Proof of the Kakutani metrization theorem

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Theorem 0.1. *Let G be a group, let (U_n) be a decreasing family of symmetric subsets of G containing 1_G such that for all $n \in \mathbb{N}$,*

$$U_{n+1}^{\cdot 2} \subseteq U_n.$$

Then there is a pseudo-metric d on G such that

(i) The pseudo-metric d is left invariant.

(ii) For all $x \in U_n$, $d(1, x) \leqslant 3 \cdot 2^{-n}$

(iii) For all $x \notin U_n$, $d(1, x) \geqslant 2^{-n}$

Proof. First note that by using our hypothesis that for all $n \in \mathbb{N}$, $U_{n+1}^{\cdot 2} \subseteq U_n$, a straightforward induction on k yields that for every $n_0 < n_1 < \cdots < n_k$ we have

$$U_{n_0} \supseteq U_{n_1} U_{n_2} \cdots U_{n_k}. \tag{*}$$

We will now build a natural analogue of a norm in our group G using a construction reminiscent of the Urysohn construction of a continuous test function on a regular space.

For each $n \in \mathbb{N}^*$ Let $V_{2^{-n}} := U_n$. We will extend this definition so that V_r makes sense for every r positive dyadic fraction. First, for all $r \geqslant 1$ we let $V_r = G$. Next, for $r \in \mathbb{N}^*[1/2]$ and $r < 1$ write the dyadic expansion of r as $r = 2^{-n_1} + \cdots + 2^{-n_k}$ with $n_1 < \cdots < n_k$. Then let

$$V_r = V_{2^{-n_1}} \cdots V_{2^{-n_k}}.$$

Given $r, s \in \mathbb{N}^*[1/2]$, note that if the biggest exponent n_k in the dyadic expansion of r satisfies $2^{-n_k} > s$ then

$$V_{r+s} = V_r V_s$$

Now property (*) may be restated as: for every $n_0 < n_1 < \cdots < n_k$, if we let $t = 2^{-n_1} + 2^{-n_2} + \cdots + 2^{-n_k}$ then

$$V_{2^{-n_0}} \supseteq V_t.$$

We will use this property to show that the V_r's are nested, i.e.

$$\forall r < s, V_r \subseteq V_s \tag{1}$$

So suppose $r < s$, write $r = 2^{-n_1} + \cdots + 2^{-n_k}$ with $n_1 < \cdots < n_k$ and $s = 2^{-m_1} + \cdots + 2^{-m_l}$ with $m_1 < \cdots < m_l$. Let i be the first integer such that $m_i \neq n_i$, then since $r < s$ we must have $m_i < n_i$. Since $m_i < n_i < n_{i+1} < \cdots n_k$, property (*) yields

$$V_{2^{-m_i}} \supseteq V_{2^{-n_i}} \cdots V_{2^{-n_k}}.$$

Multiplying each side by $V_{2^{-n_1}} \cdots V_{2^{-n_{i-1}}}$ on the left, we finally obtain

$$V_s \supseteq V_{2^{-n_1}} \cdots V_{2^{-n_{i-1}}} V_{2^{-m_i}} \supseteq V_r$$

We now define a function which will behave almost like a norm on G: for $g \in G$ we let

$$\phi(g) := \inf\{r : g \in V_r\}.$$

Then clearly $\phi(g) \leqslant 2^{-n}$ for all $x \in U_n$, and $\phi(g) \geqslant 2^{-n}$ for all $g \notin U_n$. Moreover since each V_r contains 1_G, we have $\phi(1_G) = 0$.

We will now see that ϕ satisfies a uniform continuity-like inequality, which will allow us to build a left-invariant pseudo metric with nice properties out of it.

The key to this inequality an estimate on products of the V_r's, namely for all $r \in \mathbb{N}[1/2]$ and $n \in \mathbb{N}$,

$$V_r V_{2^{-n}} \subseteq V_{r+3\cdot 2^{-n}} \tag{2}$$

First note that if n is stricly larger than the biggest exponent in the dyadic expansion of r, then we have

$$V_r V_{2^{-n}} = V_{r+2^{-n}} \subseteq V_{r+3\cdot 2^{-n}}.$$

If not, write $r = 2^{-n_1} + \cdots + 2^{-n_k}$ with $n_1 < \cdots < n_k$. Let $i \in \mathbb{N}$ such that $n_i < n \leqslant n_{i+1}$ (if $n \leqslant n_1$ we take $i = 0$). Then consider $r' = 2^{-n_1} + \cdots + 2^{-n_i} + 2 \cdot 2^{-n}$. Observe that $r' > r$, $r' - r \leqslant 2 \cdot 2^{-n}$ and n is stricly larger than the biggest exponent in the dyadic expansion of r'. We thus have

$$V_r V_{2-n} \subseteq V_{r'} V_{2^{-n}} = V_{r'+2^{-n}}.$$

Since $r \leqslant 2 \cdot 2^{-n} + r$ we conclude $V_{r'+2^{-n}} \subseteq V_{r+3\cdot 2^{-n}}$ as wanted.

We can now use 2 to establish the desired uniform continuity-like inequality, namely

$$\text{for all } x \in V_{2^{-n}} \text{ and all } g \in G, \quad |\phi(gx) - \phi(g)| \leqslant 3 \cdot 2^{-n} \tag{3}$$

Equation 2 implies that for all $g \in G$ and all $x \in V_{2^{-n}}$, if $\phi(g) < r$, then $\phi(gx) \leqslant r + 3 \cdot 2^{-n}$. Taking the infimum over all r such that $\phi(g) < r$, we conclude that

$$\phi(gx) \leqslant \phi(g) + 3 \cdot 2^{-n}$$

for all $g \in G$ and all $x \in V_{2^{-n}}$. Since $V_{2^{-n}} = U_n$ is symetric, we deduce that for all $g \in G$ and all $x \in V_{2^{-n}}$, we have $\phi(gx^{-1}) \leqslant \phi(g) + 2^{-n}$. Replacing g by gx in the above equation ($g \mapsto gx$ is a bijection of G!), we conclude that for all $g \in G$ and $x \in V_{2^{-n}}$,

$$\phi(g) \leqslant \phi(gx) + 3 \cdot 2^{-n},$$

which finishes the proof of equation 3.

We have a nice pseudometric for points close to the identity given by $(x, y) \mapsto |\phi(x) - \phi(y)|$. We will now propagate it to the whole group by letting

$$d(x, y) = \sup_{g \in G} |\phi(gx) - \phi(gy)| .$$

Let us now check our function d is a pseudo-metric. Clearly $d(x, x) = 0$ for all $x \in G$ and $d(x, y) = d(y, x)$ for all $x, y \in G$. For the triangle inequality, note that for all $g, x, y, z \in G$

$$|\phi(gx) - \phi(gz)| \leqslant |\phi(gx) - \phi(gy)| + |\phi(gy) - \phi(gz)| \leqslant d(x, y) + d(y, z).$$

Taking the supremum over $g \in G$ establishes the triangle inequality $d(x, z) \leqslant d(x, y) + d(y, z)$.

Finally, let us check that d has the desired properties.

(i) Given $x, y, h \in G$ we have

$$d(hx, hy) = \sup_{g \in G} d(ghx, ghx).$$

Since $g \mapsto g$ is a bijection of G we can replace gh by g in the right-hand term, which establishes left-invariance.

(ii) Given $x \in U_n = V_{2-n}$ and $h \in G$, equation 3 implies $|\phi(gx) - \phi(g)| \leqslant 3 \cdot 2^{-n}$ for all $g \in G$, so by taking the supremum $d(1_G, x) \leqslant 3 \cdot 2^{-n}$.

(iii) Given $x \notin U_n = V_{2^{-n}}$ we have $\phi(x) \geqslant 2^{-n}$ and since $\phi(1_G) = 0$ we deduce $|\phi(x) - \phi(1_G)| \geqslant 2^{-n}$. We conclude that $d(1_G, x) \geqslant 2^{-n}$. □