1. INTRODUCTION

Let $X$ be a smooth projective variety over $\mathbb{Q}$ and let $n \geq 1$ be an integer. The Hasse-Weil $L$-function $L(s, H^{2n-2}(X))$ associated to the cohomology of $X$ in degree $2n - 2$ is defined by an Euler product which is absolutely convergent for $\Re(s) > n$ and which is expected to have a meromorphic continuation to the whole complex plane, the only possible pole occuring at $n$. Let $N^{n-1}(X)$ be the $\mathbb{Q}$-vector space of cycles of codimension $n - 1$ on $X$ modulo homological equivalence.

Conjecture. (Tate)

$$- \text{ord}_{s=n} L(s, H^{2n-2}(X)) = \dim_{\mathbb{Q}} N^{n-1}(X).$$

To give the geometric interpretation of the first non-zero term $L^*(n, H^{2n-2}(X))$ in the Taylor expansion of $L(s, H^{2n-2}(X))$ at $s = n$, let $H_{\mathcal{M}}^{2n-1}(X, \mathbb{Q}(n))$ and $H_{\mathcal{D}}^{2n-1}(X/\mathbb{R}, \mathbb{R}(n))$
denote the integral part of the motivic cohomology group $H^{2n-1}_M(X, \mathbb{Q}(n))$ and the real Deligne-Beilinson cohomology respectively and let

$$r_D : H^{2n-1}_M(X, \mathbb{Q}(n))_{\mathbb{Z}} \oplus N^{n-1}(X) \rightarrow H^{2n-1}_D(X/\mathbb{R}, \mathbb{R}(n))$$

denote the thickened regulator. It is known that $H^{2n-1}_D(X/\mathbb{R}, \mathbb{R}(n))$ is a finite dimensional $\mathbb{R}$-vector space and that there is a canonical isomorphism

$$\text{Ext}^1_{\text{MHS}_R^+}(\mathbb{R}(0), H^{2n-2}_B(X, \mathbb{R}(n))) \simeq H^{2n-1}_D(X/\mathbb{R}, \mathbb{R}(n)).$$

where $\text{MHS}_R^+$ denotes the abelian category of real mixed $\mathbb{R}$-Hodge structures, whose definition is recalled in the body of the article (Def. 2.5). Let $\mathcal{D}(n, H^{2n-2}(X))$ denote the Deligne $\mathbb{Q}$-structure on the highest exterior power of $H^{2n-1}_D(X/\mathbb{R}, \mathbb{R}(n))$.

**Conjecture.** *(Beilinson)*

i. The map $r_D$ induces an isomorphism

$$(H^{2n-1}_M(X, \mathbb{Q}(n))_{\mathbb{Z}} \oplus N^{n-1}(X)) \otimes_{\mathbb{Q}} \mathbb{R} \tilde{\rightarrow} H^{2n-1}_D(X/\mathbb{R}, \mathbb{R}(n)),$$

ii. $\text{ord}_{s=n-1} L(s, H^{2n-2}(X)) = \dim_{\mathbb{Q}} H^{2n-1}_M(X, \mathbb{Q}(n))_{\mathbb{Z}},$

iii. $\det(\text{Im } r_D) = L^*(n, H^{2n-2}(X))\mathcal{D}(n, H^{2n-2}(X)).$

For an introduction to this circle of ideas and for more details on the conjectures, the reader is referred to [Ne94], [DS] and [Sc88].

To state our two main results, which are motivated by the conjectures above, let us denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$ and let us consider irreducible cuspidal automorphic representations $\Pi = \bigotimes' \Pi_v$ and $\sigma = \bigotimes' \sigma_v$ of $\text{GSp}(4, \mathbb{A})$ and $\text{GL}(2, \mathbb{A})$ respectively. Let $V$ be the finite set of places where $\Pi$ or $\sigma$ is ramified, together with the infinite place. We are interested in the $L$-function

$$L_V(s, \Pi \times \sigma) = \prod_{v \notin V} L(s, \Pi_v \times \sigma_v)$$

which is associated to the tensor product eight-dimensional representation of the Langlands dual of $\text{GSp}(4) \times \text{GL}(2)$, which is $\text{GSp}(4, \mathbb{C}) \times \text{GL}(2, \mathbb{C})$. This partial Euler product converges for $\text{Re}(s)$ big enough. To study special values of $L_V(s, \Pi \times \sigma)$ by motivic methods, we assume that the non-archimedean components $\Pi_f$ and $\sigma_f$ contribute to the cohomology of a Siegel threefold $S$ and of a modular curve $Y$ respectively, with trivial coefficient system. To use Novodvorsky’s integral representation [Mo09], [So84] we need to assume that $\Pi$ is globally generic. Examples of such automorphic representations are obtained as Weil liftings from $\text{GSO}(2, 2)$ (see [HK92]). Under the assumption that $\Pi$ is globally generic, the main result of [Mo09] applies, hence the partial Euler product $L_V(s, \Pi \times \sigma)$ can be completed to a product over all places which has a meromorphic continuation to the whole complex plane and a functional equation relating $s$ and $1-s$. Let $L(s, \Pi \times \sigma)$ denote the product completed to all non-archimedean places as defined in [Mo09], [So84]. Because $\Pi_f$ and $\sigma_f$ are cohomological, they are defined over a number field. Let $E$ denote a fixed
number over which both $\Pi_f$ and $\sigma_f$ are defined. For any $r \in \mathbb{Z}$, let $H^{1}_{B,!}(S \times Y, \mathbb{Q}(r))$ denote the image of compactly supported Betti cohomology in the cohomology without support, at infinite level, with coefficients in $\mathbb{Q}(r)$. Let

$$M_{B}(\Pi_f \times \sigma_f, r) = \text{Hom}_{E[G(A_f)]}(\Pi_f \times \sigma_f, H^{1}_{B,!}(S \times Y, \mathbb{Q}(r)) \otimes \mathbb{Q} E)$$

where we denote by $G$ the reductive group $\text{GSp}(4) \times \text{GL}(2)$. This is a pure $\mathbb{Q}$-Hodge structure of weight $4 - 2r$, with coefficients in $E$. Let us denote by $(M_{B}(\Pi_f \times \sigma_f, 2)_{\mathbb{R}} \cap M^{0,0})^+$ the vectors of $M_{B}(\Pi_f \times \sigma_f, 2)_{\mathbb{R}}$ which have Hodge type $(0,0)$ and which are fixed by complex conjugation. We have a canonical isomorphism of $\mathbb{R} \otimes \mathbb{Q} E$-modules

$$\text{Ext}^{1}_{\text{MHS}_{\mathbb{R}}}(\mathbb{R}(0), M_{B}(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \simeq (M_{B}(\Pi_f \times \sigma_f, 2)_{\mathbb{R}} \cap M^{0,0})^+$$

and we will deduce from the main result of [JSo07] that these $\mathbb{R} \otimes \mathbb{Q} E$-modules have rank one. Let $\mathcal{D}(\Pi_f \times \sigma_f)$ denote the Deligne $E$-structure on $\text{Ext}^{1}_{\text{MHS}_{\mathbb{R}}}(\mathbb{R}(0), M_{B}(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$. Let us consider the group $\text{GL}(2) \times_{\mathbb{G}_{m}} \text{GL}(2) = \{(g, h) \in \text{GL}(2) \times \text{GL}(2) \mid \det(g) = \det(h)\}$ and let

$$\iota : \text{GL}(2) \times_{\mathbb{G}_{m}} \text{GL}(2) \rightarrow \text{GSp}(4) \times \text{GL}(2)$$

be the embedding defined by

$$\iota \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = \begin{pmatrix} a & b \\ a' & b' \\ c & d \\ c' & d' \end{pmatrix}.$$

As explained in section 2.2.2 below, the morphism $\iota$ induces an embedding of the product of two modular curves into the Shimura variety $S \times Y$ whose cohomology class generates an $E$-subspace

$$\mathcal{Z}(\Pi_f \times \sigma_f) \subset (M_{B}(\Pi_f \times \sigma_f, 2)_{\mathbb{R}} \cap M^{0,0})^+.$$

Let $p(\Pi)$, resp. $p(\sigma)$, be the de Rham-Whittaker periods attached to $\Pi$, resp. $\sigma$, defined and suitably normalized in section 3.3 and let $p(\Pi \times \sigma)$ denote the product $p(\Pi)p(\sigma)$.

**Theorem 1.1.** Assume that $\Pi$ and $\sigma$ have trivial central characters and that $L(s, \Pi \times \sigma)$ has a pole at $s = 1$. Then

$$\mathcal{Z}(\Pi_f \times \sigma_f) = p(\Pi \times \sigma) \text{Res}_{s=1} L(s, \Pi \times \sigma) \mathcal{D}(\Pi_f \times \sigma_f).$$

Three remarks are in order. The first is that the poles of $L(s, \Pi \times \sigma)$ are at most simple ([Mo09] Thm. 1.1). As a consequence, the residue in Thm. 1.1 is nothing but the special value at 1. The second is that, according to [AS06] and [Sha81] Thm. 5.2, we have $L(0, \Pi \times \sigma) \neq 0$. As a consequence it follows from the second point of Beilinson’s conjecture that the integral motivic cohomology space corresponding to $\Pi \times \sigma$ is zero. This explains why we only need one cycle class in the above theorem. The third remark is that according to Beilinson’s conjecture for $\text{Res}_{s=1} L(s, \Pi \times \sigma)$, we should have $p(\Pi \times \sigma) \in E^\times$. We hope to address this problem in a future work. Let us also point out the following straightforward corollary of the theorem, in the spirit of the conjecture of Tate.

**Corollary 1.2.** Assume that $\Pi$ and $\sigma$ have trivial central characters and that $L(s, \Pi \times \sigma)$ has a pole at $s = 1$. Then $\mathcal{Z}(\Pi_f \times \sigma_f) \neq 0$. 

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Let us briefly outline the contents of the paper. In section 2.1, we review what is needed about the de Rham and Betti cohomologies of the Shimura varieties $S$ and $Y$. In section 2.2, we recall the definition of the Deligne rational structure $D(\Pi_f \times \sigma_f)$ and give the precise definition of $Z(\Pi_f \times \sigma_f)$. In section 2.3, we explain the computation of some Poincaré duality pairings that turn out to be crucial in the proofs of our main theorems. In section 3 we state and prove results about the integral representation of the $L(s, \Pi \times \sigma)$ and in section 4 we prove the two theorems stated above.

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2. Motives for $\text{GSp}(4) \times \text{GL}(2)$

2.1. Cohomology of Shimura varieties. Let us briefly recall the definition of the Shimura varieties we are interested in. Details can be found in [La05] Partie I.3. Let $I_2$ be the identity matrix of size two and let $J$ be the symplectic form on $\mathbb{Z}^4$ whose matrix in the canonical basis is

$$J = \begin{pmatrix} -I_2 & I_2 \end{pmatrix}.$$ 

The symplectic group $\text{GSp}(4)$ is defined as

$$\text{GSp}(4) = \left\{ g \in \text{GL}(4) \mid g^t J g = \nu(g) J, \, \nu(g) \in \mathbb{G}_m \right\}.$$ 

Then the map $\nu : \text{GSp}(4) \to \mathbb{G}_m$ is a character. Let $S = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, \mathbb{C}$ be the Deligne torus and let $\mathcal{H}$ be the $\text{GSp}(4, \mathbb{R})$-conjugacy class of the morphism $h : S \to \text{GSp}(4)_{\mathbb{R}}$ given on $\mathbb{R}$-points by

$$x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$ 

The pair $(\text{GSp}(4), \mathcal{H})$ is a pure Shimura datum in the sense of [P90] 2.1 and $\mathcal{H} \simeq \mathcal{H}^+ \sqcup \mathcal{H}^-$ is isomorphic to the disjoint union of the Siegel upper and lower half space of genus 2. It is easy to see that the reflex field of $(\text{GSp}(4), \mathcal{H})$ is the field of rational numbers. For any neat compact open subgroup $K$ of $\text{GSp}(4, \mathbb{A}_f)$, let us denote by $S^K$ the Shimura variety at level $K$ associated to $(\text{GSp}(4), \mathcal{H})$. It is a smooth quasi-projective variety over $\mathbb{Q}$ such that, as complex analytic varieties, we have

$$S^K(\mathbb{C}) = \text{GSp}(4, \mathbb{Q}) \backslash (\mathcal{H} \times \text{GSp}(4, \mathbb{A}_f)/K).$$

If $K = K(N) \subset \text{GSp}(4, \mathbb{Z})$ is the principal congruence subgroup of level $N \geq 3$, then

$$S^{K(N)}(\mathbb{C}) \simeq \bigcup_{(\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N) \backslash \mathcal{H}^+$$.
where $\Gamma(N) \subset \text{Sp}(4, \mathbb{Z})$ is the principal congruence subgroup of level $N$. In particular, the complex analytic varieties $S^K(\mathbb{C})$ have dimension 3. For $g \in \text{GSp}(4, \mathbb{A}_f)$ and $K, K'$ two neat compact open subgroups of $\text{GSp}(4, \mathbb{A}_f)$ such that $g^{-1}K'g \subset K$, right multiplication by $g$ on $S^K(\mathbb{C})$ descends to a morphism $[g] : S^{K'} \rightarrow S^K$ of $\mathbb{Q}$-schemes, which is finite and étale. This implies that there is an action of $\text{GSp}(4, \mathbb{A}_f)$ on the projective system $(S^K)$ indexed by neat compact open subgroups of $\text{GSp}(4, \mathbb{A}_f)$. In what follows, all compact open subgroups of $\text{GSp}(4, \mathbb{A}_f)$ and of $\text{GL}(2, \mathbb{A}_f)$ will be assumed to be neat and we will not mention this fact anymore. Similarly, we have the projective system of modular curves $(Y^L)$ indexed by the set of compact open subgroups $L \subset \text{GL}(2, \mathbb{A}_f)$ and which is endowed of an action of $\text{GL}(2, \mathbb{A}_f)$. Let $G$ denote the product $G = \text{GSp}(4) \times \text{GL}(2)$. This discussion shows that we have an action of $G(\mathbb{A}_f)$ on the projective system $(S^K \times Y^L)$ indexed by the set of pairs $(K, L)$ where $K$ is a compact open subgroup of $\text{GSp}(4, \mathbb{A}_f)$ and $L$ is a compact open subgroup of $\text{GL}(2, \mathbb{A}_f)$.

Let $H^*_{dR,c}(S^K \times Y^L)$ and $H^*_{dR}(S^K \times Y^L)$ be the de Rham cohomology with compact support and without support respectively, with complex coefficients. Let

$$H^*_d R!(S^K \times Y^L) = \text{Im}(H^*_d R,c(S^K \times Y^L) \rightarrow H^*_d R(S^K \times Y^L))$$

and let us define $H^*_d R!(S^K)$ and $H^*_d R!(Y^L)$ similarly.

**Lemma 2.1.** The graded $\mathbb{C}$-vector space $H^*_d R!(Y^L)$ vanishes outside degree 1.

**Proof.** Note that $H^0_{dR,c}(Y^L) = 0$ because the connected components of $Y^L$ are noncompact. As a consequence $H^2_{dR}(Y^L) = 0$ by Poincaré duality. This implies the statement. \hfill \Box

Let

$$H^1_{dR,!}(S \times Y) = \lim_K \lim_L H^1_{dR,!}(S^K \times Y^L)$$

where the limit is indexed by the compact open subgroups $K$ of $\text{GSp}(4, \mathbb{A}_f)$ and $L$ of $\text{GL}(2, \mathbb{A}_f)$. The action of $G(\mathbb{A}_f)$ on the projective system $(S^K \times Y^L)$ induces a structure of $\mathbb{C}[G(\mathbb{A}_f)]$-module on $H^1_{dR,!}(S \times Y)$. Let us define $H^3_{dR,!}(S)$ and $H^1_{dR,!}(Y)$ similarly as $H^4_{dR,!}(S \times Y)$. By the Künneth formula and Lem. 2.1, we have a canonical isomorphism of $\mathbb{C}[G(\mathbb{A}_f)]$-modules

$$H^1_{dR,!}(S \times Y) \simeq H^1_{dR,!}(S) \otimes H^1_{dR,!}(Y).$$

To relate $H^1_{dR,!}(S \times Y)$ to automorphic representations, let $P_3 = \{\Pi_{\infty,1}^{3,0}, \Pi_{\infty,1}^{1,0}, \Pi_{\infty,2}^{1,1}, \Pi_{\infty,2}^{1,2}\}$ denote the set of isomorphism classes of discrete series of $\text{GSp}(4, \mathbb{R})_+ = \nu^{-1}(\mathbb{R}_+^*)$ with the same central and infinitesimal characters as the trivial representation. Here $\Pi_{\infty,3}^{3,0}$ is holomorphic, $\Pi_{\infty,1}^{2,1}$ and $\Pi_{\infty,2}^{1,2}$ are generic, which means that they have a Whittaker model in the sense of section 3.1, and are determined by their Hodge types as indicated by the formulae displayed on p.8 and finally $\Pi_{\infty,2}^{1,3}$ is antiholomorphic. The fact that $P_3$ has 4 elements follows from Harish-Chandra’s classification, as explained in [Le17] Prop. 3.1 in the particular case where $k = k' = 0$, with the notation of loc. cit. In what follows, we will
denote by $\Pi^H_\infty$ and $\Pi^W_\infty$ the discrete series of $\text{GSp}(4, \mathbb{R})$ defined as
\[
\Pi^H_\infty = \text{Ind}_{\text{GSp}(4, \mathbb{R})_+}^{\text{GSp}(4, \mathbb{R})} \Pi_\infty^{3,0} = \text{Ind}_{\text{GSp}(4, \mathbb{R})_+}^{\text{GSp}(4, \mathbb{R})} \Pi_\infty^{0,3}, \\
\Pi^W_\infty = \text{Ind}_{\text{GSp}(4, \mathbb{R})_+}^{\text{GSp}(4, \mathbb{R})} \Pi_\infty^{2,1} = \text{Ind}_{\text{GSp}(4, \mathbb{R})_+}^{\text{GSp}(4, \mathbb{R})} \Pi_\infty^{1,2}.
\]
In particular, we have
\[
\Pi^H_\infty|_{\text{GSp}(4, \mathbb{R})_+} = \Pi_\infty^{3,0} \oplus \Pi_\infty^{0,3}, \\
\Pi^W_\infty|_{\text{GSp}(4, \mathbb{R})_+} = \Pi_\infty^{2,1} \oplus \Pi_\infty^{1,2}.
\]

Similarly, let $P_2 = \{ \sigma^{1,0}_\infty, \sigma^{0,1}_\infty \}$ denote the set of isomorphism classes of discrete series of $\text{GL}(2, \mathbb{R})_+$, which follows from (1) and the analogous statements for $\sigma^{1,0}_\infty$ is holomorphic and $\sigma^{0,1}_\infty$ is antiholomorphic. Let $\sigma_\infty$ denote the induced representation
\[
\sigma_\infty = \text{Ind}_{\text{GL}(2, \mathbb{R})_+}^{\text{GL}(2, \mathbb{R})} \sigma^{1,0}_\infty = \text{Ind}_{\text{GL}(2, \mathbb{R})_+}^{\text{GL}(2, \mathbb{R})} \sigma^{0,1}_\infty
\]
so that $\sigma_\infty|_{\text{GL}(2, \mathbb{R})_+} = \sigma^{1,0}_\infty \oplus \sigma^{0,1}_\infty$. Let $\mathfrak{sp}_4$ and $\mathfrak{gl}_2$ denote the complex Lie algebras of $\text{GSp}(4)$ and $\text{GL}(2)$ respectively and let $K_\infty$ and $L_\infty$ denote respectively the group $\mathbb{R}^+ \times U(2, \mathbb{R})$, regarded as a maximal compact modulo the center subgroup of $\text{GSp}(4, \mathbb{R})_+$, and the group $\mathbb{R}^+ \times \text{SO}(2, \mathbb{R})$, regarded as a maximal compact modulo the center subgroup of $\text{GL}(2, \mathbb{R})_+$, in the standard way. Using the relative Lie algebra cohomology groups, we have the following result.

**Proposition 2.2.** There is a canonical $G(\mathbb{A}_f)$-equivariant isomorphism
\[
H^4_{dR, !}(S \times Y) \simeq \bigoplus_{\Pi, \sigma} \bigoplus_{\Pi, \sigma} H^3(\mathfrak{sp}_4, K_\infty, \Pi_\infty)^{m(\Pi)} \otimes H^1(\mathfrak{gl}_2, L_\infty, \sigma_\infty)^{m(\sigma)} \otimes (\Pi_f \times \sigma_f)
\]
where the direct sum is indexed by equivalence classes of cuspidal automorphic representations $\Pi = \Pi_\infty \otimes \Pi_f$ of $\text{GSp}(4, \mathbb{A})$ and $\sigma = \sigma_\infty \otimes \sigma_f$ of $\text{GL}(2, \mathbb{A})$ such that $\Pi_\infty|_{\text{GSp}(4, \mathbb{R})_+} \in P_4$ and $\sigma_\infty|_{\text{GL}(2, \mathbb{R})_+} \in P_2$ and where $m(\Pi)$ and $m(\sigma)$ denote the cuspal multiplicities of $\Pi$ and $\sigma$ respectively.

**Proof.** This follows from (1) and the analogous statements for $H^3_{dR, !}(S)$, which follows from [Le17] (8) and (9), and for $H^1_{dR, !}(Y)$, which follows from [Sch11] Lem. 12.3 once noticed that for the smooth curves $Y^L$, one has a canonical isomorphism $H^1_{dR, !}(Y^L) \simeq H^1_{(2)}(Y^L)$, where $H^1_{(2)}(Y^L)$ denotes $L^2$-cohomology.

Let $\mathfrak{k}$ denote the complex Lie algebra of $K_\infty$ and $\mathfrak{l}$ denote the complex Lie algebra of $L_\infty$.

**Proposition 2.3.** For any $\Pi^{0,q}_\infty \in P_4$, resp. $\sigma^{r,s}_\infty \in P_2$, we have
\[
H^3(\mathfrak{sp}_4, K_\infty, \Pi^{0,q}_\infty) = \text{Hom}_{K_\infty} \left( \bigwedge_3 \mathfrak{sp}_4/\mathfrak{k}, \Pi^{p,q}_\infty \right), \\
H^1(\mathfrak{gl}_2, L_\infty, \sigma^{r,s}_\infty) = \text{Hom}_{L_\infty} (\mathfrak{gl}_2/\mathfrak{l}, \sigma^{r,s}_\infty)
\]
and these $\mathbb{C}$-vector spaces are one-dimensional.
Proof. According to [BoW80] II. §3, Prop. 3.1, for any Π^{\mathfrak{q},q}_\infty \in P_4, \text{ resp. } \sigma^{\mathfrak{q},q}_\infty \in P_2, the \((\mathfrak{sp}_4, K_\infty)\)-complex of Π^{\mathfrak{q},q}_\infty, \text{ resp. } the \((\mathfrak{gl}_2, L_\infty)\)-complex of \sigma^{\mathfrak{q},q}_\infty, has zero differential. This implies the two equalities in the statement of the Proposition. The statement about the dimensions is a particular case of loc. cit. II. Thm. 5.3.

Let us fix a cuspidal automorphic representation \( \Pi = \Pi_\infty \otimes \Pi_f \), \text{ resp. } \( \sigma = \sigma_\infty \otimes \sigma_f \), of \( \text{GSp}(4, A) \), \text{ resp. } \( \text{GL}(2, A) \). Note that, by definition of the relative Lie algebra cohomology complex

\[
H^3(\mathfrak{sp}_4, K_\infty, \Pi_\infty) = H^3(\mathfrak{sp}_4, K_\infty, \Pi_\infty|_{\text{GSp}(4, \mathbb{R})_+}).
\]

Assume that \( \Pi_\infty|_{\text{GSp}(4, \mathbb{R})_+} \in P_4, \text{ resp. } \sigma_\infty|_{\text{GL}(2, \mathbb{R})_+} \in P_2 \). As the non-archimedean part \( \Pi_f \) of \( \Pi \) contributes to coherent cohomology, it is defined over a number field (see [BHR94] for more details). The fact that the analogous statement for \( \sigma_f \) is true is proved in [Wa85]. Let us denote by \( E \) a fixed number field over which both \( \Pi_f \) and \( \sigma_f \) are defined. Let \( H^4_B(S \times Y, \mathbb{Q}) \) denote the image of the Betti cohomology with compact support in the Betti cohomology without support, with rational coefficients. We will use the following notation

\[
M_B(\Pi_f \times \sigma_f) = \text{Hom}_{E[G(A_f)]}(\Pi_f \times \sigma_f, H^4_B(S \times Y, \mathbb{Q}) \otimes_{\mathbb{Q}} E).
\]

In the next result, we need to assume that \( \Pi \) is globally generic. Let us recall that this means that there exists \( \Psi \in \Pi \) such that the function \( W_\Psi : \text{GSp}(4, A) \to \mathbb{C} \) defined by (11) in section 3.1 is not identically zero.

**Proposition 2.4.** Assume that \( \Pi \) is globally generic. Then, the natural inclusion of inner cohomology in usual cohomology \( H^4_B(S \times Y, \mathbb{Q}) \subset H^4_B(S \times Y, \mathbb{Q}) \) induces an isomorphism

\[
M_B(\Pi_f \times \sigma_f) = \text{Hom}_{E[G(A_f)]}(\Pi_f \times \sigma_f, H^4_B(S \times Y, \mathbb{Q}) \otimes_{\mathbb{Q}} E).
\]

**Proof.** As \( \Pi \) is globally generic, it is not CAP according to [PSS87] Thm. 1.1. Hence, we can apply [We09] Thm. 1.1 which implies that

\[
\text{Hom}_{E[\mathfrak{sp}(4, A)]}(\Pi_f, H^3_B(S, \mathbb{Q}) \otimes_{\mathbb{Q}} E) = \text{Hom}_{E[\mathfrak{gsp}(4, A)]}(\Pi_f, H^3_B(S, \mathbb{Q}) \otimes_{\mathbb{Q}} E)
\]

\[
\text{Hom}_{E[\mathfrak{sp}(4, A)]}(\Pi_f, H^3_B(S, \mathbb{Q}) \otimes_{\mathbb{Q}} E) = 0.
\]

Note that [We09] Thm. 1.1 assumes that \( \Pi_\infty \simeq \Pi_\infty^H \), but that this fact is not used in the proof and so the same proof is true when \( \Pi_\infty \simeq \Pi_\infty^W \). Furthermore, as there are no CAP representations for \( \text{GL}(2) \) the same argument as in the proof of [We09] Thm. 1.1 shows that

\[
\text{Hom}_{E[\text{GL}(2, A)]}(\Pi_f \times \sigma_f, H^1_B(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} E) = \text{Hom}_{E[\text{GL}(2, A)]}(\Pi_f \times \sigma_f, H^1_B(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} E).
\]

Hence, by the Künneth formula, we have

\[
\text{Hom}_{E[G(A_f)]}(\Pi_f \times \sigma_f, H^4_B(S \times Y, \mathbb{Q}) \otimes_{\mathbb{Q}} E) = M_B(\Pi_f \times \sigma_f).
\]

Let \( M_B(\Pi_f) \) and \( M_B(\sigma_f) \) be defined similarly as \( M_B(\Pi_f \times \sigma_f) \) via the Betti cohomology of \( S \) and \( Y \) respectively. Then, the Künneth formula implies that

\[
M_B(\Pi_f \times \sigma_f) = M_B(\Pi_f) \otimes_E M_B(\sigma_f).
\]
According to Prop. 2.2, Prop. 2.3 and the comparison isomorphism between de Rham and Betti cohomology, these are finite dimensional $\mathbb{Q}$-vector spaces endowed with a $\mathbb{Q}$-linear action of $E$ and additional structures as follows. Let $M_B(\Pi_f)_C$ and $M_B(\sigma_f)_C$ denote the vector spaces obtained after extending the scalars from $\mathbb{Q}$ to $\mathbb{C}$. Then we have the Hodge decompositions

\begin{align*}
M_B(\Pi_f)_C &= M(\Pi_f)^{3,0} \oplus M(\Pi_f)^{2,1} \oplus M(\Pi_f)^{1,2} \oplus M(\Pi_f)^{0,3}, \\
M_B(\sigma_f)_C &= M(\sigma_f)^{1,0} \oplus M(\sigma_f)^{0,1}
\end{align*}

where

\begin{align*}
M(\Pi_f)^{3,0} &= \bigoplus_{\sigma : E \rightarrow C} H^3(\mathfrak{g} \mathfrak{sp}_4, K_\infty, \Pi_\infty^{3,0})^m(\Pi_\infty^{3,0} \otimes \Pi_f), \\
M(\Pi_f)^{2,1} &= \bigoplus_{\sigma : E \rightarrow C} H^3(\mathfrak{g} \mathfrak{sp}_4, K_\infty, \Pi_\infty^{2,1})^m(\Pi_\infty^{2,1} \otimes \Pi_f), \\
M(\Pi_f)^{1,2} &= \bigoplus_{\sigma : E \rightarrow C} H^3(\mathfrak{g} \mathfrak{sp}_4, K_\infty, \Pi_\infty^{1,2})^m(\Pi_\infty^{1,2} \otimes \Pi_f), \\
M(\Pi_f)^{0,3} &= \bigoplus_{\sigma : E \rightarrow C} H^3(\mathfrak{g} \mathfrak{sp}_4, K_\infty, \Pi_\infty^{0,3})^m(\Pi_\infty^{0,3} \otimes \Pi_f), \\
M(\sigma_f)^{1,0} &= \bigoplus_{\sigma : E \rightarrow C} H^1(\mathfrak{gl}_2, L_\infty, \sigma_\infty^{1,0})^m(\sigma_\infty^{1,0} \otimes \sigma_f), \\
M(\sigma_f)^{0,1} &= \bigoplus_{\sigma : E \rightarrow C} H^1(\mathfrak{gl}_2, L_\infty, \sigma_\infty^{0,1})^m(\sigma_\infty^{0,1} \otimes \sigma_f).
\end{align*}

Furthermore $M_B(\Pi_f \times \sigma_f)$ has the tensor product Hodge structure

\begin{align*}
M_B(\Pi_f \times \sigma_f)_C = M^{4,0}_B \oplus M^{3,1}_B \oplus M^{2,2}_B \oplus M^{1,3}_B \oplus M^{0,4}_B.
\end{align*}

The following definition is taken from [Be86] §7.

**Definition 2.5.** Let $A$ be a subring of $\mathbb{R}$. A real mixed $A$-Hodge structure is a mixed $A$-Hodge structure whose underlying $A$-vector space is endowed with an involution $F_\infty$ stabilizing the weight filtration and whose $\mathbb{C}$-antilinear complexification $\overline{F}_\infty$ stabilizes the Hodge filtration.

Let $\text{MHS}^+_A$ denote the abelian category of real mixed $A$-Hodge structures.

**Definition 2.6.** Let $F$ be a ring and let $A$ be a subring of $\mathbb{R}$. A real mixed $A$-Hodge structure with coefficients in $F$ is a pair $(M, s)$ where $M$ is an object of $\text{MHS}^+_A$ and $s : F \rightarrow \text{End}_{\text{MHS}^+_A}(M)$ is a ring homomorphism.

For any ring $F$, let $\text{MHS}^+_F$ denote the abelian category of real mixed $\mathbb{R}$-Hodge structures with coefficients in $F$. The proof of the following result is straightforward.

**Proposition 2.7.** Let $F_\infty$ be the involution on $M_B(\Pi_f \times \sigma_f)$ induced by the complex conjugation on $S(\mathbb{C}) \times Y(\mathbb{C})$. Then $(M_B(\Pi_f \times \sigma_f), F_\infty)$ is an object of $\text{MHS}^+_Q,E$ which is pure of weight 4.
Furthermore, it follows from [H94] Cor. 2.3.1 that there exists a filtered $E$-vector space $(M_{dR}(\Pi_f \times \sigma_f), F^n M_{dR}(\Pi_f \times \sigma_f))$ and a comparison isomorphism
\begin{equation}
I_\infty : M_B(\Pi_f \times \sigma_f)_C \rightarrow M_{dR}(\Pi_f \times \sigma_f)_C
\end{equation}
such that the Hodge filtration of $M_B(\Pi_f \times \sigma_f)$, defined as
\[
F^p_C = \bigoplus_{p' \geq p} M^{p',q},
\]
satisfies $I_\infty(F^p_C) = F^p M_{dR}(\Pi_f \times \sigma_f)_C$. A similar statement holds for $M_B(\Pi_f)$ and $M_B(\sigma_f)$.

2.2. The Deligne rational structure and the cycle class.

2.2.1. The Deligne rational structure. Let $\Pi = \Pi_\infty \otimes \Pi_f$ and $\sigma = \sigma_\infty \otimes \sigma_f$ be irreducible cuspidal automorphic representations of $\text{GSp}(4, A)$ and $\text{GL}(2, A)$ respectively. Assume that $\Pi_\infty|_{\text{GSp}(4, R)_+} \in P_1$ and $\sigma_\infty|_{\text{GL}(2, R)_+} \in P_2$. For any integer $n$, let $M_B(\Pi_f \times \sigma_f, n)$ denote the object of $\text{MHS}_{Q, E}$ defined as
\[
M_B(\Pi_f \times \sigma_f, n) = M_B(\Pi_f \times \sigma_f) \otimes Q(n)
\]
where $Q(n)$ is the $n$-th tensor power of the Tate object. Let $M_B(\Pi_f \times \sigma_f, n)_{\pm}$ denote the subspace of $M_B(\Pi_f \times \sigma_f, n)$ where $F_\infty$ acts as $\pm 1$. The comparison isomorphism $I^{-1}_\infty$ (see (5)) between de Rham and Betti cohomology, sends the real structure $M_{dR}(\Pi_f \times \sigma_f)_R$ of $M_{dR}(\Pi_f \times \sigma_f)_C$ to the real structure $M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}} \oplus M_B(\Pi_f \times \sigma_f)^-_{\mathbb{R}}(-1)$ of $M_B(\Pi_f \times \sigma_f)_C$, where $M_B(\Pi_f \times \sigma_f)^-_{\mathbb{R}}(-1)$ simply denotes the sub-$\mathbb{R} \otimes Q E$-module $M_B(\Pi_f \times \sigma_f)^-_{\mathbb{R}} \otimes i \mathbb{R}$ of $M_B(\Pi_f \times \sigma_f)^-_{\mathbb{C}}$. In particular, we have a natural $\mathbb{R} \otimes Q E$-linear map
\[
F^3 M_{dR}(\Pi_f \times \sigma_f)_R \rightarrow M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}}
\]
defined as the composition of the natural inclusion $F^3 M_{dR}(\Pi_f \times \sigma_f)_R \subset M_{dR}(\Pi_f \times \sigma_f)_R$, of $I^{-1}_\infty$ and of the natural projection $M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}} \oplus M_B(\Pi_f \times \sigma_f)^-_{\mathbb{R}}(-1) \rightarrow M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}}$. Composing with the canonical isomorphism $M_B(\Pi_f \times \sigma_f)^+_{\mathbb{R}} \simeq M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}}$ given by multiplication by $(2\pi i)^2$, we obtain the natural map
\[
F^3 M_{dR}(\Pi_f \times \sigma_f)_R \rightarrow M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}}.
\]
Proposition 2.8. We have the following canonical short exact sequence of $\mathbb{R} \otimes Q E$-modules
\[
0 \rightarrow F^3 M_{dR}(\Pi_f \times \sigma_f)_R \rightarrow M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}} \rightarrow \text{Ext}^1_{\text{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_R) \rightarrow 0
\]
where the second map is the map defined above.

Proof. As $M_B(\Pi_f \times \sigma_f, 3)$ is pure of weight $-2$, it follows for example from [Le17] Lem. 4.11 (see also [Ne94] sections 2.2 and 2.3) that we have the canonical short exact sequence
\[
0 \rightarrow F^0 M_{dR}(\Pi_f \times \sigma_f, 3)_R \rightarrow M_B(\Pi_f \times \sigma_f, 3)^+_{\mathbb{R}}(-1) \rightarrow \text{Ext}^1_{\text{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_R) \rightarrow 0
\]
where the second map is defined similarly as above. Furthermore, we have the following canonical isomorphisms $F^3 M_{dR}(\Pi_f \times \sigma_f)_R \simeq F^0 M_{dR}(\Pi_f \times \sigma_f, 3)_R$ and $M_B(\Pi_f \times \sigma_f, 2)^+_{\mathbb{R}} \simeq M_B(\Pi_f \times \sigma_f, 3)^-_{\mathbb{R}}(-1)$. The conclusion follows. \qed
Definition 2.12. The Deligne \( E \)-structure is independent of the choice of the basis up to right multiplication by an element of \( E \). As \( \dim \) computed in basis defined on \( \mathfrak{b} \), \( \text{det} \) is defined as

\[
\text{det} = \text{det}^{\text{max}}.
\]

The Beilinson \( E \)-structure is an isomorphism. The following definition is taken from [DS] (2.3.2).

Proposition 2.10. The ranks of the \( \mathbb{R} \otimes \mathbb{Q} E \)-modules \( F^3 M_{dr}(\Pi_f \times \sigma_f)_{\mathbb{R}}, M_B(\Pi_f \times \sigma_f, 2)_{\mathbb{R}}^{\perp} \) and \( \text{Ext}^1_{\MHS_{\mathbb{R}}}^{\mathbb{R}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \) are finite and equal to \( 2m(\Pi_H \otimes \Pi_f) + m(\Pi_E \otimes \Pi_f), 2m(\Pi_{\infty}^2 \otimes \Pi_f) + 2m(\Pi_{\infty}^W \otimes \Pi_f) \) and \( m(\Pi_{\infty}^W \otimes \Pi_f) \) respectively.

Proof. The existence of the short exact sequence of Prop. 2.8 implies that it is enough to prove

\[
\text{rk} F^3 M_{dr}(\Pi_f \times \sigma_f)_{\mathbb{R}} = 2m(\Pi_H \otimes \Pi_f) + m(\Pi_E \otimes \Pi_f),
\]

\[
\text{rk} M_B(\Pi_f \times \sigma_f, 2)_{\mathbb{R}}^{\perp} = 2m(\Pi_H \otimes \Pi_f) + 2m(\Pi_{\infty}^W \otimes \Pi_f).
\]

According to [Sh74] Thm. 5.5, we have \( m(\sigma_{\infty} \otimes \sigma_f) = 1 \). Hence both statements are direct consequences of Prop. 2.3 and equality (4).

The short exact sequence of Prop. 2.8 induces the canonical isomorphism of rank one \( \mathbb{R} \otimes \mathbb{Q} E \)-modules

\[
\text{det}_{\mathbb{R} \otimes \mathbb{Q} E} F^3 M_{dr}(\Pi_f \times \sigma_f)_{\mathbb{R}} \otimes_{\mathbb{R} \otimes \mathbb{Q} E} \text{det}_{\mathbb{R} \otimes \mathbb{Q} E} \text{Ext}^1_{\MHS_{\mathbb{R}}^{\mathbb{R}}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})
\]

\[
\simeq \text{det}_{\mathbb{R} \otimes \mathbb{Q} E} M_B(\Pi_f \times \sigma_f, 2)_{\mathbb{R}}^{\perp},
\]

where we denote by \( \text{det} \) the highest exterior power. Let \( \text{det} E F^3 M_{dr}(\Pi_f \times \sigma_f)^{\vee} \) denote the \( E \)-module dual of \( \text{det} E F^3 M_{dr}(\Pi_f \times \sigma_f) \). The evaluation map

\[
\text{det} E F^3 M_{dr}(\Pi_f \times \sigma_f) \otimes E \text{det} E F^3 M_{dr}(\Pi_f \times \sigma_f)^{\vee} \rightarrow E
\]

is an isomorphism. The following definition is taken from [DS] (2.3.2).

Definition 2.11. The Beilinson \( E \)-structure on \( \text{det}_{\mathbb{R} \otimes \mathbb{Q} E} \text{Ext}^1_{\MHS_{\mathbb{R}}}^{\mathbb{R}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \) is defined as

\[
\mathcal{B}(\Pi_f \times \sigma_f) = \text{det} E F^3 M_{dr}(\Pi_f \times \sigma_f)^{\vee} \otimes E \text{det} E M_B(\Pi_f \times \sigma_f, 2)_{\mathbb{R}}^{\perp}.
\]

Let us denote by \( \delta(\Pi_f \times \sigma_f, 3) \) the determinant of the comparison isomorphism

\[
M_B(\Pi_f \times \sigma_f, 3) \rightarrow M_{dr}(\Pi_f \times \sigma_f, 3)
\]

computed in basis defined on \( E \) on both sides. Then \( \delta(\Pi_f \times \sigma_f, 3) \) is an element of \( (\mathbb{C} \otimes E)^{\times} \), which, as \( \dim E M_B(\Pi_f \times \sigma_f)^{-} \) is even, belongs to \( (\mathbb{R} \otimes \mathbb{Q} E)^{\times} \) (see [De79] p. 320) and which is independent of the choice of the basis up to right multiplication by an element of \( E^{\times} \).

Definition 2.12. The Deligne \( E \)-structure on \( \text{det}_{\mathbb{R} \otimes \mathbb{Q} E} \text{Ext}^1_{\MHS_{\mathbb{R}}}^{\mathbb{R}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}}) \) is defined as

\[
\mathcal{D}(\Pi_f \times \sigma_f) = (2\pi i)^{\dim E M_B(\Pi_f \times \sigma_f, 3)}^{-} \delta(\Pi_f \times \sigma_f, 3)^{-1} \mathcal{B}(\Pi_f \times \sigma_f).
\]
2.2.2. **The cycle class.** The reductive group $G$ contains
\[ H = \text{GL}(2) \times_{\mathbb{G}_m} \text{GL}(2) = \{ (g, h) \in \text{GL}(2) \times \text{GL}(2) \mid \det(g) = \det(h) \} \]
as a closed subgroup via the embedding $\iota : H \to G$ defined by
\[
\iota \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = \begin{pmatrix} a & b \\ a' & b' \\ c & d \\ c' & d' \end{pmatrix}.
\]
For any integer $N \geq 3$, this group homomorphism induces the closed embedding
\[
\iota^N : Y(N) \times_{\mathbb{Q}(\mu_N)} Y(N) \to S(N) \times Y(N)
\]
between the corresponding Shimura varieties. Let us fix an integer $N$ and let
\[
Z_N \in H^1_B(S(N) \times Y(N), \mathbb{Q}(2))
\]
be the cohomology class of the image of $\iota^N$, which we regard as an element of the $E[G(\mathbb{A}_f)]$-module $H^1_B(S \times Y, \mathbb{Q}(2)) \otimes_{\mathbb{Q}} E = \varinjlim M H^1_B(S(M) \times Y(M), \mathbb{Q}(2)) \otimes_{\mathbb{Q}} E$. Let
\[
\tilde{Z}(\Pi_f \times \sigma_f) = \text{Hom}_{E[G(\mathbb{A}_f)]}(\Pi_f \times \sigma_f, E[G(\mathbb{A}_f)] Z_N).
\]
This is a sub-$E$-vector space of $\text{Hom}_{E[G(\mathbb{A}_f)]}(\Pi_f \times \sigma_f, H^1_B(S \times Y, \mathbb{Q}(2)) \otimes_{\mathbb{Q}} E)$. As we assume that $\Pi$ is globally generic, the latter coincides with $M_B(\Pi_f \times \sigma_f, 2)$ (see Prop. 2.4). Note that as the cycle $Y(N) \times_{\mathbb{Q}(\mu_N)} Y(N)$ is defined over $\mathbb{Q}$, we have $\tilde{Z}(\Pi_f \times \sigma_f) \subset M_B(\Pi_f \times \sigma_f, 2)^+$. We shall denote again by
\[
Z(\Pi_f \times \sigma_f) \subset \text{Ext}^1_{\text{MHS}_{\mathbb{R}}^+}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3))
\]
the sub-$E$-vector space defined as the image of $\tilde{Z}(\Pi_f \times \sigma_f)$ by the natural map
\[
M_B(\Pi_f \times \sigma_f, 2)^+ \to M_B(\Pi_f \times \sigma_f, 2)^+ \to \text{Ext}^1_{\text{MHS}_{\mathbb{R}}^+}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)).
\]
where the first map is the canonical inclusion and the second is the third map in the short exact sequence of Prop. 2.8.

2.3. **Computation of Poincaré duality pairings.** From now on, we fix two irreducible cuspidal automorphic representations $\Pi = \Pi_{\infty} \otimes \Pi_f$ and $\sigma = \sigma_{\infty} \otimes \sigma_f$ of $\text{GSp}(4, \mathbb{A}_f)$ and $\text{GL}(2, \mathbb{A}_f)$ respectively. We assume that $\Pi_{\infty} |_{\text{GSp}(4, \mathbb{R})_{+}} \in P_4$, that $\sigma_{\infty} |_{\text{GL}(2, \mathbb{R})_{+}} \in P_2$ and that $\Pi$ is globally generic. This implies that $\Pi_{\infty} \simeq \Pi_{\infty}^W$. The following result will be useful later.

**Proposition 2.13.** [JS07] \[ m(\Pi_{\infty}^W \otimes \Pi_f) = 1. \]
2.3.1. Cohomological interpretation of a period integral. The previous result implies that the $\mathbb{C} \otimes E$-module $M(\Pi_f)^{2,1}$ has rank one. Let us explain how to attach a generator of this module to certain cusp forms in the representation space of $\Pi$. We will freely use some standard results and notations from the section 3.1 of [Le17]. In particular, if $(k, k') \in \mathbb{Z}^2$ is a pair of integers such that $k \geq k'$, let $\tau_{(k,k')}$ denote the irreducible $\mathbb{C}[K_\infty]$-module of highest weight $(k, k')$ with the conventions of loc. cit. Then, the generic member $\Pi_\infty^{2,1}$ of the discrete series $L$-packet $P_4$ contains with multiplicity one $\tau_{(3, -1)}$ as a minimal $K_\infty$-type (see loc. cit. Prop. 3.1). Furthermore, we have the Cartan decomposition $\mathfrak{gsp}_4 = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where

\[
\mathfrak{p}^\pm = \left\{ \left( \begin{array}{cc} Z & \pm iZ \\ \pm iZ & -Z \end{array} \right) \right\}, \quad Z' = Z \in \mathfrak{gl}_2.
\]

For each symmetric matrix $Z \in \mathfrak{gl}_2$, define the element $p_\pm(Z)$ of $\mathfrak{gsp}_4$ by

\[
p_\pm(Z) = \left( \begin{array}{cc} Z & \pm iZ \\ \pm iZ & -Z \end{array} \right).
\]

Let $X_{(\alpha_1, \alpha_2)} \in \mathfrak{gsp}_4$ be defined as

\[
X_{\pm(2,0)} = p_\pm \left( \begin{array}{c} 1 \\ \end{array} \right), \quad X_{\pm(1,1)} = p_\pm \left( \begin{array}{c} 1 \\ \end{array} \right), \quad X_{\pm(0,2)} = p_\pm \left( \begin{array}{c} 1 \\ \end{array} \right).
\]

It follows from an easy computation that $X_{(\alpha_1, \alpha_2)}$ is a root vector corresponding to the root $(\alpha_1, \alpha_2)$ with the conventions of loc. cit. Recall from Prop. 2.3 that $H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty^{2,1}) = \text{Hom}_{K_\infty} \left( \bigwedge^3 \mathfrak{gsp}_4 / \mathfrak{k}, \Pi_\infty^{2,1} \right)$.

Lemma 2.14. Let $\Psi_\infty \in \Pi_\infty^{2,1}$ be a highest weight vector of the minimal $K_\infty$-type. Then, there exists a unique element $\Omega_{\Psi_\infty} \in H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty^{2,1})$ such that

\[
\Omega_{\Psi_\infty}(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}) = \Psi_\infty.
\]

Proof. We have

\[
\bigwedge^3 \mathfrak{gsp}_4 / \mathfrak{k} = \bigoplus_{p+q=3} \mathfrak{p}^+ \otimes \bigwedge^q \mathfrak{p}^-.
\]

and by a weight computation, we find

\[
\bigwedge^2 \mathfrak{p}^+ \otimes \mathfrak{p}^- = \tau_{(3,-1)} \oplus \tau_{(2,0)} \oplus \tau_{(1,1)}
\]

as $\mathbb{C}[K_\infty]$-modules. As a consequence, the existence of $\Omega_{\Psi_\infty}$ follows from the fact that the vector $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$, as $\Psi_\infty$, is a vector of highest weight $(3, -1)$. Its unicity follows from the fact that, according to Prop. 2.3, the $\mathbb{C}$-vector space $H^3(\mathfrak{gsp}_4, K_\infty, \Pi_\infty^{2,1})$ has dimension one.

Let $\Psi = \Psi_\infty \otimes \Psi_f$ be a cusp form in the space of $\Pi = \Pi_\infty^{W} \otimes \Pi_f$. Note that we have an equivalence $\Pi_\infty^{W}|_{\text{GSp}(4, \mathbb{R})_+} \simeq \Pi_\infty^{2,1} \oplus \Pi_\infty^{3,2}$. Assume that $\Psi_\infty$ is a highest weight vector.
of the minimal $K_\infty$-type of $\Pi_{1,2}^\infty$ and such that $\Psi_f$ is invariant by the principal level $N$ congruence subgroup of $GSp(4, \hat{\mathbb{Z}})$. Define

$$\Omega_{\Psi} = (\Omega_{\Psi, \infty})_{\sigma : E \to \mathbb{C}} \otimes \Psi_f \in H^3_{BD,t}(S, \mathbb{C}) \otimes_{\mathbb{Q}} E.$$ 

Let

$$\omega_{\Psi} \in M_B(\Pi_f)_C = \text{Hom}_{E[GSp(4, \mathbb{A})]}(\Pi_f, H^3_{BD,t}(S, \mathbb{C}) \otimes_{\mathbb{Q}} E)$$

be the unique element sending $\Psi_f$ to $\Omega_{\Psi}$. Then $\omega_{\Psi} \in M^{2,1}(\Pi_f)$. Let $\Psi_{\infty} \in \Pi_{1,2}^\infty$ be defined as in [Le17] Rem. 3.2. It follows from loc. cit., Prop 3.13 that $\omega_{\Psi} \in M^{1,2}(\Pi_f)$ corresponds to the element

$$\Omega_{\Psi, \infty} \in H^3(\mathfrak{sp}_4, K_\infty, \Pi_{1,2}^\infty) = \text{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{sp}_4/\mathfrak{l}, \Pi_{1,2}^\infty\right)$$

classified by the identity

$$\Omega_{\Psi, \infty}(X_{(0,-2)} \wedge X_{(-1,-1)} \otimes X_{(2,0)}) = \Psi_{\infty}$$

in the same way as $\omega_{\Psi}$ corresponds to $\Omega_{\Psi, \infty}$. Let us denote by $\Psi$ the cusp form $\Psi_{\infty} \otimes \Psi_f$ which belongs to the representation space of $\Pi_{1,2}^\infty \otimes \Pi_f$.

We have the Cartan decomposition $\mathfrak{gl}_2 = \mathfrak{h} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where

$$\mathfrak{p}^\pm = \left\{ \begin{pmatrix} z & \pm iz \\ \pm iz & -z \end{pmatrix} : z \in \mathfrak{gl}_1 \right\}.$$

Let

$$v^\pm = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \in \mathfrak{p}^\pm.$$

Let $\Phi = \Phi_{\infty} \otimes \Phi_f$ be a cusp form in the space of $\sigma = \sigma_{\infty} \otimes \sigma_f$ such that $\Phi_{\infty}$ is a generator of the minimal $L_\infty$-type of $\sigma_\infty$ and such that $\Phi_f$ is invariant by the principal level $N$ congruence subgroup of $\text{GL}(2, \hat{\mathbb{Z}})$. Then, as in Lem. 2.14, we define the element of $H^1(\mathfrak{gl}_2, L_\infty, \sigma_\infty) = \text{Hom}_{L_\infty}(\mathfrak{gl}_2/\mathfrak{l}, \sigma_\infty)$ by prescribing that it sends $v^+$ to $\Phi_{\infty}$. As above, we associate to $\Phi$ the harmonic differential form

$$\Omega_{\Phi} = (\Omega_{\Phi, \infty})_{\sigma : E \to \mathbb{C}} \otimes \Phi_f \in H^1_{BD,t}(Y, \mathbb{C}) \otimes_{\mathbb{Q}} E$$

which gives rise to a generator $\eta_{\Phi}$ of $M^{1,0}(\sigma_f)$.

Let us introduce the Poincaré duality pairing

$$H^4_{BD}(S \times Y, \mathbb{Q}) \otimes_{\mathbb{Q}} H^4_{BD}(S \times Y, \mathbb{Q}) \to \mathbb{Q}(-4).$$

This becomes a perfect pairing of pure Hodge structures when restricted to invariants by any neat compact open subgroup of $G(\mathbb{A}_f)$. Furthermore this is a $G(\mathbb{A}_f)$-equivariant map when $\mathbb{Q}(-4)$ is endowed with the trivial action. This statement is easily deduced from the K"unneth formula and from the analogous statements for $S$, see [Ta93] p. 295, and for $Y$, which are similar to loc. cit. Then (8) induces a perfect pairing

$$M_B(\Pi_f \times \sigma_f) \otimes_{E} M_B(\Pi_f \times \tilde{\sigma}_f) \xrightarrow{(-1)_B} E(-4)_B$$
where \( \tilde{\Pi}_f \) and \( \tilde{\sigma}_f \) denote the representations contragredient to \( \Pi_f \) and \( \sigma_f \), respectively, and where \( E(-4)_B \) denotes the Betti realization of the \((-4)\)-th power of the Tate motive with coefficients in \( E \). Furthermore \( \langle , \rangle_B \) is a morphism of Hodge structure and has a de Rham analogue

\[
M_{dR}(\Pi_f \times \sigma_f) \otimes_E M_{dR}(\tilde{\Pi}_f \times \tilde{\sigma}_f) \xrightarrow{\langle , \rangle_{dR}} E(-4)_{dR}.
\]

We assume that the central characters \( \omega_{\Pi} \) and \( \omega_{\sigma} \) of \( \Pi \) and \( \sigma \) are trivial. Hence, because of the canonical isomorphisms \( \tilde{\Pi}_f \simeq \Pi_f \otimes (\omega_{\Pi} \circ \nu)^{-1} \) ([We05] Lem. 1.1) and \( \sigma_f \simeq \sigma_f \otimes (\omega_{\sigma} \circ \det)^{-1} \) ([JL70] Thm. 2.18 (i)), the pairings \( \langle , \rangle_B \) and \( \langle , \rangle_{dR} \) can be regarded as a pairings

\[
M_B(\Pi_f \times \sigma_f) \otimes E M_B(\Pi_f \times \sigma_f) \xrightarrow{\langle , \rangle_B} E(-4)_B
\]

\[
M_{dR}(\Pi_f \times \sigma_f) \otimes E M_{dR}(\Pi_f \times \sigma_f) \xrightarrow{\langle , \rangle_{dR}} E(-4)_{dR}
\]

whose complexifications are part of the commutative diagram

\[
\begin{array}{ccc}
M_B(\Pi_f \times \sigma_f)_C \otimes_E M_B(\Pi_f \times \sigma_f)_C & \xrightarrow{\langle , \rangle_{B,C}} & C \otimes Q E(-4)_B \\
\downarrow & & \downarrow \\
M_{dR}(\Pi_f \times \sigma_f)_C \otimes_Q E M_{dR}(\Pi_f \times \sigma_f)_C & \xrightarrow{\langle , \rangle_{dR,C}} & C \otimes Q E(-4)_{dR}
\end{array}
\]

where the vertical lines are the comparison isomorphisms.

**Lemma 2.15.** The pairing

\[
M_B(\Pi_f \times \sigma_f, 2)_C \xrightarrow{\langle \varpi_\Psi \otimes \eta_\Phi \rangle_{B,C}} C \otimes Q E(-2)_B
\]

with \( \varpi \otimes \eta \), induces an \( R \otimes Q E \)-linear map

\[
\text{Ext}^1_{\text{MHS}_k}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_R) \xrightarrow{\langle \varpi_\Psi \otimes \eta_\Phi \rangle_{B,C}} C \otimes Q E(-2)_B.
\]

**Proof.** In the Hodge decomposition (4), we have \( \varpi_\Psi \otimes \eta_\Phi \in M^{2,2} \). Moreover the image of the natural map \( F^3 M_{dR}(\Pi_f \times \sigma_f)_R \to M_B(\Pi_f \times \sigma_f, 2)_R \) (see Prop. 2.8) lies in \( M^{4,0} \oplus M^{3,1} \oplus M^{1,3} \oplus M^{0,4} \). As the Poincaré duality pairing is a morphism of Hodge structures, the restriction of

\[
M_B(\Pi_f \times \sigma_f, 2)_R \xrightarrow{\langle \varpi_\Psi \otimes \eta_\Phi \rangle_{B,C}} C \otimes Q E(-2)_B
\]

to \( F^3 M_{dR}(\Pi_f \times \sigma_f)_R \) is zero. Hence we get the induced map

\[
\text{Ext}^1_{\text{MHS}_k}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_R) \xrightarrow{\langle \varpi_\Psi \otimes \eta_\Phi \rangle_{B,C}} C \otimes Q E(-2)_B.
\]

\[\square\]

Recall that we have fixed vectors \( \Psi_f \) and \( \Phi_f \) in the representation space of \( \Pi_f \) and \( \sigma_f \) respectively. Hence, we can consider the image \( v_Z \) of the cycle class \( Z_N \) (6) by the maps

\[
H^1_B(S(N) \times Y(N), Q(2)) \to H^1_B(S \times Y, Q(2)) \otimes E \to M_B(\Pi_f \times \sigma_f, 2).
\]
where the second map sends a vector $x \in H_B^4(S \times Y, \mathbb{Q}(2))^+ \otimes E$ to the unique element of $M_B(\Pi_f \times \sigma_f, 2)^+$ sending $\Psi_f \times \sigma_f$ to $x$. It is easy to see that the vector $v_Z$ is mapped to the $E$-subspace $Z(\Pi_f \times \sigma_f)$ by the right hand map

$$M_B(\Pi_f \times \sigma_f, 2)^+ \to \text{Ext}^1_{\text{MHS}}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_R)$$

of the short exact sequence of Prop. 2.8.

We now define a Haar measure on $H(\mathbb{A})/\mathbb{R}^\times$ as follows. For every prime number $p$, we endow $H(\mathbb{Q}_p)$ with the unique Haar measure $dh_p$ for which $H(\mathbb{Z}_p)$ has volume one. Let us consider

$$U_\infty = (\mathbb{R}^\times SO(2, \mathbb{R}) \times \mathbb{R}^\times SO(2, \mathbb{R})) \cap H(\mathbb{R}).$$

Then $U_\infty$ is a subgroup of $H(\mathbb{R})_+ = \{(h_1, h_2) \in H(\mathbb{R}) \mid \det h_1 = \det h_2 > 0\}$ which is maximal compact modulo the center. Let $\mathfrak{h}$ and $u$ be the complex Lie algebras of $H(\mathbb{R})$ and $U_\infty$ respectively. Then

$$1 = (v^+, 0) \wedge (0, v^+) \wedge (v^-, 0) \wedge (0, v^-)$$

is a generator of the highest exterior power $\bigwedge^4 \mathfrak{h}/u$ and it determines a left invariant measure $dx$ on $H(\mathbb{R})/U_\infty$. Together with the Haar measure $dk$ on $SO(2, \mathbb{R}) \times SO(2, \mathbb{R})$ whose total mass is one, this defines a measure $dh_\infty = dxdk$ on $H(\mathbb{R})/\mathbb{R}^\times$. Let $dh$ denote the product measure $\prod_{v \leq \infty} dh_v$ on $H(\mathbb{A})/\mathbb{R}^\times$.

**Definition 2.16.** For two elements $x, y \in \mathbb{C} \otimes \mathbb{Q} E$, we write $x \sim y$ if $x$ and $y$ belong to the same orbit under right multiplication by $E^\times$.

Let $Z$ denote the center of $H$. In the following proposition, we regard a complex number as an element of $\mathbb{C} \otimes \mathbb{Q} E = \prod_{\sigma : E \to \mathbb{C}} \mathbb{C}$ via the diagonal inclusion $\mathbb{C} \to \prod_{\sigma : E \to \mathbb{C}} \mathbb{C}$.

**Proposition 2.17.**

$$\langle v_Z, \overline{\omega}_\Psi \otimes \eta_\Phi \rangle_{B, C} \sim \int_{Z(\mathbb{A})H(\mathbb{Q}) \backslash H(\mathbb{A})} \langle X_{(-1, 1)}, \overline{\Psi}(h_1, h_2)\Phi(h_2) \rangle dh$$

**Proof.** Recall that we denote by $Z_N \in H_B^4(S(N) \times Y(N), \mathbb{Q}(2))^+$ the cohomology class of $\iota_N(Y(N) \times \mathbb{Q}(\mu_N) Y(N))$. According to [GH78] Ch. 3, example 1 p. 386, the image of $Z_N$ in the de Rham cohomology space $H^4_{dR}(S(N) \times Y(N), \mathbb{C}(2))$ is given by the integration current along the subvariety $Y(N) \times \mathbb{Q}(\mu_N) Y(N)$. This means that for any closed differential form with compact support $\Omega$ of degree 4 on $S(N) \times Y(N)$, the Poincaré duality pairing $\langle Z_N, \Omega \rangle$ is given by

$$\langle Z_N, \Omega \rangle = \int_{Y(N) \times \mathbb{Q}(\mu_N) Y(N)} \Omega.$$

By the invariance properties of $\Psi_f$ and $\Phi_f$, the element $\overline{\Omega}_\Psi \otimes \Omega_\Phi$ defines a harmonic differential form on $S(N) \times Y(N)$. It is not compactly supported, however it is cuspidal. As a consequence, by [Bo81] Cor. 5.5, there exists a rapidly decreasing differential form $\Omega'$ such that the difference $\overline{\Omega}_\Psi \otimes \Omega_\Phi - d\Omega'$ is compactly supported. The integral

$$\int_{Y(N) \times \mathbb{Q}(\mu_N) Y(N)} d\Omega'$$
converges and is equal to zero. This follows from the fact that, as $\Omega'$ is rapidly decreasing, it extends to a smooth differential form on a smooth compactification of $Y(N) \times_{Q(\mu_N)} Y(N)$ which is zero on the boundary and from the Stokes theorem. As a consequence

$$\langle v_Z, \bar{\Omega}_\Psi \otimes \eta_\Phi \rangle_{dR, C} = \int_{Y(N) \times_{Q(\mu_N)} Y(N)} (\bar{\Omega}_\Psi \otimes \Omega_\Phi - dY)$$

$$= \int_{Y(N) \times_{Q(\mu_N)} Y(N)} \bar{\Omega}_\Psi \otimes \Omega_\Phi.$$

Let $U(N)$ denote the principal level $N$ congruence subgroup of $H(\hat{\mathbb{Z}})$. As complex analytic varieties, we have

$$Y(N) \times_{Q(\mu_N)} Y(N) \simeq H(\mathbb{Q}) \backslash H(\mathfrak{A}) / U(\infty) U(N).$$

Let $h_N$ denote the cardinality of the finite group $Z(\mathbb{Q}) \backslash Z(\mathfrak{A}_f) / U(\infty) \cap Z(\mathfrak{A}_f)$. We have

$$\int_{Y(N) \times_{Q(\mu_N)} Y(N)} \bar{\Omega}_\Psi \otimes \Omega_\Phi = \int_{H(\mathbb{Q}) \backslash H(\mathfrak{A}) / U(\infty) U(N)} (\bar{\Omega}_\Psi \otimes \Omega_\Phi)(1) dh$$

$$= \int_{H(\mathbb{Q}) \backslash H(\mathfrak{A}) / U(\infty) U(N)} \bar{\Omega}_\Psi ((v^-, 0) \land (0, v^-) \otimes (v^+, 0)) \Omega_\Phi(v^+) dh$$

$$= \int_{\mathbb{R}^+ \backslash H(\mathbb{Q}) \backslash H(\mathfrak{A}) / U(\infty) U(N)} \bar{\Omega}_\Psi ((v^-, 0) \land (0, v^-) \otimes (v^+, 0)) \Omega_\Phi(v^+) dh$$

$$= 4h_N \int_{Z(\mathfrak{A}) H(\mathbb{Q}) \backslash H(\mathfrak{A}) / U(N)} \bar{\Omega}_\Psi ((v^-, 0) \land (0, v^-) \otimes (v^+, 0)) \Omega_\Phi(v^+) dh$$

$$= \frac{4h_N}{\text{vol}(U(N))} \int_{Z(\mathfrak{A}) H(\mathbb{Q}) \backslash H(\mathfrak{A})} \bar{\Omega}_\Psi ((v^-, 0) \land (0, v^-) \otimes (v^+, 0)) \Omega_\Phi(v^+) dh$$

By an explicit computation similar to the ones conducted in the proof of [Le17] Lem. 4.27, one proves that

$$(v^-, 0) \land (0, v^-) \otimes (v^+, 0) = r \text{ ad}_{X_{(-1,1)}} (X_{(0, -2)} \land X_{(-1, -1)} \otimes X_{(2, 0)})$$

for some explicit $r \in \mathbb{Q}^\times$ that we do not need to compute as we work up to $\sim$. Note that $\text{vol}(U(N))$ is a non-zero rational number with our choice of measure. So the conclusion follows.

2.3.2. The pairing associated to the Deligne rational structure.

**Proposition 2.18.** If the complete L-function $L(s, \Pi \times \sigma)$ has a pole at $s = 1$ then $\Pi$ is obtained as a Weil lifting from $\text{GSO}(2, 2, \mathfrak{A})$ of a pair $(\sigma_1, \sigma_2)$ of inequivalent irreducible cuspidal automorphic representations of $\text{GL}(2, \mathfrak{A})$ with the same central characters.

**Proof.** This follows from [PSS84] Thm. 1.3 and [We09] Lem. 5.2 1) as we assume that $\Pi$ is globally generic hence is not CAP. 

**Proposition 2.19.** Assume that $L(s, \Pi \times \sigma)$ has a pole at $s = 1$. Then

$$m(\Pi^H_\infty \otimes \Pi_f) = 0.$$
Proof. This follows from the previous result and from [We09] Thm. 5.2. (4) and Lem. 5.2.

In the rest of section 2.3.2, we assume that $L(s, \Pi \times \sigma)$ has a pole at $s = 1$ so that the conclusion of Prop. 2.19 holds. Hence, by the Hodge decompositions (2) and (3), we have the decompositions

$$M_B(\Pi_f) = M_B(\Pi_f)^+ \oplus M_B(\Pi_f)^-,$$
$$M_B(\sigma_f) = M_B(\sigma_f)^+ \oplus M_B(\sigma_f)^-,$$

where $M_B(\Pi_f)^\pm$ and $M_B(\sigma_f)^\pm$ are one-dimensional $E$-vector spaces. Let $v^\pm$ and $w^\pm$ be generators of $M_B(\Pi_f)^\pm$ and $M_B(\sigma_f)^\pm$ respectively. By a slight abuse of notation, we regard $(v^+, v^-)$ as a basis of $M_B(\Pi_f)_C$ and $(w^+, w^-)$ as a basis of $M_B(\sigma_f)_C$. Let $\omega$ be a generator of the one dimensional $E$-vector space $F^2M_{dR}(\Pi_f)$ and let $\eta$ be a generator of the one dimensional $E$-vector space $F^1M_{dR}(\sigma_f)$. Then $\omega \otimes \eta$ is a generator of $F^3M_{dR}(\Pi_f \times \sigma_f)$. Via the comparison isomorphisms, we have

$$\omega = \alpha^+ v^+ + \alpha^- v^-,$$
$$\eta = \beta^+ w^+ + \beta^- w^-,$$

for some $\alpha^+, \alpha^-, \beta^+, \beta^- \in \mathbb{C} \otimes E$. Note that as $\omega$ and $\eta$ are defined over $E$ in the de Rham rational structure, we have $\alpha^+, \beta^+ \in \mathbb{R} \otimes Q E$ and $\alpha^-, \beta^- \in \mathbb{R} i \otimes E$. The image of $\omega \otimes \eta$ by the natural map $\phi : F^3M_{dR}(\Pi_f \times \sigma_f)_\mathbb{R} \to M_B(\Pi_f \times \sigma_f, 2)^+_\mathbb{R}$ in the exact sequence of Prop. 2.8 is

$$\phi(\omega \otimes \eta) = (2\pi i)^2(\alpha^+ \beta^+ v^+ \otimes w^+ + \alpha^- \beta^- v^- \otimes w^-).$$

As $\phi$ is injective, at least one of the two real numbers $\alpha^+ \beta^+$ and $\alpha^- \beta^-$ is non-zero.

Lemma 2.20. Let $\circ \in \{\pm\}$ be such that $\alpha^\circ \beta^\circ \not= 0$ then

$$v_B = \frac{1}{(2\pi i)^2 \alpha^\circ \beta^\circ} v^{-\circ} \otimes w^{-\circ}$$

maps to a generator of $\mathcal{B}(\Pi_f \times \sigma_f)$ by the right hand map

$$M_B(\Pi_f \times \sigma_f, 2)^+ \to \text{Ext}^1_{\text{MHS}_3}(\mathbb{R}(0), M_B(\Pi_f \times \sigma_f, 3)_{\mathbb{R}})$$

of the short exact sequence of Prop. 2.8.

Proof. For any $\lambda^+, \lambda^- \in \mathbb{R} \otimes Q E$, the vector $\lambda^+ v^+ \otimes w^+ + \lambda^- v^- \otimes w^- \in M_B(\Pi_f \times \sigma_f, 2)^+$ maps to a generator $\mathcal{B}(\Pi_f \times \sigma_f)$ if and only if $(2\pi i)^2(\lambda^+ \alpha^- \beta^- - \lambda^- \alpha^+ \beta^+) \in E^\times$. This can be proved in exactly the same way as [Le17] Lem. 6.1. The conclusion follows.

Let $\varpi$ be the vector in $M_B(\Pi_f \times \sigma_f)_\mathbb{C}$ obtained by applying the complex conjugation on the coefficients. Then $\varpi = \alpha^+ \bar{v}^+ - \alpha^- \bar{v}^-$. Let us consider

$$\varpi \otimes \eta = \alpha^+ \beta^+ v^+ \otimes w^+ + \alpha^+ \beta^- v^- \otimes w^- - \alpha^- \beta^+ v^+ \otimes w^+ - \alpha^- \beta^- v^- \otimes w^-.$$

Lemma 2.21. We have $\langle v_B, \varpi \otimes \eta \rangle_{\mathcal{B}, \mathbb{C}} \sim (2\pi i)^{-6}$. 17
Proof. Poincaré duality for $M_B(P\Pi)$ is a perfect $F_\infty$-equivariant pairing

$$M_B(P\Pi) \otimes_E M_B(P\Pi, 3) \to E.$$ 

As a consequence $v^\pm$ is dual to $(2\pi i)^3 v^\mp$ for this pairing, up to multiplication by an element of $E^\times$. Similarly, the vector $w^\pm$ is dual to $(2\pi i)^3 w^\mp$ for the pairing

$$M_B(\sigma_f) \otimes_E M_B(\sigma_f, 1) \to E,$$

up to multiplication by an element of $E^\times$. Then the conclusion follows. \hspace{1cm} \Box

**Proposition 2.22.** $\delta(\Pi_f \times \sigma_f, 3) \sim (2\pi i)^4$

*Proof.* Let $e_1, e_2, e_3, e_4$ be the vectors of a basis of $M_B(\Pi_f \times \sigma_f)$, let $f_1, f_2, f_3, f_4$ be the vectors of a basis of $M_{dR}(\Pi_f \times \sigma_f)$, let $P$ be the matrix of the vectors $I_{\infty}(e_i)$ in the basis $(f_j)$ and let us denote $\delta(\Pi_f \times \sigma_f) = \det P$. Let us consider the matrices $J_B = \langle (e_i, e_j)_B \rangle$ and $J_{dR} = \langle (f_i, f_j)_{dR} \rangle$. Let $i_\infty$ be the comparison isomorphism which is the right vertical line of the diagram (9). Then we have $P^t J_{dR} (J_B) = i_\infty(J_B)$ where $i_\infty(J_B)$ denotes the matrix obtained by applying $i_\infty$ to the coefficients of $J_B$. Note that the matrices $J_{dR}$ and $(2\pi i)^4 i_{\infty}(J_B)$ have coefficients in $E$. The last statement follows from the computation of the periods of the Tate motive as explained in [De79] 3. Taking determinants, we see that $\delta(\Pi_f \times \sigma_f)^2 \sim (2\pi i)^{-16}$ so that we have $\delta(\Pi_f \times \sigma_f) \sim (2\pi i)^{-8}$. As $M(\Pi_f \times \sigma_f)$ has rank four, it follows from [De79] (5.1.9) that $\delta(\Pi_f \times \sigma_f, 3) \sim (2\pi i)^{12} \delta(\Pi_f \times \sigma_f)$ and the conclusion follows. \hspace{1cm} \Box

**Corollary 2.23.** Let $\omega$ be a generator of $F^2 M_{dR}(\Pi_f)$ and let $\eta$ be a generator of $F^1 M_{dR}(\sigma_f)$. Then

$$Z(\Pi_f \times \sigma_f) = (2\pi i)^8 \langle v_Z, \overline{\omega} \otimes \eta \rangle_{B, E} D(\Pi_f \times \sigma_f).$$

*Proof.* Recall that, by definition

$$D(\Pi_f \times \sigma_f) = (2\pi i)^{\dim_E M_B(\Pi_f \times \sigma_f)} \delta(\Pi_f \times \sigma_f, 3)^{-1} B(\Pi_f \times \sigma_f) = (2\pi i)^{-2} B(\Pi_f \times \sigma_f)$$

where the last equality follows from Prop. 2.22. Let $\overline{\sigma}_Z$ and $\overline{\sigma}_B$ be the images of $v_Z$ and $v_B$ by the third map of the short exact sequence of Prop. 2.8. There exists $\lambda \in \mathbb{R} \otimes_E ^\ast$ such that $\overline{\sigma}_Z = \lambda \overline{\sigma}_B$ and then

$$Z(\Pi_f \times \sigma_f) = \lambda B(\Pi_f \times \sigma_f) \sim (2\pi i)^2 \lambda D(\Pi_f \times \sigma_f).$$

Pairing with $\overline{\omega} \otimes \eta$ and using Lem. 2.21, we obtain

$$\langle v_Z, \overline{\omega} \otimes \eta \rangle = \lambda \langle v_B, \overline{\omega} \otimes \eta \rangle \sim (2\pi i)^{-6} \lambda.$$

\hspace{1cm} \Box

**3. Zeta integrals**

3.1. **The global integral.** Let $\psi : \mathbb{Q} \setminus \mathbb{A} \to \mathbb{C}^\times$ be the non-trivial additive character characterized by $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. We consider the maximal unipotent subgroup $N \subset \text{GSp}(4)$ defined by

$$\begin{pmatrix} x_0 & 1 & 0 \\ 1 & 0 & 0 \\ -x_0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, x_0, x_1, x_2, x_3 \in \mathbb{G}_a$$

$$N = \left\{ n(x_0, x_1, x_2, x_3) \right\}$$

\begin{equation}
\begin{pmatrix} x_0 & 1 & 0 \\ 1 & 0 & 0 \\ -x_0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, x_0, x_1, x_2, x_3 \in \mathbb{G}_a
\end{equation}
and the character $\psi_N : N(\mathbb{Q}) \backslash N(\mathbb{A}) \to \mathbb{C}$ defined by $\psi_N(n(x_0, x_1, x_2, x_3)) = \psi(-x_0 - x_3)$.

Let $\Pi = \bigotimes_v \Pi_v$ be an irreducible cuspidal automorphic representation of $\text{GSp}(4, \mathbb{A})$. The global Whittaker function $W_{\Psi}$ on $\text{GSp}(4, \mathbb{A})$ attached to a cusp form $\Psi \in \Pi$ is

$$W_{\Psi}(g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \Psi(n g) \psi_N(n^{-1}) \, dn.$$  

Assume that the global Whittaker function $W_{\Psi}$ does not vanish for some cusp form $\Psi \in \Pi$. This assumption implies that for each place $v$, the representation $\Pi_v$ of $\text{GSp}(4, \mathbb{Q}_v)$ can be realized as a subspace of

$$\{ W : \text{GSp}(4, \mathbb{Q}_v) \to \mathbb{C} \mid \text{smooth}, \, W(n g) = \psi_N(n) W(g), \, \forall (n, g) \in N(\mathbb{Q}_v) \times \text{GSp}(4, \mathbb{Q}_v) \}.$$

We denote this subspace by $W(\Pi_v, \psi_v)$ and call it the local Whittaker model of $\Pi_v$. If $\sigma$ is a cuspidal automorphic representation of $\text{GL}(2, \mathbb{A})$ and if $\Phi \in \sigma$ is a cusp form, the global Whittaker function on $\text{GL}(2, \mathbb{A})$ attached to $\Phi$ is

$$W_{\Phi}(g) = \int_{Q \backslash \mathbb{A}} \Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-x) \, dx.$$  

It is well-known that $W_{\Phi}$ does not vanish for $\Phi \neq 0$. The local Whittaker model $W(\sigma_v, \psi_v)$ of $\sigma_v$ is defined similarly as above.

Let us now introduce Eisenstein series on $\text{GL}(2, \mathbb{A})$. Let $dt_\infty$ be the Lebesgue measure on the additive group $\mathbb{R}$. If $v$ is a non-archimedean place of $\mathbb{Q}$, let $dt_v$ be the Haar measure on $\mathbb{Q}_v$ for which $\mathbb{Z}_v$ has volume one. Let $d^\times t_v$ be the Haar measure on $\mathbb{Q}_v^\times$ defined by

$$d^\times t_v = \begin{cases} \frac{dt_v}{|v|} & \text{if } v \text{ is archimedean,} \\ \frac{p-1 dt_v}{|v|} & \text{if } v \text{ is } p\text{-adic.} \end{cases}$$

Let $dt$, resp. $d^\times t$, denote the product measure $\prod_v dt_v$ on $\mathbb{A}$, resp. $\prod_v d^\times t_v$ on $\mathbb{A}^\times$. Let $\mathcal{S}(\mathbb{A}^2)$ be the space of Schwartz-Bruhat functions on $\mathbb{A}^2$. For $\varphi \in \mathcal{S}(\mathbb{A}^2)$, let us define the global Jacquet section

$$f_\varphi(s, h_1) = |\det h_1|^s \int_{\mathbb{A}^\times} \varphi((0, t)h_1)|t|^{2s} \, d^\times t$$

for $h_1 \in \text{GL}(2, \mathbb{A})$. This integral converges for $\text{Re}(s) > 1/2$. We can define the Eisenstein series

$$E(s, h_1, f_\varphi) = \sum_{\gamma \in B(\mathbb{Q}) \backslash \text{GL}(2, \mathbb{Q})} f_\varphi(s, \gamma h_1)$$

which converges absolutely and uniformly on every compact subset in $\text{Re}(s) > 1$ except for the poles of $f_\varphi(s, h_1)$ and is continued to a meromorphic function on the whole complex plane. The global zeta integral we are interested in is defined as follows: for $\Psi \in \Pi$, $\Phi \in \sigma$ and $\varphi \in \mathcal{S}(\mathbb{A}^2)$ let

$$Z(s, \Psi, \Phi, f_\varphi) = \int_{Z(\mathbb{A})H(\mathbb{Q}) \backslash H(\mathbb{A})} \Psi(h) \Phi(h_2) E(s, h_1, f_\varphi) \, dh.$$
Here and in what follows, we regard the group \( H \) as embedded in \( \text{GSp}(4) \) via \( p \circ \iota \), where \( p : G \to \text{GSp}(4) \) is the first projection. This integral converges absolutely except for the poles of the Eisenstein series and defines a meromorphic function in \( s \in \mathbb{C} \).

**Proposition 3.1.** The function \( s \mapsto Z(s, \Psi, \Phi, f_\varphi) \) is holomorphic except for possible simple poles at \( s = 1 \) and \( 0 \). Moreover, we have

\[
\text{Res}_{s=1} Z(s, \Psi, \Phi, f_\varphi) = \frac{\hat{\varphi}(0)}{2} \int_{Z(\mathbb{A}) H(\mathbb{Q}) \backslash H(\mathbb{A})} \Psi(h) \Phi(h_2) dh
\]

where

\[
\hat{\varphi}(0) = \int_{\mathbb{H}^2} \varphi(s, t) ds dt.
\]

**Proof.** This is a particular case of [Mo09] Prop. 3.1. Note that the constant \( c \) appearing in the statement of loc. cit., Prop. 3.1 is equal to \( \frac{1}{2} \) when the ground field is \( \mathbb{Q} \). \( \square \)

Let us suppose that the cusp forms \( \Psi = \bigotimes_v \Psi_v \) and \( \Phi = \bigotimes_v \Phi_v \) are factorizable. Then, the local multiplicity one property implies that \( W_\Psi \) and \( W_\Phi \) are decomposed into a product of local Whittaker functions:

\[
W_\Psi(g) = \prod_v W_{\Psi_v}(g_v), g \in \text{GSp}(4, \mathbb{A})
\]

\[
W_\Phi(h_2) = \prod_v W_{\Phi_v}(h_{2,v}), h_2 \in \text{GL}(2, \mathbb{A}).
\]

Moreover, let us assume that the Schwartz-Bruhat function \( \varphi = \prod_v \varphi_v \) is factorizable. Then the global Jacquet section \( f_\varphi \) factorizes as \( f_\varphi(s, h_1) = \prod_v f_{\varphi_v}(s, h_{1,v}) \) where

\[
f_{\varphi_v}(s, h_{1,v}) = | \det h_{1,v} |_v^{s} \int_{\mathbb{Q}_v^*} \varphi_v((0, t_v) h_{1,v}) | t_v |_v^{2s} dt_v.
\]

For each place \( v \) of \( \mathbb{Q} \), we define the local zeta integral \( Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) \) by

\[
Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) = \int_{Z(\mathbb{Q}_v) N_H(\mathbb{Q}_v) \backslash H(\mathbb{Q}_v)} W_{\Psi_v}(h_v) W_{\Phi_v}(h_{2,v}) f_{\varphi_v}(s, h_{1,v}) dh_v
\]

where \( N_H \) denotes the maximal unipotent subgroup of \( H \) defined as \( N_H = N \cap H \).

**Proposition 3.2.** [Mo09] Prop. 3.2. Suppose that \( Z_\infty(s, W_{\Psi_\infty}, W_{\Phi_\infty}, f_{\varphi_\infty}) \) converges absolutely for \( \text{Re}(s) > e_\infty \). Then, the integral

\[
\int_{Z(\mathbb{A}) N_H(\mathbb{A}) \backslash H(\mathbb{A})} W_\Psi(h) W_\Phi(h_2) f_\varphi(s, h_1) dh
\]

converges absolutely for \( \text{Re}(s) > \max\{3, e_\infty \} \) and is equal to \( Z(s, \Psi, \Phi, f_\varphi) \).

This proposition implies that

\[
Z(s, \Psi, \Phi, f_\varphi) = \prod_v Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})
\]

for any complex number \( s \) such that \( \text{Re}(s) > \max\{3, e_\infty \} \). Let \( V \) be a finite set of places satisfying the following condition: if \( v \notin V \), then \( v \) is a finite place such that \( \psi_v \) is unramified, the representations \( \Pi_v \), resp. \( \sigma_v \), has a vector fixed by \( \text{GSp}(4, \mathbb{Z}_v) \), resp.
GL(2, Z_v) and \( \varphi_v \in S(Q_v^2) \) is the characteristic function \( \varphi_{v,0} \) of \( Z_v^2 \). Fix a place \( v \notin V \). Let \( W_0 \in W(\Pi_v, \psi_v) \) and \( W'_0 \in W(\sigma_v, \psi_v) \) be the \( \text{GSp}(4, Z_v) \)-fixed local Whittaker function normalized so that \( W_0(I_4) = 1 \) and the \( \text{GL}(2, Z_v) \)-fixed Whittaker function normalized so that \( W'_0'(I_2) = 1 \), respectively. Then, it is proved in section 3.4 of [Mo09] that

\[
Z_v(s, W_0, W'_0, f_{\varphi_{v,0}}) = L(s, \Pi_v \times \sigma_v)
\]

where the right-hand side is the Langlands degree eight local \( L \)-factor. As a consequence, for \( \mathrm{Re}(s) > \max\{3, e_\infty\} \), we have

\[
Z(s, \Psi, \Phi, f_\varphi) = \prod_{v \in V} Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) L_V(s, \Pi \times \sigma).
\]

In the following result, we denote by \( S(Q_v^2) \) the space of \( \mathbb{C} \)-valued locally constant compactly supported functions on \( Q_v^2 \).

**Proposition 3.3.** [So84] section 2. Let \( v \) be a non-archimedean place. There exists a unique polynomial \( P_v(X) \in \mathbb{C}[X] \) such that \( P_v(0) = 1 \) and that the \( \mathbb{C} \)-vector space generated by the \( Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) \) for \( W_{\Psi_v} \in W(\Pi_v, \psi_v) \), \( W_{\Phi_v} \in W(\sigma_v, \psi_v) \) and \( \varphi_v \in S(Q_v^2) \) is \( P_v(p^{-s}) \mathbb{C}[p^{-s}, p^s] \).

For any non-archimedean place \( v \in V \) let \( L_v(s, \Pi \times \sigma) = P_v(p^{-s})^{-1} \).

### 3.2. Archimedean computation.

The result of this section is a particular case of [Mo09], that we wish to report for convenience of the reader. First we need to fix the notation for our archimedean representations.

Recall that we denote by \( \sigma_{\infty}^{1,0} \), resp. \( \sigma_{\infty}^{0,1} \), the holomorphic, resp. antiholomorphic, discrete series of \( \text{GL}(2, \mathbb{R})_+ \) with the same central and infinitesimal characters as the trivial representation. With the notation of section 1.1 (i) of loc. cit., we have

\[
\sigma_{\infty}^{1,0}|_{\text{SL}(2, \mathbb{R})} = D_2,
\]

\[
\sigma_{\infty}^{0,1}|_{\text{SL}(2, \mathbb{R})} = D_{-2}.
\]

Recall also that we consider the discrete series \( \Pi_{\infty}^{2,1} \) of \( \text{GSp}(4, \mathbb{R})_+ \) with trivial central character which contains with multiplicity one the irreducible \( \mathbb{C}[K_{\infty}] \)-module \( \tau_{(3,-1)} \) as a minimal \( K_{\infty} \)-type. In other words \( \Pi_{\infty}^{2,1} \) has Blattner parameter \( (3,-1) \) and the discrete series \( \Pi_{\infty}^{1,2} \) has Blattner parameter \( (1,-3) \). In the notation of the section 1.2 (i) of loc. cit. we have

\[
\Pi_{\infty}^{2,1}|_{\text{Sp}(4, \mathbb{R})} = D_{(3,-1)},
\]

\[
\Pi_{\infty}^{1,2}|_{\text{Sp}(4, \mathbb{R})} = D_{(1,-3)}.
\]

Let \( \Psi_{\infty} \) denote a highest weight vector of the minimal \( K_{\infty} \)-type of \( \Pi_{\infty}^{2,1} \). Then \( \overline{\Psi}_{\infty} \), which is defined as in [Le17] Rem. 3.2, is a highest weight vector of the minimal \( K_{\infty} \)-type of \( \Pi_{\infty}^{1,2} \).

Let \( \Phi_{\infty} \) denote a generator of the minimal \( L_{\infty} \)-type of \( \sigma_{\infty} \). Let us fix an isomorphism \( \Pi_{\infty}^W \simeq W(\Pi_{\infty}^W, \psi_{\infty}) \) and let \( W_{X_{(-1,1)}\Psi_{\infty}} \) be the image of \( X_{(-1,1)}\overline{\Psi}_{\infty} \) under this isomorphism. Similarly let \( W_{\Phi_{\infty}} \) denote the image of \( \Phi_{\infty} \) under a fixed isomorphism \( \sigma_{\infty} \simeq W(\sigma_{\infty}, \psi_{\infty}) \).
Proposition 3.4. Let $\Gamma_C(s)$ denote the function $\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s)$. Let $\varphi_\infty \in S(\mathbb{R}^2)$ be the Schwartz-Bruhat function defined as $\varphi_\infty(x, y) = \exp(-(\pi(x^2 + y^2)))$. Normalize the functions $W_{X_{(-1,1)}^{\infty}}$ and $W_{\Phi}$ by $W_{X_{(-1,1)}^{\infty}}(I_4) = 1$ (see [Mo09] (5.4)) and $W_{\Phi}(I_2) = c_2$, where $I_4$ and $I_2$ denote the identity matrices of size 4 and 2 and where $c_2$ is the unique non-zero complex number such that the constant $C$ appearing in [Mo09] (5.11) is equal to 1. Then, for any $\Re(s) > e_\infty$, we have

$$Z_\infty(s, W_{X_{(-1,1)}^{\infty}}, W_{\Phi}, f_{\varphi_\infty}) = \Gamma_C(s + 2) \Gamma_C(s + 1)^2 \Gamma_C(s).$$

Proof. This is a particular case of the equality (5.11) in [Mo09] with $\lambda_1 = 3$, $\lambda_2 = -1$, $l = 2$. □

Remark 3.5. It would be possible to determine $c_2$ explicitly by unravelling carefully the computations of [Mo09] 5.3.

3.3. De Rham-Whittaker periods and rationality of $p$-adic integrals. In this section, we define de Rham-Whittaker periods and prove an algebraicity result for $p$-adic integrals. De Rham-Whittaker periods are analogous to the occult period invariant introduced in [H04], when the Bessel model is replaced by the Whittaker model. Note that in this section, we do not assume that $L(s, \Pi \times \sigma)$ has a pole at $s = 1$.

3.3.1. Definition of the periods. We start by defining an action of $\text{Aut}(\mathbb{C})$ on the Whittaker model. Let $\mathbb{Q}(\mu_\infty)$ denote the extension of $\mathbb{Q}$ generated by roots of unity of arbitrary order. By choosing a compatible system of roots of unity, we obtain a natural morphism $\text{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \rightarrow \hat{\mathbb{Z}}^\times$ which in fact is an isomorphism. Let us denote by $\sigma \mapsto t_\sigma = \prod p t_{\sigma,p}$ the following composite

$$\text{Aut}(\mathbb{C}) \rightarrow \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \rightarrow \hat{\mathbb{Z}}^\times \simeq \prod p \mathbb{Z}_p^\times$$

Let $W \in W(\Pi_p, \psi_p)$ be a Whittaker function and let $\sigma \in \text{Aut}(\mathbb{C})$. We define $^\sigma W$ by $^\sigma W(g) = \sigma(W(T_{\sigma,p}g))$ where $T_{\sigma,p} \in \text{GSp}(4, \mathbb{Z}_p)$ is defined by

$$T_{\sigma,p} = \begin{pmatrix}
-t_{\sigma,p}^{-3} & t_{\sigma,p}^{-2} & 1 \\
t_{\sigma,p}^{-1} & 1 & 0 \\
t_{\sigma,p}^{-2} & t_{\sigma,p} & 1
\end{pmatrix}$$

Note that for any $x_0, x_1, x_2, x_3 \in \mathbb{Q}_p$, we have

$$T_{\sigma,p}n(x_0, x_1, x_2, x_3)T_{\sigma,p}^{-1} = n(t_{\sigma,p}^{-1}x_0, t_{\sigma,p}^{-3}x_1, t_{\sigma,p}^{-2}x_2, t_{\sigma,p}^{-1}x_3)$$

where $n(x_0, x_1, x_2, x_3) \in N(\mathbb{Q}_p)$ is the element defined in (10). Using the fact that $\sigma(\psi_p(t_{\sigma,p}^{-1}x)) = \psi_p(x)$ for any $x \in \mathbb{Q}_p$, this shows that $^\sigma W \in W(\Pi_p, \psi_p)$. The map $W \mapsto W$ defines a $\sigma$-linear intertwining operator $W(\Pi_p, \psi_p) \rightarrow W(\sigma \Pi_p, \psi_p)$. Let $W(\Pi, \psi) = W(\Pi_\infty, \psi_\infty) \otimes W(\Pi_f, \psi_f)$ be the Whittaker model of $\Pi$. Similarly as above, we define a $\sigma$-linear intertwining operator $\tilde{\sigma} : W(\Pi_f, \psi_f) \rightarrow W(\sigma \Pi_f, \psi_f)$. Note that if $\Pi_p$ is unramified, then $W \mapsto W$ sends a spherical vector to a spherical vector and if we normalize the spherical vector to take value 1 on identity, then $W \mapsto W$ fixes this vector. This makes
the local and global actions of $\sigma$ compatible.

Applying the functor $X \mapsto \text{Hom}_{K_{\infty}}(\wedge^2 p^+ \otimes p^-, X)$ to the isomorphism $\Pi \sim \to W(\Pi, \psi)$ defined by $\Psi \mapsto W_{\Psi}$ and using the fact that $\text{Hom}_{K_{\infty}}(\wedge^2 p^+ \otimes p^-, \Pi_{\infty}^W) = H^3(\text{gs}_4, K_{\infty}, \Pi_{\infty}^{2,1})$ (see Prop. 2.3 and the proof of Lem. 2.14), we obtain the canonical isomorphism

$$H^3(\text{gs}_4, K_{\infty}, \Pi_{\infty}^{2,1}) \otimes \Pi_f \sim \to \text{Hom}_{K_{\infty}}\left(\wedge^2 p^+ \otimes p^-, W(\Pi_{\infty}, \psi_{\infty})\right) \otimes W(\Pi_f, \psi_f).$$

Recall that $\Pi_{\infty}^W|_{\text{GSp}(4,R)} = \Pi_{2,1}^2 \oplus \Pi_{\infty}^{1,2}$. Let $W_{\Psi_{\infty}} \in W(\Pi_{\infty}, \psi_{\infty})$ be the image of a highest weight vector $\Psi_{\infty}$ of the minimal $K_{\infty}$-type of $\Pi_{2,1}^{2,1}$ under the isomorphism $\Pi_{\infty}^W \sim \to W(\Pi_{\infty}, \psi_{\infty})$ induced by the isomorphism $\Pi \sim \to W(\Pi, \psi)$ fixed above. We normalize $W_{\Psi_{\infty}}$ so that

$$(12) \quad W_{X_{(-1,1)}}(I_4) = 1$$

as in Prop. 3.4. Like in Lem. 2.14, the normalized element $W_{\Psi_{\infty}}$ defines a normalized generator of the one-dimensional $\mathbb{C}$-vector space $\text{Hom}_{K_{\infty}}\left(\wedge^2 p^+ \otimes p^-, W(\Pi_{\infty}, \psi_{\infty})\right)$. Hence we obtain an isomorphism

$$i : H^3(\text{gs}_4, K_{\infty}, \Pi_{\infty}^{2,1}) \otimes \Pi_f \sim \to W(\Pi_f, \psi_f).$$

The left hand term has an $E$-structure $H^2_{dR}(\Pi_f)$ given by coherent cohomology of automorphic vector bundles (see [H94] 3. and in particular Prop. 3.3.9 for details). The following result can be proved exactly as [GrS17] Prop. 3.3.12 be replacing the Bessel model in loc. cit. by the Whittaker model.

**Proposition 3.6.** There exists $p(\Pi) \in \mathbb{C}^\times$, whose image in $\mathbb{C}^\times/E^\times$ is uniquely defined, such that for any $\sigma \in \text{Aut}(\mathbb{C}/E)$ the diagram

$$\begin{array}{ccc}
W(\Pi_f, \psi_f) & \xrightarrow{p(\Pi)^{-1} - 1} & H^2_{dR}(\Pi_f) \otimes E \mathbb{C} \\
\sigma \downarrow & & \downarrow 1 \otimes \sigma \\
W(\Pi_f, \psi_f) & \xrightarrow{p(\Pi)^{-1} - 1} & H^2_{dR}(\Pi_f) \otimes E \mathbb{C}
\end{array}$$

commutes.

**Remark 3.7.** Using the matrix

$$T'_{\sigma, p} = \begin{pmatrix} t_{\sigma, p}^{-1} \\ 1 \end{pmatrix}$$

instead of $T_{\sigma, p}$, a similar construction obviously exists for GL(2) and we denote by $p(\sigma)$ the corresponding $dR$-Whittaker period attached to the Whittaker function $W_{\Phi_{\infty}}$ normalized as in Prop. 3.4. By the $q$-expansion principle, it could be possible to compute $p(\sigma)$ up to $E^\times$ multiples. In the case of the group GL(2, $F$), where $F$ is a totally real quadratic number field, this kind of computation is performed in [K98] 3.5.
3.3.2. Rationality of local Rankin-Selberg integrals.

**Proposition 3.8.** Let \( v \) be a non-archimedean place. Assume that \( W_{\Psi_v} \) and \( W_{\Phi_v} \) satisfy \( \sigma W_{\Psi_v} = W_{\Psi_v} \), \( \sigma W_{\Phi_v} = W_{\Phi_v} \) and \( \sigma \circ \varphi_v = \varphi_v \) for any \( \sigma \in \text{Aut}(\mathbb{C}/E) \). Then

\[
Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) \bigg|_{s=1} \in E.
\]

Furthermore, there exist \( W_{\Psi_v}, W_{\Phi_v} \) and \( \varphi_v \) satisfying the condition above and such that

\[
Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) \bigg|_{s=1} \in E^\times
\]

and

\[
\int_{\mathbb{Q}_p^3} \varphi_v(x, y) \, dx \, dy \in E^\times.
\]

**Proof.** Following [So84] p. 380, we write for \( \text{Re}(s) \) big enough

\[
Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) = \sum_{j=-\infty}^{+\infty} A_j(W_{\Psi_v}, W_{\Phi_v}, \varphi_v) p^{js}
\]

where

\[
A_j(W_{\Psi_v}, W_{\Phi_v}, \varphi_v) = \int_{(h_1, h_2) \in N_H(\mathbb{Q}_p) \setminus H(\mathbb{Q}_p), |\det h_1| = p^j} W_{\Psi_v}(h_1, h_2) W_{\Phi_v}(h_2) \varphi_v((0, 1)h_1) \, dh.
\]

This integral is absolutely convergent and reduces to a finite sum because the integrated function has compact support modulo \( N_H(\mathbb{Q}_p) \) in the set \( \{(h_1, h_2) \in H(\mathbb{Q}_p), |\det h_1| = p^j\} \) ([CS80] Prop. 6.1) and is invariant by right translation by a sufficiently small open subgroup. Furthermore it vanishes for \( j \) big enough. Let \( \sigma \in \text{Aut}(\mathbb{C}) \). Fixing \( j \) and computing integrals over \( \{(h_1, h_2) \in N_H(\mathbb{Q}_p) \setminus H(\mathbb{Q}_p), |\det h_1| = p^j\} \), we have

\[
A_j(\sigma W_{\Psi_v}, \sigma W_{\Phi_v}, \sigma \circ \varphi_v)
\]

where the first equality follows from the definition of \( \sigma W_{\Psi_v} \) and \( \sigma W_{\Phi_v} \), the second from the fact that \( W_{\Phi_v} \) has trivial central character, and the last from an easy change of variable. In particular, under the assumptions of the Proposition, the function \( Z_v(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v}) \)
is an element of \( E((p^{-s})) \) and it follows from [So84] Lem. 3.2 that this function extends to an element of \( E(p^{-s}) \). This implies the first statement. The non-vanishing of \[
\frac{Z_{\nu}(s, W_{\Psi_v}, W_{\Phi_v}, f_{\varphi_v})}{L(s, \Pi_v \times \sigma_v)} \bigg|_{s=1}
\]
for well chosen functions \( W_{\Psi_v}, W_{\Phi_v} \) and \( \varphi_v \) satisfying the assumptions of the Proposition is a direct consequence of Prop. 3.3 and of the fact that functions satisfying \( \sigma W_{\Psi_v} = W_{\Psi_v}, \sigma W_{\Phi_v} = W_{\Phi_v} \) and \( \sigma \circ \varphi_v \) for any \( \sigma \in \text{Aut}(\mathbb{C}/E) \) define \( E \)-structures in the corresponding \( \mathbb{C} \)-vector spaces. The existence of \( W_{\Psi_v}, W_{\Phi_v} \) and \( \varphi_v \) such that in addition one has
\[
\int_{\mathbb{Q}_v^2} \varphi_v(x, y) \, dx \, dy \in E^\times
\]
is a consequence of the following elementary result: let \( k \) be a field and let \( V \) be a \( k \)-vector space with two non-zero linear functionals \( l_1, l_2 : V \to k \); then \( \ker l_1 \cup \ker l_2 \neq V \). \( \square \)

4. PROOF OF THE MAIN RESULT

Let \( p(\Pi) \), resp. \( p(\sigma) \), be the de Rham-Whittaker periods attached to \( \Pi \), resp. \( \sigma \), defined and normalized in section 3.3. Let \( p(\Pi \times \sigma) \) denote the product \( p(\Pi)p(\sigma) \).

**Theorem 4.1.** Assume that \( L(s, \Pi \times \sigma) \) has a pole at \( s = 1 \) and that \( \Pi \) and \( \sigma \) have trivial central characters. Then
\[
Z(\Pi_f \times \sigma_f) = p(\Pi \times \sigma) \text{Res}_{s=1} L(s, \Pi \times \sigma)\mathcal{D}(\Pi_f \times \sigma_f).
\]

**Proof.** Let \( \Psi = \bigotimes_v \Psi_v \) be factorizable a cusp form in the representation space of \( \Pi \) such that \( \Psi_\infty \) is a highest weight vector of the minimal \( K_\infty \)-type normalized as in (12) and such that for any non-archimedean place \( v \), the Whittaker function \( W_{\Psi_v} \) satisfies \( \sigma W_{\Psi_v} = W_{\Psi_v} \) for any element \( \sigma \) of \( \text{Aut}(\mathbb{C}/E) \). Let \( \Phi = \bigotimes_v \Phi_v \) be a factorizable cusp form in the representation space of \( \sigma \) such that \( \Phi_\infty \) is a generator of the minimal \( L_\infty \)-type of \( \sigma_\infty \) normalized in such a way that \( W_{\Phi_v}(I_2) = c_2 \) as in Prop. 3.4 and such that for any non-archimedean place \( v \), the Whittaker function \( W_{\Phi_v} \) satisfies \( \sigma W_{\Phi_v} = W_{\Phi_v} \) for any \( \sigma \in \text{Aut}(\mathbb{C}/E) \). It follows from Prop. 2.19 and the Hodge decomposition (2) that we have the isomorphism \( F^2M_{dR}(\Pi_f)_C \simeq M^{2,1}(\Pi_f) \) induced by the comparison isomorphism. Hence, by Prop. 3.6, the vector \( p(\Pi)\omega_\Psi \) is a generator of \( F^2M_{dR}(\Pi_f) \) and, similarly, the vector \( p(\sigma)\eta_\Phi \) is a generator of \( F^1M_{dR}(\sigma_f) \). According to Cor. 2.23, we have
\[
Z(\Pi_f \times \sigma_f) = \pi^8 p(\Pi \times \sigma) (v_\Psi, \omega_\Psi \otimes \eta_\Phi)_{B,C} \mathcal{D}(\Pi_f \times \sigma_f).
\]
Applying to Prop. 2.17 and Cor. 3.1 we have
\[
Z(\Pi_f \times \sigma_f) = \pi^8 p(\Pi \times \sigma) \tilde{\varphi}(0)^{-1} \text{Res}_{s=1} Z \left( s, X_{(-1,1)} \Psi, \Phi, f_{\varphi} \right) \mathcal{D}(\Pi_f \times \sigma_f)
\]
where in the last equality \( \varphi = \prod_v \varphi_v \) is any factorizable Schwartz-Bruhat function on \( \mathbb{A}_k^2 \) whose archimedean component is \( \varphi_\infty(x, y) = \exp(-\pi(x^2+y^2)) \) and whose non-archimedean components at ramified places are given by Prop. 3.8. For such a choice of \( \varphi \), we have \( \tilde{\varphi}(0) \in E^\times \). Hence the statement follows from the combination of Prop. 3.4 and Prop. 3.8. \( \square \)
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