CLASSICAL EISENSTEIN SERIES AND JACQUET EISENSTEIN SERIES

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Abstract. In this short note, we compare some classical real analytic Eisenstein series to Eisenstein series defined by Jacquet in [2].

1. Real analytic Eisenstein series

1.1. Real analytic Eisenstein series. Let $\omega : (\mathbb{Z} / N \mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be a Dirichlet character and let $E$ be a number field containing the values of $\omega$. We denote as usual by $\Gamma_0(N)$ the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$$

Let $\tau \in \mathcal{H}$ and $s \in \mathbb{C}$ such that $\Re(s) > 1$. Following [7] 2, we define the Eisenstein series

$$E_N(\tau, s, \omega) = \sum'_{(m,n) \in \mathbb{Z}^2} \frac{\omega(n)}{|Nm\tau + n|^{2s}}$$

where the superscript $\sum'$ denotes that the term $(m, n) = (0, 0)$ is omitted. We also define the Eisenstein series

$$E_N^*(\tau, s, \omega) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \frac{\omega(d)}{|c\tau + d|^{2s}}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}$. We have

$$E_N(\tau, s, \omega) = 2L_N(2s, \omega)E_N^*(\tau, s, \omega)$$

where $L_N(2s, \omega)$ is the partial Dirichlet $L$-function. As a function of $s$, the function $\Gamma(s)E_N(\tau, s, \omega)$ can be continued to a meromorphic function, which is entire if $\omega$ is non-trivial (see [7] 2 and its references). Let $\alpha \in \mathbb{Q} / \mathbb{Z}$. We define the standard complex valued real-analytic Eisenstein series

$$E_\alpha(\tau, s) = \pi^{-s} \Gamma(s) \sum'_{(m,n) \in \mathbb{Z}^2} \frac{\Im(\tau)^s}{|m\tau + n + \alpha|^{2s}}$$

$$F_\alpha(\tau, s) = \pi^{-s} \Gamma(s) \sum''_{(m,n) \in \mathbb{Z}^2} e^{2\pi i m \alpha} \frac{\Im(\tau)^s}{|m\tau + n + \alpha|^{2s}}$$

where $\sum'$, resp. $\sum''$, denotes that the term $(m, n) = (0, 0)$ is omitted if $\alpha = 0$, resp. always omitted. These are denoted by $E^{(0)}_\alpha(\tau, s)$ and $F^{(0)}_\alpha(\tau, s)$ in [6] Def. 4.2.1 (1). In
For any $\Phi$ and $s$ definitions given above, in particular, they are invariant under the action of $\Gamma \in \mathbb{S}_\Phi$ conductor of $\omega$ denote the product measure $\prod_v$ denote the Haar measure on $\mathbb{Q}$ be the Haar measure on $\mathbb{A}$ imply that $\alpha$ by multiplication on the obvious way ([6] Prop. 4.2.2 (i)). Moreover, according to [6] Prop. 4.2.2 (iv), for fixed $\tau$ these Eisenstein series have a meromorphic continuation to the whole $s$-plane and satisfy the functional equation

$$E_\alpha(\tau, s) = F_\alpha(\tau, 1 - s).$$

Note that

$$\sum_{\alpha \in (\mathbb{Z}/N\mathbb{Z})^\times} \omega(\alpha)E_\alpha(\tau, s) = \pi^{-s} \Gamma(s) N^{2s} L_N(2s, \omega) \Im(\tau)^s E_N^*(\tau, s, \omega).$$

2. Jacquet Eisenstein series

We introduce adelic Eisenstein series following [2] and compare them to the classical definitions given above.

Let $dt_\infty$ be the Lebesgue measure on $\mathbb{R}$. If $v$ is a non-archimedean place of $\mathbb{Q}$, let $dt_v$ be the Haar measure on $\mathbb{Q}_v$ for which $\mathbb{Z}_v$ has volume one. We normalize the multiplicative measures as follows:

$$d^x t_v = \begin{cases} \frac{dt_v}{|v|} & \text{if } v \text{ is archimedean}, \\ \frac{p^{-1} dt_v}{|v|} & \text{if } v \text{ is } p\text{-adic}. \end{cases}$$

In particular, the measure of $\mathbb{Z}_v^\times$ is 1, for any non-archimedean $v$. Let $dt$, respectively $d^x t$, denote the product measure $\prod_v dt_v$ on $\mathbb{A}$, respectively the product measure $\prod_v d^x t_v$ on $\mathbb{A}^\times$.

Let $\omega : \mathbb{Q}_v^\times \backslash \mathbb{A}_v^\times \rightarrow \mathbb{C}^\times$ be a finite order idele class character. We denote by $N$ the conductor of $\omega$. Denote by $\mathcal{S}(\mathbb{A}^2)$ the space of Schwartz-Bruhat functions on $\mathbb{A}^2$. For any $\Phi \in \mathcal{S}(\mathbb{A}^2)$ and $(s, h) \in \mathbb{C} \times \text{GL}(2, \mathbb{A})$ with $\Re(s) > 1$ we define

$$g_\Phi(h, s, \omega) = |\det(h)|^s \int_{\mathbb{A}^\times} \Phi((0, t)h)|t|^{2s} \omega^{-1}(t)d^x t.$$

For any $\Phi$ and $s$ as above, the $\mathbb{C}$-valued function defined by $h \mapsto g_\Phi(h, s, \omega)$ belongs to

$$I(1, s, \omega) = \left\{ g : \text{GL}(2, \mathbb{A}) \rightarrow \mathbb{C} \mid f \left( \begin{pmatrix} a & b \\ d & s \end{pmatrix} h \right) = |a|^s_\omega(d)|g(h)| \right\}.$$

This vector space is endowed with the action of $\text{GL}(2, \mathbb{A})$ by right translation and we have the restricted product factorization $I(1, s, \omega) = \bigotimes_v I_v(1, s, \omega_v)$, with obvious notations. If $\Phi = \bigotimes_v \Phi_v$ is factorizable, then $g_\Phi(h, s, \omega) = \bigotimes_v g_{\Phi_v}(h_v, s, \omega_v)$ where

$$g_{\Phi_v}(h_v, s, \omega_v) = |\det(h_v)|^s_v \int_{\mathbb{Q}_v^\times} \Phi_v((0, t_v)h_v)|t_v|^{2s} \omega_v^{-1}(t_v)d^x t_v.$$

The infinite sum

$$E_\Phi(h, s, \omega) = \sum_{\gamma \in B(\mathbb{Q}) \backslash \text{GL}(2, \mathbb{Q})} g_\Phi(\gamma h, s, \omega)$$

converges absolutely in the right half plane $\Re(s) > 1$. For any $h \in \text{GL}(2, \mathbb{A})$ and any $\Phi$, the Eisenstein series $E_\Phi(h, s, \omega)$ has a meromorphic continuation to the whole $s$-plane which is entire if $\omega$ is non-trivial ([2] Prop. 19.3, [3] Lem. 4.2).
Lemma 2.1. Assume $\omega_\infty(-1) = 1$. Let $\Phi_\infty \in S(\mathbb{R}^2)$ be defined by $\Phi_\infty(x, y) = e^{-\pi(x^2+y^2)}$. Then

$$g_{\Phi_\infty}\left(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, s, \omega_\infty\right) = \pi^{-s} \Gamma(s)$$

for any $\theta \in \mathbb{R}$.

Proof. Left to the reader. □

Lemma 2.2. Let $p$ be a prime which does not divide $N$. If $\Phi_p \in S(\mathbb{Q}_p^2)$ is the characteristic function of $\mathbb{Z}_p^2$, then $h_p \mapsto g_{\Phi_p}(h_p, s, \omega_p)$ is the unique element of $I_p(1, s, \omega_p)$ whose restriction to $\text{GL}(2, \mathbb{Z}_p)$ is constant and equal to $L_p(2s, \omega_p^{-1})$.

Proof. Note that the statement makes sense by the Iwasawa decomposition $\text{GL}(2, \mathbb{Q}_p) = B(\mathbb{Q}_p)\text{GL}(2, \mathbb{Z}_p)$. Let $h_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}_p)$. Then $c$ and $d$ are coprime and so

$$g_{\Phi_p}(h_p, s, \omega_p) = \int_{\mathbb{Q}_p^\times} \Phi_p(tc, td)|t|^{2s}\omega_p^{-1}(t)d^\times t = \int_{\mathbb{Z}_p^\times -\{0\}} |t|^{2s}\omega_p^{-1}(t)d^\times t = L_p(2s, \omega_p^{-1}).$$

Let us denote by $K_{0,p}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}_p) | c \in N\mathbb{Z}_p \right\}$.

Lemma 2.3. Let $p$ be a prime which divides $N$. If $\Phi_p \in S(\mathbb{Q}_p^2)$ is the characteristic function of $N\mathbb{Z}_p \times (1 + N\mathbb{Z}_p)$ then $h_p \mapsto g_{\Phi_p}(h_p, s, \omega_p)$ is the unique element of $I_p(1, s, \omega_p)$ whose restriction to $\text{GL}(2, \mathbb{Z}_p)$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{p^{\nu_p(N)-1}(p-1)}\omega_p(d)1_{K_{0,p}(N)}(a \begin{pmatrix} b \\ c \end{pmatrix} d)$$

where $1_{K_{0,p}(N)}$ is the characteristic function of $K_{0,p}(N)$.

Proof. Let us denote by $\alpha > 0$ the $p$-adic valuation of $N$. Let $h_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}_p)$. The integral to be computed is

$$g_{\Phi_p}(h_p, s, \omega_p) = \int_{\mathbb{Q}_p^\times} \Phi_p(tc, td)|t|^{2s}\omega_p^{-1}(t)d^\times t = \sum_{m=-\infty}^{+\infty} \int_{p^m\mathbb{Z}_p^\times} \Phi_p(tc, td)|t|^{2s}\omega_p^{-1}(t)d^\times t.$$

Assume that $h_p \in K_{0,p}(N)$ so that $d \in \mathbb{Z}_p^\times$. By definition of $\Phi_p$, the only term in the above sum which is non zero is the term indexed by $m = 0$ and this term is equal to

$$\int_{\{t \in \mathbb{Z}_p^\times | td \equiv 1 \pmod{p^\alpha}\}} \omega_p^{-1}(t)d^\times t = \omega_p(d) \int_{\{t \in \mathbb{Z}_p^\times | t \equiv 1 \pmod{p^\alpha}\}} \omega_p^{-1}(t)d^\times t.$$

As $\omega_p$ is trivial on $\{t \in \mathbb{Z}_p^\times | t \equiv 1 \pmod{p^\alpha}\}$, we have

$$\int_{\{t \in \mathbb{Z}_p^\times | t \equiv 1 \pmod{p^\alpha}\}} \omega_p^{-1}(t)d^\times t = \frac{1}{(\mathbb{Z}_p/p^\alpha\mathbb{Z}_p)^\times} = \frac{1}{p^\alpha-1(p-1)}.$$
Now assume that \( h_p = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \notin K_{0,p}(N) \). This means that the \( p \)-adic valuation \( \mu \) of \( c \) verifies \( 0 \leq \mu < \alpha \). Then for any \( t \in \mathbb{Q}_p^* \) we have
\[
tc \in N \mathbb{Z}_p = p^n \mathbb{Z}_p \implies v_p(t) \geq \alpha - \mu > 0 \implies td \notin 1 + N \mathbb{Z}_p.
\]
As a consequence \( g_{\Phi_p}(h_p, s, \omega_p) = 0 \). This concludes the proof.

**Proposition 2.4.** Assume that \( \omega \) is even. Let \( \omega_{Dir} \) be the primitive Dirichlet character attached to \( \omega \) and let \( N \) be its conductor. Let \( \Phi_\infty \) be the Schwartz-Bruhat function on \( \mathbb{R}^2 \) be defined by \( \Phi_\infty(x, y) = e^{-\pi(x^2 + y^2)} \) and let \( \Phi_p \), for any prime number \( p \), be the Schwartz-Bruhat function on \( \mathbb{Q}_p^2 \) which is the characteristic function of \( p^{\mu_p(N)-1}(p-1)I_{N \mathbb{Z}_p \times (1+N \mathbb{Z}_p)} \) if \( p | N \) and which is the characteristic function of \( \mathbb{Z}_p^2 \) if \( p \nmid N \). Define \( \Phi \) as the restricted tensor product \( \Phi = \bigotimes_v \Phi_v \). Then, for any \( s \in \mathbb{C} \) such that \( \Re(s) > 1 \) and any \( h = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}(2, \mathbb{R})^+ \) we have
\[
E_\Phi(h, s, \omega) = \pi^{-s} \Gamma(s) L_N(2s, \omega) \Im(\tau)^s E^*_N(\tau, s, \omega_{Dir})
\]
where \( \tau = \frac{ai+b}{a+d} \).

**Proof.** We have
\[
E_\Phi(h, s, \omega) = \sum_{\gamma \in B(\mathbb{Q}) \setminus \text{GL}(2, \mathbb{Q})} g_\Phi(\gamma h, s, \omega)
\]
\[
= \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}(2, \mathbb{Z})} g_\Phi(\gamma h, s, \omega)
\]
\[
= L_N(2s, \omega) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \omega_{Dir}(d) g_{\Phi_\infty}(\gamma h, s, \omega_\infty)
\]
where the second equality follows from the bijection \( \Gamma_\infty \setminus \text{SL}(2, \mathbb{Z}) \simeq B(\mathbb{Q}) \setminus \text{GL}(2, \mathbb{Q}) \) and the third equality from Lem 2.2 and Lem. 2.3. According to the Iwasawa decomposition for \( \text{GL}(2, \mathbb{R}) \), there exist positive real numbers \( u, x, y, \theta \), with \( y > 0 \), such that
\[
\gamma h = \left( \begin{array}{cc} u & \frac{1}{u} \\ y^{1/2} & x y^{-1/2} \end{array} \right) \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right)
\]
By assumption \( \omega \) is even, so \( \omega_\infty \) is trivial. Then, it follows from Lem. 2.1 that
\[
g_{\Phi_\infty}\left( \left( \begin{array}{cc} u & \frac{1}{u} \\ y^{1/2} & x y^{-1/2} \end{array} \right) \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right), s, \omega_\infty \right) = y^s \pi^{-s} \Gamma(s)
\]
\[
= \Im(\tau)^s \pi^{-s} \Gamma(s)
\]
\[
= \pi^{-s} \Gamma(s) \frac{y^s}{|\tau + d|^{2s}}.
\]
REFERENCES


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