\text{Sh}_{G_n}(U), \overline{Q}(t_n))$ to be the composite of these morphisms.

Definition 2.6. When $\varepsilon_n = 2$, we define $\text{Eis}_{M,n} : \mathcal{B}^0_{V_1} \otimes \mathcal{B}^0_{V_2} \xrightarrow{\text{Eis}_M \otimes \text{Eis}_M} H^1_M(\text{Sh}_{\text{GL}_2}(V_1), \overline{Q}(1)) \otimes H^1_M(\text{Sh}_{\text{GL}_2}(V_2), \overline{Q}(1)) \xrightarrow{p_1^* \otimes p_2^*} H^1_M(\text{Sh}_{H_n}(V), \overline{Q}(1)) \otimes H^1_M(\text{Sh}_{H_n}(V), \overline{Q}(1)) \xrightarrow{\cup} H^2_M(\text{Sh}_{H_n}(V), \overline{Q}(2)) \xrightarrow{\Delta_n^2} H^{d_n+1}_M(\text{Sh}_{G_n}(U), \overline{Q}(t_n))$.

where the third morphism is the cup-product in motivic cohomology.

Remark 2.7. The notation $\text{Eis}_{M,n}$ is slightly abusive as these morphisms depend also on $U$, $V$ and the data entering in the definition of $t_n$.

3. Cohomology of locally symmetric spaces

3.1. Representation theory. We set the notations for the representation theory background needed to describe the component at infinity of the automorphic representations under consideration.
3.1.1. **Cartan decomposition.** The maximal compact subgroup $K_\infty$ of $Sp_{2n}(\mathbb{R})$ is described as

$$K_\infty = \{( \begin{pmatrix} A & B \\ -B & A \end{pmatrix} ) \mid AA^t + BB^t = 1, AB^t = BA^t \}.$$  

It is isomorphic to $U(n)$ via the map $( \begin{pmatrix} A & B \\ -B & A \end{pmatrix} ) \mapsto A + iB$ and its Lie algebra is

$$\mathfrak{t} = \{( \begin{pmatrix} A & B \\ -B & A \end{pmatrix} ) \mid A = -A^t, B = B^t \}.$$  

Letting

$$\mathfrak{p}_c^\pm = \left\{ \left( \begin{array}{cc} A & \pm iA \\ \pm A & -A \end{array} \right) \in M(2n, \mathbb{C}) \mid A = A^t \right\},$$

one has a Cartan decomposition

$$\mathfrak{g}_c = \mathfrak{t} \oplus \mathfrak{p}_c^+ \oplus \mathfrak{p}_c^-.$$  

3.1.2. **Root system.** For $1 \leq j \leq n$, let $D_j \in M(n, n)$ be the matrix with entry $1$ at position $(j, j)$ and zero elsewhere. Define

$$T_j = -i \left( \begin{array}{cc} 0 & D_j \\ -D_j & 0 \end{array} \right).$$

Then $\mathfrak{h} = \oplus_j \mathbb{R} \cdot T_j$ is a compact Cartan subalgebra of $\mathfrak{g}_c$. We let $(e_j)_j$ denote the basis of $\mathfrak{h}_c^*$ dual to $(T_j)_j$. A system of positive roots for $(\mathfrak{g}_c, \mathfrak{h}_c)$ is then given by

$$2e_j, \quad 1 \leq j \leq n,$$

$$e_j + e_k, \quad 1 \leq j < k \leq n,$$

$$e_j - e_k, \quad 1 \leq j < k \leq n.$$  

The simple roots are $e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n$. We note that $\mathfrak{p}_c^+$ is spanned by the root spaces corresponding to the positive roots of type $2e_j$ and $e_j + e_k$. We denote $\Delta = \{ \pm 2e_j, \pm (e_j + e_k) \}$ the set of all roots, $\Delta_c = \{ \pm (e_j - e_k) \}$ the set of compact roots and $\Delta_{nc} = \Delta - \Delta_c$ the non-compact roots. Finally, we note $\Delta^+, \Delta_c^+$ and $\Delta_{nc}^+$ the set of positive, positive compact and positive non-compact roots, respectively.

The corresponding root vectors for each root space are given as follows:

- For $1 \leq j \leq n$, the element $X_{\pm 2e_j} = (D_j, \pm iD_j)$ spans the root space of $\pm 2e_j$.
- For $1 \leq j < k \leq n$, letting $E_{jk}$ be the matrix with entry $1$ at positions $(j, k)$ and $(k, j)$ and zeroes elsewhere, the elements $X_{\pm (e_j + e_k)} = (E_{jk}, \pm iE_{jk})$ spans the root space for of $e_j + e_k$.
- Finally, for $1 \leq j < k \leq n$, letting $F_{jk}$ be the matrix with entry $1$ at position $(j, k)$, $-1$ at position $(k, j)$ and zeroes elsewhere, the element $X_{\pm (e_j - e_k)} = (\pm F_{jk}, \pm iE_{jk})$ spans the root space of the compact root $\pm (e_j - e_k)$.

3.1.3. **Weyl groups.** Recall that the Weyl group of $G$ is given by $\mathcal{W}_G = \{ \pm 1 \}^n \rtimes S_n$. The reflection $\sigma_j$ in the orthogonal hyperplane of $2e_j$ simply reverses the sign of $e_j$ while leaving the other $e_k$ fixed. The reflection $\sigma_{jk}$ in the orthogonal hyperplane of $e_j - e_k$ exchanges $e_j$ and $e_k$ and leaves the remaining $e_l$ fixed. The Weyl group $\mathcal{W}_K$ of $K_\infty \cong U(n)$ is isomorphic to $S_n$ and, via the embedding into $G$, identifies with the subgroup of $\mathcal{W}_G$ generated by the $\sigma_{jk}$. With the identification $\mathcal{W}_G = N(T)/Z(T)$, an explicit description of $\mathcal{W}_G$ and $\mathcal{W}_K$ is given as follows. The matrices corresponding to the reflections $\sigma_{jk}$ are

$$\begin{pmatrix} S_{jk} & 0 \\ 0 & -S_{jk} \end{pmatrix},$$
where $S_{jk}$ is the matrix with entry 1 at places $(\ell, \ell)$, $\ell \neq j, k$, $(k, j)$ and $(j, k)$ and zeroes elsewhere. The matrices corresponding to the reflection $\sigma_j$ in the hyperplane orthogonal to $2e_j$ are of the form

$$
\left( \begin{array}{cc}
0 & T_j \\
-T_j & 0
\end{array} \right),
$$

where $T_j$ denotes the diagonal matrix with $-1$ at the place $(j, j)$ and ones at the other entries of the diagonal.

3.1.4. $K_{\infty}$-types. We previously defined the maximal compact subgroup $K_{\infty} \simeq U(n)$ of $\text{Sp}_{2n}(\mathbb{R})$, with Lie algebra $\mathfrak{k}$, and we considered the Cartan decomposition $\mathfrak{sp}_{2n, \mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^+ \oplus \mathfrak{p}_{\mathbb{C}}^-$. Recall that $\mathfrak{p}_{\mathbb{C}}^\pm = \bigoplus_{\alpha \in \Delta^+_c \lor \Delta^-_c} X_{\pm \alpha}$.

Denote by $(k_1, \cdots, k_n) = k_1e_1 + \cdots + k_ne_n$, with $k_i \in \mathbb{Z}$ the integral weights. Integral weights are dominant for our choice of $\Delta^+_c$ if $k_1 \geq k_2 \geq \cdots \geq k_n$. Recall that there is a bijection between isomorphism classes of finite dimensional irreducible complex representations of $K_{\infty}$ and dominant integral weights, given by assigning to the representation $\tau(k_1, \cdots, k_n)$ its highest weight $(k_1, \cdots, k_n)$.

3.2. Lie algebra cohomology. Let $A_G = \mathbb{R}^*_+$ denote the identity component of the center of $G(\mathbb{R})$ and let $K_G = A_G K_{\infty} \subset G(\mathbb{R})$. The embedding $\mathfrak{sp}_{2n, \mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ induces an isomorphism

$$
\mathfrak{sp}_{2n, \mathbb{C}}/\mathfrak{k} \simeq \mathfrak{g}_{\mathbb{C}}/(\text{Lie}(K_G))_{\mathbb{C}}.
$$

By [4, II. Proposition 3.1], for any discrete series $\pi_{\infty}$ associated to the trivial representation (cf. §3.3 below), we have

$$
H^d(\mathfrak{g}, K_G; \pi_{\infty}) = \text{Hom}_{K_G}(\bigwedge^d \mathfrak{sp}_{2n, \mathbb{C}}/\mathfrak{k}; \pi_{\infty}),
$$

where $d = n(n+1)/2$. By using the Cartan decomposition above, we get

$$
\bigwedge^d \mathfrak{sp}_{2n, \mathbb{C}}/\mathfrak{k} = \bigoplus_{p+q=d} \bigwedge^p \mathfrak{p}_{\mathbb{C}}^+ \otimes C \bigwedge^q \mathfrak{p}_{\mathbb{C}}^-.
$$

One can easily decompose each term of the sum above in its irreducible constituents (if treated as a $K_{\infty}$-representation via the adjoint action). This will be helpful for writing explicit elements in $H^d(\mathfrak{g}, K_G; \pi_{\infty})$ according to the minimal $K_{\infty}$-type of $\pi_{\infty}$.

Example 3.1. Let $n = 3$. Then $\mathfrak{p}_{\mathbb{C}}^+$ (resp. $\mathfrak{p}_{\mathbb{C}}^-$) is the six dimensional irreducible representation of $K_{\infty}$ of weight $(2, 0, 0)$ (resp. $(0, 0, -2)$). Using Sage package for Lie groups, one
can see that
\[
\bigwedge^6 p_+^C = \tau(4,4,4)
\]
\[
\bigwedge^5 p_+^C \otimes C \bigwedge^5 p_-^C = \tau(4,2,2) \oplus \tau(4,3,1) \oplus \tau(4,4,0)
\]
\[
\bigwedge^4 p_+^C \otimes C \bigwedge^4 p_-^C = \tau(2,1,1) \oplus \tau(2,2,0) \oplus \tau(2,3,0) \oplus \tau(2,3,1) \oplus \tau(2,3,2) \oplus \tau(2,3,3) \oplus \tau(2,4,0)
\]
\[
\bigwedge^3 p_+^C \otimes C \bigwedge^3 p_-^C = 2 \cdot \tau(1,0,0) \oplus \tau(1,1,0) \oplus \tau(1,1,1) \oplus \tau(2,1,1) \oplus \tau(2,1,2) \oplus \tau(2,2,0) \oplus \tau(2,2,1) \oplus \tau(2,2,2) \oplus \tau(2,2,3) \oplus \tau(2,3,0) \oplus \tau(2,3,1) \oplus \tau(2,3,2) \oplus \tau(2,4,0)
\]
\[
\bigwedge^2 p_+^C \otimes C \bigwedge^2 p_-^C = \tau(-1,1,-1) \oplus \tau(-1,2,-1) \oplus \tau(-2,1,-1) \oplus \tau(-2,2,-1) \oplus \tau(-2,3,-1) \oplus \tau(-2,4,-1) \oplus \tau(-3,1,-1) \oplus \tau(-3,2,-1) \oplus \tau(-3,3,-1) \oplus \tau(-4,1,-1) \oplus \tau(-4,2,-1) \oplus \tau(-4,3,-1) \oplus \tau(-4,4,-1)
\]
\[
\bigwedge^1 p_+^C \otimes C \bigwedge^1 p_-^C = \tau(-2,-2,0) \oplus \tau(-1,-3,-1) \oplus \tau(0,-4,0)
\]
\[
\bigwedge^0 p_+^C \otimes C \bigwedge^0 p_-^C = \tau(-4,-4,-4).
\]

It will be useful to have some explicit description of the components \(\tau(2,2,-4)\) and \(\tau(4,-2,-2)\) of \(\bigwedge^3 p_+^C \otimes C \bigwedge^3 p_-^C\). We have a decomposition of \(K_\infty\) representations \(\bigwedge^3 p_+^C = \tau(3,3,0) \oplus \tau(4,1,1)\), \(\bigwedge^3 p_-^C = \tau(-1,-1,-4) \oplus \tau(0,-3,-3)\). Since each of the four summands have multiplicity-free weights (i.e. every weight space has dimension at most one), then one can easily check that the vector

\[
X_{(2,2,-4)} := (X_{2e_1} \wedge X_{2e_2} \wedge X_{e_1+e_2}) \otimes (X_{-e_1-e_3} \wedge X_{-e_2-e_3} \wedge X_{-2e_3})
\]

is a highest weight vector of \(\tau(2,2,-4) \subset \tau(3,3,0) \otimes \tau(-1,-1,-4)\) (indeed \(\tau(2,2,-4)\) is the Cartan component of the tensor product and each of the terms in the tensor product defining \(X_{(2,2,-4)}\) is a highest weight vector of its corresponding representation). Analogously, a highest weight vector of \(\tau(4,-2,-2) \subset \tau(4,1,1) \otimes \tau(0,-3,-3)\) is given by

\[
X_{(4,-2,-2)} := (X_{e_1+e_3} \wedge X_{e_2+e_3} \wedge X_{2e_3}) \otimes (X_{-e_2-e_3} \wedge X_{-2e_2} \wedge X_{-2e_3}).
\]

Observe finally that we can pass from \(\tau(2,2,-4)\) to \(\tau(-4,-2,-2)\) by the action of the complex conjugation.

3.3. Discrete series \(L\)-packets. We recall some standard facts on discrete series. For any non-singular weight \(\Lambda \in \Delta\), define

\[
\Delta^+(\Lambda) := \{\alpha \in \Delta : \langle \alpha, \Lambda \rangle > 0\}, \quad \Delta^+(\Lambda) = \Delta^+(\Lambda) \cap \Delta_c,
\]

where \(\langle , \rangle\) is the standard scalar product on \(\mathbb{R}^3\).

Let \(\lambda\) be a dominant weight for \(G_0 = Sp_{2n}\) (with respect to the complexification \(h_C\) of the compact Cartan algebra \(h\)) and let \(\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = (n, n - 1, \ldots, 1)\). As \(|W_{G_0}/W_{K_\infty}| = 2^n\), the set of equivalence classes of irreducible discrete series representations of \(G_0(\mathbb{R})\) with Harish-Chandra parameter \(\lambda + \rho\) contains \(2^n\) elements. More precisely, let us choose representatives \(\{w_1, \ldots, w_{2^n}\}\) of \(W_{G_0}/W_{K_\infty}\) of increasing length and such that for any \(1 \leq i \leq 2^n\), the weight \(w_i(\lambda + \rho)\) is dominant for \(K_\infty\). Then for any \(1 \leq i \leq 2^n\) there exists an irreducible discrete series \(\pi^\Lambda_{i,\infty}\), where \(\Lambda = w_i(\lambda + \rho)\), of Harish-Chandra parameter \(\Lambda\) and
containing with multiplicity 1 the minimal \( K_{\infty} \)-type with highest weight \( \Lambda + \delta_{G_0} - 2\delta_{K_{\infty}} \) where \( \delta_{G_0} \), resp. \( \delta_{K_{\infty}} \), is the half-sum of roots, resp. of compact roots, which are positive with respect to the Weyl chamber in which \( \Lambda \) lies, i.e.,

\[
2\delta_{G_0} := \sum_{\alpha \in \Delta^+(\Lambda)} \alpha, \quad 2\delta_{K_{\infty}} := \sum_{\alpha \in \Delta^+_{\text{c}}(\Lambda)} \alpha.
\]

Moreover, for \( i \neq j \), \( \Lambda = w_i(\lambda + \rho) \), \( \Lambda' = w_j(\lambda + \rho) \), the representations \( \pi_{\infty} \) and \( \pi_{\infty}' \) are not equivalent and any discrete series of \( G_0 \) is obtained in this way ([21 Theorem 9.20]). We define the discrete series \( L \)-packet \( P(V^\lambda) \) associated to \( \lambda \) to be the set of isomorphism classes of discrete series of \( G_0(\mathbb{R}) \) whose Harish-Chandra parameter is of the form \( \Lambda = w_i(\lambda + \rho) \), for some \( 1 \leq i \leq 2^n \).

**Lemma 3.2.** Let \( n = 3 \). There exist two irreducible discrete series representations \( \pi_{\infty}^{3,3} \) and \( \pi_{\infty}^{3,3} \) of \( \text{Sp}_6(\mathbb{R}) \) with Harish-Chandra parameter \((2, 1, -3)\) and \((3, -1, -2)\), and trivial central character. On the other hand, by definition of cuspidal cohomology we have

\[
\pi_{\infty}^{3,3}, \pi_{\infty}^{3,3} \ \text{for some} \ \Lambda = \pi \ \text{with respect to the Weyl chamber in which} \ \pi \ \text{lies, i.e.,}
\]

\[
\sum_{\alpha \in \Delta^+(\Lambda)} \alpha, \quad \sum_{\alpha \in \Delta^+_{\text{c}}(\Lambda)} \alpha.
\]

Using this formula, one easily checks that the minimal \( K_{\infty} \)-type of the discrete series \( \pi_{\infty}^{3,3} \) is given by the formula

\[
\Lambda + \delta_{G_0} - 2\delta_{K_{\infty}} = 2w_i\rho - 2\delta_{K_{\infty}}.
\]

Using this formula, one easily checks that the minimal \( K_{\infty} \)-types corresponding to each of representative \( w_i \), \( i = 1, \ldots, 8 \) described above are, respectively, \( \tau_1 = \tau(4,4,4), \tau_2 = \tau(4,4,0), \tau_3 = \tau(4,2,-2), \tau_4 = \tau(2,2,-4), \tau_5 = \tau(4,-2,-2), \tau_6 = \tau(2,-2,-4), \tau_7 = \tau(0,-4,-4) \) and \( \tau_8 = \tau(-4,-4,-4) \). The result follows by considering \( i = 4, 5 \).

**Remark 3.3.** Observe that each minimal \( K_{\infty} \)-type of each discrete series, except for the ones stated in Lemma 3.2, corresponds to a different component of \( \Lambda^0 \mathfrak{g}_C/Lie(K_G)C \) of Example 3.1.

### 3.4. Cohomological automorphic representations

For any neat compact open subgroup \( U \subset G(A_f) \), let \( H^d_{(2)}(\text{Sh}_G(U), \mathbb{C}) \) denote the \( L^2 \)-cohomology of \( \text{Sh}_G(U) \) with coefficients in \( \mathbb{C} \) in the middle degree. There is a canonical Hecke equivariant isomorphism of \( C \)-vector spaces

\[
H^d_{(2)}(\text{Sh}_G(U), \mathbb{C}) \simeq \bigoplus_{\pi = \pi_{\infty} \otimes \overline{\pi}_f} H^d(\mathfrak{g}, K; \pi_{\infty})^m_{\text{disc}}(\pi) \otimes \pi_f^U
\]

where \( \pi \) runs over the discrete spectrum of \( L^2(\mathbb{Z}(A)G(Q)\backslash G(A), 1) \), where 1 is the trivial character. On the other hand, by definition of cuspidal cohomology we have

\[
H^d_{\text{cusp}}(\text{Sh}_G(U), \mathbb{C}) = \bigoplus_{\pi = \pi_{\infty} \otimes \overline{\pi}_f} H^d(\mathfrak{g}, K; \pi_{\infty})^m_{\text{cusp}}(\pi) \otimes \pi_f^U
\]

where \( \pi \) runs over cuspidal automorphic representations.
Let $\Pi^{\text{triv}}$ be the discrete series $L$-packet attached to the trivial representation, i.e. the set of isomorphism classes of irreducible discrete series of $G(\mathbb{R})$ with trivial central character and with Harish-Chandra parameter $w_i \rho$ for some $w_i \in \mathfrak{W}_G/\mathfrak{W}_{K_\infty}$. It follows from Harish-Chandra’s classification that $|\Pi^{\text{triv}}| = 2^{n-1}$ (cf. also [36, §2.5] to see how to pass from $G_0$ to $G$).

Let $\pi = \pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation of $G(\mathbb{A})$. We make the following hypothesis.

1. $\pi_\infty \in \Pi^{\text{triv}}$.
2. for any $\sigma_\infty \in \Pi^{\text{triv}}$ we have $m_0(\sigma_\infty \otimes \pi_f) = 1$ where $m_0(\sigma)$ denotes the multiplicity of $\sigma$ in the space of cuspidal automorphic forms,
3. for any cuspidal representation $\sigma = \sigma_\infty \otimes \pi_f$ such that $\sigma_f \simeq \pi_f$ and $H^d(\mathfrak{g}, K_G, \sigma_\infty) \neq 0$ we have $\sigma_\infty \in \Pi^{\text{triv}}$.

**Remark 3.4.** The first two points are a stability at infinity assumption. The third assumption is likely to be true as for any $\sigma_\infty \in \Pi^{\text{triv}}$ we have dim $H^d(\mathfrak{g}, K_G; \sigma_\infty) = 2$ and we expect a rank $2^n$ motive.

**Lemma 3.5.** Under the hypothesis (1), (2) and (3), we have

$$H^d_{(1)}(\text{Sh}_G(U), \mathbb{C})[\pi_f] = H^d_{\text{cusp}}(\text{Sh}_G(U), \mathbb{C})[\pi_f] = H^d(\text{Sh}_G(U), \mathbb{C})[\pi_f]$$

**Proof.** The proof is as the one of [28 Proposition 1]. Observe that the regularity assumption of loc. cit. on the weight is only used to deduce that $\pi_\infty$ is a discrete series representation, and this forms part of our set of hypothesis.

**Corollary 3.6.** Under the hypotheses (1), (2) and (3) we have

$$H^d(\text{Sh}_G(U), \mathbb{C})[\pi_f] \simeq \bigoplus_{\pi_\infty \in \Pi^{\text{triv}}} \text{Hom}_K(\Lambda^d \mathfrak{g}/\mathfrak{t}, \pi_\infty).$$

Moreover, for any $\pi_\infty \in \Pi^{\text{triv}}$, the $\mathbb{C}$-vector space $\text{Hom}_K(\Lambda^d \mathfrak{g}/\mathfrak{t}, \pi_\infty)$ has dimension 2.

**Proof.** See [28 Theorem 5.2].

**Remark 3.7.** If $\pi_\infty \in \Pi^{\text{triv}}$ is a discrete series for $G(\mathbb{R})$, then its restriction to $G_0(\mathbb{R})$ decomposes as $\pi_\infty^1 \oplus \pi_\infty^2$, where $\pi_\infty^1$ is a discrete series for $G_0(\mathbb{R})$ (with trivial coefficients), and both conjugate one to the other. In this case, each of the spaces $\text{Hom}_K(\Lambda^d \mathfrak{g}/\mathfrak{t}, \pi_\infty^i)$ has dimension one.

### 3.5. Test vectors

**Lemma 3.8.** Let $p, q \geq 0$ be two integers such that $p + q = d$. Let $\pi = \pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation of $G(\mathbb{A})$ such that $\pi_\infty|_{G_0(\mathbb{R})} \simeq \pi_\infty^1 \oplus \pi_\infty^2$ is a discrete series such that $\text{Hom}_{K_G}(\Lambda^p \mathfrak{p}_C^+ \otimes \Lambda^q \mathfrak{p}_C^-, \pi_\infty^1) \neq 0$ and such that $\pi_f^U \neq 0$. Let $\Psi = \Psi_\infty \otimes \Psi_f$ be a cusp form in the space of $\pi$ such that $\Psi_\infty$ is a highest weight vector of the minimal $K_\infty$-type $\tau_\infty^1$ of $\pi_\infty$ and such that $\Psi_f$ is a non-zero vector in $\pi_f^U$. Let $X_{1\infty}$ be a highest weight vector in the $K_\infty$-type $\tau_\infty^1 \subset \Lambda^p \mathfrak{p}_C^+ \otimes \Lambda^q \mathfrak{p}_C^-$ (this inclusion is assured by the hypothesis that $\pi_\infty$ contributes to the $(p, q)$-part of the cohomology). Then there exists up to scalars a unique non-zero $\omega_\Psi \in \text{Hom}_{K_G}(\Lambda^p \mathfrak{p}_C^+ \otimes \Lambda^q \mathfrak{p}_C^-, \pi_\infty^1) \otimes \pi_f^U$ differential form.
on $\text{Sh}_G(U)$ such that $\omega_\Psi(X^1_\infty) = \Psi$. Moreover, the cohomology class of $\omega_\Psi$ belongs to $H^d_{dR,!}(\text{Sh}_G(U), \mathbb{C})$.

Proof. The results follows basically from [3, Theorem II.5.3]. The fact that the harmonic form $\omega_\Psi \in H^d_{dR,!}(\text{Sh}_G(U), \mathbb{C})$ follows from the cuspidality of $\Psi$ and from [3, Corollary 5.5]. □

Remark 3.9. When $n = 3$ and $\pi_\infty|_{\text{Sp}_4(\mathbb{R})} \simeq \pi_\infty^{3,3} \oplus \pi_\infty^{3,3}$, we have that $\tau^1_\infty = \tau(2,2,-4)$ and we can (and do) choose $X^1_\infty$ as constructed in Example 3.1.

3.6. **Archimedean $L$-functions and Deligne cohomology.** We end up this chapter by recalling some classical results on the relation between Deligne cohomology groups and the $L$-function of a motive, which explains when one expects to have non-trivial motivic cohomology.

3.6.1. **Hodge decomposition.** In this section, we describe the Hodge decomposition of the motive

$$M(\pi_f)^{\sigma} = H^d_{dR}(\pi^{\sigma})[\pi_f]''',$

which will allows us to describe $\Gamma$-factor of its $L$-function and its simple poles.

Recall from [12], [28] that $M(\pi_f)$ is pure of weight $w = \frac{n(n+1)}{2} - |\lambda|$. The Hodge weights lie in the set of pairs $(p, q)$ where $p$ is of the form

$$\sum_{i \in B} (n - i + 1) - \sum_{i \in B} \lambda_i,$$

and where $B$ runs over all subsets of $\{1, 2, \ldots, n\}$. Moreover, under a stability assumption, all those weights will appear.

**Example 3.10.** When $n = 3$, the Hodge weights are given by

$$(-|\lambda|, 6), (1 - \lambda_1 - \lambda_2, 5 - \lambda_3), (2 - \lambda_1 - \lambda_3, 4 - \lambda_2), (3 - \lambda_2 - \lambda_3, 3 - \lambda_1),$$

and their symmetric. The ordered values of $p$ in the weights $(p, q)$ of the decomposition are therefore

$$-|\lambda| \leq 1 - \lambda_1 - \lambda_2 \leq 2 - \lambda_1 - \lambda_3 \leq t_1 \leq t_2 \leq 4 - \lambda_2 \leq 5 - \lambda_3 \leq 6,$$

where $\{t_1, t_2\} = \{3 - \lambda_1, 3 - \lambda_2 - \lambda_3\}$.

3.6.2. **Archimedean $L$-functions and Deligne cohomology.** We recall now, following [38], the definition of the $\Gamma$-factor of $M(\pi_f)$. The simple poles of this factor determine (in the range of convergence of the Euler product or under some assumption) for which twists of the motive $M(\pi_f)$ its Deligne cohomology is of dimension 1, and it will turn out that it is precisely for these twists that the methods we use allow us to construct motivic cohomology classes.

Let us suppose that $\pi_\infty \in \Pi^{triv}$. The Betti realisation of $M(\pi_f)$ admits a Hodge decomposition

$$M_B(\pi_f) \otimes \mathbb{C} = \bigoplus_{p+q = w} H^{p,q}$$

and it is equipped with an involution $\sigma$ such that $\sigma(H^{p,q}) = H^{q,p}$. Denote $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ the Betti numbers. For any $p$, write $H^{p,p} = H^{p,+} \oplus H^{p,-}$, where $H^{p,\pm} = \{x \in H^{p,p}| \sigma(x) = \pm(-1)^p x\}$, and let $h^{p,\pm} = \dim_{\mathbb{C}} H^{p,\pm}$. Let $\Gamma_R(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s)$ be the real and complex Gamma factors, so that we have $\Gamma_C(s) = \Gamma_R(s)\Gamma_R(s+1)$. The archimedean factor of the $L$-function of $M(\pi_f)$ is then defined as

$$\Gamma(M(\pi_f), s) = \prod_{p < q} \Gamma_C(s-p)^{h^{p,q}} \cdot \prod_p \Gamma_R(s-p)^{h^{p,+}} \Gamma_R(s-p+1)^{h^{p,-}}.$$
Since the gamma function $\Gamma(s)$ has simple poles at $s = -n$, $n \in \mathbb{N}$, a simple calculation shows the following.

**Lemma 3.11.** The order of the pole of $\Gamma(M(\pi_f), s)$ at $s = m$, $m \in \mathbb{Z}$, is given by

$$\sum_{m \leq p < q} h^{p,q} + \sum_{2p = w, m \leq p} h^{p,(-1)^{m-p}}.$$

Immediately from this lemma, we get:

**Lemma 3.12.** Let $n \geq 2$, then $\Gamma(M(\pi_f), s)$ has a pole of order $h^{p_0,1}$ (resp. $h^{p_0,0}$) at $s = w/2$ (resp. $s = \frac{w-1}{2}$), where $(p_0, q_0) = (w/2, w/2)$ (resp. $(p_0, q_0) = (\frac{w-1}{2}, \frac{w+1}{2})$) if $n \equiv 0, 3 \pmod{4}$ (resp. $n \equiv 1, 2 \pmod{4}$).

Finally, recall the following result:

**Proposition 3.13** ([37] §2, Proposition]. We have

$$\dim_{\mathbb{R}} H^1_{BD}(M(\pi_f)(w+1-m)) = \begin{cases} \text{ord}_{s=m} L(M(\pi_f), s) & \text{if } m < w/2 \\ \text{ord}_{s=m} L(M(\pi_f), s) - \text{ord}_{s=m+1} L(M(\pi_f), s) & \text{if } m = w/2. \end{cases}$$

Throughout this text we will make the following two assumptions:

1. $h^{p_0,1} \neq 0$ whenever $\varepsilon_n = 1$ and $h^{p_0,0} \neq 0$ in the case $\varepsilon_n = 2$.
2. When $\varepsilon_n = 1$, we will assume that $L(M(\pi_f), s)$ does not have a pole at $s = w/2 + 1$.

The first assumption is equivalent to the existence of a non-zero test vector, which we need to evaluate the regulator map in Deligne cohomology. The second assumption is in accordance with Tate’s conjecture, as we will require that the contribution to Tate’s conjecture will come from Deligne cohomology and not from the cycle class map.

According to Beilinson’s conjectures, Lemma 3.12 and Proposition 3.13 imply that one expect to construct non-trivial motivic cohomology classes in $H^1(M(\pi_f)(\frac{w}{2} + \frac{\varepsilon_n+1}{2}))$, which is the precise twist where we constructed the classes in [2,3].

4. Computation of the regulator

4.1. Deligne-Beilinson cohomology. Let $X$ denote a complex analytic variety which is smooth, quasi-projective and of pure dimension $d$. Let $\overline{X}$ be a smooth compactification of $X$ such that $Y = \overline{X} - X$ is a simple normal crossing divisor. Let $j : X \to \overline{X}$ be the canonical embedding. For $p \in \mathbb{Z}$, let $R(p)$ denote the subgroup $(2\pi i)^p \mathbb{R}$ of $\mathbb{C}$. We will denote by the same symbol the constant sheaf with value $R(p)$ on $X$. Let $\Omega^*_X$ be the sheaf of holomorphic differential forms on $X$ and let $\Omega^*_X(\log Y)$ be the sheaf of holomorphic differential forms on $X$ with logarithmic poles along $Y$ (see [3] §3.1). The Hodge filtration on $\Omega^*_X(\log Y)$ is defined as $F^p\Omega^*_X(\log Y) = \bigoplus_{p \geq p} \Omega^p_X(\log Y)$.

There are natural morphisms of complexes $R_j_* R(p) \to R_j_* \Omega^*_X$ and $F^p\Omega^*_X(\log Y) \to R_j_* \Omega^*_X$. The Deligne-Beilinson complex is defined as

$$R(p)_D := \text{cone}(R_j_* R(p) \oplus F^p\Omega^*_X(\log Y) \to R_j_* \Omega^*_X)[-1],$$

and the Deligne-Beilinson cohomology groups $H^*_D(X, R(p))$ are then defined as the hypercohomology groups $H^*_D(X, R(p)_D)$ of the complex of sheaves $R(p)_D$. It can be shown that these are independent of the choice of $\overline{X}$. From now on, assume that $X$ is defined as the analytification of the base change to $\mathbb{C}$ of a smooth, quasi-projective $\mathbb{R}$-scheme. The complex conjugation $F_\infty$ is an antiholomorphic involution on $X$ and we define real Deligne-Beilinson cohomology to be $H^*_D(X, R(p)) = H^*_D(X, R(p))\overline{F}_\infty = 1$, where $\overline{F}_\infty$ denotes the composition of the morphism induced by $F_\infty$ and of complex conjugation on coefficients.
We will need various more explicit descriptions of these cohomology groups and compatibility results between these descriptions. Let \( S^m(X/R, R(n)) \) be the vector space of \( C^\infty \) differential forms of degree \( m \) on \( X \) with values in \( R(n) \) and which are fixed by the morphism \( F_\infty \).

**Proposition 4.1.** \([20] (7.3.1)\). We have a canonical isomorphism

\[
H^m_D(X/R, R(n)) \simeq \frac{\{(\phi, \omega) \in S^{m-1}(X/R, R(n-1)) \oplus H^0(X, \Omega^1_X(\log Y)) \mid d\phi = \pi_{n-1}(\omega)\}}{dS^{n-2}(X/R, R(n-1))},
\]

where \( \pi_{n-1} : C \to R(n-1) \) is the natural projection defined by \( \pi_{n-1}(z) = \frac{z^{n-1}}{2}. \)

**Remark 4.2.** Let \( r_{1,1} : H^1_M(X, Q(1)) \to H^1_D(X, R(1)) \) be Beilinson’s regulator. Recall the canonical isomorphism \( O(X)^\times \otimes Q \simeq H^1_M(X, Q(1)). \) Then for \( u \in O(X)^\times \), the Deligne cohomology class \( r_{1,1}(u \otimes 1) \) is represented by \( \log |u| \in S^0(X/R, R(0)) \), the corresponding \( \omega \in H^0(X, \Omega^1_X(\log Y)) \) being \( d \log u \).

The following result gives the explicit description of the external cup-product in Deligne-Beilinson cohomology via the isomorphism of Proposition 4.1.

**Proposition 4.3.** Let \( X \) and \( Y \) be the base changes to \( C \) of two smooth, quasi-projective \( R \)-schemes. Let \( p_X : X \times Y \longrightarrow X \) and \( p_Y : X \times Y \longrightarrow Y \) be the canonical projections. Then, via the isomorphism of Proposition 4.1, the external cup-product

\[
\sqcup : H^m_D(X/R, R(m)) \otimes H^{m'}_D(Y/R, R(m')) \longrightarrow H^{m+m'}_D(X \times Y/R, R(m+m'))
\]

is

\[
(\phi, \omega) \sqcup (\phi', \omega') = (p_X^*\phi \wedge p_Y^*(\pi_m\omega) + (-1)^m p_X^*(\pi_m\omega) \wedge p_Y^*\phi') + p_X^*\phi \wedge p_Y^*\omega \wedge p_Y^*\phi'
\]

for any \( m, m' \).

**Proof.** The external cup-product is by definition \( x \sqcup y = p_X^*(x) \cup p_Y^*(y) \), where \( \sqcup \) denotes the usual cup-product. Hence, the statement follows from the explicit formulas for the usual cup-product given in \([10] \) \( \S 2.5 \) (see also \([11] \) \S 3.10)). \( \square \)

To give an explicit description of the Gysin morphism in Deligne-Beilinson cohomology, we need to use Deligne-Beilinson homology and currents as in \([20] \). For the definition of real Deligne-Beilinson homology, see \([18] \) Definition 1.9. Let \( T^*(X/R, R(m)) \) denote the complex of \( R(m) \)-valued currents which are fixed by the involution \( F_\infty \). Let \( T^{log}_*(X/R, C) \) denote the complex of currents on \( X \) with logarithmic poles along \( Y \) which are fixed by the involution \( F_\infty \) and endowed with its Hodge filtration (see \([18] \) Definition 1.4).

**Proposition 4.4.** \([20] \) Lemma 6.3.9\] Let \( i \) and \( j \) be two integers. We have a canonical isomorphism

\[
H^i_D(X/R, R(j)) = \{(S, T) \in T^{-i-1}(X/R, R(j-1)) \oplus F^j T^{log}_{i-1}(X/R, C) \mid dS = \pi_{j-1}(T)\} / \{d(S, T) \mid (S, T) \in T^{-i-2}(X/R, R(j-1)) \oplus F^j T^{log}_{i-1}(X/R, C)\},
\]

As currents are covariant for proper maps, the proposition below gives an explicit description of the Gysin morphism in Deligne-Beilinson cohomology.

**Proposition 4.5.** The following statements hold:

1. \([18] \) Theorem 1.15\] There is a canonical isomorphism between Deligne-Beilinson homology and cohomology

\[
H^i_D(X/R, R(j)) \simeq H^{d-i}_D(X/R, R(d-j)).
\]
where the superscript \(\rightarrow\) defined as the localization of the \(\text{log}\) classes given in Proposition 4.4, the Gysin morphism

\[ i_\ast : H^0_D(Y/R, R(n)) \to H^{m+2c}(X/R, R(n+c)) \]

is induced by the map \((S, T) \mapsto (i_\ast S, i_\ast T)\).

**Remark 4.6.** Let \(S_i^j(X, R(j)) \subset S^i(X, R(j))\) be the subspace of compactly supported differential forms. Let \(\phi \in S^i(X/R, R(j))\). Following \[28\] p. 119, let \(T_\phi \in T^{i-2d}(X/R, R(j-d))\) denote the current defined by

\[ \eta \in S^{2d-i}(X, R(d-j)) \mapsto T_\phi(\eta) = \frac{1}{(2\pi i)^d} \int_X \eta \wedge \phi. \] (5)

Similarly, for \(\omega \in H^0(X, \omega_X(\log Y))\) denote by \(T_\omega\) the integration current along \(\omega\).

### 4.2. A pairing on Deligne-Beilinson cohomology

Let \(\pi\) be a fixed cuspidal automorphic representation of \(G(A)\) having non-zero fixed vectors by a neat open compact group \(U \subseteq G(A_f)\) as in §3.4. We are going to use the main result of \[28\], so we assume that \(\pi\) satisfies the conditions denoted by (Gal) (existence of a Galois representation associated to \(\pi\) with the expected properties), (GO) (ordinarity at \(p\) of the Galois representation) and (RLI) (large image of the residual Galois representation) in [28 1.3]. It follows from [22] that under the additional assumption that there exists a finite place \(\ell\) such that \(\pi_\ell\) is the Steinberg representation of \(G(Q_\ell)\) twisted by a character, then (Gal) is true. Let \(K\) be a number field containing the Hecke eigenvalues of \(\pi\). Let \(N\) be the smallest positive integer such that the principal congruence subgroup \(U(N)\) is contained in \(U\) and let \(\mathcal{H}_K\) be the abstract spherical Hecke algebra outside \(N\) with coefficient in \(K\). This is a commutative \(K\)-algebra by the Satake isomorphism. Let \(\theta_\pi : \mathcal{H}_K \to K\) be the character determined by \(\pi\) and let \(m = \ker \theta_\pi\) the maximal ideal of the Hecke algebra associated to \(\pi\).

In what follows, given an \(R\)-vector space \(V\), we will denote \(V_m\) the \(R \otimes \mathbb{Q} \mathcal{H}^N_{K, m}\)-module defined as the localization \(V \otimes \mathbb{Q} \mathcal{H}^N_{K, m}\).

**Lemma 4.7.** We have a canonical short exact sequence of \(R \otimes \mathbb{Q} \mathcal{H}^N_{K, m}\)-modules

\[ 0 \to F^t H^{d+1}_{dR, !}(\text{Sh}_G(U), R)_m \to H^d_{B, !}(\text{Sh}_G(U), R(t-1))(-1)^{t-1} \to H^d_{D} \to 0 \]

where the superscript \((-1)^{t-1}\) in \(H^d_{B, !}(\text{Sh}_G(U), R(t-1))(-1)^{t-1}\) denotes the elements where \(F_{\infty}\) acts by \((-1)^{t-1}\).

**Proof.** As explained in [4] §5.1, Equation (5.4)], it follows easily from the definition of \(H^d_{D}(\text{Sh}_G(U), R(t))\) given at the beginning of the previous section that we have a long exact sequence

\[ \ldots \to H^d_B(\text{Sh}_G(U), C)/F^t H^d_B(\text{Sh}_G(U), C) \to H^{d+1}_D(\text{Sh}_G(U), R(t)) \to H^{d+1}_B(\text{Sh}_G(U), R(t)) \to \ldots \]

By taking the fixed points by the involution \(F^*_\infty\), we obtain the long exact sequence

\[ \ldots \to H^d_B(\text{Sh}_G(U), R(t))(-1)^t \to H^d_B(\text{Sh}_G(U), C)F^*_\infty / F^t H^d_{dR, !}(\text{Sh}_G(U), R) \to H^{d+1}_D(\text{Sh}_G(U)/R, R(t)) \to H^{d+1}_B(\text{Sh}_G(U), R(t))(-1)^t \to \ldots \]

Localizing at \(m\) and applying [28 Theorem 2], we obtain the short exact sequence

\[ 0 \to H^d_{B, !}(\text{Sh}_G(U), R(t))_m(-1)^t \to H^d_{B, !}(\text{Sh}_G(U), C)F^*_m / F^t H^d_{dR, !}(\text{Sh}_G(U), R)_m \to H^{d+1}_D(\text{Sh}_G(U)/R, R(t))_m \to 0. \]
The conclusion now follows from the equality

\[ H^d_{B,!}(\text{Sh}_G(U), C)^{\mathcal{F}^c} = H^d_{B,!}(\text{Sh}_G(U), R(t))^{(-1)^t} \oplus H^d_{B,!}(\text{Sh}_G(U), R(t-1))^{(-1)^{t-1}}. \]

\[ \square \]

Poincaré duality is a perfect pairing

\[ H^d_B(\text{Sh}_G(U), \mathbb{Q}) \times H^d_{B,c}(\text{Sh}_G(U), \mathbb{Q}) \to \mathbb{Q}(-d), \]

which is a morphism of mixed \( \mathbb{Q} \)-Hodge structures. It follows from a straightforward extension of the arguments of [24, Lemma 4.10] to our context that this pairing induces a morphism of pure \( \mathbb{Q} \)-Hodge structures

\[ H^d_{B,!}(\text{Sh}_G(U), \mathbb{Q}) \times H^d_{B,!}(\text{Sh}_G(U), \mathbb{Q}) \to \mathbb{Q}(-d). \]

of weight 2d, that we denote by \( (x, y) \mapsto (x, y) \). By a slight abuse of notation, we will denote in the same way any pairing deduced from the one above after extending the scalars from \( \mathbb{Q} \) and the analogous pairing in de Rham cohomology.

**Lemma 4.8.** Let \( \Psi \) be a cusp form in the space of \( \pi \) satisfying the conditions of Lemma 3.8 and let \( \omega_\Psi \in H^d_{dR,!}(\text{Sh}_G(U), C) \) be given by Lemma 3.8. Assume that \( \omega_\Psi \) belongs to \( H^{d/2, d/2} \) (resp. \( H^{(d+1)/2, (d-1)/2} \oplus H^{(d-1)/2, (d+1)/2} \) if \( \varepsilon = 1 \) (resp. \( \varepsilon = 2 \)). Then the morphism

\[ H^d_{B,!}(\text{Sh}_G(U), R(t-1))^{(-1)^{t-1}} \to C, \]

defined by \( y \mapsto \langle \omega_\Psi, y \rangle \) induces a morphism

\[ \langle \omega_\Psi, \cdot \rangle : H^{d+1}_D(\text{Sh}_G(U)/R, R(t))_m \to C \otimes_{\mathbb{Q}} K. \]

**Proof.** As we are considering Hecke operators which are prime to \( N \), we have \( \langle Tx, y \rangle = \langle x, Ty \rangle \) for any \( x, y \in H^d_{B,!}(\text{Sh}_G(U), K) \) and any \( T \in \mathcal{H}_K^N \). Hence, it follows from the universal property of the localization functor that the morphism \( H^d_{B,!}(\text{Sh}_G(U), R(t-1))^{(-1)^{t-1}} \to C \) defined by \( y \mapsto \langle \omega_\Psi, y \rangle \) induces a morphism

\[ H^d_{B,!}(\text{Sh}_G(U), R(t-1))^{(-1)^{t-1}}_m \to C \otimes_{\mathbb{Q}} K. \]

Notice that, by the choice of the Hodge type of \( \omega_\Psi \), this morphism vanishes on the image of \( F^t H^d_{dR,!}(\text{Sh}_G(U), R)_m \) in \( H^d_{B,!}(\text{Sh}_G(U), R(t-1))^{(-1)^{t-1}}_m \) under the second morphism of the exact sequence of Lemma 4.7 as Poincaré duality is a morphism of Hodge structures. Hence the conclusion follows from Lemma 4.7. \[ \square \]

### 4.3. Integral expression for the pairing

In this section we state and prove the first main result of this article, which relates the complex regulators of the motivic cohomology classes to an adelic integral.

#### 4.3.1. Preliminary lemmas

Let us start with some lemmas that will be useful in the proof of the main theorem.

**Lemma 4.9.** Let \( r_D : H^{d+1}_M(\text{Sh}_G(U), \mathbb{Q}(t)) \to H^{d+1}_D(\text{Sh}_G(U), R(t)) \otimes_{\mathbb{Q}} \mathbb{Q} \) be Beilinson’s higher regulator. Let \( \overline{\phi} := \phi \otimes_{\mathcal{O}^{0, V_1}} (\mathcal{O}^{0, V_1})^{(i)} \) if \( \varepsilon = 1 \) (resp. \( \varepsilon = 2 \)). Then, via the isomorphisms given by Proposition 4.4 and 4.5 (i), the cohomology
class \( r_D(Eis_{M,n}^1(\phi_f)) \) (resp. \( r_D(Eis_{M,n}^2(\phi_f \otimes \phi'_f)) \)) is represented by the pair of currents \((\iota_{n,*}T_{\xi}, \iota_{n,*}T'_{\xi'})\), where

\[
\xi = \begin{cases} 
\text{pr}_1^* \log |u(\phi_f)| & \text{if } \varepsilon = 1, \\
\text{pr}_1^* (\log |u(\phi_f)|) \text{pr}_2^* (\pi_1(d \log u(\phi'_f)) - \text{pr}_2^* (\log |u(\phi'_f)|) \text{pr}_1^* (\pi_1(d \log u(\phi_f))) & \text{if } \varepsilon = 2.
\end{cases}
\]

\[
\xi' = \begin{cases} 
\text{pr}_1^* d \log u(\phi_f) & \text{if } \varepsilon = 1, \\
\text{pr}_1^* (d \log u(\phi_f)) \wedge \text{pr}_2^* (d \log u(\phi'_f)) & \text{if } \varepsilon = 2.
\end{cases}
\]

**Proof.** Let us first treat the case \( \varepsilon = 1 \). According to [18, §3.7], the regulator maps are morphisms between twisted Poincaré duality theories. As a consequence, we have the commutative diagram

\[
\begin{array}{ccc}
H^*_M(\text{Sh}_{\text{GL}_2}(U_1), Q(1)) & \xrightarrow{p^*_n,M} & H^*_M(\text{Sh}_H(U'), Q(1)) \\
\downarrow r^*_D & & \downarrow r^*_D \\
H^*_D(\text{Sh}_{\text{GL}_2}(U_1)/R, R(1)) & \xrightarrow{p^*_n,D} & H^*_D(\text{Sh}_H(U')/R, R(1))
\end{array}
\]

Via the isomorphism of Proposition [4.1], the morphism \( p^*_n,D \) is given by the pullback of differential forms. The morphism \( \iota_{n,D,*} \) is defined as the composition

\[
H^*_D(\text{Sh}_H(U')/R, R(1)) \xrightarrow{\iota_{n,R,*}} H^*_D(\text{Sh}_G(U)/R, R(t)) \approx H^*_{d+1}(\text{Sh}_G(U)/R, R(t)),
\]

where the first isomorphism is induced by \((\phi, \omega) \mapsto (T_\phi, T_\omega)\) (recall that the real dimension of \( H \) is \( \frac{1}{2}(d + \varepsilon - 1) \)) and where \( \iota_{n,D,*} \) is induced by \((S, T) \mapsto (\iota_* S, \iota_* T)\). Hence, the statement follows from Remark [4.2]. The case \( \varepsilon = 2 \) follows similarly by writing down the obvious diagram and using Proposition [4.3].

**Lemma 4.10.** There exists a sequence \((K_n)_{n \geq 0}\) of compact subsets of \( \text{Sh}_G(U) \) and functions \( \sigma_n \in C^\infty(\text{Sh}_G(U), R) \) such that \( K_n \subset \bar{K}_{n+1}, \bigcup_n K_n = \text{Sh}_G(U), \sigma_n = 1 \) in a neighborhood of \( K_n \), the support of \( \sigma_n \) is a subset of \( \bar{K}_{n+1} \), we have \( 0 \leq \sigma_n \leq 1 \) and for any \( x \in \text{Sh}_G(U) \) we have \( |d\sigma_{n,x}| \leq 2^{-n} \), where \(|.|\) is the norm on the cotangent space induced by the \( G(R) \)-invariant hermitian metric on \( \text{Sh}_G(U) \).

**Proof.** The Siegel upper half-plane is a symmetric space when endowed of its \( G(R) \)-invariant hermitian metric, so it is geodesically complete (see [26, Proposition 1.11]). It follows that \( \text{Sh}_G(U) \) is also geodesically complete. Hence we can apply [9] VII Lemma 2.4] and the proof is complete.

**Lemma 4.11.** Let \( \theta \) be a rapidly decreasing differential form on \( \text{Sh}_H(U') \). Then the function \( x \mapsto |\theta_x| \) is rapidly decreasing, hence integrable on \( \text{Sh}_H(U') \).

**Proof.** This follows from [2, Proposition 5.5] as explained in [3, p. 48].

The ideas of the proof of the following result are the same as the ones of [23, Proposition 4.24]. Note however that the last argument in the proof of [23, Proposition 4.24] is a bit sketchy whereas the proof presented here is fully detailed.
Furthermore, as Poincaré duality is induced by the pairing between currents and compactly and by definition of the Poincaré duality pairing on interior cohomology, we have

Moreover, via these isomorphisms, the canonical morphism

that there exists a compactly supported differential form

to approximate it with compactly supported differential forms. It follows from [3, Corollary H_1^d(X, \mathbb{R}(t-1))^{F_{-1}} \subset H_0^d(X, \mathbb{R}(t-1)) \simeq H_0^d \big( X, \mathbb{R}(t-1) \big),

where the last isomorphism is well known (cf. for example, [39]). Hence, we can represent cohomology classes in \( H^d_B(X, \mathbb{R}(t-1))^{F_{-1}} \) by closed differential forms or, equivalently (cf. Remark 4.6), by currents.

Recall now from Proposition 4.5 that \( H^d_{B}(X/\mathbb{R}, \mathbb{R}(t)) \cong H^d_{\mathbb{D}}(X/\mathbb{R}, \mathbb{R}(c-1)) \) and that, by Proposition 4.4, we have

Moreover, via these isomorphisms, the canonical morphism

is given by sending \([T_\rho] \mapsto [(T_\rho, 0)]\), where \([T_\rho]\) is the de Rham cohomology class of a closed current \( T_\rho \in \mathcal{T}^{-d}(X/\mathbb{R}, \mathbb{C}) \) and where \([(T_\rho, 0)]\) denotes the cohomology class in \( H^{d-1}_{\mathbb{D}}(X/\mathbb{R}, \mathbb{R}(c-1)) \). According to Lemma 4.9 and Lemma 4.7, this implies the existence of currents \( T_\rho \in \mathcal{T}^{-d}(X/\mathbb{R}, \mathbb{R}(c-2)) \) and \( \tau \in \mathcal{T}^{-d-1}(X/\mathbb{R}, \mathbb{R}(c-2)) \) such that

Since we want to evaluate currents on our differential form \( \omega_\Psi \) associated to \( \Psi \), we need to approximate it with compactly supported differential forms. It follows from [3 Corollary 5.5] that there exists a compactly supported differential form \( \omega'_c \) of degree \( d \) on \( X \) and a rapidly decreasing differential form \( \epsilon' \) on \( X \) such that

As a consequence \( \omega_\Psi = \omega_c + d\epsilon \) where \( \omega_c = \omega'_c + \bar{\alpha}'_c \) and \( \epsilon = \epsilon' + \bar{\epsilon}' \). By definition of the pairing \( \langle \cdot, [\omega_\Psi] \rangle : H^1_D(M(\pi_f)) \to \mathbb{C} \) (induced by the one in Lemma 4.8), we have

and by definition of the Poincaré duality pairing on interior cohomology, we have

Furthermore, as Poincaré duality is induced by the pairing between currents and compactly supported differential forms, we have

\[
\langle [T_\rho], [\omega_c] \rangle = T_\rho(\omega_c) = \iota_{n,*}T_\xi(\omega_c) + d\tau(\omega_c).
\]
We have \( d\tau(\omega_c) = \pm \tau(d\omega_c) = 0 \) as \( \omega_c \) is closed and we conclude that
\[
\iota_n^* T_\xi(\omega_c) = T_\xi(\iota_n^*(\omega_c)) = \int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^*(\omega_c),
\]
where \( U' = U \cap H(A_f) \).

We have reduced to prove the following

**Lemma 4.13.**
\[
\int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^*(\omega_c) = \int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^*(\omega_\Psi).
\]

**Proof.** Note first that the right hand integral is convergent. Indeed, the differential form \( \omega_\Psi \) is rapidly decreasing as it is cuspidal, the form \( \xi \) is slowly increasing because Eisenstein series are slowly increasing and by the equality of Proposition 4.15, and the product of a slowly increasing and by a rapidly decreasing differential form is rapidly decreasing.

Since \( \omega_\Psi = \omega_c + d\varepsilon \), to prove (6) amounts to prove that
\[
\int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^* d\varepsilon = 0.
\]

To prove this equality, note first that, for any compactly supported real valued differential form \( \eta \) on \( \text{Sh}_G(U) \), we have
\[
\int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^* d\eta = T_\rho(d\eta) - (d\tau)(d\eta) = 0,
\]
since \( T_\rho \) is a closed current. Let \( (\sigma_k)_{k \geq 0} \) be the sequence of functions given by Lemma 4.10.

For any \( k \geq 0 \), the differential form \( \sigma_k \varepsilon \) is compactly supported. Hence
\[
\int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^* d\varepsilon = \int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^* (d\varepsilon - \sigma_k \varepsilon).
\]

Let us prove that \( \int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^* d(\varepsilon - \sigma_k \varepsilon) \to 0 \) as \( k \to +\infty \). We have \( d(\sigma_k \varepsilon) = d\sigma_k \wedge \varepsilon + \sigma_k \varepsilon \).

Hence
\[
\int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^* d(\varepsilon - \sigma_k \varepsilon) = \int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^* (d\varepsilon - \sigma_k \varepsilon) + \int_{\text{Sh}_\mathbb{H}(U')} \xi \wedge \iota_n^* (d\sigma_k \wedge \varepsilon).
\]

We have \( \iota_n^* d\varepsilon = f \cdot \text{vol} \) for a rapidly decreasing function \( f \), where \( \text{vol} \) is the volume form on \( \text{Sh}_\mathbb{H}(U') \). Then we have
\[
\xi \wedge \iota_n^* (d\varepsilon - \sigma_k \varepsilon) = \xi \wedge (f - \sigma_k f) \cdot \text{vol}.
\]

The form \( \xi \wedge (f - \sigma_k f) \) is bounded by the rapidly decreasing function \( 2 |\xi f| \) on \( \text{Sh}_\mathbb{H}(U') \) and vanishes on \( K_k \cap \text{Sh}_\mathbb{H}(U') \). As the union of the \( K_k \cap \text{Sh}_\mathbb{H}(U') \) is \( \text{Sh}_\mathbb{H}(U') \) this implies that \( \lim_{k \to +\infty} \int \xi \wedge \iota_n^* (d\varepsilon - \sigma_k \varepsilon) = 0 \). Moreover the differential form \( \xi \wedge \iota_n^* \varepsilon \) is rapidly decreasing, hence the function \( x \mapsto |(\xi \wedge \iota_n^* \varepsilon)_x| \) is integrable on \( \text{Sh}_\mathbb{H}(U') \) by Lemma 4.11.

As a consequence, similarly as in the proof of (2) Proposition 2.2, we have
\[
\left| \int_{\text{Sh}_\mathbb{H}(U')} \iota_n^* d\sigma_k \wedge \xi \wedge \iota_n^* \varepsilon \right| \leq \int_{\text{Sh}_\mathbb{H}(U')} c|\iota_n^* d\sigma_k||\xi \wedge \iota_n^* \varepsilon|\text{vol}
\leq c \cdot 2^{-k} \int_{\text{Sh}_\mathbb{H}(U')} |\xi \wedge \iota_n^* \varepsilon|\text{vol}
\]
for some constant \( c \in \mathbb{R} \). As a consequence \( \lim_{k \to +\infty} \int_{\text{Sh}_\mathbb{H}(U')} \iota_n^* d\sigma_k \wedge \xi \wedge \iota_n^* \varepsilon = 0 \), as desired. \( \square \)
This completes the proof of the theorem. □

4.4. Kronecker limit formula. Let us recall, following [31], Kronecker limit formula, relating the logarithm of the absolute value of modular units to classical Eisenstein series.

4.4.1. Eisenstein series. Let $\mathcal{S}(\mathbb{A}^2)$ denote the space of Schwartz-Bruhat functions $\Phi = \Phi_f \otimes \Phi_\infty$ on $\mathbb{A}^2$. Given $\Phi \in \mathcal{S}(\mathbb{A}^2)$, denote by

$$f^1(g, \Phi, s) := |\det(g)|^s \int_{\text{GL}_1(\mathbb{A})} \Phi((0, t)g)|t|^{2s}dt$$

the normalised Siegel section in $\text{Ind}_{\text{B}_2(\mathbb{A})}^\text{GL}_2(\mathbb{A})(|\lambda|^s)$ and define the associated Eisenstein series

$$E^1(g, \Phi, s) := \sum_{\gamma \in \text{B}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{Q})} f^1(\gamma g, \Phi, s).$$

(7)

Similarly, if $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{A}^2)$, we denote by

$$f^2(g_1, g_2, \Phi_1, \Phi_2, s) := |\det(g_1)|^s \int_{\text{GL}_1(\mathbb{A})} \int_{\text{GL}_1(\mathbb{A})} \Phi_1((0, t_1)g_1) \Phi_2((0, t_2)g_2)|t_1|^{2s}|t_2|^{2s}dt_1dt_2,$$

denote the associated Eisenstein series

$$E^2(h, \Phi_1, \Phi_2, s) := \sum_{\gamma \in \text{B}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{Q})} f^2(\gamma h, \Phi_1, \Phi_2, s).$$

These Eisenstein series are absolutely convergent for $\text{Re}(s)$ big enough and they satisfy a functional equation.

4.4.2. Second Kronecker limit formula. For the remainder of the section, fix the function $\Phi_\infty$ on $\mathbb{R}^2$ defined by the rule $(x, y) \mapsto e^{-\pi(x^2+y^2)}$ and, for each $\mathbb{Q}$-valued function $\Phi_f \in \mathcal{S}(\mathbb{A}^2_f, \mathbb{Q})$, a positive integer $N_{\Phi_f}$ such that $\Phi_f$ is constant modulo $N_{\Phi_f}\mathbb{Z}^2$. Finally, denote $S_0(\mathbb{A}^2_f, \mathbb{Q}) \subset \mathcal{S}(\mathbb{A}^2_f, \mathbb{Q})$ the space of elements $\Phi_f$ such that $\Phi_f((0,0)) = 0$. A simple calculation gives the following.

Lemma 4.14. Let $\Phi_f, \Phi'_f \in S_0(\mathbb{A}^2_f, \mathbb{Q})$ be factorizable. Then, we have $f^1(g, \Phi_f, 0) \in \mathcal{B}_0^{0, K(N_{\Phi_f})}$ and $f^2(g_1, g_2, \Phi_f, \Phi'_f, 0) \in \mathcal{B}_0^{0, K(N_{\Phi_f})} \otimes \mathcal{B}_0^{0, K(N_{\Phi'_f})}$.}

We can now state the following (classical) result, which relates modular units to values of the adelic Eisenstein series defined in [7].

Theorem 4.15 ([31 Corollary 5.6]). Let $\Phi_f \in S_0(\mathbb{A}^2_f, \mathbb{Q})$ with $N_{\Phi_f} \geq 3$, $\phi_f := f^1(g, \Phi_f, 0)$ be as in Lemma 4.14, then $u(\phi_f)$ defines a $\mathbb{Q}$-valued function on the $\mathbb{Q}$-scheme $\text{Sh}_{\text{GL}_2}(K(N_{\Phi_f}))$, such that

$$E^1(g, \Phi, s) = \log|u(\phi_f)| + O(s),$$

where $\Phi = \Phi_f \otimes \Phi_\infty$.

Example 4.16. When $\Phi_f = \text{char}((0,1) + N\mathbb{Z}^2)$ for $N \geq 4$, the corresponding $u(\phi_f) \in \mathcal{O}(\text{Sh}_{\text{GL}_2}(K(N)))^\times \otimes \mathbb{Q}$ is given by $\prod_{b \in (\mathbb{Z}/N\mathbb{Z})^\times} g_{0,b/N}^{\text{char}(\mathbb{Z})}$, where $g_{0,b/N}$ is the Siegel unit as in [19 §1.4]. Indeed, $\text{Sh}_{\text{GL}_2}(K(N))$ is a disjoint union of connected components all isomorphic to the modular curve $Y(N)$, which are indexed by the class of $|\det(k)| \in \mathbb{Z}^\times/(1 + N\mathbb{Z})$, for $k \in \text{GL}_2(\hat{\mathbb{Z}})$. Choose a system of representatives given by the elements $k_d = (\begin{smallmatrix} 1 \\ d \end{smallmatrix})$, as $d$
Proof. explained above. This last step produces the volume factor using the equivalence between top differential forms on

\[ \prod_{b \in \mathbb{Z}/N\mathbb{Z}^\times} g^{(N)}_{(b, N)}(z) = \prod_{b \in \mathbb{Z}/N\mathbb{Z}^\times} g^{(N)}_{0, br_d/N}(z), \]

where \( r_d \) denotes the inverse of \( d \) modulo \( N \). \( \varphi \) is Euler’s totient function, and \( g_{0, *}/N \) is the Siegel unit as in [19 §1.4]. Thus, \( u(\phi_f) \) descends to an element of \( \mathcal{O}(\text{Sh}_{\text{GL}_2}(K(1)(N)))^\times \otimes \mathbb{Q} \), as each \( g_{b/N} \) does.

4.5. The adelic integral. We finish the chapter using Kronecker limit formula to rewrite Theorem 4.12 in terms of classical Eisenstein series.

Fix the choice of a measure on \( \mathbf{H}(\mathbb{A}) \) as follows. For each finite place \( p \) of \( \mathbb{Q} \), we take the Haar measure \( dh_p \) on \( \mathbf{H}(\mathbb{Q}_p) \) that assigns volume one to \( \mathbf{H}(\mathbb{Z}_p) \). For the archimedean place, we fix a generator \( X_0 \) of the highest exterior power of \( b_{\mathbb{C}} \Sigma_{\mathbb{H}, \mathbb{C}} \), which induces an equivalence between top differential \( \omega \) forms on \( X_{\mathbb{H}} = \mathbf{H}(\mathbb{R})/K_{\mathbb{H}, \infty} \) and invariant measures \( \mu_{\omega, \infty} \) on \( \mathbf{H}(\mathbb{R}) \) assigning measure one to \( K_{\mathbb{H}, \infty} \) (cf. [16 p. 83] for details). We then define \( dh = d_{\mu, \infty} \prod_p dh_p \).

Before stating the main result of the section, we introduce the following notation. We work with \( \Phi_f, \Phi'_f \in S_0(\mathbb{A}_f, \overline{\mathbb{Q}}) \) and let \( \overline{\phi} := f^1(g, \Phi_f, 0) \in \mathcal{R}_0^V \) (resp. \( \overline{\varphi} := f^2(g_1, g_2, \Phi_f, \Phi'_f, 0) \in \mathcal{R}_0^{V_1} \otimes \overline{\mathbb{Q}} \mathcal{R}_0^{V_2} \)) if \( \varepsilon = 1 \) (resp. \( \varepsilon = 2 \)), for suitable sufficiently small level subgroups \( V_1, V_2 \subset \text{GL}_2(A_f) \). Moreover, denote by \( E^\varepsilon(\overline{\varphi}, s) \) the pull-back to \( \mathbf{H} \) of \( E^1(\Phi, s) \) (resp. \( E^2(\Phi, s) \)) if \( \varepsilon = 1 \) (resp. \( \varepsilon = 2 \)), where \( \Phi = \Phi_f \otimes \Phi_\infty \) and \( \Phi' = \Phi'_f \otimes \Phi_\infty \) with \( \Phi_\infty \) as in [14.2]

After applying Theorem 4.15 and Lemma 4.9, Theorem 4.12 now reads as follows.

**Theorem 4.17.** Let \( \Phi_1, \Phi_2 \in S_0(\mathbb{A}_f, \overline{\mathbb{Q}}) \) be factorizable and let \( \Psi, \omega \Psi \) be as in Lemma 4.8. We have

\[
\langle r_D(\text{Eis}_M(\overline{\phi})), [\omega \Psi] \rangle = \frac{2h_{U'}}{\text{vol}(U')} \int_{\mathbf{H}(\mathbb{Q})\mathcal{Z}_G(\mathbb{A}) \mathbf{H}(\mathbb{A})} E^\varepsilon(h, \overline{\phi}, 0) \omega \Psi(X_0)(h) \cdot dh,
\]

where \( h_{U'} = |\mathcal{Z}_G(\mathbb{Q})\mathcal{Z}_G(A_f)/(\mathcal{Z}_G(A_f) \cap U')| \).

**Proof.** We first treat the case of \( \varepsilon = 1 \). Recall that Theorem 4.12 gives

\[
\langle r_D(\text{Eis}_M(\overline{\phi})), [\omega \Psi] \rangle = \int_{\text{Sh}(U')} \xi \wedge \omega \Psi,
\]

where, by Lemma 4.9 \( \xi = \text{pr}_1^* \log |u(\overline{\phi})| \). Recall that \( \text{Sh}_{\text{GL}_2}(V_1) \), with \( V_1 = \text{pr}_1(U') \), is a disjoint union of connected components all isomorphic to a modular curve of certain level. These components are indexed by the class of \( \det(k) \in \pi_0(\text{Sh}_{\text{GL}_2}(V_1)) = \mathbb{Q}_{>0} \mathbb{A}_f^\times /\det(V_1) \), for \( k \in \text{GL}_2(A_f) \). Choose a system of representatives given by elements \( k_d \in \text{GL}_2(\mathbb{Z}) \), as \( d \) varies in \( \pi_0(\text{Sh}_{\text{GL}_2}(V_1)) \). Then, a point in \( \text{Sh}_{\text{GL}_2}(V_1)(\mathbb{C}) \) is represented by a pair \( (z, k_d) \), with \( z \) in the upper half plane and \( k_d \in \text{GL}_2(\mathbb{Z}) \). Thus, if we write \( g \in \text{GL}_2(\mathbb{A}) \) as the product \( g_Q g_{\infty} r \), with \( g_Q \in \text{GL}_2(\mathbb{Q}) \), \( g_{\infty} \in \text{GL}_2(\mathbb{R}) \), and \( r \in \text{GL}_2(\mathbb{Z}) \), Theorem 4.15 gives that

\[
E^1(g, \Phi, 0) = \log |u(\overline{\phi})(g_{\infty} : i, k_{\det(r)})|,
\]

and, as a result, that \( \xi \) equals to the pull-back to \( \mathbf{H}(\mathbb{A}) \) of \( E^1(g, \Phi, 0) \). We get the desired formula by passing from integrating over \( \text{Sh}(U') \) to integrating over \( \mathbf{H}(\mathbb{Q})\mathcal{Z}_G(\mathbb{A}) \mathbf{H}(\mathbb{A}) \) using the equivalence between top differential forms on \( X_{\mathbb{H}} \) and invariant measures on \( \mathbf{H}(\mathbb{R}) \) explained above. This last step produces the volume factor \( \frac{2h_{U'}}{\text{vol}(U')} \) appearing in (8).
The proof of the case of $\epsilon = 2$ is almost identical, with the only difference that Theorem 4.15 (with Proposition 4.3) identifies $\xi$ with the pull-back of $E^2(g_1, g_2, \Phi_f, \Phi'_f, 0)$ to $H(A)$.

\[\square\]

5. Connection with non-critical values of the Spin $L$-function for $\text{GSp}_6$

In this section, we use the main result of 32 to write the adelic integral calculating the archimedean regulator in terms of a special value of a Spin $L$-function for $\text{GSp}_6$. In the rest of the text, we denote $G := \text{GSp}_6$ and $H := H_3$.

5.1. The pairing for $\text{GSp}_6$. We now specialise to the case of $n = 3$. We continue with the calculation of the integrand in (8), by studying the value $\omega_\phi(X_0)(\overline{h})$. The differential form $\omega_\phi$ was defined by the value taken at a fixed highest weight vector $X_{(2,2,-4)}$ of the $K_\infty$-component $\tau_{(2,2,-4)}$ in the decomposition of $\wedge^3 p_C^+ \otimes \wedge^3 p_C^-$ of Example 3.1. We set $X_0 := (X_{2e_1} \wedge X_{2e_2} \wedge X_{2e_3}) \otimes (X_{-2e_1} \wedge X_{-2e_2} \wedge X_{-2e_3}) \in \wedge^6 h_C \otimes h_C = \wedge^3 p_C^+ \otimes \wedge^3 p_C^-$ as the basis of the 1-dimensional subspace $\wedge^3 p_C^+ \otimes \wedge^3 p_C^-$.

**Lemma 5.1.** Up to renormalizing $X_0$ by an explicit non-zero rational factor, the projections of $X_0$ to $\tau_{(2,2,-4)}$ and $\tau_{(4,-2,-2)}$ are given, respectively, by $A \cdot X_{(2,2,-4)}$ and $A' \cdot X_{(4,-2,-2)}$, where $A = \text{Ad}_{X_{3e_2-e_2}} \circ \text{Ad}_{X_{3e_3-e_3}}$, $A' = \text{Ad}_{X_{3e_2-e_2}} \circ \text{Ad}_{X_{3e_3-e_3}}$.

**Proof.** Recall that $X_0$ is a weight $(0,0,0)$-vector (with respect to the action of $h_C$) in $\wedge^3 p_C^+ \otimes \wedge^3 p_C^-$. Thus, we may write $X_0 = Y \oplus \alpha x_{(2,2,-4)}$, where $Y$ belongs to the complement of $\tau_{(2,2,-4)}$ in the decomposition of $\wedge^3 p_C^+ \otimes \wedge^3 p_C^-$ as the sum of its weight subspaces (cf. Example 3.1), the vector $x_{(2,2,-4)}$ is a generator of the one dimensional weight $(0,0,0)$-eigenspace of $\tau_{(2,2,-4)}$, and $\alpha$ is a scalar. We can assume $x_{(2,2,-4)} = \text{Ad}_{X_{3e_2-e_2}} \circ \text{Ad}_{X_{3e_3-e_3}}(X_{(2,2,-4)})$, where $X_{(2,2,-4)} = X_{2e_1} \wedge X_{2e_2} \wedge X_{e_1+e_2} \otimes X_{-e_1-e_3} \wedge X_{-e_2-e_3} \wedge X_{-2e_3}$ is the highest weight vector for $\tau_{(2,2,-4)}$. Since the weight $(2,2,-4)$ has multiplicity one in $\wedge^3 p_C^+ \otimes \wedge^3 p_C^-$, we have that $\text{Ad}_{X_{2e_1-e_2}} \circ \text{Ad}_{X_{2e_2-e_3}}(Y) = 0$ and hence

$$\text{Ad}_{X_{2e_1-e_2}} \circ \text{Ad}_{X_{2e_2-e_3}}(X_0) = \alpha \text{Ad}_{X_{2e_1-e_2}} \circ \text{Ad}_{X_{2e_2-e_3}} \circ \text{Ad}_{X_{3e_2-e_2}} \circ \text{Ad}_{X_{3e_3-e_3}}(X_{(2,2,-4)}).$$

A direct computation 3 shows that

- $\text{Ad}_{X_{2e_1-e_2}} \circ \text{Ad}_{X_{2e_2-e_3}}(X_0) = 2^6 \cdot X_{(2,2,-4)}$,
- $\text{Ad}_{X_{2e_1-e_2}} \circ \text{Ad}_{X_{2e_2-e_3}} \circ \text{Ad}_{X_{3e_2-e_3}} \circ \text{Ad}_{X_{3e_3-e_1}}(X_{(2,2,-4)}) = 2^{10} \cdot 3^2 \cdot 5^2 \cdot X_{(2,2,-4)}$.

Therefore, the projection of $X_0$ to $\tau_{(2,2,-4)}$ is $\frac{1}{5000} x_{(2,2,-4)}$. The other projection follows (with the same coefficient) by applying the action of complex conjugation. This finishes the proof of the Lemma. \[\square\]

**Corollary 5.2.** We keep the notation of Theorem 4.17. Then,

$$\langle r_D(\text{Ein}^3_{M,3}(\phi)), [\omega_\phi] \rangle = \frac{2h_U}{\text{vol}(U')} \int_{H(Q)Z_G(A) \setminus H(A)} E^1(h, \phi, 0)(A \cdot \Psi)(h) \cdot dh,$$

with $h_U = |Z_G(Q)Z_G(A_f)/(Z_G(A_f) \cap U')|$ and $A$ as in Lemma 5.1.

1by a slight abuse of language, we denote by $X_0$ the vector in $\wedge^6 p_{H,C}$ as well as its image in $\wedge^3 p_C^+ \otimes \wedge^3 p_C^-$ by $\iota$, this should cause no confusion.

2The authors have found Sage package for Lie groups very useful for these computations.
Proof. Equation (8) gives

\[ \langle r_D(Eis_{\mathcal{M},3}(\overline{\phi})), [\omega_\Psi] \rangle = \frac{2h_{\nu'}}{\text{vol}(U')} \int_{\mathbf{H}(\mathbf{Q})Z_{\mathbf{G}}(\mathbf{A})\backslash \mathbf{H}(\mathbf{A})} E^1(h, \overline{\phi}, 0)\omega_\Psi(X_0)(h) \cdot dh, \] (9)

But

\[ \omega_\Psi(X_0)(h) = \omega_\Psi(A \cdot X_{(2,2,-4)})(h) = (A \cdot \Psi)(h), \]

where the first equality follows from Lemma 5.1. \qed

5.2. The Spin $L$-function. Recall the following.

**Definition 5.3.** For a character $\chi$ of $\mathbf{Q}_\ell^\times$, define

\[ L(\chi, s) := \begin{cases} (1 - \chi(\ell)^{-s})^{-1} & \text{if } \chi|_{\mathbf{Q}_\ell^\times} = 1 \\ 1 & \text{otherwise.} \end{cases} \]

Let $\chi_0, \chi_1, \chi_2, \chi_3$ be smooth characters of $\mathbf{Q}_\ell^\times$. They define an unramified character $\Delta$ of the Borel $B_\ell = T_\ell \cdot U_{B,\ell}$ of $\mathbf{G}(\mathbf{Q}_\ell)$, which is trivial on the unipotent radical $U_{B,\ell}$, and on the diagonal torus $T_\ell$ is

\[ d = \text{diag}(a, b, c, \mu a^{-1}, \mu b^{-1}, \mu c^{-1}) \mapsto \Delta(d) := \chi_1(a)\chi_2(b)\chi_3(c)\chi_0(\mu). \]

The modular character of the Borel subgroup $\delta_{B_\ell} : T_\ell \to \mathbf{C}$ is given by

\[ \text{diag}(a, b, c, \mu a^{-1}, \mu b^{-1}, \mu c^{-1}) \mapsto \frac{|a|^2|b|^2|c|^2}{|\mu|^6}. \]

**Definition 5.4.** The (normalized) principal series representation $\pi(\chi) = \pi(\chi_0, \chi_1, \chi_2, \chi_3)$ is the representation of $\mathbf{G}(\mathbf{Q}_\ell)$ whose underlying vector space is the space of functions $f : \mathbf{G}(\mathbf{Q}_\ell) \to \mathbf{C}$ satisfying

\[ f(\text{dn} g) = \frac{|a|^2|b|^2|c|^2}{|\mu|^6} \Delta(d)f(g), \]

for every $d = \text{diag}(a, b, c, \mu a^{-1}, \mu b^{-1}, \mu c^{-1})$ and $u \in U_{B,\ell}$, and where the action of $\mathbf{G}(\mathbf{Q}_\ell)$ is given by right-translation.

**Definition 5.5.** Let $\pi = \pi(\chi)$ be an irreducible principal series. It’s Spin $L$-factor is defined as

\[ L(\pi, \text{Spin}, s) := L(\chi_0, s) \prod_{k=1}^{3} \prod_{1 \leq i_1 < \cdots < i_k \leq 3} L(\chi_0\chi_{i_1} \cdots \chi_{i_k}, s). \]

5.3. The integral representation of the Spin $L$-function. Let $\pi$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character. Denote by $I(\Phi, \Psi, s)$ the integral

\[ \int_{\mathbf{H}(\mathbf{Q})Z_{\mathbf{G}}(\mathbf{A})\backslash \mathbf{H}(\mathbf{A})} E^1(h_1, \Phi, s)\Psi(h)dh, \]

where $\Phi \in \mathcal{S}(\mathbf{A}^2)$ and $\Psi$ is a cusp form in the space of $\pi$. Here, we assume that $\Phi$ and $\Psi$ are factorizable.
5.3.1. Fourier coefficients of type \((4 2)\). Let \(\mathcal{O}\) be the unipotent orbit of \(\mathbf{G}\) associated to the partition \((4 2)\). To such \(\mathcal{O}\) one can define a set of Fourier coefficients as follows. Denote by \(h_{\mathcal{O}}\) the one-dimensional torus
\[
t \mapsto \text{diag}(t^3, t, t^{-3}, t^{-1}, t^{-1})
\]
attached to \(\mathcal{O}\) (cf. \[21\] p. 82]). Given any positive root \(\alpha\) (for the action of the diagonal torus of \(\mathbf{G}\)), there is a non-negative integer \(n\) such that
\[
h_{\mathcal{O}}(t)x_{\alpha}(u)h_{\mathcal{O}}(t)^{-1} = x_{\alpha}(t^n u),
\]
where \(x_{\alpha}\) denotes the one-parameter subgroup associated to \(\alpha\). Let \(U_{2}(\mathcal{O})\) denote the subgroup of the unipotent radical \(U_{B}\) of the standard Borel \(B\) of \(\mathbf{G}\) generated by the \(x_{\alpha}\) such that \(n \geq 2\). If \(\alpha \neq e'_2 - e'_3\), then \(n \geq 2\), thus \(U_{2}(\mathcal{O})\) coincides with the unipotent radical \(U_{P}\) of the standard parabolic \(P\) with Levi \(\mathbf{GL}_{2}^{2} \times \mathbf{GL}_{2}\), given by
\[
\left\{ \begin{pmatrix} a & g \\ \mu a^{-1} & \mu' g^{-1} \end{pmatrix} : a, \mu \in \mathbf{GL}_{1}, \ g \in \mathbf{GL}_{2} \right\}.
\]
Let \(\chi : U_{P}(\mathbb{Q}) \backslash U_{P}(\mathbb{A}) \to \mathbb{C}^{\times}\) be the non-degenerate unitary character as in \[32\] §2.1.

**Definition 5.6.** Let \(\Psi\) be a cusp form in the space of \(\pi\). Define the Fourier coefficient
\[
\Psi_{\chi, U_{P}}(g) := \int_{U_{P}(\mathbb{Q}) \backslash U_{P}(\mathbb{A})} \chi^{-1}(u)\Psi(ug)du.
\]

**Remark 5.7.** According to \[15\] Theorem 2.7], every cuspidal automorphic representation of \(\mathbf{GSp}_{4}\) has a non-zero Fourier coefficient associated to a partition with only even numbers appearing (i.e. \((6), (4 2)\), \((2 2 2)\)). For instance, the representations with non-zero Fourier coefficient corresponding to the unipotent orbit \((6)\) are the generic ones.

5.3.2. The unfolding. Recall the following result.

**Proposition 5.8** (\[13\], Proposition 7.1). The integral \(I(\Phi, \Psi, s)\) unfolds to
\[
\int_{U_{B_{H}}(\mathbb{A}) \backslash Z_{\mathbf{G}}(\mathbb{A}) \backslash H(\mathbb{A})} f^{1}(h_{1}, \Phi, s)\Psi_{\chi, U_{P}}(h)dh,
\]
where \(U_{B_{H}}\) is the unipotent radical of the Borel \(B_{H}\) of \(H\) and \(f^{1}(h_{1}, \Phi, s)\) is the normalised section defined in \(\S 4.3\).

As explained in \[32\], the Fourier coefficient \(\Psi_{\chi, U_{P}}\) might not factorise in general, thus Proposition 5.8 does not imply that \(I(\Phi, \Psi, s)\) has the structure of an Euler product; however, in \[32\], the authors define and study local integrals corresponding to this unfolded integral, and use them to relate the global integral \(I(\Phi, \Psi, s)\) to values of the Spin \(L\)-function of \(\pi\), as we now recall.

5.3.3. Connection with values of the Spin \(L\)-function. Recall that we have taken \(\pi = \pi_{\infty} \otimes \otimes_{p} \pi_{p}\) to be a cuspidal automorphic representation of \(\mathbf{G}(\mathbb{A})\) with trivial central character. We further suppose that \(\pi\) supports a Fourier coefficient of type \((4 2)\); this means that there is a cusp form \(\Psi\) in the space of \(\pi\) such that \(\Psi_{\chi, U_{P}}\) is not identically zero.

We now recall following \[32\] the definition of the local integrals corresponding to \(I(\Phi, \Psi, s)\) and their properties. We start with the following definition.

**Definition 5.9.** A \((U_{P}, \chi)\)-model for \(\pi_{P}\) is a linear functional \(\Lambda : \pi_{P} \to \mathbb{C}\) such that
\[
\Lambda(u \cdot v) = \chi(u)\Lambda(v),
\]
for all \(v \in \pi_{p}\) and \(u \in U_{P}\).
For a \((U_p, \chi)\)-model \(\Lambda\) for \(\pi_p\), a vector \(v\) in the space of \(\pi_p\), and \(\Phi_p \in S(Q_p^2, C)\), define

\[
I_p(\Phi_p, v, s) := \int_{U_{B\mathbb{H}}(Q_p)\mathbb{Z}_G(Q_p)\backslash H(Q_p)} f(g, \Phi_p, s) \Lambda(g \cdot v) \cdot dg,
\]

where \(f(g, \Phi_p, s) \in \text{Ind}_{B\mathbb{GL}_2(Q_p)}^{\mathbb{GL}_2(Q_p)}(\delta_{B\mathbb{GL}_2})\) denotes the function

\[
|\det(g)|^s \int_{\mathbb{GL}_1(Q_p)} \Phi_p((0, t)g)|t|^{2s} \cdot dt.
\]

One has the following.

**Theorem 5.10** ([32] Theorem 1.1, Proposition 5.1).

1. If \(p\) is a finite odd prime and \(\pi_p\) is unramified, let \(v_0 \in \pi_p\) be a spherical vector and let \(\Phi_p = \text{char}(Z_p^2)\); then, for any \((U_p, \chi)\)-model \(\Lambda\) for \(\pi_p\), we have

\[
I_p(\Phi_p, v_0, s) = \Lambda(v_0) \cdot L(\pi_p, \text{Spin}, s).
\]

2. If \(\pi_p\) is ramified and \(v_0\) is a vector in \(\pi_p\), then there exists a vector \(v\) in \(\pi_p\) and a function \(\Phi_p \in S_0(Q_p^2, C)\) such that for all \((U_p, \chi)\)-models \(\Lambda\)

\[
I_p(\Phi_p, v, s) = \Lambda(v_0).
\]

**Remark 5.11.** As explained in the proof of [32] Proposition 5.1, in the case of a finite bad place \(p\), one can choose \(\Phi_p\) to be \(\text{char}((0, 1) + p^n\mathbb{Z}_p)\), with \(n\) a suitable positive integer depending on \(v_0\).

Finally, consider the archimedean integral

\[
I_\infty(\Phi_\infty, \Psi, s) := \int_{U_{B\mathbb{H}}(R)\mathbb{Z}_G(R)\backslash H(R)} f(h_1, \Phi_\infty, s) \Psi_{\chi, U_p}(h) dh.
\]

This integral has been studied in [13]. The following lemma, which shows that it can be made non-zero at arbitrary \(s = s_0\) if one has some freedom on the choice of \(\Phi_\infty\) and \(\Psi_\infty\), will not be used.

**Lemma 5.12** ([13] Proposition 12.1]). Fix \(s_0 \in C\); then, there exists a Siegel section \(f(h_1, \Phi_\infty, s_0)\) and a cusp form \(\Psi_\infty\) such that

\[
I_\infty(\Phi_\infty, \Psi_\infty, s_0) \neq 0.
\]

### 5.4. The regulator computation.

Let us fix a neat open compact subgroup \(U = \prod_p U_p\) of \(G(A_f)\) and \(\Phi_f = \otimes_p \Phi_p \in S_0(A_f^2, Q)\) as follows. Let \(\Sigma\) be a finite set of primes containing all the bad finite primes for \(\pi\) and \(\infty\); we assume that

- For every prime \(p \notin \Sigma\), we set \(U_p = G(Z_p)\) and \(\Phi_p = \text{char}(Z_p^2)\);
- If \(p \in \Sigma\), we let \(\Phi_p = \text{char}((0, 1) + p^nZ_p^2)\), and

\[
U_p = \{g \in G(Z_p) : g \equiv I \pmod{p^n}\},
\]

with \(n\) a suitable positive integer as in the proof of [32] Proposition 5.1].

Finally, we denote \(\overline{\delta} = f^1(g, \Phi_f, 0)\) and fix the non-degenerate unitary character \(\chi : U_p(Q) \backslash U_p(A) \to C^\times\) associated (as in [32] §2.1]) to the real étale quadratic extension of \(Q\) defining the group \(H\).

Let \(\pi = \pi_\infty \otimes (\otimes_p \pi_p)\) be a cuspidal automorphic representation of \(G(A)\) with trivial central character, which is stable at infinity (see Remark 3.4). Then, we have the following.
Indeed, notice that by Theorem 2.7, if

$$\langle D(E_{\text{is}}), [\omega] \rangle \in \overline{Q}^I I_{\infty}(\Phi_{\infty}, A \cdot \Psi, 0) \cdot L^\Sigma(\pi, \text{Spin}, 0),$$

where $A$ is the operator defined in Lemma 5.1, $L^\Sigma(\pi, \text{Spin}, s) = \prod_{p \in \Sigma} L(\pi_p, \text{Spin}, s),$ and

$$I_{\infty}(\Phi_{\infty}, A \cdot \Psi, s) = \int_{U_BH(A)} E(h_1, \Phi, 0) \tilde{\Psi}(h) dh.$$

Proof. This follows directly from Corollary 5.2 and Theorem 5.10. For the sake of clarity, we give a sketch of its proof.

Recall from Corollary 5.2 and Proposition 5.8, we have

$$\langle r_D(E_{\text{is}}), [\omega] \rangle = C_{U_\text{R}} \int_{H(Q)Z_G(A)} E(h_1, \Phi, 0) \tilde{\Psi}(h) dh$$

$$= C_{U_\text{R}} \int_{U_BH(A)Z_G(A)} f^1(h_1, \Phi, 0) \tilde{\Psi}_{\chi,U}(h) dh$$

$$= C_{U_\text{R}} I(\Phi, \tilde{\Psi}, 0),$$

where $C_{U_\text{R}} = \frac{2h_{\text{tr}}}{\text{vol}(U_f)}$ and $\tilde{\Psi} = (A \cdot \Psi_\infty) \otimes \Psi_f.$

We now study the quantity $I(\Phi, \tilde{\Psi}, 0).$ Given a finite set of primes $S$ containing $\Sigma,$ define

$$I_S(\Phi, \tilde{\Psi}, s) := \int_{U_BH(A)Z_G(A)} f^1(h_1, \Phi, 0) \tilde{\Psi}_{\chi,U}(h) dh,$$

where $\Phi_S = \prod_{v \in S} \Phi_v.$ Then, in the range of convergence

$$I(\Phi, \tilde{\Psi}, s) = \lim_{S \subseteq S} I_S(\Phi, \tilde{\Psi}, s).$$

Notice that, by Theorem 2.7, if $p \not\in S,$

$$I_{S \cup \{p\}}(\Phi, \tilde{\Psi}, s) = L(\pi_p, \text{Spin}, s) \cdot I_S(\Phi, \tilde{\Psi}, s).$$

Indeed,$$ I_{S \cup \{p\}}(\Phi, \tilde{\Psi}, s)$$
equals

$$\int_{U_BH(A)Z_G(A)} f^1(h_1, \Phi, s) \left( \int_{U_BH(Q_p)Z_G(Q_p)} f^1(h_{1,p}, \Phi, s) \tilde{\Psi}_{\chi,U}(h_p h_p) dh_p \right) dh.$$

As $p \not\in \Sigma,$ $\pi_p$ is unramified at $p.$ Fix a spherical vector $v_0$ for $\pi_p,$ as $\tilde{\Psi}$ is invariant under the action of $G(\mathbb{Z}_p),$ there is a $G(\mathbb{Z}_p)$-equivariant map $\varphi_p : \pi_p \to \pi$ sending $v_0$ to $\tilde{\Psi},$ $\varphi_p.$ For a fixed $h \in G(A_S),$ the functional $\Lambda_h : \pi_p \to \overline{Q}$ defined by $\Lambda_h(v) := \varphi_p(v)_{\chi,U}(h)$ is clearly a $(U_p, \chi)$-model and $\Lambda_h(h_p \cdot v_0) = \tilde{\Psi}_{\chi,U}(h_p h).$ Theorem 5.10(1) implies then

$$\int_{U_BH(Q_p)Z_G(Q_p)} f^1(h_{1,p}, \Phi, s) \tilde{\Psi}_{\chi,U}(h_p h_p) dh_p$$

$$= \int_{U_BH(Q_p)Z_G(Q_p)} f^1(h_{1,p}, \Phi, s) \Lambda_h(h_p \cdot v_0) dh_p$$

$$= L(\pi_p, \text{Spin}, s) \cdot \Lambda_h(v_0)$$

$$= L(\pi_p, \text{Spin}, s) \cdot \tilde{\Psi}_{\chi,U}(h),$$
which implies the desired equality.

Taking the limit varying the set $S \supseteq \Sigma$, we get

$$I(\Phi, \hat{\Psi}, s) = I_{\Sigma}(\Phi, \hat{\Psi}, s) \cdot L^{\Sigma}(\pi, \text{Spin}, s)$$

where $L^{\Sigma}(\pi, \text{Spin}, s) = \prod_{p \in \Sigma} L(\pi_p, \text{Spin}, s)$. Moreover, by a similar argument as above and by Theorem 5.10 (and its proof in [32]), there exists a cusp form $\Psi'$ in the space of $\pi$, which coincides with $\hat{\Psi}$ at all but the ramified primes, such that

$$I_{\Sigma}(\Phi, \Psi', s) = C_{\Sigma}I_{\infty}(\Phi, \hat{\Psi}, s),$$

where

$$I_{\infty}(\Phi, \hat{\Psi}, s) = \int_{U_{\text{H}}(\mathbb{R})Z_{\text{G}}(\mathbb{R})/\text{H}(\mathbb{R})} f^1(h_1, \Phi_{\infty}, s) \hat{\Psi}_{\chi, U_p}(h)dh,$$

and the constant $C_{\Sigma}$ lies in $\overline{\mathbb{Q}}^\times$ because of our assumption on the Fourier coefficient $\Psi_{\chi, U_p}$.

Summing all up, we have proved that there exists a cusp form $\Psi'$ in the space of $\pi$, which coincides with $\hat{\Psi} = (A \cdot \Psi_{\infty}) \otimes \Psi_f$ at all but the ramified primes, such that

$$\langle r_D(Eis_{A, 3}(\phi)), [\omega_{\Psi'}] \rangle = C_{U \cap \text{H}} \cdot C_{\Sigma} \cdot I_{\infty}(\Phi, \hat{\Psi}, 0) \cdot L^{\Sigma}(\pi, \text{Spin}, 0).$$

Remark 5.14. We expect to express the archimedean factor $I_{\infty}(\Phi, \hat{\Psi}, 0)$ as a product of certain Gamma factors, by giving explicit formulae of the Fourier coefficients of type (42) for cusp forms in the discrete series $\pi_\infty$, whose restriction to $\text{Sp}_6(\mathbb{R})$ is $\pi^{3,3}_\infty \oplus \pi^{3,3}_\infty$. This can be achieved, as in [27], [29], by using the differential equations that arise from Schmid’s realisation of discrete series in [35]. This calculation is quite involved, but we expect to come back to it in the near future.

5.4.1. A remark on the non-vanishing of the regulator. We have the following direct consequence of Theorem 5.13.

Corollary 5.15. We assume that there exists an automorphic representation $\pi$ of $G(\mathbb{A})$ and a cusp form $\Psi$ in $\pi$, which satisfy all the running assumptions of Theorem 5.13 and such that

$$I_{\infty}(\Phi_{\infty}, A \cdot \Psi, 0) \cdot L^{\Sigma}(\pi, \text{Spin}, 0) \neq 0.$$

Then, the class $Eis_{A, 3}(\phi)$ is non-trivial and thus $H^T_M(\text{Sh}_G(U), \mathbb{Q}(4))$ is non-zero.

By assuming spin functoriality for $\pi$ and by using a generalisation of the prime number theorem due to Jacquet and Shalika [17], we can improve the corollary by establishing that the non-vanishing of our pairing relies on the non-vanishing of the archimedean integral, as we now explain.

Let $\text{Spin} : LG \to LGL_8$ denote the homomorphism between $L$-groups induced by the spin representation $\text{Spin} : GSpin_7 \to GL_8$. Roughly, Langlands’ functoriality would predict the existence of a spin lift to $GL_8$ for $\pi$, i.e. the existence of an automorphic representation $\Pi$ of $GL_8(\mathbb{A})$ whose $L$-parameter $\phi_{\Pi_v}$ at each place $v$ is obtained from composing the $L$-parameter $\phi_{\pi_v}$ of $\pi_v$ with $\text{Spin}$, thus implying that

$$L(\pi, \text{Spin}, s) = L(\Pi, s),$$

where the latter denotes the standard $L$-function associated to $\Pi$. 
Remark 5.16. In [22], a potential version of spin functoriality is discussed and proved. If we assume that there exists a prime \( \ell \) such that \( \pi_\ell \) is the Steinberg representation of \( G(\mathbb{Q}_\ell) \), the result [22, Theorem C], which builds upon [1 Theorem A], produces a cuspidal automorphic representation \( \Pi \) of \( \text{GL}_s(\mathbb{A}_F) \), over a finite totally real extension \( F/\mathbb{Q} \), with the desired properties. For instance, at each finite place \( v \) of \( F \) above an odd prime \( p \not\in \Sigma \) one has \( L(\Pi_v, s) = L(\pi_p, \text{Spin}, s) \).

**Corollary 5.17.** Suppose that there exists a spin lift \( \Pi \) of \( \pi \), which is cuspidal, and that

\[
I_\infty(\hat{\Phi}_\infty, A \cdot \Psi, 1) \neq 0,
\]

where \( \hat{\Phi} \) denotes the Fourier transform of \( \Phi \). Then, the class \( \text{Eis}^1_{\mathcal{M}, 3}(\hat{\phi}) \) is non-trivial.

**Proof.** Recall that the Eisenstein \( E^1(g, \Phi, s) \) satisfies the functional equation

\[
E^1(g, \Phi, s) = E^1(g, \hat{\Phi}, 1 - s).
\]

This implies that

\[
I(\Phi, \Psi, s) = I(\hat{\Phi}, \Psi, 1 - s) = I_\Sigma(\hat{\Phi}, \Psi, 1 - s) L_\Sigma(\pi, \text{Spin}, 1 - s).
\]

Since \( L_\Sigma(\pi, \text{Spin}, s) = L_\Sigma(\Pi, s) \), by Theorem 5.13 we get that

\[
\langle r_\mathcal{D}(1), [\omega_\Psi] \rangle = \frac{1}{\mathbb{Q}} I_\infty(\hat{\Phi}_\infty, A \cdot \Psi, 1) \cdot L_\Sigma(\Pi, 1).
\]

We now claim that \( L_\Sigma(\Pi, 1) \neq 0 \). In [17], it is shown that \( L(\Pi, s) \neq 0 \) for any \( s \) with \( \text{Re}(s) = 1 \). Writing

\[
L_\Sigma(\Pi, s) = L(\Pi, s) \prod_{p \in \Sigma} L(\Pi_p, s)^{-1},
\]

our claim follows from the fact that each Euler factor \( L(\Pi_p, s) \) has no pole in the region \( \text{Re}(s) \geq \frac{1}{2} \) (eg. [34 p. 317]).

**References**


