ALGEBRAIC CYCLES AND RESIDUES OF DEGREE EIGHT L-FUNCTIONS OF $\mathrm{GSp}(4) \times \mathrm{GL}(2)$

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Abstract. By embedding the product of two modular curves into the product of a Siegel threefold and of a modular curve, we construct a cohomology class whose non-triviality is equivalent to the existence of a pole for a degree eight automorphic L-function, under mild local assumptions. We also prove a cohomological formula for non-critical residues of these L-functions in the spirit of Beilinson conjecture. These results rely on the cohomological interpretation of an automorphic period integral and on the study of an integral representation of the L-functions.

Contents

1. Introduction

Let X be a smooth projective variety over $\mathbb{Q}$ and let $n \geq 1$ be an integer. The Hasse-Weil L-function $L(s, H^{2n-2}(X))$ associated to the cohomology of X in degree $2n-2$ is expected to be defined by an Euler product which is absolutely convergent for $\mathrm{Re}(s) > n$ and to have a meromorphic continuation to the whole complex plane, the only possible pole occuring at n. Let $N^{n-1}(X)$ be the $\mathbb{Q}$-vector space of cycles of codimension $n-1$ on X modulo homological equivalence.

Conjecture. *(Tate)*

$$-\operatorname{ord}_{s=n} L(s, H^{2n-2}(X)) = \dim_{\mathbb{Q}} N^{n-1}(X).$$

To give the geometric interpretation of the first non-zero term $L^*(n, H^{2n-2}(X))$ in the Taylor expansion of $L(s, H^{2n-2}(X))$ at $s = n$, let $H^{2n-1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$ and $H^{2n-1}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$

denote the integral motivic cohomology and the real Deligne-Beilinson cohomology respectively and let

$$r_{\mathcal{D}}: H^{2n-1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}} \oplus N^{n-1}(X) \longrightarrow H^{2n-1}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$$

denote the thickened regulator. It is known that $H^{2n-1}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$ is a finite dimensional $\mathbb{R}$-vector space and that there is a canonical isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^{2n-2}_B(X, \mathbb{R}(n))) \simeq H^{2n-1}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)).$$

where $\mathrm{MHS}^+_{\mathbb{R}}$ denotes the abelian category of mixed real $\mathbb{R}$-Hodge structures, whose definition is recalled in the body of the article. Let $\mathcal{D}(n, H^{2n-2}(X))$ denote the Deligne $\mathbb{Q}$-structure on the highest exterior power of $H^{2n-1}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$.

Conjecture. *(Beilinson)*

i. The map $r_{\mathcal{D}}$ induces an isomorphism

$$(H^{2n-1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}} \oplus N^{n-1}(X)) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\ \sim\ } H^{2n-1}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)),$$

ii. $\mathrm{ord}_{s=n-1} L(s, H^{2n-2}(X)) = \dim_{\mathbb{Q}} H^{2n-1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$,

iii. $\det(\mathrm{Im}\, r_{\mathcal{D}}) = L^*(n, H^{2n-2}(X))\mathcal{D}(n, H^{2n-2}(X))$.

For an introduction to this circle of ideas and for more details on the conjectures, the reader is referred to [Ne94] and [Sc88].

To state our main result, which is motivated by the conjectures above, let us consider irreducible cuspidal automorphic representations $\Pi = \bigotimes'_v \Pi_v$ and $\sigma = \bigotimes'_v \sigma_v$ of $\mathrm{GSp}(4, \mathbb{A})$ and $\mathrm{GL}(2, \mathbb{A})$ respectively, over the field of rational numbers. Let V be the finite set of places where Π or σ is ramified, together with the infinite place. We are interested in the L-function

$$L_V(s, \Pi \times \sigma) = \prod_{v \notin V} L(s, \Pi_v \times \sigma_v)$$

which is associated to the tensor product eight-dimensional representation of the Langlands dual of $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, which is $\mathrm{GSp}(4, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$. This partial Euler product converges for $\mathrm{Re}(s)$ big enough. To study special values of $L_V(s, \Pi \times \sigma)$ by motivic methods, we assume that the non-archimedean components Π_f and σ_f contribute to the cohomology of a Siegel threefold S and of a modular curve Y respectively, with trivial coefficient system. To use Novodvorsky's integral representation [Mo09], [So84] we need to assume that Π is globally generic and in order to avoid that Π contributes to the Eisenstein cohomology of S, we assume that Π is not CAP. Examples of such automorphic representations are obtained as Weil liftings from $\mathrm{GSO}(2, 2)$ (see [HK92], [Teh13]). Under the assumption that Π is globally generic, the main result of [Mo09] applies, hence the partial Euler product $L_V(s, \Pi \times \sigma)$ can be completed to a product over all places which has a meromorphic continuation to the whole complex plane and a functional equation relating s and $1 - s$. Let $L(s, \Pi \times \sigma)$ denote the product completed to all non-archimedean places as defined in [Mo09], [So84]. Because Π_f and σ_f are cohomological, they are defined over their rationality field, which is a number field. Let E denote a suffciently large number field containing the rationality fields of Π_f and σ_f. Let $H^4_{B,!}(S\times Y,3)$ denote the image of compactly supported Betti cohomology in the cohomology without support, with coefficients in $\mathbb{Q}(3)$, and let

$$M_B(\Pi_f\times\sigma_f,3) \quad = \quad \mathrm{Hom}_{E[G(\mathbb{A}_f)]}(\Pi_f\times\sigma_f, H^4_{B,!}(S\times Y,3)\otimes_{\mathbb{Q}} E)$$

where we denote by G the reductive group $\mathrm{GSp}(4)\times\mathrm{GL}(2)$. This is a pure $\mathbb{Q}$-Hodge structure of weight -2, with coefficients in E. Let us denote by $(M_B(\Pi_f\times\sigma_f,2)_{\mathbb{R}}\cap M^{2,2})^+$ the vectors of $M_B(\Pi_f\times\sigma_f,2)_{\mathbb{R}}$ which have Hodge type $(2,2)$ and which are fixed by complex conjugation. We have a canonical isomorphism of $\mathbb{R}\otimes E$-modules

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f\times\sigma_f,3)_{\mathbb{R}}) \simeq (M_B(\Pi_f\times\sigma_f,2)_{\mathbb{R}}\cap M^{2,2})^+$$

and we will deduce from the main result of [JSo07] that these $\mathbb{R}\otimes E$-modules have rank one. Let $\mathcal{D}(\Pi_f\times\sigma_f)$ denote the Deligne E-structure on $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f\times\sigma_f,3)_{\mathbb{R}})$. Let us consider the group $\mathrm{GL}(2)\times_{\mathbb{G}_m}\mathrm{GL}(2)=\{(g,h)\in\mathrm{GL}(2)\times\mathrm{GL}(2)\mid \det(g)=\det(h)\}$ and let

$$\iota:\mathrm{GL}(2)\times_{\mathbb{G}_m}\mathrm{GL}(2)\to\mathrm{GSp}(4)\times\mathrm{GL}(2)$$

be the embedding defined by

$$\iota\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\right)=\left(\begin{pmatrix}a&&b&\\&a'&&b'\\c&&d&\\&c'&&d'\end{pmatrix},\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\right).$$

As explained in section 2.2 below, the morphism ι induces an embedding of the product of two modular curves into the Shimura variety $S\times Y$ whose cohomology class generates a E-subspace

$$\mathcal{Z}(\Pi_f\times\sigma_f)\subset(M_B(\Pi_f\times\sigma_f,2)_{\mathbb{R}}\cap M^{2,2})^+.$$

To state our first main result, we need the local Weil lifting from $\mathrm{GSO}(2,2)$ to $\mathrm{GSp}(4)$, for which we refer the reader to [HK92] and [Teh13]. Let us simply recall the isomorphism $\mathrm{GSO}(2,2)\simeq(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathbb{G}_m$ where the embedding $\mathbb{G}_m\to\mathrm{GL}(2)\times\mathrm{GL}(2)$ is given by $x\mapsto(xI_2,x^{-1}I_2)$. As a consequence irreducible representations of $\mathrm{GSO}(2,2)$ are pairs of irreducible representations of $\mathrm{GL}(2)$ with the same central characters.

Theorem 1.1. *Assume that for every non-archimedean place $v\in V$ the representation Π_v is obtained as a Weil lifting from $\mathrm{GSO}(2,2)$ of a pair (σ_1,σ_2). Assume that if σ_v is supercuspidal, then the contragredient $\check{\sigma}_v$ is not isomorphic to any representation of the form $|\,|^z\otimes\sigma_i$, for $i=1,2$. Then*

$$\mathcal{Z}(\Pi_f\times\sigma_f)\neq 0 \iff L(s,\Pi\times\sigma) \text{ has a pole at } s=1.$$

Remark 1.2. *The first assumption of Thm. 1.1 allows to use computations of* [So84] *of some ramified p-adic integrals. The second assumption ensures that Thm. 6.1 in loc. cit. is true. We need this result to prove that for $v\in V$ the ramified local L-factor $L(s,\Pi_v\times\sigma_v)$ does not have a pole at $s=1$. This should be true in general, according to the conjecture on the purity of the monodromy filtration (see* [Ne94] *§1, 1.4).*

To go further, let us assume that $L(s, \Pi \times \sigma)$ has a pole at $s = 1$. Then, by a result of Piatetski-Shapiro and Soudry, the automorphic representation Π is obtained as a global Weil lifting from $\mathrm{GSO}(2,2)$ of a pair (σ_1, σ_2) of two inequivalent irreducible cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ with the same central characters. Let $p(\Pi \times \sigma)$ be the product of the de Rham-Whittaker periods attached to Π and σ and defined in section 3.2. In the next result, as in Thm. 1.1, the assumption allows us to ensure that for non-archimedean $v \in V$, the local L-factor $L(s, \Pi_v \times \sigma_v)$ does not have a pole at $s = 1$.

Theorem 1.3. *Assume that for every non-archimedean place $v \in V$ the representation σ_v is not supercuspidal. Then*

$$\mathcal{Z}(\Pi_f \times \sigma_f) = p(\Pi \times \sigma)\,\mathrm{Res}_{s=1} L(s, \Pi \times \sigma)\mathcal{D}(\Pi_f \times \sigma_f).$$

Remark 1.4. *Note that the poles of $L(s, \Pi \times \sigma)$ are at most simple according to* [Mo09] *Thm. 1.1. As a consequence, the residue in Thm. 1.3 is nothing but the special value at* 1.

Let us briefly outline the contents of the paper. In section 2.1, we review what is needed about the de Rham and Betti cohomologies of the Shimura varieties S and Y. In section 2.2, we recall the definition of the Deligne rational structure $\mathcal{D}(\Pi_f \times \sigma_f)$ and give the precise definition of $\mathcal{Z}(\Pi_f \times \sigma_f)$. In section 2.3, we explain the computation of some Poincaré duality pairings that turn out to be crucial in the proofs of our main theorems. In section 3 we state and prove results about the integral representation of the $L(s, \Pi \times \sigma)$ and in section 4 we prove the two theorems stated above.

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2. Motives for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$

2.1. **Cohomology of Shimura varieties.** Let us briefly recall the definition of the Shimura varieties we are interested in. Details can be found in [La05] Partie I.3. Let I_2 be the identity matrix of size two and let $\mathcal{J}$ be the symplectic form on $\mathbb{Z}^4$ whose matrix in the canonical basis is

$$\mathcal{J} = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}.$$

The symplectic group $\mathrm{GSp}(4)$ is defined as

$$\mathrm{GSp}(4) = \left\{ g \in \mathrm{GL}(4) \,|\, g^t \mathcal{J} g = \nu(g)\psi,\ \nu(g) \in \mathbb{G}_m \right\}.$$

Then the map $\nu : \mathrm{GSp}(4) \to \mathbb{G}_m$ is a character. Let $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ be the Deligne torus and let $\mathcal{H}$ be the $\mathrm{GSp}(4, \mathbb{R})$-conjugacy class of the morphism $h : \mathbb{S} \to \mathrm{GSp}(4)_{\mathbb{R}}$ given on $\mathbb{R}$-points by

$$x + iy \longmapsto \begin{pmatrix} x & & y & \\ & x & & y \\ -y & & x & \\ & -y & & x \end{pmatrix}.$$

The pair $(\mathrm{GSp}(4), \mathcal{H})$ is a pure Shimura datum in the sense of [P90] 2.1. It is easy to see that the reflex field of $(\mathrm{GSp}(4), \mathcal{H})$ is the field of rational numbers. For any neat compact open subgroup K of $\mathrm{GSp}(4, \mathbb{A}_f)$, let us denote by S^K the Shimura variety at level K associated to $(\mathrm{GSp}(4), \mathcal{H})$. It is a smooth quasi-projective variety over $\mathbb{Q}$ such that, as complex analytic varieties, we have

$$S^K(\mathbb{C}) = \mathrm{GSp}(4, \mathbb{Q}) \backslash (\mathcal{H} \times \mathrm{GSp}(4, \mathbb{A}_f)/K).$$

If $K = K(N) \subset \mathrm{GSp}(4, \hat{\mathbb{Z}})$ is the principal congruence subgroup of level N, then

$$S^{K(N)}(\mathbb{C}) = \bigsqcup_{(\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N) \backslash \mathcal{H}$$

where $\mathcal{H}$ is the Siegel upper half space and $\Gamma(N) \subset \mathrm{Sp}(4, \mathbb{Z})$ is the principal congruence subgroup of level N. In particular, the complex analytic varieties $S^K(\mathbb{C})$ have dimension 3. For $g \in \mathrm{GSp}(4, \mathbb{A}_f)$ and K, K' two neat compact open subgroups of $\mathrm{GSp}(4, \mathbb{A}_f)$ such that $g^{-1}K'g \subset K$, right multiplication by g on $S^K(\mathbb{C})$ descends to a morphism $[g] : S^{K'} \to S^K$ of $\mathbb{Q}$-schemes, which is finite and étale. This implies that there is an action of $\mathrm{GSp}(4, \mathbb{A}_f)$ on the projective system (S^K) indexed by neat compact open subgroups of $\mathrm{GSp}(4, \mathbb{A}_f)$. In what follows, all compact open subgroups of $\mathrm{GSp}(4, \mathbb{A}_f)$ and of $\mathrm{GL}(2, \mathbb{A}_f)$ will be assumed to be neat and we will not mention this fact anymore. Similarly, we have the projective system of modular curves (Y^L) indexed by the set of compact open subgroups $L \subset \mathrm{GL}(2, \mathbb{A}_f)$ and which is endowed of an action of $\mathrm{GL}(2, \mathbb{A}_f)$. Let G denote the product $G = \mathrm{GSp}(4) \times \mathrm{GL}(2)$. This discussion shows that we have an action of $G(\mathbb{A}_f)$ on the projective system $(S^K \times Y^L)$ indexed by the set of pairs (K, L) where K is a compact open subgroup of $\mathrm{GSp}(4, \mathbb{A}_f)$ and L is a compact open subgroup of $\mathrm{GL}(2, \mathbb{A}_f)$.

Let $H^*_{dR,c}(S^K \times Y^L)$ and $H^*_{dR}(S^K \times Y^L)$ be the de Rham cohomology with compact support and without support respectively, with complex coefficients. Let

$$H^*_{dR,!}(S^K \times Y^L) = \mathrm{Im}(H^*_{dR,c}(S^K \times Y^L) \to H^*_{dR}(S^K \times Y^L))$$

and let us define $H^*_{dR,!}(S^K)$ and $H^*_{dR,!}(Y^L)$ similarly.

Proposition 2.1. *The graded $\mathbb{C}$-vector spaces $H^*_{dR,!}(S^K)$ and $H^*_{dR,!}(Y^L)$ are concentrated in degree 3 and 1 respectively.*

Proof. The first statement is proved in [MT02] Prop. 1. For the second statement, note that $H^0_{dR,c}(Y^L) = 0$ because the connected components of Y^L are non-compact. As a consequence $H^2_{dR}(Y^L) = 0$ by Poincaré duality. □

Let

$$H^4_{dR,!}(S \times Y) = \varinjlim_K \varinjlim_L H^4_{dR,!}(S^K \times Y^L)$$

where the limit is indexed by the compact open supbgroups K of $\mathrm{GSp}(4, \mathbb{A}_f)$ and L of $\mathrm{GL}(2, \mathbb{A}_f)$. The action of $G(\mathbb{A}_f)$ on the projective system $(S^K \times Y^L)$ induces a structure of $\mathbb{C}[G(\mathbb{A}_f)]$-module on $H^4_{dR,!}(S \times Y)$. Let us define $H^3_{dR,!}(S)$ and $H^1_{dR,!}(Y)$ similarly as

$H^4_{dR,!}(S \times Y)$. By the Künneth formula and Prop. 2.1, we have a canonical isomorphism of $\mathbb{C}[G(\mathbb{A}_f)]$-modules

$$H^4_{dR,!}(S \times Y) \simeq H^3_{dR,!}(S) \otimes H^1_{dR,!}(Y). \tag{1}$$

To relate $H^4_{dR,!}(S \times Y)$ to automorphic representations, let $P_4 = \{\Pi^H_\infty, \Pi^W_\infty, \overline{\Pi}^W_\infty, \overline{\Pi}^H_\infty\}$ denote the set of isomorphism classes of discrete series of $\mathrm{GSp}(4,\mathbb{R})_+ = \nu^{-1}(\mathbb{R}^\times_+)$ with the same central and infinitesimal characters as the trivial representation, where Π^H_∞ is holomorphic, Π^W_∞ and $\overline{\Pi}^W_\infty$ are generic and $\overline{\Pi}^H_\infty$ is antiholomorphic. The fact that P_4 has 4 elements follows from Harish-Chandra's classification, as explained in [Le17] Prop. 3.1 in the particular case where $k = k' = 0$, with the notation of loc. cit. Similarly, let $P_2 = \{\sigma_\infty, \overline{\sigma}_\infty\}$ denote the set of isomorphism classes of discrete series of $\mathrm{GL}(2,\mathbb{R})_+ = \det^{-1}(\mathbb{R}^\times_+)$ with the same central and infinitesimal characters as the trivial representation, where σ_∞ is holomorphic and $\overline{\sigma}_\infty$ is anti-holomorphic. Let $\mathfrak{gsp}_4$ and $\mathfrak{gl}_2$ denote the complex Lie algebras of $\mathrm{GSp}(4)$ and $\mathrm{GL}(2)$ respectively and let K_∞ and L_∞ denote respectively the group $\mathrm{U}(2,\mathbb{R})$, regarded as a maximal compact subgroup of $\mathrm{GSp}(4,\mathbb{R})$, and the group $\mathrm{SO}(2,\mathbb{R})$, regarded as a maximal compact subgroup of $\mathrm{GL}(2,\mathbb{R})$, in the standard way. Using the relative Lie algebra cohomology groups, we have the following result.

Proposition 2.2. *There is a canonical $G(\mathbb{A}_f)$-equivariant isomorphism*

$$H^4_{dR,!}(S \times Y) \simeq \bigoplus_{\Pi,\sigma} H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty)^{m(\Pi)} \otimes H^1(\mathfrak{gl}_2, L_\infty, \sigma_\infty)^{m(\sigma)} \otimes (\Pi_f \times \sigma_f)$$

where the direct sum is indexed by equivalence classes of cuspidal automorphic representations $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathrm{GSp}(4,\mathbb{A})$ and $\sigma = \sigma_\infty \otimes \sigma_f$ of $\mathrm{GL}(2,\mathbb{A})$ such that $\Pi_\infty \in P_4$ and $\sigma_\infty \in P_2$ and where $m(\Pi)$ and $m(\sigma)$ denote the cuspidal multiplicities of Π and σ respectively.

Proof. This follows from (1) and the analogous statements for $H^3_{dR,!}(S)$ (see [Le17] (8) and (9)) and $H^1_{dR,!}(Y)$ (see [Sch11] Lem. 12.3). □

Proposition 2.3. *For any $\Pi_\infty \in P_4$, resp. $\sigma_\infty \in P_2$, we have*

$$\begin{aligned} H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty) &= \mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{sp}_4/\mathfrak{k}, \Pi_\infty\right), \\ H^1(\mathfrak{gl}_2, L_\infty, \sigma_\infty) &= \mathrm{Hom}_{L_\infty}(\mathfrak{sl}_2/\mathfrak{l}, \sigma_\infty) \end{aligned}$$

and these $\mathbb{C}$-vector spaces are one-dimensional.

Proof. According to [BoW80] II. §3, Prop. 3.1, for any $\Pi_\infty \in P_4$, resp. $\sigma_\infty \in P_2$, the $(\mathfrak{gsp}_4, K_\infty)$-complex of Π_∞, resp. the $(\mathfrak{gl}_2, L_\infty)$-complex of σ_∞, has zero differential. This implies the two equalities in the statement of the Proposition. The statement about the dimensions is a particular case of loc. cit. II. Thm. 5.3. □

Let us fix a cuspidal automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$, resp. $\sigma = \sigma_\infty \otimes \sigma_f$, of $\mathrm{GSp}(4,\mathbb{A})$, resp. $\mathrm{GL}(2,\mathbb{A})$ such that $\Pi_\infty \in P_4$, resp. $\sigma_\infty \in P_2$. Then, it is one of the main results of [BHR94] that the non-archimedean part Π_f of Π is defined over its rationality field, which is a number field. The fact that the analogous statement for σ_f is true is proved in [Wa85]. Let us denote by E the composite of the rationality fields of Π_f and σ_f. Let $H^4_{B,!}(S\times Y)$ denote the image of the Betti cohomology with compact support in the Betti cohomology without support, with rational coefficients. We will use the following notation

$$M_B(\Pi_f\times\sigma_f) \quad = \quad \mathrm{Hom}_{E[G(\mathbb{A}_f)]}(\Pi_f\times\sigma_f, H^4_{B,!}(S\times Y)\otimes_{\mathbb{Q}} E).$$

Proposition 2.4. *Assume that* Π *is not CAP. Then, the natural inclusion* $H^4_{B,!}(S\times Y)\subset H^4_B(S\times Y)$ *induces an isomorphism*

$$M_B(\Pi_f\times\sigma_f) = \mathrm{Hom}_{E[G(\mathbb{A}_f)]}(\Pi_f\times\sigma_f, H^4_B(S\times Y)\otimes_{\mathbb{Q}} E).$$

Proof. As Π is not CAP, we can apply [We09] Thm. 1.1 which implies that

$$\begin{aligned}\mathrm{Hom}_{E[\mathrm{GSp}(4,\mathbb{A}_f)]}(\Pi_f, H^3_B(S,\mathbb{Q})\otimes_{\mathbb{Q}} E) &= \mathrm{Hom}_{E[\mathrm{GSp}(4,\mathbb{A}_f)]}(\Pi_f, H^3_{B,!}(S,\mathbb{Q})\otimes_{\mathbb{Q}} E)\\ \mathrm{Hom}_{E[\mathrm{GSp}(4,\mathbb{A}_f)]}(\Pi_f, H^4_B(S,\mathbb{Q})\otimes_{\mathbb{Q}} E) &= 0.\end{aligned}$$

Note that [We09] Thm. 1.1 assumes that Π_∞ is holomorphic, but that this is not used in the proof and so the same proof is true for generic Π_∞. Furthermore, as there are no CAP representations for GL(2) the same argument as in the proof of [We09] Thm. 1.1 shows that

$$\mathrm{Hom}_{E[\mathrm{GL}(2,\mathbb{A}_f)]}(\sigma_f, H^1_B(Y,\mathbb{Q})\otimes_{\mathbb{Q}} E) = \mathrm{Hom}_{E[\mathrm{GL}(2,\mathbb{A}_f)]}(\sigma_f, H^1_{B,!}(Y,\mathbb{Q})\otimes_{\mathbb{Q}} E).$$

Hence, by the Künneth formula, we have

$$\mathrm{Hom}_{E[G(\mathbb{A}_f)]}(\Pi_f\times\sigma_f, H^4_B(S\times Y,\mathbb{Q})\otimes_{\mathbb{Q}} E) = M_B(\Pi_f\times\sigma_f).$$

□

Let $M_B(\Pi_f)$ and $M_B(\sigma_f)$ be defined similarly as $M_B(\Pi_f\times\sigma_f)$ via the Betti cohomology of S and Y respectively. Then, the Künneth formula implies that

$$M_B(\Pi_f\times\sigma_f) = M_B(\Pi_f)\otimes M_B(\sigma_f).$$

According to Prop. 2.2, Prop. 2.3 and the comparison isomorphism between de Rham and Betti cohomology, these are finite dimensional $\mathbb{Q}$-vector spaces endowed with a $\mathbb{Q}$-linear action of E and additional structures as follows. Let $M_B(\Pi_f)_{\mathbb{C}}$ and $M_B(\sigma_f)_{\mathbb{C}}$ denote the vector spaces obtained after extending the scalars from $\mathbb{Q}$ to $\mathbb{C}$. Then we have the Hodge decompositions

$$M_B(\Pi_f)_{\mathbb{C}} = M(\Pi_f)^{3,0}\oplus M(\Pi_f)^{2,1}\oplus M(\Pi_f)^{1,2}\oplus M(\Pi_f)^{1,3}, \tag{2}$$

$$M_B(\sigma_f)_{\mathbb{C}} = M(\sigma_f)^{1,0}\oplus M(\sigma_f)^{0,1} \tag{3}$$

where

$$\begin{aligned} M(\Pi_f)^{3,0} &= \bigoplus_{\sigma:E\to\mathbb{C}} H^3(\mathfrak{gsp}_4, K_\infty, \pi_\infty^H)^{m(\Pi_\infty^H\otimes\Pi_f)}, \\ M(\Pi_f)^{2,1} &= \bigoplus_{\sigma:E\to\mathbb{C}} H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty^W)^{m(\Pi_\infty^W\otimes\Pi_f)}, \\ M(\Pi_f)^{1,2} &= \bigoplus_{\sigma:E\to\mathbb{C}} H^3(\mathfrak{gsp}_4, K_\infty, \overline{\Pi}_\infty^W)^{m(\overline{\Pi}_\infty^W\otimes\Pi_f)}, \\ M(\Pi_f)^{0,3} &= \bigoplus_{\sigma:E\to\mathbb{C}} H^3(\mathfrak{gsp}_4, K_\infty, \overline{\Pi}_\infty^H)^{m(\overline{\Pi}_\infty^H\otimes\Pi_f)}, \\ M(\sigma_f)^{1,0} &= \bigoplus_{\sigma:E\to\mathbb{C}} H^1(\mathfrak{gl}_2, L_\infty, \sigma_\infty)^{m(\sigma_\infty\otimes\sigma_f)}, \\ M(\sigma_f)^{0,1} &= \bigoplus_{\sigma:E\to\mathbb{C}} H^1(\mathfrak{gl}_2, L_\infty, \overline{\sigma}_\infty)^{m(\overline{\sigma}_\infty\otimes\sigma_f)} \end{aligned}$$

and $M_B(\Pi_f \times \sigma_f)$ has the tensor product Hodge structure

$$M_B(\Pi_f \times \sigma_f)_{\mathbb{C}} = M^{4,0} \oplus M^{3,1} \oplus M^{2,2} \oplus M^{1,3} \oplus M^{0,4}. \tag{4}$$

The following definition is taken from [Be86] §7.

Definition 2.5. *A real mixed $\mathbb{R}$-Hodge structure is a mixed $\mathbb{R}$-Hodge structure whose underlying $\mathbb{R}$-vector space is endowed with an involution F_∞ stabilizing the weight filtration and whose $\mathbb{C}$-antilinear complexification $\overline{F}_\infty$ stabilizes the Hodge filtration.*

Let $\mathrm{MHS}_{\mathbb{R}}^+$ denote the abelian category of real mixed $\mathbb{R}$-Hodge structures.

Definition 2.6. *Let F be a ring. A real mixed $\mathbb{R}$-Hodge structure with coefficients in F is a pair (M, s) where M is an object of $\mathrm{MHS}_{\mathbb{R}}^+$ and $s : F \to \mathrm{End}_{\mathrm{MHS}_{\mathbb{R}}^+}(M)$ is a ring homomorphism.*

For any ring F, let $\mathrm{MHS}_{\mathbb{R},F}^+$ denote the abelian category of real mixed $\mathbb{R}$-Hodge structures with coefficients in F.

Proposition 2.7. *Let F_∞ be the involution on $M_B(\Pi_f \times \sigma_f)$ induced by the complex conjugation on $S(\mathbb{C}) \times Y(\mathbb{C})$. Then $(M_B(\Pi_f \times \sigma_f), F_\infty)$ is an object of $\mathrm{MHS}_{\mathbb{Q},E}^+$ which is pure of weight 4.*

Furthermore, it follows from [H94] Cor. 2.3.1 that there exists a filtered E-vector space $(M_{dR}(\Pi_f \times \sigma_f, F^* M_{dR}(\Pi_f \times \sigma_f))$ and a comparison isomorphism

$$I_\infty : M_B(\Pi_f \times \sigma_f)_{\mathbb{C}} \to M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{C}} \tag{5}$$

such that the Hodge filtration of $M_B(\Pi_f \times \sigma_f)$, defined as

$$F_{\mathbb{C}}^p = \bigoplus_{p' \geq p} M^{p',q},$$

satisfies $I_\infty(F_{\mathbb{C}}^p) = F^p M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{C}}$. A similar statement holds for $M_B(\Pi_f)$ and $M_B(\sigma_f)$.

2.2. **The Deligne rational structure and the cycle class.**

2.2.1. *The Deligne rational structure.* Let $\Pi = \Pi_\infty \otimes \Pi_f$ and $\sigma = \sigma_\infty \otimes \sigma_f$ be irreducible cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A})$ and $\mathrm{GL}(2, \mathbb{A})$ respectively. Assume that $\Pi_\infty \in P_4$ and $\sigma_\infty \in P_2$. For any integer n, let $M_B(\Pi_f \times \sigma_f, n)$ denote the object of $\mathrm{MHS}^+_{\mathbb{Q},E}$ defined as

$$M_B(\Pi_f \times \sigma_f, n) = M_B(\Pi_f \times \sigma_f) \otimes_{\mathbb{Q}} \mathbb{Q}(n)$$

where $\mathbb{Q}(n)$ is the n-th tensor power of the Tate object. Let $M_B(\Pi_f \times \sigma_f, n)^\pm$ denote the subspace of $M_B(\Pi_f \times \sigma_f, n)$ where F_∞ acts as ± 1. The comparison isomorphism I_∞^{-1} (see (5)) between de Rham and Betti cohomology, sends the real structure $M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}}$ of $M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{C}}$ to the real structure $M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}} \oplus M_B(\Pi_f \times \sigma_f)^-_{\mathbb{R}}(-1)$ of $M_B(\Pi_f \times \sigma_f)_{\mathbb{C}}$, where $M_B(\Pi_f \times \sigma_f)^-_{\mathbb{R}}(-1)$ simply denotes the sub-$\mathbb{R} \otimes E$-module $M_B(\Pi_f \times \sigma_f)^-_{\mathbb{R}} \otimes i\,\mathbb{R}$ of $M_B(\Pi_f \times \sigma_f)^-_{\mathbb{C}}$. In particular, we have a natural $\mathbb{R} \otimes E$-linear map

$$F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}} \to M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}}$$

defined as the composition of the natural inclusion $F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}} \subset M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}}$, of I_∞^{-1} and of the natural projection $M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}} \oplus M_B(\Pi_f \times \sigma_f)^-_{\mathbb{R}}(-1) \to M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}}$. Composing with the canonical isomorphism $M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}} \simeq M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}}$ given by multiplication by $(2\pi i)^2$, we obtain the natural map

$$F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}} \to M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}}.$$

Proposition 2.8. *We have the following short exact sequence of $\mathbb{R} \otimes E$-modules*

$$0 \to F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}} \to M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}} \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \to 0$$

where the second map is the map defined above.

Proof. As $M_B(\Pi_f \times \sigma_f, 3)$ is pure of weight -2, it follows from [Le17] Lem. 4.11 that we have the short exact sequence

$$0 \to F^0 M_{dR}(\Pi_f \times \sigma_f, 3)_{\mathbb{R}} \to M_B(\Pi_f \times \sigma_f, 3)^-_{\mathbb{R}}(-1) \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \to 0$$

where the second map is defined similarly as above. Furthermore, we have the following canonical isomorphisms $F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}} \simeq F^0 M_{dR}(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}$ and $M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}} \simeq M_B(\Pi_f \times \sigma_f, 3)^-_{\mathbb{R}}(-1)$. The conclusion follows. □

Remark 2.9. *Let us denote by $(M_B(\Pi_f \times \sigma_f, 2)_{\mathbb{R}} \cap M^{2,2})^+$ the vectors of $M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}}$ which have Hodge type $(2, 2)$. It is straightforward to deduce from the previous result a canonical isomorphism*

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \simeq (M_B(\Pi_f \times \sigma_f, 2)_{\mathbb{R}} \cap M^{2,2})^+$$

as claimed in the introduction.

Proposition 2.10. *The ranks of the $\mathbb{R} \otimes E$-modules $F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}}$, $M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}}$ and $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$ are finite and equal to $2m(\Pi_\infty^H \otimes \Pi_f) + m(\Pi_\infty^W \otimes \Pi_f)$, $2m(\Pi_\infty^H \otimes \Pi_f) + 2m(\Pi_\infty^W \otimes \Pi_f)$ and $m(\Pi_\infty^W \otimes \Pi_f)$ respectively.*

Proof. The existence of the short exact sequence of Prop. 2.8 implies that it is enough to prove

$$\begin{aligned} rk\, F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}} &= 2m(\Pi_\infty^H \otimes \Pi_f) + m(\Pi_\infty^W \otimes \Pi_f), \\ rk\, M_B(\Pi_f \times \sigma_f, 2)_{\mathbb{R}}^+ &= 2m(\Pi_\infty^H \otimes \Pi_f) + 2m(\Pi_\infty^W \otimes \Pi_f). \end{aligned}$$

We have $m(\sigma_\infty \otimes \sigma_f) = m(\overline{\sigma}_\infty \otimes \sigma_f)$ and according to [Sh74] Thm. 5.5, this multiplicity is equal to one. Similarly $m(\Pi_\infty^H \otimes \Pi_f) = m(\overline{\Pi}_\infty^H \otimes \Pi_f)$ and $m(\Pi_\infty^W \otimes \Pi_f) = m(\overline{\Pi}_\infty^W \otimes \Pi_f)$, hence both statements are direct consequences of Prop. 2.3 and equality (4). □

The short exact sequence of Prop. 2.8 induces the canonical isomorphism of rank one $\mathbb{R} \otimes E$-modules

$$\det_{\mathbb{R}} F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}} \otimes \det_{\mathbb{R}} \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \simeq \det_{\mathbb{R}} M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}},$$

where we denote by det the highest exterior power. Let $\det_E F^3 M_{dR}(\Pi_f \times \sigma_f)^\vee$ denote the E-module dual of $\det_E F^3 M_{dR}(\Pi_f \times \sigma_f)$. The evaluation map

$$\det_E F^3 M_{dR}(\Pi_f \times \sigma_f) \otimes \det_E F^3 M_{dR}(\Pi_f \times \sigma_f)^\vee \to E$$

is an isomorphism.

Definition 2.11. *The Beilinson E-structure on* $\det_{\mathbb{R}} \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$ *is defined as*

$$\mathcal{B}(\Pi_f \times \sigma_f) = \det_E F^3 M_{dR}(\Pi_f \times \sigma_f)^\vee \otimes \det_E M_B(\Pi_f \times \sigma_f, 2)^+.$$

Let us denote by $\delta(\Pi_f \times \sigma_f, 3)$ the determinant of the comparison isomorphism

$$M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{C}} \to M_{dR}(\Pi_f \times \sigma_f, 3)_{\mathbb{C}}$$

computed in basis defined on E on both sides. Then $\delta(\Pi_f \times \sigma_f, 3)$ is an element of $(\mathbb{C} \otimes E)^\times$, which, as $\dim_E M_B(\Pi_f \times \sigma_f)^-$ is even, belongs to $(\mathbb{R} \otimes E)^\times$ (see [De79] p. 320) and which is independent of the choice of the basis up to right multiplication by an element of $E^\times$.

Definition 2.12. *The Deligne E-structure on* $\det_{\mathbb{R}} \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$ *is defined as*

$$\mathcal{D}(\Pi_f \times \sigma_f) \quad = \quad (2\pi i)^{\dim_E M_B(\Pi_f \times \sigma_f, 3)^-} \delta(\Pi_f \times \sigma_f, 3)^{-1} \mathcal{B}(\Pi_f \times \sigma_f).$$

2.2.2. *The cycle class.* The reductive group G contains

$$H = \mathrm{GL}(2) \times_{\mathbb{G}_m} \mathrm{GL}(2) = \{(g,h) \in \mathrm{GL}(2) \times \mathrm{GL}(2) \mid \det(g) = \det(h)\}$$

as a closed subgroup via the embedding $\iota : H \to G$ defined by

$$\iota\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = \left(\begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right).$$

For any integer $N \geq 3$, this group homomorphism induces the closed embedding

$$Y(N) \times_{\mathbb{Q}(\mu_N)} Y(N) \xrightarrow{\iota_N} S(N) \times Y(N)$$

between the corresponding Shimura varieties. Let us fix an integer N and let

$$\mathcal{Z}_N \in H^4_B(S(N) \times Y(N), \mathbb{Q}(2)) \tag{6}$$

be the cohomology class of the image of ι_N, which we regard as an element of the $E[G(\mathbb{A}_f)]$-module $H^4_B(S \times Y, \mathbb{Q}) \otimes_{\mathbb{Q}} E = \varinjlim_M H^4_B(S(M) \times Y(M), \mathbb{Q}) \otimes_{\mathbb{Q}} E$. Let

$$\widetilde{\mathcal{Z}}(\Pi_f \times \sigma_f) = \mathrm{Hom}_{E[G(\mathbb{A}_f)]}(\Pi_f \times \sigma_f, E[G(\mathbb{A}_f)]\mathcal{Z}_N).$$

This is a sub-E-vector space of $\mathrm{Hom}_{E[G(\mathbb{A}_f)]}(\Pi_f \times \sigma_f, H^4_B(S \times Y, \mathbb{Q}) \otimes_{\mathbb{Q}} E)$. As we assume that Π is not CAP, the latter coincides with $M_B(\Pi_f \times \sigma_f)$ (see Prop. 2.4). Note that as the cycle $Y(N) \times_{\mathbb{Q}(\mu_N)} Y(N)$ is defined over $\mathbb{Q}$, we have $\widetilde{\mathcal{Z}}(\Pi_f \times \sigma_f) \subset M_B(\Pi_f \times \sigma_f, 2)^+$. We shall denote again by

$$\mathcal{Z}(\Pi_f \times \sigma_f) \subset \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$$

the sub-E-vector space defined as the image of $\widetilde{\mathcal{Z}}(\Pi_f \times \sigma_f)$ by the natural map

$$M_B(\Pi_f \times \sigma_f, 2)^+ \to M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}} \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}).$$

where the first map is the canonical inclusion and the second is the third map in the short exact sequence of Prop. 2.8.

2.3. **Computation of Poincaré duality pairings.** From now on, we fix two irreducible cuspidal automorphic representations $\Pi = \Pi_\infty \otimes \Pi_f$ and $\sigma = \sigma_\infty \otimes \sigma_f$ of $\mathrm{GSp}(4, \mathbb{A}_f)$ and $\mathrm{GL}(2, \mathbb{A}_f)$ respectively. We assume that $\Pi_\infty \in P_4$, that $\sigma_\infty \in P_2$ and that Π is globally generic. This implies that Π_∞ is not the holomorphic nor the antiholomorphic member of P_4. In other words $\Pi_\infty \in \{\Pi^W_\infty, \overline{\Pi}^W_\infty\}$.

Proposition 2.13. [JSo07] *The multiplicity* $m(\Pi^W_\infty \otimes \Pi_f) = m(\overline{\Pi}^W_\infty \otimes \Pi_f)$ *is equal to one.*

2.3.1. *Cohomological interpretation of a period integral.* The previous result implies that the $\mathbb{C} \otimes E$-module $M(\Pi_f)^{2,1}$ has rank one. Let us explain how to attach a generator of this module to certain cusp forms in the representation space of Π. We will freely use some standard results and notations from the section 3.1 of [Le17]. In particular, if $(k, k') \in \mathbb{Z}^2$ is a pair of integers such that $k \geq k'$, let $\tau_{(k,k')}$ denote the irreducible $\mathbb{C}[K_\infty]$-module of highest weight (k, k') with the conventions of loc. cit. Then, the generic member Π^W_∞ of the discrete series L-packet P_4 contains with multiplicity one $\tau_{(3,-1)}$ as a minimal K_∞-type (see loc. cit. Prop. 3.1). Furthermore, we have the Cartan decomposition $\mathfrak{sp}_4 = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where

$$\mathfrak{p}^\pm = \left\{ \begin{pmatrix} Z & \pm iZ \\ \pm iZ & -Z \end{pmatrix}, Z^t = Z \in \mathfrak{gl}_2 \right\}.$$

For each symmetric matrix $Z \in \mathfrak{gl}_2$, define the element $p_\pm(Z)$ of $\mathfrak{sp}_4$ by

$$p_\pm(Z) = \begin{pmatrix} Z & \pm iZ \\ \pm iZ & -Z \end{pmatrix}.$$

Let $X_{(\alpha_1,\alpha_2)} \in \mathfrak{sp}_4$ be defined as

$$X_{\pm(2,0)} = p_\pm\left(\begin{pmatrix} 1 & \\ & \end{pmatrix}\right), X_{\pm(1,1)} = p_\pm\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right), X_{\pm(0,2)} = p_\pm\left(\begin{pmatrix} & \\ & 1 \end{pmatrix}\right).$$

It follows from an easy computation that $X_{(\alpha_1,\alpha_2)}$ is a root vector corresponding to the root (α_1, α_2) with the conventions of loc. cit. Recall from Prop. 2.3 that

$$H^3\left(\mathfrak{gsp}_4, K_\infty, \Pi_\infty^W\right) = \mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{sp}_4/\mathfrak{k}, \Pi_\infty^W\right).$$

Lemma 2.14. *Let $\Psi_\infty \in \Pi_\infty^W$ be a highest weight vector of the minimal K_∞-type. Then, there exists a unique element $\Omega_{\Psi_\infty} \in H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty^W)$ such that*

$$\Omega_{\Psi_\infty}(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}) = \Psi_\infty.$$

Proof. We have

$$\bigwedge^3 \mathfrak{sp}_4/\mathfrak{k} = \bigoplus_{p+q=3} \bigwedge^p \mathfrak{p}^+ \otimes \bigwedge^q \mathfrak{p}^-.$$

and by a weight computation, we find

$$\bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^- = \tau_{(3,-1)} \oplus \tau_{(2,0)} \oplus \tau_{(1,1)}$$

as $\mathbb{C}[K_\infty]$-modules. As a consequence, the exsitence of Ω_{Ψ_∞} follows from the fact that the vector $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$, as Ψ_∞, is a vector of highest weight $(3,-1)$. Its unicity follows from the fact that, according to Prop. 2.3, the $\mathbb{C}$-vector space $H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty^W)$ has dimension one. □

Let $\Psi = \Psi_\infty \otimes \Psi_f$ be a cusp form in the space of $\Pi = \Pi_\infty^W \otimes \Pi_f$ such that Ψ_∞ is a highest weight vector of the minimal K_∞-type and such that Ψ_f is invariant by the principal level N congruence subgroup of $\mathrm{GSp}(4, \widehat{\mathbb{Z}})$. Define

$$\Omega_\Psi = (\Omega_{\Psi_\infty})_{\sigma: E \to \mathbb{C}} \otimes \Psi_f \in H^3_{B,!}(S, \mathbb{C}) \otimes_{\mathbb{Q}} E. \tag{7}$$

Let

$$\omega_\Psi \in M_B(\Pi_f)_{\mathbb{C}} = \mathrm{Hom}_{E[\mathrm{GSp}(4,\mathbb{A}_f)]}(\Pi_f, H^3_{B,!}(S,\mathbb{C}) \otimes_{\mathbb{Q}} E)$$

be the unique element sending Ψ_f to Ω_Ψ. Then $\omega_\Psi \in M^{2,1}(\Pi_f)$. Let $\overline{\Psi}_\infty \in \overline{\Pi}_\infty^W$ be defined as in [Le17] Rem. 3.2. It follows from loc. cit., Prop 3.13 that $\overline{\omega}_\Psi \in M^{1,2}(\Pi_f)$ correponds to the element

$$\Omega_{\overline{\Psi}_\infty} \in H^3\left(\mathfrak{gsp}_4, K_\infty, \overline{\Pi}_\infty^W\right) = \mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{sp}_4/\mathfrak{k}, \overline{\Pi}_\infty^W\right)$$

characterized by the identity

$$\Omega_{\overline{\Psi}_\infty}(X_{(0,-2)} \wedge X_{(-1,-1)} \otimes X_{(2,0)}) = \overline{\Psi}_\infty$$

in the same way as ω_Ψ correponds to Ω_{Ψ_∞}. Let us denote by $\overline{\Psi}$ the cusp form $\overline{\Psi}_\infty \otimes \Psi_f$ which belongs to the representation space of $\overline{\Pi}_\infty^W \otimes \Pi_f$.

We have the Cartan decomposition $\mathfrak{sl}_2 = \mathfrak{l} \oplus \mathfrak{p}'^+ \oplus \mathfrak{p}'^-$ where

$$\mathfrak{p}'^\pm = \left\{ \begin{pmatrix} z & \pm iz \\ \pm iz & -z \end{pmatrix}, z \in \mathfrak{gl}_1 \right\}.$$

Let

$$v^{\pm} = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \in \mathfrak{p}'^{\pm}.$$

Let $\Phi = \Phi_\infty \otimes \Phi_f$ be a cusp form in the space of $\sigma = \sigma_\infty \otimes \sigma_f$ such that Φ_∞ is a generator of the minimal L_∞-type of σ_∞ and such that Φ_f is invariant by the principal level N congruence subgroup of $\mathrm{GL}(2, \widehat{\mathbb{Z}})$. Then, as in Lem. 2.14, we define the element of $H^1(\mathfrak{gl}_2, L_\infty, \sigma_\infty) = \mathrm{Hom}_{L_\infty}(\mathfrak{sl}_2/\mathfrak{l}, \sigma_\infty)$ by prescribing that it sends v^+ to Φ_∞. As above, we associate to Φ the harmonic differential form

$$\Omega_\Phi = (\Omega_{\Phi_\infty})_{\sigma: E \to \mathbb{C}} \otimes \Phi_f \in H^1_{B,!}(Y, \mathbb{C}) \otimes_{\mathbb{Q}} E$$

which gives rise to a generator η_Φ of $M^{1,0}(\sigma_f)$.

Let us introduce the Poincaré duality pairing

$$H^4_{B,!}(S \times Y, \mathbb{Q}) \otimes H^4_{B,!}(S \times Y, \mathbb{Q}) \to \mathbb{Q}(-4). \tag{8}$$

This becomes a perfect pairing of pure Hodge structures when restricted to invariants by any neat compact open subgroup of $G(\mathbb{A}_f)$. Furthermore this is a $G(\mathbb{A}_f)$-equivariant map when $\mathbb{Q}(-4)$ is endowed with the trivial action. This statement is easily deduced from the Künneth formula and from the analogous statements for S, see [Ta93] p. 295, and for Y, which are similar to loc. cit. Then (8) induces a perfect pairing

$$M_B(\Pi_f \times \sigma_f) \otimes M_B(\check{\Pi}_f \times \check{\sigma}_f) \xrightarrow{\langle\ ,\ \rangle_B} E(-4)_B$$

where $\check{\Pi}_f$ and $\check{\sigma}_f$ denote the representations contragredient to Π_f and σ_f, respectively, and where $E(-4)_B$ denotes the Betti realization of the (-4)-th power of the Tate motive with coefficients in E. Furthermore $\langle\ ,\ \rangle_B$ is a morphism of Hodge structure and has a de Rham analogue

$$M_{dR}(\Pi_f \times \sigma_f) \otimes M_{dR}(\check{\Pi}_f \times \check{\sigma}_f) \xrightarrow{\langle\ ,\ \rangle_{dR}} E(-4)_{dR}.$$

We assume that the central characters ω_Π and ω_σ of Π and σ are trivial. Hence, because of the canonical isomorphisms $\check{\Pi}_f \simeq \Pi_f \otimes (\omega_{\Pi_f} \circ \nu)^{-1}$ ([We05] Lem. 1.1) and $\check{\sigma}_f \simeq \sigma_f \otimes (\omega_{\sigma_f} \circ \det)^{-1}$ ([JL70] Thm. 2.18 (i)), the pairings $\langle\ ,\ \rangle_B$ and $\langle\ ,\ \rangle_{dR}$ can be regarded as a pairings

$$M_B(\Pi_f \times \sigma_f) \otimes M_B(\Pi_f \times \sigma_f) \xrightarrow{\langle\ ,\ \rangle_B} E(-4)_B$$

$$M_{dR}(\Pi_f \times \sigma_f) \otimes M_{dR}(\Pi_f \times \sigma_f) \xrightarrow{\langle\ ,\ \rangle_{dR}} E(-4)_{dR}$$

whose complexifications are part of the commutative diagram

$$\begin{array}{ccc} M_B(\Pi_f \times \sigma_f)_{\mathbb{C}} \otimes M_B(\Pi_f \times \sigma_f)_{\mathbb{C}} & \xrightarrow{\langle\ ,\ \rangle_{B,\mathbb{C}}} & \mathbb{C} \otimes E(-4)_B \\ \downarrow & & \downarrow \\ M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{C}} \otimes M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{C}} & \xrightarrow{\langle\ ,\ \rangle_{dR,\mathbb{C}}} & \mathbb{C} \otimes E(-4)_{dR} \end{array} \tag{9}$$

where the vertical lines are the comparison isomorphisms.

Lemma 2.15. *The pairing*

$$M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}} \xrightarrow{\langle\ ,\overline{\omega}_\Psi \otimes \eta_\Phi\rangle_{B,\mathbb{C}}} \mathbb{C} \otimes E(-4)_B$$

with $\overline{\omega} \otimes \eta$, induces an injective $\mathbb{R} \otimes E$-linear map

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \xrightarrow{\langle\ ,\overline{\omega}_\Psi \otimes \eta_\Phi\rangle_{B,\mathbb{C}}} \mathbb{C} \otimes E(-4)_B.$$

Proof. In the Hodge decomposition (4), we have $\overline{\omega} \otimes \eta \in M^{2,2}$. Moreover the image of the natural map $F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}} \to M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}}$ (see Prop. 2.8) lies in $M^{4,0} \oplus M^{3,1} \oplus M^{1,3} \oplus M^{0,4}$. As the Poincaré duality pairing is a morphism of Hodge structures, the restriction of

$$M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}} \xrightarrow{\langle\ ,\overline{\omega}_\Psi \otimes \eta_\Phi\rangle_{B,\mathbb{C}}} \mathbb{C} \otimes E(-4)_B$$

to $F^3 M_{dR}(\Pi_f \times \sigma_f)_{\mathbb{R}}$ is zero. Hence we get the induced map

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \xrightarrow{\langle\ ,\overline{\omega}_\Psi \otimes \eta_\Phi\rangle_{B,\mathbb{C}}} \mathbb{C} \otimes E(-4)_B.$$

It is easy to see that this map is non-zero. According to Prop. 2.10 and Prop. 2.13, the $\mathbb{R} \otimes E$-module $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$ has rank one. As a consequence, the map is injective, as claimed. □

Recall that we have fixed vectors Ψ_f and Φ_f in the representation space of Π_f and σ_f respectively. Hence, we can consider the image $v_{\mathcal{Z}}$ of the cycle class $\mathcal{Z}_N$ (6) by the maps

$$H^4_B(S(N) \times Y(N), \mathbb{Q}(2))^+ \to H^4_B(S \times Y, \mathbb{Q}(2))^+ \otimes E \to M_B(\Pi_f \times \sigma_f, 2)^+$$

where the second map sends a vector $x \in H^4_B(S \times Y, \mathbb{Q}(2))^+ \otimes E$ to the unique element of $M_B(\Pi_f \times \sigma_f, 2)^+$ sending $\Psi_f \times \sigma_f$ to x. It is easy to see that the vector $v_{\mathcal{Z}}$ is mapped to the E-subspace $\mathcal{Z}(\Pi_f \times \sigma_f)$ by the right hand map

$$M_B(\Pi_f \times \sigma_f, 2)^+ \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$$

of the short exact sequence of Prop. 2.8.

We now define a Haar measure on $H(\mathbb{A})$ as follows. For every prime number p, we endow $H(\mathbb{Q}_p)$ with the unique Haar measure dh_p for which $H(\mathbb{Z}_p)$ has volume one. Let us consider

$$U_\infty = \mathbb{R}^\times_+(\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R})) \subset H(\mathbb{R})$$

which is maximal compact modulo the center. Let $\mathfrak{h}$ and $\mathfrak{u}$ be the complex Lie algebras of $H(\mathbb{R})$ and U_∞ respectively. Then

$$\mathbf{1} = (v^+, 0) \wedge (0, v^+) \wedge (v^-, 0) \wedge (0, v^-)$$

is a generator of the highest exterior power $\bigwedge^4 \mathfrak{h}/\mathfrak{u}$ and it determines a left invariant measure dx on $H(\mathbb{R})/U_\infty$. Together with the Haar measure dk on $\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R})$ whose total mass is one, this defines a measure $dh_\infty = dxdk$ on $H(\mathbb{R})/\mathbb{R}^\times_+$. Let dh denote the product measure $\prod_{v \leq \infty} dh_v$ on $H(\mathbb{A})/\mathbb{R}^\times_+$.

Definition 2.16. *For two elements $x, y \in \mathbb{C} \otimes E$, we write $x \sim y$ if x and y belong to the same orbit under right multiplication by $E^\times$.*

In the following proposition, we regard a complex number as an element of $\mathbb{C}\otimes E = \prod_{\sigma:E\to\mathbb{C}}\mathbb{C}$ via the diagonal inclusion $\mathbb{C}\to\prod_{\sigma:E\to\mathbb{C}}\mathbb{C}$.

Proposition 2.17.

$$\langle v_{\mathcal{Z}},\overline{\omega}_\Psi\otimes\eta_\Phi\rangle_{B,\mathbb{C}}\sim\int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})}(X_{(-1,1)}\overline{\Psi})(h_1,h_2)\Phi(h_2)dh$$

Proof. Recall that we denote by $\mathcal{Z}_N\in H^4_B(S(N)\times Y(N),\mathbb{Q}(2))^+$ the cohomology class of $\iota_N(Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N))$. According to [GH78] Ch. 3, example 1 p. 386, the image of $\mathcal{Z}_N$ in the de Rham cohomology space $H^4_{dR}(S(N)\times Y(N),\mathbb{C})$ is given by the integration current along the subvariety $Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N)$. This means that for any closed differential form with compact support Ω of degree 4 on $S(N)\times Y(N)$, the Poincaré duality pairing $\langle\mathcal{Z}_N,\Omega\rangle$ is given by

$$\langle\mathcal{Z}_N,\Omega\rangle=\int_{Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N)}\Omega.$$

By the invariance properties of Ψ_f and Φ_f, the element $\overline{\Omega}_\Psi\otimes\Omega_\Phi$ defines a harmonic differential form on $S(N)\times Y(N)$. It is not compactly supported, however it is cuspidal. As a consequence, by [Bo81] Cor. 5.5, there exists a rapidly decreasing differential form Ω' such that the difference $\overline{\Omega}_\Psi\otimes\Omega_\Phi-d\Omega'$ is compactly supported. The integral

$$\int_{Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N)}d\Omega'$$

converges and is equal to zero. This follows from the fact that, as $d\Omega'$ is rapidly decreasing, it extends to a differential form on a smooth compactification of $Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N)$ and from the Stokes theorem. As a consequence

$$\begin{aligned}\langle v_{\mathcal{Z}},\overline{\omega}_\Psi\otimes\eta_\Phi\rangle_{dR,\mathbb{C}} &= \int_{Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N)}(\overline{\Omega}_\Psi\otimes\Omega_\Phi-d\Omega')\\ &= \int_{Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N)}\overline{\Omega}_\Psi\otimes\Omega_\Phi.\end{aligned}$$

Let $U(N)$ denote the principal level N congruence subgroup of $H(\widehat{\mathbb{Z}})$. As complex analytic varieties, we have

$$Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N)\simeq Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})/U_\infty U(N).$$

As a consequence

$$\int_{Y(N)\times_{\mathbb{Q}(\mu_N)}Y(N)}\overline{\Omega}_\Psi\otimes\Omega_\Phi=\int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})/U_\infty U(N)}(\overline{\Omega}_\Psi\otimes\Omega_\Phi)(\mathbf{1})dh$$

$$= [H(\widehat{\mathbb{Z}}):U(N)]^{-1}\int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})}\overline{\Omega}_\Psi((v^-,0)\wedge(0,v^-)\otimes(v^+,0))\Omega_\Phi(v^+)dh.$$

By an explicit computation similar to the ones conducted in the proof of [Le17] Lem. 4.27, one proves that

$$(v^-,0)\wedge(0,v^-)\otimes(v^+,0)=r\,\mathrm{ad}_{X_{(-1,1)}}(X_{(0,-2)}\wedge X_{(-1,-1)}\otimes X_{(2,0)})$$

for some $r \in \mathbb{Q}^\times$ that we do not need to compute as we work up to $\sim$. The conclusion follows. □

2.3.2. *The pairing associated to the Deligne rational structure.*

Proposition 2.18. *If the complete L-function $L(s, \Pi \times \sigma)$ has a pole at $s = 1$ then Π is obtained as a Weil lifting from $\mathrm{GSO}(2, 2, \mathbb{A})$ of a pair (σ_1, σ_2) of inequivalent irreducible cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ with the same central characters.*

Proof. This follows from [PSS84] Thm. 1.3. □

Proposition 2.19. *Assume that $L(s, \Pi \times \sigma)$ has a pole at $s = 1$. Then*

$$m(\Pi_\infty^H \otimes \Pi_f) = m(\overline{\Pi}_\infty^H \otimes \Pi_f) = 0.$$

Proof. This follows from the previous result and from [We09] Thm. 5.2. (4) and Lem. 5.2. □

In the rest of section 2.3.2, we assume that $L(s, \Pi \times \sigma)$ has a pole at $s = 1$ so that the conclusion of Prop. 2.19 holds. Hence, by the Hodge decompositions (2) and (3), we have the decompositions

$$\begin{aligned} M_B(\Pi_f) &= M_B(\Pi_f)^+ \oplus M_B(\Pi_f)^-, \\ M_B(\sigma_f) &= M_B(\sigma_f)^+ \oplus M_B(\sigma_f)^- \end{aligned}$$

where $M_B(\Pi_f)^\pm$ and $M_B(\sigma_f)^\pm$ are one-dimensional E-vector spaces. Let $v^\pm$ and $w^\pm$ be generators of $M_B(\Pi_f)^\pm$ and $M_B(\sigma_f)^\pm$ respectively. By a slight abuse of notation, we regard (v^+, v^-) as a basis of $M_B(\Pi_f)_\mathbb{C}$ and (w^+, w^-) as a basis of $M_B(\sigma_f)_\mathbb{C}$. Let ω be a generator of the one dimensional E-vector space $F^2 M_{dR}(\Pi_f)$ and let η be a generator of the one dimensional E-vector space $F^1 M_{dR}(\sigma_f)$. Then $\omega \otimes \eta$ is a generator of $F^3 M_{dR}(\Pi_f \times \sigma_f)$. Via the comparison isomorphisms, we have

$$\begin{aligned} \omega &= \alpha^+ v^+ + \alpha^- v^-, \\ \eta &= \beta^+ w^+ + \beta^- w^-. \end{aligned}$$

for some $\alpha^+, \alpha^-, \beta^+, \beta^- \in \mathbb{C} \otimes E$. Note that as ω and η are defined over E in the de Rham rational structure, we have $\alpha^+, \beta^+ \in \mathbb{R} \otimes E$ and $\alpha^-, \beta^- \in \mathbb{R}\, i \otimes E$. The image of $\omega \otimes \eta$ by the natural map $\phi : F^3 M_{dR}(\Pi_f \times \sigma_f)_\mathbb{R} \to M_B(\Pi_f \times \sigma_f, 2)_\mathbb{R}^+$ in the exact sequence of Prop. 2.8 is

$$\phi(\omega \otimes \eta) = (2\pi i)^2 (\alpha^+ \beta^+ v^+ \otimes w^+ + \alpha^- \beta^- v^- \otimes w^-).$$

As ϕ is injective, at least one of the two real numbers $\alpha^+ \beta^+$ and $\alpha^- \beta^-$ is non-zero.

Lemma 2.20. *Let $\circ \in \{\pm\}$ be such that $\alpha^\circ \beta^\circ \neq 0$ then*

$$v_\mathcal{B} = \frac{1}{(2\pi i)^2 \alpha^\circ \beta^\circ} v^{-\circ} \otimes w^{-\circ}$$

maps to a generator of $\mathcal{B}(\Pi_f \times \sigma_f)$ by the right hand map

$$M_B(\Pi_f \times \sigma_f, 2)^+ \to \mathrm{Ext}^1_{\mathrm{MHS}_\mathbb{R}^+}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_\mathbb{R})$$

of the short exact sequence of Prop. 2.8.

Proof. For any $\lambda^+, \lambda^- \in \mathbb{R} \otimes E$, the vector $\lambda^+ v^+ \otimes w^+ + \lambda^- v^- \otimes w^- \in M_B(\Pi_f \times \sigma_f, 2)^+$ maps to a generator $\mathcal{B}(\Pi_f \times \sigma_f)$ if and only if $(2\pi i)^2(\lambda^+\alpha^-\beta^- - \lambda^-\alpha^+\beta^+) \in E^\times$. This can be proved in exactly the same way as [Le17] Lem. 6.1. The conclusion follows. □

Let $\overline{\omega}$ be the vector in $M_B(\Pi_f \times \sigma_f)_\mathbb{C}$ obtained by applying the complex conjugation on the coefficients. Then $\overline{\omega} = \alpha^+ v^+ - \alpha^- v^-$. Let us consider

$$\overline{\omega} \otimes \eta = \alpha^+\beta^+ v^+ \otimes w^+ + \alpha^+\beta^- v^+ \otimes w^- - \alpha^-\beta^+ v^- \otimes w^+ - \alpha^-\beta^- v^- \otimes w^-.$$

Lemma 2.21. *We have* $\langle v_\mathcal{B}, \overline{\omega} \otimes \eta \rangle_{B,\mathbb{C}} \sim (2\pi i)^{-6}$.

Proof. Poincaré duality for $M_B(\Pi_f)$ is a perfect F_∞-equivariant pairing

$$M_B(\Pi_f) \otimes M_B(\Pi_f, 3) \to E.$$

As a consequence $v^\pm$ is dual to $(2\pi i)^3 v^\mp$ for this pairing, up to multiplication by an element of $E^\times$. Similarly, the vector $w^\pm$ is dual to $(2\pi i)w^\mp$ for the pairing $M_B(\sigma_f) \otimes M_B(\sigma_f, 1) \to E$, up to multiplication by an element of $E^\times$. Then the conclusion follows. □

Proposition 2.22. $\delta(\Pi_f \times \sigma_f, 3) \sim (2\pi i)^4$

Proof. Let e_1, e_2, e_3, e_4 be the vectors of a basis of $M_B(\Pi_f \times \sigma_f)$, let f_1, f_2, f_3, f_4 be the vectors of a basis of $M_{dR}(\Pi_f \times \sigma_f)$, let P be the matrix of the vectors $I_\infty(e_i)$ in the basis (f_j) and let us denote $\delta(\Pi_f \times \sigma_f) = \det P$. Let us consider the matrices $J_B = (\langle e_i, e_j \rangle_B)$ and $J_{dR} = (\langle f_i, f_j \rangle_{dR})$. Let i_∞ be the comparison isomorphism which is the right vertical line of the diagram (9). Then we have $P^t J_{dR} P = i_\infty(J_B)$ where $i_\infty(J_B)$ denotes the matrix obtained by applying i_∞ to the coefficients of J_B. Note that the matrices J_{dR} and $(2\pi i)^4 i_\infty(J_B)$ have coefficients in E. The last statement follows from the computation of the periods of the Tate motive as explained in [De79] 3. Taking determinants, we see that $\delta(\Pi_f \times \sigma_f)^2 \sim (2\pi i)^{-16}$ so that we have $\delta(\Pi_f \times \sigma_f) \sim (2\pi i)^{-8}$. As $M(\Pi_f \times \sigma_f)$ has rank four, it follows from [De79] (5.1.9) that $\delta(\Pi_f \times \sigma_f, 3) \sim (2\pi i)^{12}\delta(\Pi_f \times \sigma_f)$ and the conclusion follows. □

Corollary 2.23. *Let* ω *be a generator of* $F^2 M_{dR}(\Pi_f)$ *and let* η *be a generator of* $F^1 M_{dR}(\sigma_f)$. *Then*

$$\mathcal{Z}(\Pi_f \times \sigma_f) = (2\pi i)^8 \langle v_\mathcal{Z}, \overline{\omega} \otimes \eta \rangle_{B,\mathbb{C}} \mathcal{D}(\Pi_f \times \sigma_f).$$

Proof. Recall that, by definition

$$\mathcal{D}(\Pi_f \times \sigma_f) = (2\pi i)^{\dim_E M_B(\Pi_f \times \sigma_f, 3)^-} \delta(\Pi_f \times \sigma_f, 3)^{-1} \mathcal{B}(\Pi_f \times \sigma_f) = (2\pi i)^{-2} \mathcal{B}(\Pi_f \times \sigma_f)$$

where the last equality follows from Prop. 2.22. Let $\overline{v}_\mathcal{Z}$ and $\overline{v}_\mathcal{B}$ be the images of $v_\mathcal{Z}$ and $v_\mathcal{B}$ by the third map of the exact sequence above. There exists $\lambda \in \mathbb{R} \otimes E$ such that $\overline{v}_\mathcal{Z} = \lambda \overline{v}_\mathcal{B}$ and then

$$\mathcal{Z}(\Pi_f \times \sigma_f) = \lambda \mathcal{B}(\Pi_f \times \sigma_f) = (2\pi i)^2 \lambda \mathcal{D}(\Pi_f \times \sigma_f).$$

Pairing with $\overline{\omega} \otimes \eta$ and using Lem. 2.21, we obtain

$$\langle v_\mathcal{Z}, \overline{\omega} \otimes \eta \rangle = \lambda \langle v_\mathcal{B}, \overline{\omega} \otimes \eta \rangle \sim (2\pi i)^{-6} \lambda.$$

□

3. Zeta integrals

3.1. **The global integral.** Let $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the non-trivial additive character characterized by $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. We consider the maximal unipotent subgroup $N \subset \mathrm{GSp}(4)$ defined by

$$N = \left\{ n(x_0, x_1, x_2, x_3) = \begin{pmatrix} 1 & x_0 & & \\ & 1 & & \\ & & 1 & \\ & & -x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix}, x_0, x_1, x_2, x_3 \in \mathbb{G}_a \right\}$$

and the character $\psi_N : N(\mathbb{Q})\backslash N(\mathbb{A}) \to \mathbb{C}$ defined by $\psi_N(n(x_0, x_1, x_2, x_3)) = \psi(-x_0 - x_3)$.

Let $\Pi = \bigotimes_v' \Pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$. The global Whittaker function W_Ψ on $\mathrm{GSp}(4, \mathbb{A})$ attached to a cusp form $\Psi \in \Pi$ is

$$W_\Psi(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \Psi(ng)\psi_N(n^{-1})dn. \tag{10}$$

Assume that the global Whittaker function W_Ψ does not vanish for some cusp form $\Psi \in \Pi$. This assumption implies that for each place v, the representation Π_v of $\mathrm{GSp}(4\,\mathbb{Q}_v)$ can be realized as a subspace of

$$\{W : \mathrm{GSp}(4, \mathbb{Q}_v) \to \mathbb{C} \mid \text{smooth}\,,\ W(ng) = \psi_N(n)W(g),\ \forall (n, g) \in N(\mathbb{Q}_v) \times \mathrm{GSp}(4, \mathbb{Q}_v)\}.$$

We denote this subspace by $W(\Pi_v, \psi_v)$ and call it the local Whittaker model of Π_v. If σ is a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ and if $\Phi \in \sigma$ is a cusp form, the global Whittaker function on $\mathrm{GL}(2, \mathbb{A})$ attached to Φ is

$$W_\Phi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \Phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-x)dx.$$

It is well-known that W_Φ does not vanish for $\Phi \neq 0$. The local Whittaker model $W(\sigma_v, \psi_v)$ of σ_v is defined similarly as above.

Let us now introduce Eisenstein series on $\mathrm{GL}(2, \mathbb{A})$. Let dt_∞ be the Lebesgue measure on the additive group $\mathbb{R}$. If v is a non-archimedean place of $\mathbb{Q}$, let dt_v be the Haar measure on $\mathbb{Q}_v$ for which $\mathbb{Z}_v$ has volume one. Let $d^\times t_v$ be the Haar measure on $\mathbb{Q}_v^\times$ defined by

$$d^\times t_v = \begin{cases} \frac{dt_v}{|t_v|} & \text{if } v \text{ is archimedean,} \\ \frac{p}{p-1}\frac{dt_v}{|t_v|} & \text{if } v \text{ is } p\text{-adic.} \end{cases}$$

Let dt, resp. $d^\times t$, denote the product measure $\prod_v dt_v$ on $\mathbb{A}$, resp. $\prod_v d^\times t_v$ on $\mathbb{A}^\times$. Let $\mathcal{S}(\mathbb{A}^2)$ be the space of Schwartz-Bruhat functions on $\mathbb{A}^2$. For $\varphi \in \mathcal{S}(\mathbb{A}^2)$, let us define the global Jacquet section

$$f_\varphi(s, h_1) = |\det h_1|^s \int_{\mathbb{A}^\times} \varphi((0, t)h_1)|t|^{2s} d^\times t$$

for $h_1 \in \mathrm{GL}(2, \mathbb{A})$. This integral converges for $\mathrm{Re}(s) > 1/2$. We can define the Eisenstein series

$$E(s, h_1, f_\varphi) = \sum_{\gamma \in B(\mathbb{Q}) \backslash \mathrm{GL}(2, \mathbb{Q})} f_\varphi(s, \gamma h_1)$$

which converges absolutely and uniformly on every compact subset in $\mathrm{Re}(s) > 1$ except for the poles of $f_\varphi(s, h_1)$ and is continued to a meromorphic function on the whole complex plane. The global zeta integral we are interested in is defined as follows: for $\Psi \in \Pi$, $\Phi \in \sigma$ and $\varphi \in \mathcal{S}(\mathbb{A}^2)$ let

$$\mathcal{Z}(s, \Psi, \Phi, f_\varphi) = \int_{Z(\mathbb{A}) H(\mathbb{Q}) \backslash H(\mathbb{A})} \Psi(h) \Phi(h_2) E(s, h_1, f_\varphi) dh.$$

Here and in what follows, we regard the group H as embedded in $\mathrm{GSp}(4)$ via $p \circ \iota$, where $p : G \to \mathrm{GSp}(4)$ is the first projection. This integral converges absolutely except for the poles of the Eisenstein series and defines a meromorphic function in $s \in \mathbb{C}$.

Proposition 3.1. *The function $s \mapsto \mathcal{Z}(s, \Psi, \Phi, f_\varphi)$ is holomorphic except for possible simple poles at $s = 1$ and 0. Moreover, we have*

$$\mathrm{Res}_{s=1} \mathcal{Z}(s, \Psi, \Phi, f_\varphi) = \frac{\widehat{\varphi}(0)}{2} \int_{Z(\mathbb{A}) H(\mathbb{Q}) \backslash H(\mathbb{A})} \Psi(h) \Phi(h_2) dh$$

where

$$\widehat{\varphi}(0) = \int_{\mathbb{A}^2} \varphi(s, t) ds dt.$$

Proof. This is a particular case of [Mo09] Prop. 3.1. Note that the constant c appearing in the statement of loc. cit., Prop. 3.1 is equal to $\frac{1}{2}$ when the ground field is $\mathbb{Q}$. □

Let us suppose that the cusp forms $\Psi = \bigotimes'_v \Psi_v$ and $\Phi = \bigotimes'_v \Phi_v$ are factorizable. Then, the local multiplicity one property implies that W_Ψ and W_Φ are decomposed into a product of local Whittaker functions:

$$\begin{aligned} W_\Psi(g) &= \prod_v W_{\Psi_v}(g_v), g \in \mathrm{GSp}(4, \mathbb{A}) \\ W_\Phi(h_2) &= \prod_v W_{\Phi_v}(h_{2,v}), h_2 \in \mathrm{GL}(2, \mathbb{A}). \end{aligned}$$

Moreover, let us assume that the Schwartz-Bruhat function $\varphi = \prod_v \varphi_v$ is factorizable. Then the global Jacquet section f_φ factorizes as $f_\varphi(s, h_1) = \prod_v f_{\varphi_v}(s, h_{1,v})$ where

$$f_{\varphi_v}(s, h_{1,v}) = |\det h_{1,v}|_v^s \int_{\mathbb{Q}_v^\times} \varphi_v((0, t_v) h_{1,v}) |t_v|_v^{2s} d^\times t_v.$$

For each place v of $\mathbb{Q}$, we define the local zeta integral $\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})$ by

$$\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) = \int_{Z(\mathbb{Q}_v) N_H(\mathbb{Q}_v) \backslash H(\mathbb{Q}_v)} W_{\Psi_v}(h_v) W_{\Phi_v}(h_{2,v}) f_{\varphi_v}(s, h_{1,v}) dh_v$$

where N_H denotes the maximal unipotent subgroup of H defined as $N_H = N \cap H$.

Proposition 3.2. [Mo09] *Prop. 3.2. Suppose that* $\mathcal{Z}_\infty(s, W_{\Psi_\infty}, W_{\Phi_\infty}, f_{\varphi_\infty})$ *converges absolutely for* $\mathrm{Re}(s) > e_\infty$*. Then, the integral*

$$\int_{Z(\mathbb{A})N_H(\mathbb{A})\backslash H(\mathbb{A})} W_\Psi(h) W_\Phi(h_2) f_\varphi(s, h_1) dh$$

converges absolutely for $\mathrm{Re}(s) > \max\{3, e_\infty\}$ *and is equal to* $\mathcal{Z}(s, \Psi, \Phi, f_\varphi)$*.*

This proposition implies that

$$\mathcal{Z}(s, \Psi, \Phi, f_\varphi) = \prod_v \mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})$$

for any complex number s such that $\mathrm{Re}(s) > \max\{3, e_\infty\}$. Let V be a finite set of places satisfying the following condition: if $v \notin V$, then v is a finite place such that ψ_v is unramified, the representations Π_v, resp. σ_v, has a vector fixed by $\mathrm{GSp}(4, \mathbb{Z}_v)$, resp. $\mathrm{GL}(2, \mathbb{Z}_v)$ and $\varphi_v \in \mathcal{S}(\mathbb{Q}_v^2)$ is the characteristic function $\varphi_{v,0}$ of $\mathbb{Z}_v^2$. Fix a place $v \notin V$. Let $W_0 \in W(\Pi_v, \psi_v)$ and $W_0' \in W(\sigma_v, \psi_v)$ be the $\mathrm{GSp}(4, \mathbb{Z}_v)$-fixed local Whittaker function normalized so that $W_0(I_4) = 1$ and the $\mathrm{GL}(2, \mathbb{Z}_v)$-fixed Whittaker function normalized so that $W_0'(I_2) = 1$, respectively. Then, it is proved in section 3.4 of [Mo09] that

$$\mathcal{Z}_v(s, W_0, W_0', f_{\varphi_{v,0}}) = L(s, \Pi_v \times \sigma_v)$$

where the right-hand side is the Langlands degree eight local L-factor. As a consequence, for $\mathrm{Re}(s) > \max\{3, e_\infty\}$, we have

$$\mathcal{Z}(s, \Psi, \Phi, f_\varphi) = \prod_{v \in V} \mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) L_V(s, \Pi \times \sigma).$$

It is explained in section 2 of [So84] that for any non-archimedean place v, the integrals $\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})$ extend to rational functions in p^{-s}. We denote by $L(s, \Pi_v \times \sigma_v)$ the polynomial with constant term 1 which is the common denominator with smallest degree of the functions $\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})$.

3.2. **De Rham-Whittaker periods and ramified p-adic integrals.** In this section, we define de Rham-Whittaker periods and prove an algebraicity result for ramified p-adic integrals. De Rham-Whittaker periods are analogous to the occult period invariant introduced in [H04], when the Bessel model is replaced by the Whittaker model. In this section, we are in the situation where $L(s, \Pi \times \sigma)$ has a pole at $s = 1$. In particular, there exist σ_1 and σ_2 two cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ satisfying the statement of Prop. 2.18. This will allow us to apply [So84] Thm. 6.1 under an additional local assumption.

Let $\Pi = \Pi_\infty^W \otimes \Pi_f$ be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ whose archimedean component is the generic member Π_∞^W of P_4. Let Ψ_∞ be a highest weight vector of the minimal K_∞-type of Π_∞^W. Via Lem. 2.14, this defines a generator of $H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty^W)$, hence a $E[\mathrm{GSp}_4(\mathbb{A}_f)]$-equivariant embedding $i_{\Psi_\infty} : \Pi_f \to H^3_{dR,!}(S, \mathbb{C})$ (see (7)). By [H94] Cor. 2.3.1, we know that the $\mathbb{C}$-vector space $H^3_{dR,!}(S, \mathbb{C})$ has a E-structure $H^3_{dR,!}(S)_E$. We wil say that a vector $\Psi_f \in \Pi_f$ is arithmetic if $i_{\Psi_\infty}(\Psi_f) \in H^3_{dR,!}(S)_E$.

Proposition 3.3. *There exists $p(\Pi) \in \mathbb{C}^\times$ such that the global Whittaker functional defined by $\Psi_f \in \Pi_f \mapsto W_{\Psi_\infty} \otimes W_{\Psi_f}$ (see (10)) sends arithmetic vectors to functions of the shape $p(\Pi)^{-1}(W_{\Psi_\infty} \otimes W_f)$ where W_f takes values in $\overline{\mathbb{Q}}$.*

Proof. This follows from the unicity of the Whittaker model as proved in [GS17] Prop. 3.3.2. □

A similar result holds for $\mathrm{GL}(2, \mathbb{A})$ and we denote by $p(\sigma)$ the corresponding de Rham-Whittaker period.

Proposition 3.4. *Let v be a non-archimedean place such that σ_v is not supercuspidal. Assume that the functions W_{Ψ_v}, W_{Φ_v} and φ_v are $\overline{\mathbb{Q}}$-valued. Then*

$$\left. \frac{\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})}{L(s, \Pi_v \times \sigma_v)} \right|_{s=1} \in \overline{\mathbb{Q}}.$$

Furthermore, there exist $\overline{\mathbb{Q}}$-valued functions W_{Ψ_v}, W_{Φ_v} and φ_v such that

$$\left. \frac{\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})}{L(s, \Pi_v \times \sigma_v)} \right|_{s=1} \in \overline{\mathbb{Q}}^\times$$

and

$$\int_{\mathbb{Q}_v^2} \varphi_v(x, y)\, dxdy \in \overline{\mathbb{Q}}^\times.$$

Proof. Following [So84] p. 380, we write for $\mathrm{Re}(s)$ big enough

$$\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) = \sum_{j=-\infty}^{+\infty} A_j(W_{\Psi_v}, W_{\Phi_v}, \varphi_v) p^{js}$$

where

$$A_j(W_{\Psi_v}, W_{\Phi_v}, \varphi_v) = \int_{(h_1,h_2) \in N_H(\mathbb{Q}_p) \backslash H(\mathbb{Q}_p), |\det h_1| = p^j} W_{\Psi_v}(h_1, h_2) W_{\Phi_v}(h_2) \varphi_v((0,1)h_1) dh.$$

This integral is absolutely convergent and reduces to a finite sum because the integrated function has compact support modulo $N_H(\mathbb{Q}_p)$ in the set $\{(h_1, h_2) \in H(\mathbb{Q}_p), |\det h_1| = p^j\}$ ([CS80] Prop. 6.1) and is invariant by right translation by a sufficiently small open subgroup. Furthermore it vanishes for j big enough. In particular $\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) \in \overline{\mathbb{Q}}((p^{-s}))$ and it follows from loc. cit. Lem. 3.2 that this function extends to an element of $\overline{\mathbb{Q}}(p^{-s})$. This implies the first statement. To prove the second, note that because σ_v is not supercuspidal, we can apply loc. cit. Thm. 6.1 which claims that

$$L(s, \Pi_v \times \sigma_v) = L(s, \sigma_{1,v} \times \sigma_v) L(s, \sigma_{2,v} \times \sigma_v).$$

As $\sigma_{i,v}$ and σ_v are unitary, generic and irreducible representations of $\mathrm{GL}(2, \mathbb{Q}_v)$, it is known that the L-factors $L(s, \sigma_{i,v} \times \sigma_v)$ do not have a pole at $s = 1$ (see [Ku94] § 3.2 for example). As a consequence, if we had

$$\left. \frac{\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})}{L(s, \Pi_v \times \sigma_v)} \right|_{s=1} = 0$$

for any $\overline{\mathbb{Q}}$-valued Whittaker functions W_{Ψ_v} W_{Φ_v} and any $\overline{\mathbb{Q}}$-valued Schwartz-Bruhat function φ_v, then $\mathcal{Z}_v(1, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) = 0$ for any $\overline{\mathbb{Q}}$-valued Whittaker functions W_{Ψ_v} W_{Φ_v}

and any $\overline{\mathbb{Q}}$-valued Schwartz-Bruhat function φ_v. As these $\overline{\mathbb{Q}}$-valued functions define $\overline{\mathbb{Q}}$-structures on the spaces of $\mathbb{C}$-valued functions, and as the integral we are interested in is trilinear in $(W_{\Psi_v}, W_{\Phi_v}, \varphi_v)$, this would imply that $\mathcal{Z}_v(1, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) = 0$ for any $\mathbb{C}$-valued W_{Ψ_v}, W_{Φ_v} and φ_v. This contradicts the computation of Step II of the proof of loc. cit. Thm. 5.1, which also shows that we can assume that φ_v is the characteristic function of $p_v^m\,\mathbb{Z}_{p_v} \times (1 + p_v^m\,\mathbb{Z}_{p_v})$, for some integer m, so that

$$\int_{\mathbb{Q}_v^2} \varphi_v(x,y)\,dxdy \in \overline{\mathbb{Q}}^\times.$$

□

3.3. **Archimedean computation.** The result of this section is a particular case of [Mo09], that we wish to report for convenience of the reader. First we need to fix the notation for our archimedean representations.

Recall that we denote by σ_∞, resp. $\overline{\sigma}_\infty$, the holomorphic, resp. antiholomorphic, discrete series of $\mathrm{GL}(2,\mathbb{R})_+$ with the same central and infinitesimal characters as the trivial representation. With the notation of section 1.1 (i) of loc. cit., we have

$$\begin{aligned} \sigma_\infty|_{\mathrm{SL}(2,\mathbb{R})} &= D_2, \\ \overline{\sigma}_\infty|_{\mathrm{SL}(2,\mathbb{R})} &= D_{-2}. \end{aligned}$$

Recall also that we consider the discrete series Π_∞^W of $\mathrm{GSp}(4,\mathbb{R})_+$ with trivial central character which contains with multiplicity one the irreducible $\mathbb{C}[K_\infty]$-module $\tau_{(3,-1)}$ as a minimal K_∞-type. In other words Π_∞^W has Blattner parameter $(3,-1)$ and the discrete series $\overline{\Pi}_\infty^W$ has Blattner parameter $(1,-3)$. In the notation of the section 1.2 (i) of loc. cit. we have

$$\begin{aligned} \Pi_\infty^W|_{\mathrm{Sp}(4,\mathbb{R})} &= D_{(3,-1)}, \\ \overline{\Pi}_\infty^W|_{\mathrm{Sp}(4,\mathbb{R})} &= D_{(1,-3)}. \end{aligned}$$

Let Ψ_∞ denote a highest weight vector of the minimal K_∞-type of Π_∞^W. Then $\overline{\Psi}_\infty$, which is defined as in [Le17] Rem. 3.2, is a highest weight vector of the minimal K_∞-type of $\overline{\Pi}_\infty^W$. Let Φ_∞ denote a generator of the minimal L_∞-type of σ_∞.

Proposition 3.5. *Let $\Gamma_\mathbb{C}(s)$ denote the function $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. Let $\varphi_\infty \in \mathcal{S}(\mathbb{R}^2)$ be the Schwartz-Bruhat function defined as $\varphi_\infty(x,y) = \exp(-\pi(x^2+y^2))$. There exists $C \in \mathbb{C}^\times$ such that for any $\mathrm{Re}(s) > e_\infty$, we have*

$$\mathcal{Z}_\infty(s, W_{X_{(-1,1)}\overline{\Psi}_\infty}, W_{\Phi_\infty}, f_{\varphi_\infty}) = C \times \Gamma_\mathbb{C}(s+2)\,\Gamma_\mathbb{C}(s+1)^2\,\Gamma_\mathbb{C}(s).$$

Proof. This is a particular case of the equality (5.11) in loc. cit. with $\lambda_1 = 3$, $\lambda_2 = -1$, $l = 2$. □

4. Proofs of main results

Theorem 4.1. *Assume that for every non-archimedean place $v \in V$ the representation Π_v is obtained as a Weil lifting from $\mathrm{GSO}(2,2)$ of a pair (σ_1, σ_2). Assume that if σ_v is supercuspidal, then the contragredient $\check{\sigma}_v$ is not isomorphic to any representation of the* *form* $|\,|^z \otimes \sigma_i$, *for* $i = 1, 2$. *Then, the* E*-vector space* $\mathcal{Z}(\Pi_f \times \sigma_f)$ *is non-zero if and only if* $L(s, \Pi \times \sigma)$ *has a pole at* $s = 1$.

Proof. Let $\overline{v}_{\mathcal{Z}}$ be the image of the vector $v_{\mathcal{Z}} \in M_B(\Pi_f \times \sigma_f, 2)^+$ defined just after the proof of Lem. 2.15 by the right hand map

$$M_B(\Pi_f \times \sigma_f, 2)^+ \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$$

of the short exact sequence of Prop. 2.8. Then $\overline{v}_{\mathcal{Z}} \in \mathcal{Z}(\Pi_f \times \sigma_f)$ and it is clear that

$$\mathcal{Z}(\Pi_f \times \sigma_f) \neq 0 \iff \overline{v}_{\mathcal{Z}} \neq 0.$$

According to Lem. 2.15, we have

$$\overline{v}_{\mathcal{Z}} \neq 0 \iff \langle \overline{v}_{\mathcal{Z}}, \overline{\omega}_\Psi \otimes \eta_\Phi \rangle \neq 0 \iff \langle v_{\mathcal{Z}}, \overline{\omega}_\Psi \otimes \eta_\Phi \rangle \neq 0.$$

Let $m \geq 0$ be an integer. Let $\varphi = \prod_v \varphi_v$ be the factorizable Schwartz-Bruhat function whose archimedean component is $\varphi_\infty(x, y) = \exp(-\pi(x^2 + y^2))$, such that for every non-archimedean place $p \in V$, the p-component φ_p is the characteristic function of the subset $p^m \mathbb{Z}_p \times (1 + p^m \mathbb{Z}_p)$ of $\mathbb{Q}_p^2$ and such that for every $p \notin V$, the p-component φ_p is the characteristic function of $\mathbb{Z}_p^2$. Then $\widehat{\varphi}(0) \in \mathbb{Q}^\times$ and according to Prop. 2.17 and Prop. 3.1, we have

$$\langle v_{\mathcal{Z}}, \overline{\omega}_\Psi \otimes \eta_\Phi \rangle \sim \mathrm{Res}_{s=1}\, \mathcal{Z}\left(s, X_{(-1,1)}\Psi, \Phi, f_\varphi\right).$$

As we assume that for every non-archimedean place $v \in V$ the representation Π_v is obtained as a Weil lifting from $\mathrm{GSO}(2, 2, \mathbb{Q}_v)$, step II of the proof of [So84] Thm. 5.1 implies

$$\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) = 1$$

for suitable Whittaker functions W_{Ψ_v} and W_{Φ_v}, provided that m is big enough. Furthermore, our second assumption implies that Thm. 6.1 in loc. cit. holds. As a consequence, the local L-factor $L(s, \Pi_v \times \sigma_v)$ does not have a pole at $s = 1$, as already explained in the proof of Prop. 3.4. This implies that

$$\left.\frac{\mathcal{Z}_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})}{L(s, \Pi_v \times \sigma_v)}\right|_{s=1} \in \mathbb{C}^\times$$

for suitable Whittaker functions W_{Ψ_v} and W_{Φ_v}, provided that m is big enough. According to Prop. 3.5 the archimedean integral $\mathcal{Z}_\infty(s, W_{X_{(-1,1)}\overline{\Psi}_\infty}, W_{\Phi_\infty}, f_{\varphi_\infty})$ extends to a meromorphic function which is holomorphic and non-zero at $s = 1$. As a consequence, we can choose Ψ, Φ and φ in such a way that

$$\mathrm{Res}_{s=1}\, \mathcal{Z}\left(s, X_{(-1,1)}\Psi, \Phi, f_\varphi\right) = C' \times \mathrm{Res}_{s=1}\, L(s, \Pi \times \sigma)$$

for some $C' \in \mathbb{C}^\times$. As the poles of $L(s, \Pi \times \sigma)$ are at most simple according to [Mo09] Thm. 1.1, the conclusion follows. □

Theorem 4.2. *Assume that* $L(s, \Pi \times \sigma)$ *has a pole at* $s = 1$ *and that for any ramified non-archimedean place* $v \in V$*, the representation* σ_v *is not supercuspidal. Then*

$$\mathcal{Z}(\Pi_f \times \sigma_f) = p(\Pi \times \sigma)\, \mathrm{Res}_{s=1}\, L(s, \Pi \times \sigma) \mathcal{D}(\Pi_f \times \sigma_f).$$

Proof. Let $\Psi = \bigotimes'_v \Psi_v$ be factorizable a cusp form in the representation space of Π such that Ψ_∞ is a highest weight vector of the minimal K_∞-type normalized in such a way that the constant C appearing in Prop. 3.5 is equal to one and such that for any non-archimedean place v, the Whittaker function W_{Ψ_v} is $\overline{\mathbb{Q}}$-valued. Let $\Phi = \bigotimes'_v \Phi_v$ be a factorizable cusp form in the representation space of σ such that Φ_∞ is a generator of the minimal L_∞-type of σ_∞ and such that for any non-archimedean place v, the Whittaker function W_{Φ_v} is $\overline{\mathbb{Q}}$-valued. It follows from Prop. 2.19 and the Hodge decomposition (2) that we have the isomorphism $F^2 M_{dR}(\Pi_f)_\mathbb{C} \simeq M^{2,1}(\Pi_f)$ induced by the comparison isomorphism. Hence, by Prop. 3.3, the vector $p(\Pi)\omega_\Psi$ is a generator of $F^2 M_{dR}(\Pi_f)$ and, similarly, the vector $p(\sigma)\eta_\Phi$ is a generator of $F^1 M_{dR}(\sigma_f)$. According to Cor. 2.23, we have

$$
\begin{aligned}
\mathcal{Z}(\Pi_f \times \sigma_f) &= \pi^8 p(\Pi \times \sigma) \langle v_\mathcal{Z}, \overline{\omega}_\Psi \otimes \eta_\Phi \rangle_{B,\mathbb{C}} \mathcal{D}(\Pi_f \times \sigma_f) \\
&= \pi^8 p(\Pi \times \sigma) \widehat{\varphi}(0)^{-1} \operatorname{Res}_{s=1} \mathcal{Z}\left(s, X_{(-1,1)}\Psi, \Phi, f_\varphi\right) \mathcal{D}(\Pi_f \times \sigma_f)
\end{aligned}
$$

where in the last equality $\varphi = \prod_v \varphi_v$ is any factorizable Schwartz-Bruhat function on $\mathbb{A}^2$ whose archimedean component is $\varphi_\infty(x,y) = \exp(-\pi(x^2+y^2))$ and whose non-archimedean components at ramified places are given by Prop. 3.4. For such a choice of φ, we have $\widehat{\varphi}(0) \in \overline{\mathbb{Q}}^\times$. Hence the statement follows from the combination of Prop. 3.4 and Prop. 3.5. □

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