Polish groups and non-free actions in the discrete or measurable context

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Introduction

In this text which presents some of the author's work in the past nine years, the main theme is the study of actions of countable groups on sets with little to no structure. The emphasis is on non-free actions since free actions are completely described by their number of orbits in the absence of an underlying structure. We adopt a descriptive set-theoretic viewpoint by considering various *Polish* spaces of actions, thus allowing the use of the Baire category theorem.

The fact that our action spaces are Polish is each time related to Polish groups: all elements of the action spaces we consider can be seen as group homomorphisms taking values in a specific Polish group. We explain in details why relevant spaces and groups are Polish in Part I. Although this part is directed towards non-specialists, our presentation contains a few facts which are hard to track in the literature. Let us now list the Polish groups and spaces which will be the protagonists of this thesis, along with a brief presentation of its content.

The group $\mathfrak{S}_\infty$ of all permutations of the integers. Presented in Section 3, $\mathfrak{S}_\infty$ plays a prominent role: it encapsulates all actions on countable discrete sets, which are the topic of Part II.

The group $\mathfrak{S}_\infty$ first arises via ample generics, which are discussed in Section 8. Indeed, $\mathfrak{S}_\infty$ is the fundamental example of a Polish non-archimedean group with ample generics. This by now well-known fact is proved in Section 8.1, where we highlight the connection with the space of actions on the integers of finitely generated free groups. We then give examples of Polish groups with ample generics which are not non-archimedean in Section 8.2, starting with Malicki's examples, and then giving the connected examples we obtained with Kaïchouh. In Section 8.3 we discuss *quasi non-archimedeanity*, a property of Polish groups which we introduced with Gelander. This property is satisfied by all known examples of Polish groups with ample generics, and it is equivalent to being totally disconnected for locally compact groups. We discuss in Section 8.4 a new negation of quasi non-archimedeanity called almost n-archimedeanity, and related properties. We finally compare ample generics to various archimedianity notions in Section 8.5.

The group $\mathfrak{S}_\infty$ also appears when considering spaces of transitive actions on countable sets, discussed in Section 9. Another natural model for such a space is the space of infinite index subgroups of a countable group Γ, and Section 9.1 is essentially devoted to the proof that these two spaces are the same from our point of view (Theorem 9.9). Section 9.2

presents an important property of transitive actions called high transitivity, which can be reformulated as having dense image in $\mathfrak{S}_\infty$. This in turn yields a group property also called high transitivity, which means being isomorphic to a dense subgroup of $\mathfrak{S}_\infty$. Other important properties of transitive actions related to the topology on the space of infinite index subgroups are discussed in Section 9.3: weak containment, high faithfulness and finally totipotency.

Section 10 is entirely devoted to high transitivity for group acting on trees. Our main goal is to present a result obtained with Fima, Moon and Stalder, namely high transitivity for an optimal subclass of groups acting on trees. This result is stated in Section 10.1. The goal of the remaining sections is to prove the result for free products. We actually give a new stronger statement for (non-amenable) free products: their generic transitive actions on a countable infinite set are highly transitive. We present the required Bass-Serre theory for the proof in Section 10.2, and some additional facts on free products and trees in Section 10.3. Finally, the proof is given in Section 10.4.

The space of Y-valued measurable maps $\mathrm{L}^0(X,\mu,Y)$. Given a Polish space Y and a standard probability space (X,μ), the space $\mathrm{L}^0(X,\mu,Y)$ of Y-valued measurable maps can be endowed with the Polish topology of convergence in measure, as explained in Section 4.

When Y is actually a Polish group, one gets a connected Polish group which plays an important role in our aforementioned work with Kaïchouh on ample generics: we proved that if G has ample generics, then so does $\mathrm{L}^0(X,\mu,G)$, thus providing the first examples of connected Polish groups with ample generics (see Section 8.2.2).

Although we barely mention it in this text, this construction has also been invaluable in the construction of Polish full groups undertaken with Carderi, and then of L^1 full groups, see Remark 7.13.

The groups of measure-preserving and non-singular transformations. Given a standard probability space (X,μ), there are two natural Polish groups to consider: the group $\mathrm{Aut}(X,\mu)$ of measure-preserving bijections and the group $\mathrm{Aut}^*(X,\mu)$ of non-singular (also called measure class-preserving) bijections.

Since $\mathrm{Aut}(X,\mu)$ being G_δ in $\mathrm{L}^0(X,\mu,X)$ was key in our work with Carderi on Polish full groups, we took the opportunity to directly derive the Polishness of $\mathrm{Aut}(X,\mu)$ from this fact in Section 5. We also explain why $\mathrm{Aut}^*(X,\mu)$ is a Polish group in Section 6, using a more conventional approach. We briefly mention how automatic continuity of $\mathrm{Aut}(X,\mu)$ was used in our proof that $\mathrm{Aut}^*(X,\mu)$ has a unique Polish group topology, and set the stage for the last protagonists by showing that the two-sided uniform metric is complete, a well-known but seldom proved fact.

Full groups of non-singular equivalence relations. Our last protagonist comes directly from orbit equivalence theory: given a non-singular action of a countable group Γ on (X,μ), the equivalence relation whose classes are the Γ-orbits is a non-singular equivalence relation $\mathcal{R}$, and its full group $[\mathcal{R}]$ is the group of non singular bijections whose graph is contained in $\mathcal{R}$. Section 7 is devoted to basic facts on these Polish groups which can be hard to find when venturing in the non probability measure-preserving world. Full groups then appear in this text in two distinct ways.

The first is ample generics: as it turns out, full groups of ergodic $\mathbb{Z}$-actions have ample generics as soon as the associated equivalence relation is not type II_1. This result was obtained with Kaïchouh in the type III case, and the remaining type II_∞ case is treated here, in Section 8.2.3. As in our joint paper, we emphasize the natural connection with $\mathfrak{S}_\infty$ by embedding full groups in $\mathrm{L}^0(X, \mu, \mathfrak{S}_\infty)$, allowing us to think of full groups elements as random permutations, and more generally of group homomorphisms $\Gamma \to [\mathcal{R}]$ as random Γ-actions on $\mathbb{N}$.

The second way full groups appear is in the study of non-free actions in the measurable context, which is the topic of Part III. Indeed, the action on the underlying probability space of dense subgroups of full groups enjoy remarkable properties, even when forgetting about the whole measurable structure. One of these is a strengthening of high transitivity we call permutational fullness, another is the fact that density is inherited by stabilizers. All these properties are studied in Section 11 and were obtained with Carderi and Gaboriau in an unpublished work. We restrict to the probability-measure preserving setup for simplicity, but some results like Theorem 11.9 hold in the non-singular setup as well. Section 12 finally presents our result with Carderi and Gaboriau: under an optimal hypothesis on the cost of a given ergodic p.m.p. equivalence $\mathcal{R}$, we construct dense free subgroups of its full group whose associated random transitive action is almost surely totipotent.

List of the author's works presented in this thesis

[T1] Adriane Kaïchouh and François Le Maître. Connected Polish groups with ample generics. *Bulletin of the London Mathematical Society*, 47(6):996–1009, 2015.

[T2] Tsachik Gelander and François Le Maître. Infinitesimal topological generators and quasi non-archimedean topological groups. *Topology and its Applications*, 218:97–113, 2017.

[T3] François Le Maître. Highly faithful actions and dense free subgroups in full groups. *Groups, Geometry, and Dynamics*, 12(1):207–230, 2018.

[T4] Pierre Fima, François Le Maître, Soyoung Moon, and Yves Stalder. A characterization of high transitivity for groups acting on trees. *Discrete Analysis*, 2022.

[T5] Alessandro Carderi, Damien Gaboriau, and François Le Maître. On dense totipotent free subgroups in full groups. *Geometry & Topology*, 27(6):2297–2318, 2023.

Contents

I Some spaces of maps and Polish groups

In this first part, we review various constructions of Polish groups and spaces which will be useful in the sequel.

1 Basic facts on Polish spaces

Before giving examples of Polish groups, we need a few facts on Polish spaces. A **Polish space** is a separable topological space whose topology admits a compatible[1] complete metric. The existence of a compatible complete metric implies that the Baire category theorem holds (any countable intersection of dense open sets must be dense itself), while the separability implies that various relevant sets can be written as countable intersections of (dense) open sets. Since countable intersection of open sets play a crucial role in the theory of Polish spaces, let us give their nickname right away.

Definition 1.1. A subset Y of a topological space X is G_δ if it can be written as a countable intersection of open sets.

An important family of G_δ subsets is provided by closed subsets of metrizable spaces. Indeed if (X, d) is a metric space and $F \subseteq X$ is closed, then F can be written as the intersection over $n \in \mathbb{N}$ of the open set of elements at distance less than $\frac{1}{n}$ from F. Another example of G_δ subset is obtained by removing countably many points from the ambient space, for instance $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ subset of the real line ($\mathbb{R} \setminus \mathbb{Q}$ is actually homeomorphic to the product space $\mathbb{N}^\mathbb{N}$ which plays a central role in the theory of Polish spaces).

Observe that a metric space is separable if and only if it is second-countable, i.e. its topology admits a countable basis. Since any subspace of a second-countable topological space is itself second-countable, subspaces of Polish spaces are always separable for the induced topology[2]. The question of which subspaces are actually Polish is thus settled by the following important result, which encompasses the well-known fact that closed subspaces of complete metric spaces are complete for the induced metric.

Theorem 1.2 (Alexandrov, see [Kec95, Thm. 3.11])**.** *A subspace of a complete metric space admits a compatible complete metric for the induced topology if and only if it is G_δ.*

Corollary 1.3. *Let X be a Polish space, let $Y \subseteq X$. Then Y is Polish for the induced topology if and only if Y is a G_δ subset of X.* □

[1] A metric is called compatible with a topology when it induces this topology.

[2] Separability does not pass to subspaces in general, e.g. the space $\{0,1\}^{[0,1]}$ of subsets of $[0,1]$ is separable for the product topology (consider finite unions of intervals with rational endpoints) but the subspace of singletons is not separable since it is discrete for the induced topology.

Observe that any countable intersection of G_δ sets is G_δ, and that in Polish spaces countable intersections of *dense* G_δ sets are also dense G_δ: they can be rewritten as countable intersections of dense open sets, so they are dense by Baire's category theorem.

If $f : X \to Y$ is a map and (U_n) is a family of subsets of Y, we always have $f^{-1}(\bigcap_n U_n) = \bigcap_n f^{-1}U_n$, so the inverse image of a G_δ set by a continuous map is G_δ. This is not true at all for direct images since the image of an open set needs not be open and moreover direct images do not commute with countable intersections. We can however get around this problem in certain specific cases, using the following fundamental result.

Theorem 1.4 (Sierpinski, see [Gao09, Thm. 2.2.9])**.** *Let X be a Polish space, let Y be a metrizable topological space, and let $f : X \to Y$ be a continuous open map. Then $f(X)$ is Polish.*

Restrictions of open maps need not be open, so in order to use this theorem, we make a small observation.

Lemma 1.5. *Let X, Y be topological spaces, let $f : X \to Y$ be open, and suppose that $Z \subseteq X$ satisfies $Z = f^{-1}(f(Z))$. Then $f_{\restriction Z} : Z \to f(Z)$ is also open.*

Proof. Let $U \subseteq Z$ be open, then $U = Z \cap V$, where V is an open subset of X. We claim that $f(U) = f(Z) \cap f(V)$, so $f(U)$ is open in $f(Z)$ as wanted. Indeed, the inclusion $f(U) \subseteq f(Z) \cap f(V)$ always hold since $U = Z \cap V$, and the reverse inclusion follows from the fact that $Z = f^{-1}(f(Z))$: if $y \in f(Z) \cap f(V)$, write $y = f(z) = f(v)$ for some $z \in Z$ and $v \in V$, we then have $v \in f^{-1}(f(Z))$ and thus $v \in Z$, yielding $v \in U$ and hence $y \in f(U)$ as wanted. □

Corollary 1.6. *Let X be a Polish space, let Y be a metrizable topological space, and let $f : X \to Y$ be a continuous open map. Then for all G_δ subset $Z \subseteq X$ such that $Z = f^{-1}(f(Z))$, we have that $f(Z)$ is G_δ in Y.*

Proof. By Corollary 1.3, the set Z is Polish, and the restriction of f to a map $f : Z \to f(Z)$ is continuous if we endow $f(Z)$ with the induced topology from X, which is metrizable.

The restriction of f to Z is continuous, and by the previous lemma it is also open. Using the above theorem, we obtain that $f(Z)$ is Polish. So $f(Z)$ is G_δ by Corollary 1.3. □

Remark 1.7. The hypothesis that $Z = f^{-1}(f(Z))$ can be restated as the fact that Z is a union of fibers of the map f. This hypothesis is fundamental: using the Baire space, one can show that when $X = \mathbb{R}^2$ and f is the projection onto the first coordinate, any analytic subset of $\mathbb{R}$ arises as the projection of a G_δ subset of $\mathbb{R}^2$.

Another key fact about Polish spaces is the following.

Proposition 1.8. *Any countable product of Polish spaces is Polish for the product topology.*

Let us give a few relevant examples of Polish spaces:

- Any discrete countable space is Polish, since its topology is induced by the complete metric
$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

- Taking advantage of Proposition 1.8, we deduce that $\mathbb{N}^{\mathbb{N}}$ is a Polish space. It is sometimes called the *Baire space*.
- For all $n \geqslant 1$, $\mathbb{R}^n$ is a Polish space. More generally any locally compact second-countable space is Polish. However $\mathbb{R}^{\mathbb{N}}$ is Polish but not locally compact, because compact sets therein must have empty interior (our previous example $\mathbb{N}^{\mathbb{N}}$ also fails to be locally compact for the same reason).

The following theorem is a cornerstone of descriptive set theory. Recall that the **Borel** σ-algebra of a topological space is the σ-algebra generated by open sets. Elements of this σ-algebra are simply called Borel sets, and a map $f : X \to Y$ is called **Borel** if $f^{-1}(U)$ is Borel whenever $U \subseteq Y$ is open (equivalently, if the preimage of any Borel set is Borel).

Theorem 1.9 (Lusin-Suslin, see [Kec95, Thm. 15.1])**.** *Let X and Y be Polish spaces, let $f : X \to Y$ be a Borel injective map. Then for every $A \subseteq X$ Borel, the set $f(A)$ is also Borel.*

We end this section with a brief discussion of **Baire measurability**, referring to [Kec95, Chapter 8] for details. The notion of a dense G_δ subset in a Polish space is a good analogue of a full measure subset in a measured space since countable intersections of dense G_δ sets are themselves dense G_δ. The following two definitions are thus a natural analogue of measure zero subset, and of Lebesgue measurable sets respectively.

Definition 1.10. Let X be a Polish space. A subset of X is called **comeager** if it contains a dense G_δ set, and **meager** if its complement is comeager. A subset $B \subseteq X$ is called **Baire measurable** if there exists an open set $U \subseteq X$ such that $B \triangle U$ is meager.

By the Baire category theorem, countable intersections of comeager sets are comeager, so by taking complements countable unions of meager sets are meager. It easily follows that countable unions of Baire-measurable sets are Baire measurable. To see that complements of Baire measurable sets are Baire-measurable, first note that whenever $U \subseteq X$ is open, the set $\bar{U} \setminus U$ is meager because it is closed with empty interior. In particular $X \setminus U$ is Baire measurable because it differs from the open set $X \setminus \bar{U}$ by a meager set. Now if A differs from U by a meager set, the complement of A differs from the complement of U by the same meager set, and since the union of two meager sets is meager, it follows that $X \setminus A$ differs from the open set $X \setminus \bar{U}$ by a meager set. We have shown the following.

Proposition 1.11. *Let X be a Polish space. The set of all Baire measurable subsets of X forms a σ-algebra. In particular, it contains all Borel subsets of X.* $\square$

Let X and Y be Polish spaces, we say that a map $f : X \to Y$ is *Baire-measurable* if for all $U \subseteq Y$ open, the set $f^{-1}(U)$ is Baire-measurable, or equivalently if for all $B \subseteq Y$ Borel, the set $f^{-1}(B)$ is Baire-measurable[3].

Using the fact that Lebesgue measurable sets differ from open sets by arbitrarily small measure subsets, one obtains Lusin's theorem: Lebesgue-measurable maps admit continuous restrictions to arbitrarily large measure subsets. For Baire-measurable maps the situation is even better.

[3]It is important to note here that, as for Lebesgue measurability, we do note require preimages of Baire-measurable sets to be Baire-measurable, as there are even continuous functions which fail this property.

Theorem 1.12. *Let X and Y be Polish spaces, let $f : X \to Y$ be Baire-measurable. Then there is a dense G_δ subset $Z \subseteq X$ such that the restriction of f to Z is continuous.*

Proof. Since Y is Polish, it is second-countable: let $(U_n)_{n\in\mathbb{N}}$ be a basis for its topology. For every $n \in \mathbb{N}$, since f is Baire-measurable we have an open set $V_n \subseteq X$ such that the set $f^{-1}(U_n) \bigtriangleup V_n$ is meager. Fix a dense G_δ set Z_n disjoint from $f^{-1}(U_n) \bigtriangleup V_n$. Then the restriction of f to the dense G_δ set $Z := \bigcap_n Z_n$ is continuous since the preimage of every U_n is now equal to the open set $Z \cap V_n$, and $(U_n)_{n\in\mathbb{N}}$ is a basis of the topology of Y. □

2 Basic facts on Polish groups.

By definition, a Polish group is a topological group whose underlying topology is Polish. Basic examples are provided by locally compact second-countable groups since their topologies are Polish. Also note that thanks to Proposition 1.8, the class of Polish groups is closed under countable products, so for instance $\mathbb{Z}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{N}}$ are Polish groups.

It is very easy to check that a group with a Polish topology is a topological group (and hence a Polish group) thanks to the following proposition.

Proposition 2.1 ([Kec95, Ex. 14.15])**.** *Let G be a group endowed with a Polish topology. Then G is a Polish group if and only if multiplication is separately continuous: whenever $g_n \to g \in G$ and $h \in G$, we both have*

$$\lim_{n\to+\infty} g_n h = gh \text{ and } \lim_{n\to+\infty} hg_n = hg.$$

Proof. Consider the action of G on itself by left translation, which is separately continuous by hypothesis. It is thus continuous by [Kec95, Thm. 9.14], which means that group multiplication is a continuous map $G \times G \to G$. Now observe that the graph of the inverse map is a closed set because it is the set of pairs (g, h) such that $gh = 1$. We deduce that the inverse map is Borel by [Kec95, Thm. 14.12]. In particular, the inverse map is Baire- measurable so by Theorem 1.12 we find a dense G_δ subset $X \subseteq G$ such that the restriction of $g \mapsto g^{-1}$ to X is continuous.

Now let $g_n \to g \in G$, consider $h \in g^{-1}X \cap \bigcap_n g_n^{-1}X$ (which exists because the latter set is a countable intersection of dense G_δ sets, hence it is dense G_δ itself). Then $gh \in X$ and $g_n h \in X$ for all n. In particular, $(g_n h)^{-1} \to (gh)^{-1}$, so $h^{-1}g_n^{-1} \to h^{-1}g^{-1}$, and so $g_n^{-1} \to g^{-1}$ as wanted. □

In the previous section, we mentioned that Polish subspaces of Polish spaces are exactly G_δ subspaces. For Polish groups, we have the following stronger result.

Proposition 2.2. *Let G be a Polish group, let $H \leqslant G$ be a subgroup. Then H is Polish for the induced topology if and only if H is closed in G.*

Proof. If H is closed, then it is clearly Polish, so we need to show that if H is Polish, then it is closed. By Corollary 1.3, H is G_δ, so it is dense G_δ in its closure $\bar{H}$ which is Polish. Now let $g \in \bar{H}$. Then gH is also dense G_δ in $\bar{H}$, so by the Baire category theorem $gH \cap H$ is not empty, which implies that $g \in H$. We conclude that $\bar{H} = H$, so H is closed. □

The following automatic continuity result is fundamental.

Theorem 2.3 (Banach, [Ban32])**.** *Let G and H be Polish groups, let $\pi : G \to H$ be a Baire-measurable group homomorphism. Then π is continuous.*

Proof. By Theorem 1.12, there is a dense G_δ subset $X \subseteq G$ such that π is continuous when restricted to X. Let $g_n \to g$ in G. As in the end of the proof of Theorem 2.1, we fix an element h in the dense G_δ set $g^{-1}X \cap \bigcap_n g_n^{-1}X$, then since $\pi_{\restriction X}$ is continuous we have

$$\pi(g_n)\pi(h) = \pi(g_n h) \to \pi(gh) = \pi(g)\pi(h),$$

so $\pi(g_n) \to \pi(g)$ as wanted. $\square$

In general, one often encounters the following situation: we have a Polish group H and a subgroup $G \leqslant H$ which is also Polish, but for another topology. The following result says in particular that if G sits reasonably well inside H (i.e. if the inclusion map is Baire-measurable), his topology has to refine that of H and is the unique Polish group topology on G such that the inclusion $G \hookrightarrow H$ is Baire-measurable.

Proposition 2.4. *Let G and H be Polish groups, let $\pi : G \to H$ be a group homomorphism, suppose that π is a Baire-measurable map. Then π is continuous, $\pi(G)$ is a Borel subgroup of H, and $\pi(G)$ admits a unique Polish group topology such that $\pi : G \to \pi(G)$ is Baire-measurable (equivalently, continuous).*

Proof. Since π is Baire-measurable it has to be continuous by Banach's theorem which we just stated. In particular its kernel is closed, so $G/\mathrm{Ker}\,\pi$ is a Polish group by [Gao09, Thm. 2.2.10]. Then π induces a continuous injective homomorphism $\bar{\pi} : G/\mathrm{Ker}\,\pi \to H$. Since $\bar{\pi}$ is continuous injective, the set $\bar{\pi}(G/\mathrm{Ker}\,\pi) = \pi(G)$ is Borel by Theorem 1.9. Denote by τ the Polish group topology on $\pi(G)$ that we obtain by identifying $\pi(G)$ to $G/\mathrm{Ker}\,\pi$ via $\bar{\pi}$, by construction $\pi : G \to \pi(G)$ is continuous and since Baire-measurable homomorphisms are continuous we have to show that τ is the only Polish group topology such that $\pi : G \to \pi(G)$ is continuous. So suppose τ' is another Polish group topology on $\pi(G)$ which makes π continuous. By the universal property of the quotient topology, τ' has to refine τ. But then the identity map $(\pi(G), \tau') \to (\pi(G, \tau))$ is continuous injective, so appealing once more to Theorem 1.9 its inverse is also Borel. By Banach's theorem, this inverse map is continuous so the identity map $(\pi(G), \tau') \to (\pi(G, \tau))$ is a homeomorphism, which means that $\tau = \tau'$ as wanted. $\square$

Remark 2.5. In this text, we will only use this proposition when π is injective, but the general form is useful to keep in mind.

Some Polish groups G satisfy the remarkable property that their Polish group topology is actually the *only* Polish group topology one can endow them with. Even more remarkable is the **automatic continuity property**, which states that given any separable topological group H, every group homomorphism $G \to H$ has to be continuous.

Proposition 2.6. *Suppose a Polish group G satisfies the automatic continuity property. Then its Polish group topology is unique.*

Proof. Let τ be the Polish group topology for which G enjoys the automatic continuity property, let τ' be another Polish group topology on G. The identity map $(G, \tau) \to (G, \tau')$ has to be continuous so $\tau = \tau'$ from the uniqueness in the previous proposition. $\square$

Finally, let us mentions that one always has a natural way of building a compatible complete metric on any Polish group G: one starts with a left-invariant metric d_l as provided by the Birkhoff-Kakutani theorem, and then the metric d defined by

$$d(g, h) \coloneqq d_l(g, h) + d_l(g^{-1}, h^{-1})$$

is automatically complete (see [Gao09, Cor. 2.2.2]).

3 The permutation group of the integers $\mathfrak{S}_\infty$

Definition 3.1. The group $\mathfrak{S}_\infty$ is the group of all permutations of the set $\mathbb{N}$ of nonnegative integers. It is endowed with the topology of pointwise convergence, viewing $\mathbb{N}$ as a discrete set.

The topology of $\mathfrak{S}_\infty$ can concretely be understood as follows: given a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of permutations, we have $\sigma_n \to \sigma$ if and only if for every $k \in \mathbb{N}$, we have $\sigma_n(k) = \sigma(k)$ for large enough $n \in \mathbb{N}$. Using this characterization, it is not hard to see that $\mathfrak{S}_\infty$ is a topological group, even without relying on Proposition 2.1.

Let us now prove that $\mathfrak{S}_\infty$ is Polish, using an approach that will appear again later: we will show that it is G_δ in a well-chosen function space.

Proposition 3.2. *$\mathfrak{S}_\infty$ is a Polish group.*

Proof. By definition, the topology on $\mathfrak{S}_\infty$ is the topology induced by $\mathbb{N}^\mathbb{N}$, viewing $\mathbb{N}$ as a discrete set. The latter is obviously a Polish space, so $\mathbb{N}^\mathbb{N}$ is Polish. In order to show that $\mathfrak{S}_\infty$ is Polish, it suffices to show that it is a G_δ subset of $\mathbb{N}^\mathbb{N}$ in view of Corollary 1.3. But this follows in a straightforward manner by unraveling the definition of a bijection. Indeed a function $\mathbb{N} \to \mathbb{N}$ is surjective if and only if it belongs to the G_δ set

$$\bigcap_{n\in\mathbb{N}} \left(\bigcup_{m\in\mathbb{N}} \{f \in \mathbb{N}^\mathbb{N} : f(m) = n\} \right),$$

and it is injective if and only if it belongs to the closed (in particular G_δ) set

$$\bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}\setminus\{n\}} \{f \in \mathbb{N}^\mathbb{N} : f(m) \neq f(n)\}.$$

So $\mathfrak{S}_\infty$ can be written as the intersection of two G_δ sets: it is itself G_δ, hence Polish. $\square$

Remark 3.3. The above approach generalizes as follows. First, given a compact Polish space X and a Polish space Y, the space $\mathcal{C}(X, Y)$ of continuous maps from X to Y is Polish for the *compact-open* topology. Taking $Y = X$, one can show that the homeomorphism group $\mathrm{Homeo}(X)$ is G_δ in $\mathcal{C}(X, X)$ and that it is a topological group for the compact-open topology.

Now, given a locally compact Polish space X, one can consider its Alexandrov compactification $X \cup \{\infty\}$ and endow its homeomorphism group with the compact open topology. Homeomorphisms of X then correspond to the closed subgroup consisting of the stabilizer of ∞, in particular they form a Polish group. Note that the compact-open topology on $\mathrm{Homeo}(X)$ is not a group topology in general (although it is Polish), and that the topology induced by the compact-open topology on $\mathrm{Homeo}(X \cup \{\infty\})$ is described by taking as a subbasis the usual compact-open subbasis to which we add sets of the form $\{g \in \mathrm{Homeo}(X)\colon g(F) \subseteq X \setminus K\}$, where F is closed and K is compact in X (for all this, see [Dij05]). In the next section, we prepare the ground for a similar construction in the measurable context.

4 Spaces of measurable maps

A probability space $(X, \mathcal{B}, \mu)$ is called **standard** when X can be endowed with a Polish topology so that $\mathcal{B}$ is the associated Borel σ-algebra and μ is atomless, e.g. $X = [0, 1]$

endowed with the Lebesgue measure. A classical result in descriptive set theory implies that all standard probability spaces are isomorphic [Kec95, Thm. 17.41], but in this text we only need the easier fact that they are isomorphic *up to a null set*. From now on, the underlying σ-algebra $\mathcal{B}$ will be implicit in our notation.

Definition 4.1. Let (X, μ) be a standard probability space, and let Y be a Polish space. Then $\mathrm{L}^0(X, \mu, Y)$ is the space of all measurable maps $f : X \to Y$ up to measure zero, i.e. we identify two such maps f_1, f_2 when $\mu(\{x \in X \colon f_1(x) \neq f_2(x)\}) = 0$. It is endowed with the **topology of convergence in measure**, defined by declaring open the following sets for every $A \subseteq X$ measurable, $U \subseteq Y$ open and $\epsilon > 0$:

$$\tilde{U}_{\epsilon,A} \coloneqq \{f \in \mathrm{L}^0(X, \mu, Y) \colon \mu(f^{-1}(U) \cap A) > \epsilon\}.$$

Using the second-countability of Y and the separability of the measure algebra[4] of (X, μ), it is not hard to see that the topology of convergence in measure is second-countable. In order to see that it is Polish, we make a small detour which will justify in passing the fact that we call it the topology of convergence of measure.

Lemma 4.2. *Let d be a compatible metric on Y. Let $f_0 \in \mathrm{L}^0(X, \mu, Y)$, then a basis of neighborhoods of f is given by*

$$\left\{f \in \mathrm{L}^0(X, \mu, Y) \colon \mu\left(\{x \in X \colon d(f(x), f_0(x)) > \epsilon\}\right) < \epsilon\right\}.$$

Proof. Let (y_n) be dense in Y, then for each n let

$$A_n \coloneqq \left\{x \in X \colon d(f_0(x), y_n) < \frac{\epsilon}{2}\right\}.$$

By density $X = \bigcup_n A_n$, in particular there is $N \in \mathbb{N}$ such that $\mu(\bigcup_{n<N} A_n) > 1 - \frac{\epsilon}{2}$. If we denote by U^n the $\frac{\epsilon}{2}$ open ball around y_n, we have by construction that the open set

$$\bigcap_{n<N} \tilde{U}^n_{\mu(A_n)-\frac{\epsilon}{2N}, A_n}$$

is contained in $\left\{f \in \mathrm{L}^0(X, \mu, Y) \colon \mu\left(\{x \in X \colon d(f(x), f_0(x)) > \epsilon\}\right) < \epsilon\right\}$.

In order to conclude, we need to show that conversely, given any f_0 in an open set for the topology of convergence in measure, there is $\epsilon > 0$ such that this open set contains

$$\left\{f \in \mathrm{L}^0(X, \mu, Y) \colon \mu\left(\{x \in X \colon d(f(x), f_0(x)) > \epsilon\}\right) < \epsilon\right\}.$$

Without loss of generality, take $f_0 \in \tilde{U}_{\epsilon,A}$, then since U is open we find $n \in \mathbb{N}$ such that $\mu(\{x \in A \colon d(f_0(x), Y \setminus U) > \frac{1}{n}\}) < \epsilon$. Let

$$\delta = \min\left(\frac{1}{n}, \epsilon - \mu(\{x \in A \colon d(f_0(x), Y \setminus U) > \tfrac{1}{n}\}\right),$$

then observe that if we have $f \in \mathrm{L}^0(X, \mu, Y)$ such that $\mu(\{x \in X \colon d(f_0(x), f(x)) > \delta\}) < \delta$ then $f \in \tilde{U}_{\epsilon,A}$. $\square$

[4]The **measure algebra** of a probability space is the algebra of measurable subsets up to measure zero, endowed with the topology induced by the complete metric $d_\mu(A, B) = \mu(A \triangle B)$. Its separability when (X, μ) is standard follows by putting a compatible Polish topology on X and checking that regularity implies density of finite unions of elements of a fixed countable basis of open sets.

Let d be a compatible *bounded* metric on Y (such a metric always exist because $\min(1,d)$ induces the same topology as d). Then we can endow $\mathrm{L}^0(X,\mu,Y)$ with the metric $\tilde{d}$ defined by

$$\tilde{d}(f,g) = \int_X d(f(x),g(x))d\mu(x). \tag{1}$$

We then have the following well-known properties of convergence in measure and $\tilde{d}$, see e.g. [Kec10, Sec. 19] (everything is stated for groups there but the relevant statements work equally well without this assumption).

Lemma 4.3. *The following hold.*

(i) The metric $\tilde{d}$ is compatible with the topology of convergence in measure.

(ii) A sequence of functions (f_n) converges to f in measure if and only if every subsequence of (f_n) converges almost surely to f.

(iii) If d is complete, then so is $\tilde{d}$.

As a consequence of item (iii) and the above-mentioned second-countability of the topology of convergence in measure, we obtain the following.

Corollary 4.4. *Whenever Y is a Polish space, $\mathrm{L}^0(X,\mu,Y)$ is Polish for the topology of convergence in measure.* □

Remark 4.5. By definition a standard probability space is atomless. However, all the definitions and facts above hold as well when there are atoms, with unchanged proofs. It is of interest to note that when (X,μ) is completely atomic and A is its (countable) set of atoms, $\mathrm{L}^0(X,\mu,Y)$ is naturally homeomorphic to Y^A. In particular, one can think of the L^0 construction as a generalization of the operation of taking a countable power of a set. As we will see in Section 8.2.2, a key point of this generalization is that the absence of atoms yields a *contractible* (in particular connected) topological space.

5 The group of probability measure-preserving bijections $\mathrm{Aut}(X,\mu)$

We now define our second favorite Polish group, which stems from ergodic theory. Let (X,μ) be a standard (atomless) probability space, we denote by $\mathrm{Aut}(X,\mu)$ the group of all *measure preserving bijections* of (X,μ) up to measure zero, i.e. the group of all $T: X\to X$ Borel bijection such that for all $A\subseteq X$ Borel, we have $\mu(A)=\mu(T^{-1}(A))$ where we identify two such bijections T_1,T_2 as soon as $\mu(\{x\in X\colon T_1(x)\neq T_2(x)\})=0$.

Example 5.1. Take $X=[0,1)$ and λ the Lebesgue measure. Then for every $\alpha\in\mathbb{R}$, the bijection T_α defined by $T_\alpha(x)=x+\alpha \mod 1$ is a measure-preserving bijection.

Since all standard atomless probability spaces are isomorphic, there is only one group $\mathrm{Aut}(X,\mu)$ up to group isomorphism. We will follow an unconventional path by defining on it a Polish group topology which depends on the choice of a Polish topology on X a priori, and then check that this topology is independent of this choice. This approach is motivated by our construction of Polish full groups with Carderi [CLM16], although we won't present this construction in this text.

Lemma 5.2. *Fix a Polish topology τ compatible with the standard Borel space structure of X. Then $\mathrm{Aut}(X,\mu)$ is G_δ in $\mathrm{L}^0(X,\mu,X)$ for the topology of convergence in measure.*

Proof. Note that $\mathrm{Aut}(X, \mu)$ can be identified with the space of Borel maps $X \to X$ which are both measure-preserving, and become injective when restricted to an appropriate full measure set. It thus suffices to check that the space of such maps is G_δ.

First we check that the space of all measure preserving maps $f \in \mathrm{L}^0(X, \mu, X)$ is closed by showing that its complement is open: suppose that $f \in \mathrm{L}^0(X, \mu, X)$ is not measure-preserving. Then by taking if needed a complement we find $A \subseteq X$ such that $\mu(f^{-1}(A)) > \mu(A)$. Using regularity, we then find U open containing A such that $\mu(f^{-1}(A)) > \mu(U)$, so in particular $\mu(f^{-1}(U)) > \mu(U)$. Then the open set

$$\{g \in \mathrm{L}^0(X, \mu, X) \colon \mu(g^{-1}(U)) > \mu(U)\}$$

is a neighborhood of f disjoint from the set of measure-preserving maps.

Let us now check that the space of injective measure-preserving maps $f \in \mathrm{L}^0(X, \mu, X)$ is G_δ. Let (A_n) be a separating family for X: as soon as $x \neq y \in X$, there is $n \in \mathbb{N}$ such that $x \in A_n$ but $y \notin A_n$. The fact that the space of injective measure-preserving maps $f \in \mathrm{L}^0(X, \mu, X)$ is G_δ will follow directly from the following claim: a measure-preserving map f is injective if and only if for every $\epsilon > 0$ and $n \in \mathbb{N}$, there is an open set U such that $\mu(U) < \mu(A_n) + \epsilon$ and $\mu(A_n \cap f^{-1}(U)) > \mu(A_n) - \epsilon$.

Let us prove the above claim: if f is injective then $A_n = f^{-1}(f(A_n))$. By regularity we find $U \supseteq f(A_n)$ which satisfies $\mu(U \setminus f(A_n)) < \epsilon$ and since f is measure-preserving, U is as wanted.

For the converse, we fix for every n a sequence $(U_n^k)_k$ such that $\mu(f^{-1}(U_n^k) \cap A_n > \mu(A_n) - 2^{-k}$ and $\mu(f^{-1}(U_{n_k}) \setminus A_n)) < 2^{-k}$. By the Borel-Cantelli lemma, after throwing away null sets, we have that for all x, $x \in A_n$ if and only if $x \in f^{-1}(U_{n_k})$ for all but finitely many $k \in \mathbb{N}$. We may finally check that f is injective: let $x \neq y$, then there is n such that $x \in A_n$ but $y \notin A_n$, in particular there must be k such that $x \in f^{-1}(U_n^k)$ but $y \notin f^{-1}(U_n^k)$, so $f(x) \in U_{n_k}$ but $f(y) \notin U_{n_k}$. We conclude that $f(x) \neq f(y)$ as wanted.

We now have that $\mathrm{Aut}(X, \mu)$ is equal to the intersection of a closed (in particular G_δ) subset with another G_δ subset, in particular it is G_δ which ends the proof. □

Theorem 5.3. *Let (X, τ) be a Polish space endowed with a Borel probability measure μ. Then $\mathrm{Aut}(X, \mu)$ is a Polish group for the topology of convergence in measure.*

Proof. Since G_δ subspaces of Polish spaces are themselves Polish and $\mathrm{Aut}(X, \mu)$ is G_δ in the Polish space $\mathrm{L}^0(X, \mu, X)$ by the previous lemma, we only need to check that $\mathrm{Aut}(X, \mu)$ is a topological group for the topology of convergence in measure. By Proposition 2.1 we only need to check that given a fixed element $h \in G$, both left and right multiplication by h are continuous.

To see that right multiplication by h is continuous, one can observe that given a compatible metric d on Y, the metric $\tilde{d}$ given by equation (1) induces the topology of convergence in measure, and since h preserves the measure we have that for all $g \in G$,

$$\tilde{d}(g_1 h, g_2 h) = \int_X d(g_1 h(x), g_2 h(x)) = \int_X d(g_1(x), g_2(x)) = \tilde{d}(g_1, g_2).$$

In other words, the right multiplication by h is an isometry of $(\mathrm{Aut}(X, \mu), \tilde{d})$, in particular it is continuous.

To see that left multiplication by h is continuous, we fix again a compatible metric d and we use the characterizations given by Lemma 4.3. Let $\epsilon > 0$, and $g_0 \in G$, consider the open neighborhood U of hg_0 given by

$$U := \{f \colon \mu(\{x \in X \colon d(hg_0(x), f(x)) < \epsilon\}) > 1 - \epsilon\}$$

By Lusin's theorem, there is a Borel set $X_0 \subseteq X$ of measure $> 1-\epsilon/2$ such that the restriction of h to X_0 is uniformly continuous. Let $\delta > 0$ such that for all $x_1, x_2 \in X_0$, $d(x_1, x_2) < \delta$ implies $d(h(x_1), h(x_2)) < \epsilon$. Now for every g such that $\mu(\{x \in X : d(g_0(x), g(x)) < \delta\}) > 1 - \epsilon/2$, the event that both $g(x) \in X_0$ and $d(g_0(x), g(x)) < \delta$ has probability at least $1 - \epsilon$ because g is measure-preserving. By construction the event $d(hg_1(x), hg_2(x)) < \epsilon$ then has probability at least $1 - \epsilon$. In other words, the neighborhood V of g_0 defined by

$$V \coloneqq \mu(\{x \in X : d(g_0(x), g(x)) < \delta\}) > 1 - \epsilon/2$$

satisfies that for all $g \in V$, we have $hg \in U$, thus finishing the proof. $\square$

We now describe the standard way of understanding the Polish group topology of $\mathrm{Aut}(X, \mu)$, which shows in particular that this topology does not depend on the Polish topology we put on X.

Definition 5.4. Given a standard probability space (X, μ), the **weak topology** on $\mathrm{Aut}(X, \mu)$ is the weakest topology such that for all $A \subseteq X$ Borel , the map $(S, T) \mapsto \mu(S(A) \bigtriangleup T(A))$ is continuous.

One can check that the weak topology is a Polish group topology, see e.g. [Kec10, Sec. 1]. Moreover, it is a result of Robert Kallman that the weak topology is the unique Polish group topology on $\mathrm{Aut}(X, \mu)$ [Kal85]. In particular, we have the following result (see also [CLM16, Prop. 2.9] for a different proof).

Proposition 5.5. *Let X be a Polish space endowed with a nonatomic probability measure μ. Then the weak topology and the topology of convergence in measure coincide on $\mathrm{Aut}(X, \mu)$.* $\square$

Let us conclude this section by mentioning that $\mathrm{Aut}(X, \mu)$ not only has a unique Polish group topology, but it actually satisfies the automatic continuity property by a result of Ben Yaacov, Berenstein and Melleray [BYBM13].

6 The group of non-singular bijections $\mathrm{Aut}^*(X, \mu)$

If μ is a σ-finite (possibly finite) Borel measure on a standard Borel space X, its **measure class** is the set $[\mu]$ of all Borel σ-finite measures ν on X such that for all $A \subseteq X$ Borel, $\nu(A) = 0$ if and only if $\mu(A) = 0$. Recall that by the Radon-Nikodym theorem, $\nu \in [\mu]$ if and only if there is $f \in \mathrm{L}^1(X, \mu)$ such that $\nu = f\mu$ and $f(x) > 0$ for almost all $x \in X$. Such a map f is unique up to measure zero, and denoted by $\frac{d\nu}{d\mu}$.

Given a standard probability space (X, μ), we then define the larger group $\mathrm{Aut}^*(X, \mu)$ of *non-singular* bijections of (X, μ), which are all Borel bijections $T : X \to X$ such that for all $A \subseteq X$ Borel, we have $\mu(A) = 0$ if and only if $\mu(T(A)) = 0$. In other words, we require that T preserves the measure-class of μ, that is $T_*\mu \in [\mu]$ or equivalently $T_*\nu \in [\mu]$ for all $\nu \in [\mu]$. Here is a straightforward consequence of the uniform continuity characterization of absolute continuity for measures (see for instance [Coh13, Lem. 4.2.1]).

Lemma 6.1. *Let $T \in \mathrm{Aut}^*(X, \mu)$. Then for every $\epsilon > 0$, there is $\delta > 0$ such that for all $A \subseteq X$ Borel, $\mu(A) < \delta$ implies $\mu(T(A)) < \epsilon$.* $\square$

The group $\mathrm{Aut}^*(X, \mu)$ can also be endowed with a Polish group topology, but to our knowledge the latter does not have a description similar to that of $\mathrm{Aut}(X, \mu)$ in terms of convergence in measure, which is why the constructions of Polish full groups from [CLM16] don't carry over to the non-singular setup a priori.

Let us describe the Polish group topology of $\mathrm{Aut}^*(X,\mu)$. Observe that by construction, for all $T \in \mathrm{Aut}^*(X,\mu)$ and $f \in \mathrm{L}^1(X,\mu)$, we have that $\int_X f d\mu = \int_X f \circ T^{-1} dT_*\mu$, in particular the map $f \mapsto f \circ T^{-1}$ induces an isometry $\mathrm{L}^1(X,\mu) \to \mathrm{L}^1(X, T_*\mu)$.

Moreover, a straightforward computation shows that for all $g \in \mathrm{L}^1(X,\mu)$ almost surely non zero, the multiplication by g induces an isometry $\mathrm{L}^1(X,\mu) \to \mathrm{L}^1(X, g\mu)$, so equivalently the multiplication by g^{-1} induces an isometry $\mathrm{L}^1(X, g\mu) \to \mathrm{L}^1(X,\mu)$

In particular, composing the map $f \mapsto f \circ T^{-1}$ and the multiplication by $\frac{dT_*\mu}{d\mu}^{-1}$, we see that each T induces an isometry[5]. $\iota(T)$ of $\mathrm{L}^1(X,\mu)$ given by

$$\iota(T)(f)(x) := f[T^{-1}(x)] \left(\frac{dT_*\mu}{d\mu}(x)\right)^{-1}.$$

Definition 6.2. The **strong topology** on $\mathrm{Aut}^*(X,\mu)$ is the weakest topology such that for all $f \in \mathrm{L}^1(X,\mu)$, the map $(S,T) \mapsto \iota(S)f - \iota(T)f$ is continuous.

Remark 6.3. The strong topology is often called the weak topology, but since the weak topology that we defined on $\mathrm{Aut}(X,\mu)$ makes sense on the whole group $\mathrm{Aut}^*(X,\mu)$ we prefer to use a different term.

The following result is due to Ionescu Tulcea, see [IT65].

Proposition 6.4. *The strong topology on* $\mathrm{Aut}^*(X,\mu)$ *is a Polish group topology.*

Sketch of proof. Let us denote by $\mathrm{Isom}(\mathrm{L}^1(X,\mu))$ the isometry group of $\mathrm{L}^1(X,\mu)$, which is a Polish group for the topology of pointwise convergence, as is every isometry group of a complete separable metric space (see [Kec95, Example 9 in Sec. 9.B]). One can then check that the map $\iota : \mathrm{Aut}^*(X,\mu) \to \mathrm{Isom}(\mathrm{L}^1(X,\mu))$ is a group homomorphism, using the chain rule. Also note that ι is injective, and then check that its image is the group of isometries of $\mathrm{L}^1(X,\mu)$ which preserve the cone of positive functions. So $\mathrm{Aut}^*(X,\mu)$ can be seen via ι as a closed subgroup of the Polish group $\mathrm{Isom}(\mathrm{L}^1(X,\mu))$, hence it is Polish. $\square$

Remark 6.5. To see that $\mathrm{Aut}(X,\mu)$ is Polish for the topology induced by the strong topology, one can now observe that it is equal to the $\mathrm{Aut}^*(X,\mu)$-stabilizer of the function constant equal to 1 in $\mathrm{L}^1(X,\mu)$, hence closed in $\mathrm{Aut}^*(X,\mu)$. * Using the density of step functions, it is then not hard to show that the strong topology induces the weak topology on $\mathrm{Aut}(X,\mu)$ (this also follows from the uniqueness of the Polish group topology of $\mathrm{Aut}(X,\mu)$). The weak topology actually makes sense the whole group of non-singular bijections $\mathrm{Aut}^*(X,\mu)$, but it fails to be a group topology as the following example shows (group multiplication is separately continuous but not continuous).

Example 6.6. We work with $X = [0,1)$ endowed with the Lebesgue measure. For every $n \in \mathbb{N}$, consider the partition $(I_n^k)_{k=1}^n$ of $[0,1)$ into half-open intervals of size $\frac{1}{n}$, and take $A_n^k \subseteq I_n^k$ of measure $\frac{1}{n^2}$. For every n, let T_n be any non-singular bijection such that $T_n(I_n^k) = I_n^k$ and $T_n(A_n^k) = I_n^k \setminus A_n^k$ for all $k \in \{1,\dots,n\}$. Observe that $T_n \to \mathrm{id}$ weakly.

Now let $A_n = \bigcup_{k=1}^n A_n^k$, and let U_n be an involution of support $[0,\frac{1}{n}) \cup A_n$ which exchanges $[0,\frac{1}{n})$ and A_n. Again $U_n \to \mathrm{id}_X$, but $\mu(T_nU_n([0,\frac{1}{2})) \triangle X) \to 0$ so T_nU_n cannot converge for the weak topology.

[5]This action can also be understood as follows: we have a natural isometric action of $\mathrm{Aut}^*(X,\mu)$ on $\mathrm{L}^\infty(X,\mu)$ by precomposition, inducing an action by isometries on its predual $\mathrm{L}^1(X,\mu)$. Such a point of view generalizes to show that automorphisms groups of von Neumann algebras with separable predual are Polish.

Although the weak topology does not define a group topology on $\mathrm{Aut}^*(X,\mu)$, it is refined by the strong topology and generates the same Borel structure as the strong topology. This observation is key to our recent proof that the strong topology of $\mathrm{Aut}^*(X,\mu)$ is its unique Polish group topology [LM22]. Our result also relies on the aforementioned automatic continuity property for the group of measure-preserving bijections of (X,μ) [BYBM13]. The following question is natural.

Question 1. Does $\mathrm{Aut}^*(X,\mu)$ have the automatic continuity property?

Remark 6.7. The unified framework that has been developed by Sabok [Sab19] and then Malicki [Mal16a] to deal with automatic continuity for $\mathrm{Aut}(X,\mu)$, $\mathcal{U}(\mathcal{H})$ and the isometry group of the Urysohn space does not apply here, which makes this question particularly appealing.

We end this section by defining another (non separable!) group topology on $\mathrm{Aut}^*(X,\mu)$ called the **uniform topology** which will be put on full groups in the next section. It is induced by the **uniform metric**

$$d_u(S,T) \coloneqq \mu(\{x \in X \colon S(x) \neq T(x)\}).$$

The following lemma is well-known but we could not find a reference for a proof.

Lemma 6.8. *The uniform topology is a group topology on* $\mathrm{Aut}^*(X,\mu)$*, and the metric* $d_u(S,T)+d_u(S^{-1},T^{-1})$ *is complete.*

Proof. First observe that d_u is left-invariant: for all $S,T_1,T_2 \in \mathrm{Aut}^*(X,\mu)$ we have $d_u(ST_1,ST_2)=d_u(T_1,T_2)$. Let us show the continuity of multiplication: take $S_n\to S$ and $T_n \to T$, then

$$\begin{aligned} d_u(S_nT_n,ST) &\leqslant d_u(S_nT_n,S_nT)+d_u(S_nT,ST)\\ &\leqslant d_u(T_n,T)+d_u(S_nT,ST)\end{aligned}$$

Now observe that by definition

$$d_u(S_nT,ST)=\mu(T^{-1}(\{x\in X\colon S_n(x)\neq S(x)\})),$$

so the second term in the previous inequality tends to zero by Lemma 6.1 and the fact that $S_n \to S$, while the first tends to zero because $T_n \to T$. We conclude that group multiplication is continuous.

For the continuity of the inverse map, suppose $T_n \to T$, then by left-invariance $d_u(T_n^{-1},T^{-1}) = d_u(\mathrm{id}_X, T_nT^{-1}) \to 0$ by continuity of multiplication.

Let us finally check completeness. Let (T_n) be a d-Cauchy sequence where $d(S,T) \coloneqq d_u(S,T)+d_u(S^{-1},T^{-1})$. It suffices to show that some subsequence of (T_n) converges so by taking a subsequence we may as well assume $d(T_n,T_{n+1})<2^{-n}$. Let $A_n=\{x\in X : T_n(x)\neq T_{n+1}(x)\}$, then because $d_u\leqslant d$, the set A_n has measure at most 2^{-n}. Since $\sum_n 2^{-n}$ is finite, the Borel-Cantelli lemma ensures us that almost every $x\in X$ belongs to finitely many A_n. For every N let B_N be the set of $x\in X$ such that $T_{N-1}(x)\neq T_N(x)$ but $T_n(x)=T_N(x)$ for all $n\geqslant N$, then up to measure zero (B_N) is a partition of X, and since each T_n was invertible, we see that $(T_N(B_N))_{N\geqslant 0}$ consists of disjoint sets. Define $T: X\to X$ by $T(x)=T_N(x)$ for all $x\in B_N$ and all $N\in\mathbb{N}$. Since each T_n was a bijection, we see that

$$T_N(B_N)=\{x\in X : T_{N-1}^{-1}(x)\neq T_N^{-1}(x) \text{ and } T_n^{-1}(x)=T_N^{-1}(x) \text{ for all } n\geqslant N\}.$$

But using now that $d_u(T_n^{-1}, T_{n+1}^{-1}) \leqslant 2^{-n}$ and the Borel-Cantelli lemma, we see that $(T_N(B_N))_{N\geqslant 0}$ is also a partition of X up to a null set, which shows that T is invertible. The fact that $d_u(T_n, T) \to 0$ (and hence $d(T_n, T) \to 0$ by continuity of the inverse) is now a direct consequence of the fact that (B_n) is a partition so $\mu(\bigcup_{n\geqslant N} B_n) \to 0$. This finishes the proof that d is complete, so the lemma is proved. □

7 Full groups of non-singular countable equivalence relations

We now introduce full groups of non-singular countable equivalence relations. Let us emphasize that apart from our results on ample generics, we will only consider measure-preserving equivalence relations, which are special cases of non-singular equivalence relations. The non-singular world is nevertheless an appealing playground for many of the constructions that we have made in the measure-preserving world (such as Polish full groups [CLM16, CLM18] and L^1 full groups [LMS21]). However, we don't know how to show in general that their natural non-singular analogues are Polish as well because $\mathrm{Aut}^*(X,\mu)$'s topology is not that of convergence in measure when seeing it inside $\mathrm{L}^0(X,\mu,X)$ (see also Remark 7.13). Our student Fabien Hoareau is nevertheless working on a family of Polish full groups which preserve an infinite σ-finite measure.

Definition 7.1. A **non-singular action** of a countable group Γ on (X,μ) is a Γ-action on X by non-singular bijections. It is **ergodic** if every Γ-invariant set is ergodic.

Given a non-singular action of a countable group $\Gamma \curvearrowright (X,\mu)$, we get an equivalence relation $\mathcal{R}_{\Gamma\curvearrowright X}$ which encodes the partition of the space into Γ-orbits, given by $(x,y) \in \mathcal{R}_{\Gamma\curvearrowright X}$ if and only if x and y lie in the same Γ-orbit. Equivalence relations on X that arise in this manner are called **non-singular equivalence relations**. Ergodicity of the action is actually encoded by the corresponding non-singular equivalence relation as follows.

Definition 7.2. A non singular equivalence relation $\mathcal{R}$ is **ergodic** when every Borel set which is a union of $\mathcal{R}$-classes has measure 0 or 1.

Example 7.3. Let us fix the Polish space $X = \{0,1\}^{\mathbb{N}}$ which is the set of all subsets of $\mathbb{N}$. Then X is a compact group for the symmetric difference operation, and the dense countable subgroup Γ of finite subsets acts by symmetric difference on X. The associated equivalence relation is

$$\mathcal{R}_0 = \{(P,Q) \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \colon P \triangle Q \text{ is finite}\}.$$

This equivalence relation is *hyperfinite*: it can be written as an increasing union of Borel subequivalence relations with finite classes $\mathcal{R}_n$, which here we can take as

$$\mathcal{R}_n = \{(P,Q) \colon P \triangle Q \subseteq \{0,\dots,n\}\}.$$

Let us now describe various measure classes on X which yield different non-singular ergodic hyperfinite equivalence relations.

- We first take
$$\mu_{1/2} = \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)^{\otimes\mathbb{N}},$$
which is the Haar measure on X. Then Γ acts in a measure-preserving manner, and this measure-preserving action is ergodic while the measure is finite. By definition $(X,\mu,\mathcal{R}_0)$ is a **type II_1** non-singular equivalence relation, which is thus actually probability measure-preserving. Using the uniqueness of the Haar measure and the density of Γ in X, one can show that $\mu_{1/2}$ is the *only* Γ-invariant probability measure.

- If we now fix $p \in (0, \frac{1}{2})$ with $p \neq \frac{1}{2}$, and consider
$$\mu_p = (p\delta_0 + (1-p)\delta_1)^{\otimes \mathbb{N}},$$
the Γ-action is non singular and it can be shown using the *ratio set* that no measure in the class of μ_p can be preserved: the action is of **type III**, and more precisely that the action is actually of type III_λ, where $\lambda = \frac{p}{1-p}$.

- Let us now see $\{0,1\}^{\mathbb{N}}$ as $\{0,1\}^{\mathbb{Z}}$ via a bijection $\mathbb{N} \to \mathbb{Z}$. Let $\mathcal{P}_f(\mathbb{Z} \setminus \mathbb{N})$ be the set of finite subsets of $\mathbb{Z} \setminus \mathbb{N}$, which we view as binary sequences $(x_i)_{i<0}$ which take the value 1 finitely many times. Let μ_∞ be the σ-finite infinite measure on $\{0,1\}^{\mathbb{Z}}$ defined by
$$\mu_\infty = \sum_{A \in \mathcal{P}_f(\mathbb{Z}\setminus\mathbb{N})} \delta_A \otimes \mu_{1/2}.$$
Via our natural bijection $\{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{N}}$ we get a measure on X which we still denote by μ_∞, and we have a measure-preserving action of Γ which is of **type $\mathbf{II}_\infty$**. An example of a probability measure in the class of μ_∞ is given by first enumerating $\mathcal{P}_f(\mathbb{Z} \setminus \mathbb{N})$ as $\mathcal{P}_f(\mathbb{Z} \setminus \mathbb{N}) = \{A_k : n \in \mathbb{N}\}$ and then taking the measure $\mu'_\infty = \sum_{k\in\mathbb{N}} \frac{1}{2^k+1}\delta_{x^k} \otimes \mu_{1/2}$.

Definition 7.4. Given a non-singular equivalence relation $\mathcal{R}$ on a standard probability space (X, μ), its **full group** $[\mathcal{R}]$ is the subgroup of $\mathrm{Aut}^*(X, \mu)$ defined by
$$[\mathcal{R}] = \{T \in \mathrm{Aut}^*(X, \mu) : \forall x \in X, (x, T(x)) \in \mathcal{R}\}.$$

Suppose that $\mathcal{R}$ is a non-singular equivalence relation, let $\Gamma \curvearrowright (X, \mu)$ be a non-singular action of a countable group Γ such that $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$. Then one can check that an element $T \in \mathrm{Aut}^*(X, \mu)$ is in $[\mathcal{R}]$ if and only if there is a partition $(A_\gamma)_{\gamma\in\Gamma}$ such that for all $x \in A_\gamma$, $T(x) = \gamma x$. It follows that if $\Gamma \curvearrowright (X, \mu)$ preserves a measure $\lambda \in [\mu]$, then the elements of the full group also preserve λ.

In particular, we see that the definitions that were given in Example 7.3 are properties of the corresponding non-singular equivalence relation. Let us state these properties for completeness.

Definition 7.5. Let $\mathcal{R}$ be an ergodic non-singular equivalence relation on a standard probability space (X, μ). We say that $\mathcal{R}$ is

- **type $\mathbf{II}_1$** if there is a finite measure $\mu' \in [\mu]$ which is preserved by every element of the full group of $\mathcal{R}$;

- **type $\mathbf{II}_\infty$** if there is a σ-finite *infinite* measure $\mu' \in [\mu]$ which is preserved by every element of the full group of $\mathcal{R}$;

- **type III** otherwise.

In order to see that these definitions are sound, we need to check that type II_1 and type II_∞ are mutually exclusive, which is a direct consequence of the following lemma.

Lemma 7.6. *Let $\mathcal{R}$ be a type* II *ergodic equivalence relation, i.e. suppose there is a σ-finite (finite or infinite) measure $\mu' \in [\mu]$ which is preserved by the full group of $\mathcal{R}$, and let μ'' be another measure in $[\mu]$ preserved by the full group of $\mathcal{R}$. Then there is a constant $C > 0$ such that $\mu'' = C\mu'$.*

Proof. Let $f = \frac{d\mu''}{d\mu'}$, then since both μ' and μ'' are $[\mathcal{R}]$-invariant, the function f is $[\mathcal{R}]$-invariant. By ergodicity f must be constant (indeed for every $t \in \mathbb{Q}$ its level set $A_t \coloneqq \{x \in X : f(x) < t\}$ is $[\mathcal{R}]$-invariant so it has measure 0 or 1, which implies that f is constant up to measure zero). $\square$

The group topology we put on full groups of non-singular equivalence relations is the one induced by the uniform topology on $\mathrm{Aut}^*(X, \mu)$, coming from the uniform metric $d_u(S, T) = \mu(\{x \in X \colon S(x) \neq T(x)\})$ defined at the end of the previous section. Observing that full groups of non-singular equivalence relations are closed in the uniform topology, the following consequence of Lemma 6.8 is immediate.

Proposition 7.7. *Let $\mathcal{R}$ be a non-singular equivalence relation. Then the metric $d_u(S, T) + d_u(S^{-1}, T^{-1})$ is complete on $[\mathcal{R}]$.* $\square$

Remark 7.8. When restricted to the group of measure-preserving bijections $\mathrm{Aut}(X, \mu)$, the inverse map is a d_u-isometry, so the metric d_u itself is complete on $\mathrm{Aut}(X, \mu)$. In particular, it is complete on full groups of probability measure-preserving equivalence relations.

In order to show that full groups of non-singular equivalence relations are Polish, we now only need to show that they are separable. We do this via the following lemma which is important on its own since it connects full groups of non-singular equivalence relations to the permutation group $\mathfrak{S}_\infty$. We first need a bit of terminology.

Definition 7.9. A non-singular equivalence relation $\mathcal{R}$ is **aperiodic** if for almost every $x \in X$, the $\mathcal{R}$-class of x is infinite.

Definition 7.10. Given an aperiodic non-singular equivalence relation $\mathcal{R}$, a **decomposition** of $\mathcal{R}$ is a sequence $\mathcal{F} = (f_n)$ of Borel maps $f_n : X \to X$ with disjoint graphs such that $f_0 = \mathrm{id}_X$ and for almost all $x \in X$,

$$[x]_{\mathcal{R}} = \{f_n(x) \colon n \in \mathbb{N}\}.$$

Every aperiodic non-singular equivalence relation admits a decomposition: if $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$, we enumerate $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$ where $\gamma_0 = 1$ and inductively define $f_n(x) = \gamma_k x$ where k is the first integer such that $\gamma_k x \notin \{f_m(x) \colon m < n\}$.

Observe that given a decomposition $\mathcal{F} = (f_n)$ of $\mathcal{R}$, for almost all $x \in X$ we have a natural bijection $\varphi_x^{\mathcal{F}} : \mathbb{N} \to [x]_{\mathcal{R}}$ given by $n \mapsto f_n(x)$. and the assumption $f_0 = \mathrm{id}_X$ guarantees that 0 is sent to x. Using these bijections, and the fact that every element of the full group acts by permutation on each equivalence class, we have another natural way of understanding full groups which was observed in [KLM15].

Lemma 7.11 (see [KLM15, Prop. 13])**.** *Let $\mathcal{R}$ be an aperiodic non-singular equivalence relation, let (f_n) be a decomposition of $\mathcal{R}$, then the map $\Phi^{\mathcal{F}}$ which takes every $T \in [\mathcal{R}]$ to the map $\Phi^{\mathcal{F}}(T) : X \to \mathfrak{S}_\infty$ defined by:*

$$\Phi^{\mathcal{F}}(T)(x) = (\varphi_x^{\mathcal{F}})^{-1} T_{\restriction [x]_{\mathcal{R}}} \varphi_x^{\mathcal{F}}$$

induces an embedding $[\mathcal{R}] \to \mathrm{L}^0(X, \mu, \mathfrak{S}_\infty)$. $\square$

Using the fact that $\mathrm{L}^0(X, \mu, \mathfrak{S}_\infty)$ is Polish (see Corollary 4.4) and Proposition 7.7, we obtain the following.

Corollary 7.12. *Full groups of non-singular equivalence relations are Polish for the uniform topology.* □

Remark 7.13. In addition to the above interpretation of the full group as a group of random permutations, let us mention that in the type II_1 case, one can understand its topology as follows. Given a type II_1 equivalence relation $\mathcal{R}$, assume μ is actually preserved and let $\Gamma \curvearrowright (X, \mu)$ such that $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$. Assuming for simplicity that Γ is acting freely, then each element of the full group T has a unique **cocycle** $c_T : X \to \Gamma$ given by $T(x) = c_T(x) \cdot x$, and the map $T \mapsto c_T$ is a topological *space* embedding of $[T]$ into $\mathrm{L}^0(X, \mu, \Gamma)$, whose image is the space of cocycles of elements of elements of $[\mathcal{R}]$. Fixing a Polish topology on X, the continuous map $\Phi : \mathrm{L}^0(X, \mu, \Gamma) \to \mathrm{L}^0(X, \mu, X)$ which takes $f : X \to \Gamma$ to $\Phi(f) : x \mapsto \gamma \cdot x$ satisfies by definition that $\Phi^{-1}(\mathrm{Aut}(X\mu))$ is the space of cocycles of elements of $[\mathcal{R}]$, which is Polish since $\mathrm{Aut}(X, \mu)$ is G_δ in $\mathrm{L}^0(X, \mu, X)$ by Lemma 5.2. We thus recover in a third way the fact that the full group $[\mathcal{R}]$ is Polish. This observation was the starting point of our work with Carderi where we constructed a Polish topology on the full group of the action of any Polish group by measure-preserving bijections [CLM16, CLM18]. Our later work on L^1 full groups of graphings [LM18b, LM21], and with Slutsky on L^1 full groups of actions of Polish normed groups [LMS21] also makes a crucial use of this basic observation.

We finally introduce pseudo full groups of non-singular equivalence relations, which are an important tool in order to build elements of the full group.

Definition 7.14. Let $\mathcal{R}$ be a non-singular equivalence relation. Its **pseudo full group** $[[\mathcal{R}]]$ is the set of all partially defined maps $\varphi : A \to B$, where $A, B \subseteq X$ are Borel, such that φ is bijective and for all $x \in A$, we have $(x, \varphi(x)) \in \mathcal{R}$. The set A is called the domain of φ, denoted by $\operatorname{dom} \varphi$, while B is the range of φ, denoted by $\operatorname{rng} \varphi$.

Observe that if $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$ is type II and μ is preserved by Γ, then again since every $\varphi \in [[\mathcal{R}]]$ is obtained by gluing together restrictions of elements of Γ, every φ must satisfy $\mu(\operatorname{dom} \varphi) = \mu(\operatorname{rng} \varphi)$. The following converse is very useful and well-known. We provide the proof for completeness.

Lemma 7.15. *Let $\mathcal{R}$ be a non-singular ergodic equivalence relation of type* II *on (X, μ), where μ is σ-finite and $[\mathcal{R}]$-invariant. Suppose that $A, B \subseteq X$ are Borel subsets of X such that $\mu(A) = \mu(B)$. Then there is $\varphi \in [[\mathcal{R}]]$ such that* $\operatorname{dom} \varphi = A$ *and* $\operatorname{rng} \varphi = B$ *(up to measure zero).*

Proof. We first treat the case where $\mu(A) = \mu(B) < +\infty$. Let Γ be a countable group acting on X so that $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$. Let us fix an enumeration $\Gamma = \{\gamma_n : n \geqslant 0\}$ of Γ. We recursively define $\varphi_n \in [[\mathcal{R}]]$ with domain contained in A, range contained in B, by $\varphi_0 = \emptyset$ and

$$\varphi_{n+1}(x) = \begin{cases} \varphi_n(x) & \text{if } x \in \operatorname{dom} \varphi_n; \\ \gamma_n x & \text{if } x \in A \setminus \operatorname{dom} \varphi_n \text{ and } \gamma_n x \in B \setminus \operatorname{rng} \varphi_n. \end{cases}$$

Let $\varphi = \bigcup_n \varphi_n$, assume by contradiction that either $\mu(A \setminus \operatorname{dom} \varphi) > 0$ or $\mu(B \setminus \operatorname{rng} \varphi) > 0$. Then since μ is preserved and $\mu(A) = \mu(B)$, we *both* have $\mu(A \setminus \operatorname{dom} \varphi) > 0$ and $\mu(B \setminus \operatorname{rng} \varphi) > 0$. Since $\mathcal{R}$ is ergodic, the Γ-invariant positive measure set $\bigcup_n \gamma_n^{-1}(B \setminus \operatorname{rng} \varphi)$ has full measure. We thus find n such that

$$\mu\big((A \setminus \operatorname{dom} \varphi) \cap \gamma_n^{-1}(B \setminus \operatorname{rng} \varphi)\big) > 0,$$

and observe that the definition of φ_{n+1} is contradicted since its domain should have contained this positive measure set. This finishes the proof when both $\mu(A)$ and $\mu(B)$ are finite.

For the case $\mu(A) = \mu(B) = +\infty$, we first use the fact that μ is atomless to write both A and B are disjoint unions of measure 1 sets $A = \bigsqcup_n A_n$, $B = \bigsqcup_n B_n$. Using the previous case, we find $\varphi_n \in [[\mathcal{R}]]$ with domain A_n and range B_n up to null sets. We finally take $\varphi \coloneqq \bigcup_n \varphi_n$, which has domain A and range B up to measure zero. $\square$

II Non-free actions in the discrete context

8 Ample generics and quasi non-archimedean groups

8.1 Ample generics for $\mathfrak{S}_\infty$

Given a countable group Γ, it is natural to try to classify its actions on countable infinite sets up to *conjugacy*. Recall that two actions $\alpha : \Gamma \curvearrowright X$ and $\beta : \Gamma \curvearrowright Y$ are **conjugate** if there is a bijection $f : X \to Y$ such that for all $\gamma \in \Gamma$ and all $x \in X$, $\beta(\gamma)f(x) = f(\alpha(\gamma)x)$, or equivalently for all $\gamma \in \Gamma$,

$$\beta(\gamma) = f \circ \alpha(\gamma) \circ f^{-1}.$$

Since all countable infinite sets are in bijection with $\mathbb{N}$, we may then focus on Γ-actions on $\mathbb{N}$, which are the same thing as group homomorphisms $\Gamma \to \mathfrak{S}_\infty$. In view of the following lemma, the space $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ of Γ-actions on $\mathbb{N}$ is a Polish space.

Lemma 8.1. *Let Γ be a countable group, let G be a Polish group. Then the space of group homomorphisms $\Gamma \to G$ is a Polish space for the topology of pointwise convergence.*

Proof. Being a countable product of Polish spaces, the space G^Γ of all maps $\Gamma \to G$ is Polish. The space of homomorphisms $\Gamma \to G$ is easily seen to be a closed subspace thereof, so it is Polish as well. $\square$

The topology on $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ can be described as follows: a basic open neighborhood of a homomorphism α is obtained from a finite set $F \Subset \mathbb{N}$ and a finite set $S \subseteq \Gamma$ as the set $\mathcal{U}_{S,F}(\alpha)$ of all actions α' such that for all $\gamma \in S$ and all $x \in F$

$$\alpha'(\gamma)(x) = \alpha(\gamma)(x).$$

Remark 8.2. When Γ is finitely generated, observe that one can simply fix once and for all a finite generating set S, and then by continuity of the product map on $\mathfrak{S}_\infty$, the set of all $\mathcal{U}_{S,F}$ where F ranges over finite subsets of $\mathbb{N}$ is a neighborhood basis of α.

The fact that we are looking at Γ-actions up to conjugacy is formalized by noting that we have a natural $\mathfrak{S}_\infty$-action on $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ which is given by pointwise conjugacy: $\sigma \cdot \alpha(\gamma) = \sigma\alpha(\gamma)\sigma^{-1}$, so that conjugacy classes are exactly $\mathfrak{S}_\infty$-orbits on $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$. A natural question then arises: is there an orbit which is comeager, i.e. which contains a dense $\mathcal{G}_\delta$ set? As we will see, answering this question when Γ ranges over finitely generated free groups amounts to checking whether $\mathfrak{S}_\infty$ has ample generics. In particular, the following central theorem will be reformulated as the fact that $\mathfrak{S}_\infty$ has ample generics.

Theorem 8.3 (Folklore)**.** *Let $n \in \mathbb{N}$, let $\mathbb{F}_n$ denote the free group over n generators. Consider the set $\mathcal{A}_n \subseteq \mathrm{Hom}(\mathbb{F}_n, \mathfrak{S}_\infty)$ consisting of $\mathbb{F}_n$-actions α on $\mathbb{N}$ satisfying the following conditions:*

(1) α only has finite orbits;

(2) for every transitive action β of $\mathbb{F}_n$ on a finite set, there are infinitely many α-orbits such that the restriction of α to each of these orbits is conjugate to β.

Then $\mathcal{A}_n$ is dense G_δ in $\mathrm{Hom}(\mathbb{F}_n, \mathfrak{S}_\infty)$, and consists of a single conjugacy class.

Remark 8.4. As a warm-up, observe that for $n = 1$ we are stating that the permutations which have only finite orbits, and which have infinitely many orbits of size k for every $k \geqslant 1$, are all conjugate and form a dense G_δ set.

Proof. The fact that $\mathcal{A}_n$ consists of a single conjugacy class is left as an exercise. Let us then show that $\mathcal{A}_n$ is G_δ.

First, let β be a $\mathbb{F}_n$-action on a finite set of the form $\{1, ..., k\}$. Denote by $a_1, ..., a_n$ the standard generators of $\mathbb{F}_n$.

Observe that given $x \in \mathbb{N}$, the restriction of α to the $\mathbb{F}_n$-orbit of x is conjugate to β if and only if there are $x_1, ..., x_k \in \mathbb{N}$ such that $x = x_i$ for some $i \in \{1, ..., k\}$ and $\alpha(a_m)x_j = x_{\beta(a_m)j}$ for all $m \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$. If we denote by $\mathcal{A}_{\beta,x}$ the set of such actions, it is then clear that $\mathcal{A}_{\beta,x}$ is open.

Now $\alpha \in \mathcal{A}_n$ if and only if for every $x \in \mathbb{N}$, there is $k \geqslant 1$ and a transitive β-action on $\{1, ..., k\}$ such that α restricted to the $\mathbb{F}_n$-orbit of x is conjugate to β, and for every $k \geqslant 1$ and every transitive β-action on $\{1, ..., k\}$, there are infinitely many $y \in \mathbb{N}$ such that α restricted to the $\mathbb{F}_n$-orbit of y is conjugate to β. In other words, if we denote by $\mathbb{N}^{(N)}$ the set of N-tuples of pairwise distinct elements of $\mathbb{N}$ and by $\mathcal{B}$ the (countable) set of transitive $\mathbb{F}_n$-actions on sets of the form $\{1, ..., k\}$ for some $k \geqslant 1$, then

$$\mathcal{A}_n = \left(\bigcap_{x \in \mathbb{N}} \bigcup_{\beta \in \mathcal{B}} \mathcal{A}_{\beta,x} \right) \cap \left(\bigcap_{\beta \in \mathcal{B}} \bigcap_{N \in \mathbb{N}} \bigcup_{(y_1, \dots, y_N) \in \mathbb{N}^{(N)}} \bigcap_{j=1}^{N} \mathcal{A}_{\beta, y_j} \right). \tag{2}$$

Since each $\mathcal{A}_{\beta,x}$ is open, this shows that $\mathcal{A}$ is G_δ.

As we will see shortly, proving that $\mathcal{A}_n$ is dense essentially boils down to proving the following claim, where the **support** of an action $\alpha : \Gamma \curvearrowright X$ is the set of points $x \in X$ such that there is $\gamma \in \Gamma$ satisfying $\alpha(\gamma)x \neq x$.

Claim. For all $x \in \mathbb{N}$, the set of $\mathbb{F}_n$-actions on $\mathbb{N}$ with finite support is dense.

Proof of the claim. Let α be any $\mathbb{F}_n$-action on $\mathbb{N}$, let $S = \{a_1, ..., a_n\}$ be the standard generating set of $\mathbb{F}_n$, and let $F \Subset \mathbb{N}$. We need to find an element α' of $\mathcal{U}_{S,F}(\alpha)$ with finite support, i.e. an action α' with finite support such that $\alpha'(a_i)y = \alpha(a_i)y$ for all $y \in F$ and $i \in \{1, ..., n\}$.

To this end, let $F' = F \cup \bigcup_{i=1}^n \alpha(a_i)F$. For every $i \in \{1, ..., n\}$, the bijection $\alpha(a_i)$ induces a partial bijection φ_i on the finite set F' which by construction of F' satisfies $\varphi_i(y) = \alpha(a_i)y$ for all $y \in F$. Let $\sigma_i : F' \to F'$ be an extension of this partial bijection to a bijection of F'. We finally define α' by

$$\alpha'(a_i)y = \begin{cases} \sigma_i(y) & \text{if } y \in F'; \\ y & \text{otherwise.} \end{cases}$$

By construction α' has its support contained in F' and $\alpha' \in \mathcal{U}_{S,F}$, which finishes the proof. $\square_{\text{claim}}$

Let us now explain why $\mathcal{A}_n$ is dense. By the claim it suffices to show given any action α with finite support, every neighborhood of α intersects $\mathcal{A}_n$. If we let F be a finite set defining the neighborhood $\mathcal{U}_{S,F}(\alpha)$, since α has finite support the set $F' := \bigcup_{\gamma\in\mathbb{F}_n} \alpha(\gamma F)$ is finite. It is then not hard to modify α outside of the finite $\alpha(\mathbb{F}_n)$-invariant set F' so that it belongs to $\mathcal{A}_n$. $\square$

We now give the original definition of ample generics and explain how it connects to the previous result.

Definition 8.5. A Polish group G has **ample generics** if for every $n \geqslant 1$, the diagonal conjugacy action $G \curvearrowright G^n$ given by $g\cdot(g_1, ..., g_n) = (gg_1g^{-1}, \dots, gg_ng^{-1})$ admits a comeager orbit.

Lemma 8.6. *Let G be a Polish group, let $a_1, ..., a_n$ denote the standard generators of $\mathbb{F}_n$. Then the map $\Phi : \mathrm{Hom}(\mathbb{F}_n, G) \to G^n$ which maps α to $(\alpha(a_1), ..., \alpha(a_n))$ is a homeomorphism which is G-equivariant when we endow $\mathrm{Hom}(\mathbb{F}_n, G)$ with the conjugacy action $g \cdot \alpha(\gamma) = g\alpha(\gamma)g^{-1}$ and G^n with the diagonal conjugacy action described in the previous definition.*

Proof. The continuity and equivariance of Φ are clear, and the fact that it is injective follows from the fact that $\{a_1, ..., a_n\}$ is a generating set for $\mathbb{F}_n$. Surjectivity is a consequence of the universal property of $\mathbb{F}_n$ and openness follows the continuity of group operations in G. $\square$

In view of the previous lemma a Polish group G has ample generics if and only if the conjugacy action of G on $\mathrm{Hom}(\mathbb{F}_n, G)$ has a comeager orbit for every $n \geqslant 1$.

Example 8.7. By Theorem 8.3, the Polish group $\mathfrak{S}_\infty$ has ample generics.

Remark 8.8. Glasner, Kitroser and Melleray have obtained a neat characterization of the countable groups Γ satisfying that $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ has a comeager orbit in terms of isolated subgroups in the space of subgroups of Γ [GKM16, Thm. 1.3].

Ample generics arose in the work of Hodges, Hodkinson, Lascar and Shelah [HHLS93]. It was used to show that some automorphism groups of countable structures essentially remember the underlying structure as abstract groups. In general, automorphism groups of countable structures are exactly the Polish groups which are *non-archimedean*. Recall that a topological group is called **archimedean** if for every neighborhood of the identity U and every $g \in G$, there is $h \in U$ and $n \in \mathbb{N}$ such that $g = h^n$. The definition of a non-archimedean group is stronger than the negation of being archimedean.

Definition 8.9. A topological group G is **non-archimedean** if its identity admits a basis of neighborhoods consisting of open subgroups.

The prototypical example of a non-archimedean Polish group is the group $\mathfrak{S}_\infty$ of permutations of $\mathbb{N}$: a basis of neighborhoods of the identity is given by the family of open subgroups

$$G_n := \{\sigma \in \mathfrak{S}_\infty \colon \forall i \in \{0, \dots, n\}, \sigma(i) = i\},$$

where $n \in \mathbb{N}$. As a matter of fact, every Polish non-archimedean group is isomorphic to a closed subgroup of $\mathfrak{S}_\infty$ (see [BK96, Thm. 1.5.1] for this and the fact that automorphism groups of countable structures are exactly Polish non-archimedean groups).

In an influential paper where they isolate the property of ample generics and study some model-theoretic ways to understand it, Kechris and Rosendal show that Polish groups with ample generics share the following remarkable property.

Theorem 8.10 (Kechris-Rosendal [KR07]). *Let G be a Polish group with ample generics. Then G has the **automatic continuity property**: for every separable topological group H, every group homomorphism $G \to H$ has to be continuous.*

In the same paper, they ask whether there exists Polish groups with ample generics which are not non-archimedean. In the next section, we will discuss three breeds of examples which were found in 2015.

8.2 Ample generics outside non-archimedean groups

The following three types of examples do yield Polish groups with ample generics which are not non-archimedean.

8.2.1 Tsankov's group The first example that we will discuss was constructed by Tsankov. Malicki then proved that it has ample generics[6]. As we will discuss in the next section, one of its main features is that it is a totally disconnected group which is not non-archimedean (see Example 8.22).

Definition 8.11 (Tsankov, see [Tsa06, Sec. 5]). Endow $\mathbb{N}$ with the measure μ defined by $\mu(\{n\}) = \frac{1}{n}$, and consider the subgroup G_μ of $\mathfrak{S}_\infty$ consisting of all permutations $\sigma \in \mathfrak{S}_\infty$ whose **support** (defined as $\operatorname{supp}\sigma \coloneqq \{n \in \mathbb{N} \colon \sigma(n) \neq n\}$) has finite measure. Endow this group with the left-invariant metric d_μ defined by

$$d_\mu(\sigma, \tau) \coloneqq \mu(\{n \in \mathbb{N} \colon \sigma(n) \neq \tau(n)\}).$$

By [Tsa06, Thm. 5.3] G_μ is then a Polish group for the topology induced by d_μ.

Theorem 8.12 (Malicki [Mal16b]). *The above-defined group G_μ has ample generics.*

Proof. We follow again the strategy of the proof of Theorem 8.3. Fix $n \in \mathbb{N}$. By definition, the group G_μ is a subgroup of $\mathfrak{S}_\infty$, and the inclusion map is continuous. It then follows from Theorem 8.3 that the space $\mathcal{A}$ of homomorphism $\alpha : \mathbb{F}_n \to G_\mu$ such that

(i) α only has finite orbits;

(ii) for every transitive action β of $\mathbb{F}_n$ on a finite set, there are infinitely many α-orbits such that the restriction of α to each of these orbits is conjugate to β.

is G_δ. The fact that all homomorphisms $\mathbb{F}_n \to G_\mu$ are actions whose support has finite measure allows one to show that $\mathcal{A}$ consists of a single G_μ-conjugacy class. Finally, to see that $\mathcal{A}$ is dense, we first establish the following claim.

Claim. For all $x \in \mathbb{N}$, the set of homomorphisms $\mathbb{F}_n \to G_\mu$ with finite support is dense in $\operatorname{Hom}(\mathbb{F}_n, G_\mu)$

Proof of the claim. Let $\alpha : \mathbb{F}_n \to G_\mu$, let $S = \{a_1, ..., a_n\}$ be the standard generating set of $\mathbb{F}_n$, let $\epsilon > 0$ and let $F \Subset \mathbb{N}$ large enough so that $\mu(\operatorname{supp}\alpha \setminus F) < \epsilon$.

We then repeat the proof of the claim from the proof of Theorem 8.3: let $F' = F \cup \bigcup_{i=1}^n \alpha(a_i)F \subseteq \operatorname{supp}\alpha$. For every $i \in \{1, ..., n\}$, the bijection $\alpha(a_i)$ induces a partial bijection φ_i on the finite set F' which by construction of F' satisfies $\varphi_i(y) = \alpha(a_i)y$ for

[6] As we briefly discuss in Remark 8.13 this example belongs to a much larger class, but we prefer to stick to a concrete example for readability.

all $y \in F$. Let $\sigma_i : F' \to F'$ be an extension of this partial bijection to a bijection of F'. We finally define α' by

$$\alpha'(a_i)y = \begin{cases} \sigma_i(y) & \text{if } y \in F'; \\ y & \text{otherwise.} \end{cases}$$

By construction α' has its support contained in F' and $d_\mu(\alpha(a_i), \alpha'(a_i)) < \epsilon$ for all $i = 1, \dots, n$ because $\alpha(a_i)$ and $\alpha'(a_i)$ only differ on $\operatorname{supp}\alpha \setminus F$. $\square_{\text{claim}}$

Density of $\mathcal{A}$ now easily follows, using the fact that $\mu(\{n\})$ tends to zero. We have shown that for every $n \in \mathbb{N}$, $G_\mu \curvearrowright \operatorname{Hom}(\mathbb{F}_n, G_\mu)$ has a dense G_δ orbit, so by Lemma 8.6 the proof that G_μ has ample generics is complete. $\square$

Remark 8.13. The only two features of the measure μ that we used were that $\mu(\{n\})$ is positive but tends to zero while $\sum_n \mu(\{n\}) = +\infty$. Even more generally, Tsankov's groups make sense more generally when one picks an *ideal* of $\mathcal{P}(\mathbb{N})$, and one can then give a very large class of totally disconnected groups satisfying ample generics while not being non archimedean, see the main result from [Mal16b].

8.2.2 Groups of measurable maps

Our second class of examples was obtained together with Kaïchouh [KLM15]: we observed that if G has ample generics, then $\mathrm{L}^0(X, \mu, G)$ also has ample generics, a natural generalization of the easy fact that $G^{\mathbb{N}}$ has ample generics. Since the proof is rather straightforward, we give it in full details.

We start by observing the following general fact about spaces of measurable maps, where we make the following slight abuse of notation. If Y is a Polish space and $Z \subseteq Y$, we identify $\mathrm{L}^0(X, \mu, Z)$ to the subspace of $\mathrm{L}^0(X, \mu, Y)$ consisting of maps $f : X \to Y$ such that $f(x) \in Z$ for almost every $x \in X$.

Proposition 8.14. *Let Y be a Polish space, and let $Z \subseteq Y$ be G_δ. The following hold.*

(i) If Z is G_δ in Y then $\mathrm{L}^0(X, \mu, Z)$ is G_δ in $\mathrm{L}^0(X, \mu, Y)$.

(ii) If Z is dense in Y then $\mathrm{L}^0(X, \mu, Z)$ is dense in $\mathrm{L}^0(X, \mu, Y)$.

Proof. We first prove (i): assume Z is G_δ. By Alexandrov's theorem Polish subspaces of Polish spaces are exactly G_δ subspaces, see Corollary 1.3. We thus first see that since Z is Polish, $\mathrm{L}^0(X, \mu, Z)$ is Polish for the topology of convergence in measure. Observe that the latter is equal to the topology induced by the Polish topology of convergence in measure of $\mathrm{L}^0(X, \mu, Y)$. Using Alexandrov's theorem once more, we conclude that $\mathrm{L}^0(X, \mu, Z)$ is G_δ.

Let us prove (ii). Assume that Z is dense, let $\{z_n \colon n \in \mathbb{N}\} \subseteq Z$ be countable dense and let d be a compatible metric on Y. We work with the neighborhood basis given by lemma 4.2. Let us fix $\epsilon > 0$ and $f \in \mathrm{L}^0(X, \mu, Y)$, we need to find $g \in \mathrm{L}^0(X, \mu, Z)$ such that

$$\mu\left(\{x \in X \colon d(g(x), f(x)) > \epsilon\}\right) < \epsilon.$$

For every $x \in X$, let $g(x) \coloneqq z_n$ where n is the least integer such that $d(f(x), z_n) < \epsilon$. Then we actually have $d(f(x), g(x)) < \epsilon$ for *every* $x \in X$, so in particular g is as wanted. $\square$

Remark 8.15. Here is a more concrete proof of the first part that we gave in [KLM15]. First note that $\mathrm{L}^0(X, \mu, Z)$ is G_δ whenever Z is open, using the fact that it is the intersection over $n \in \mathbb{N}$ of the open sets

$$\left\{ f \in \mathrm{L}^0(X, \mu, Y) \colon \mu\big(f^{-1}(Z)\big) > 1 - \frac{1}{n} \right\}.$$

Then observe that if $Z = \bigcap_n Z_n$ with Z_n open, then $\mathrm{L}^0(X,\mu,Z) = \bigcap_n \mathrm{L}^0(X,\mu,Z_n)$ so it has to be G_δ as well.

The key tool to our proof is the following version of the Jankov-von Neumann theorem, which allows us to *choose elements in a measurable way.*

Theorem 8.16. *Let (X,μ) be a standard probability space and Y be a Polish space, let $A \subseteq X \times Y$ be Borel, suppose that $\pi_X(A) = X$ where $\pi_X : X \times Y \to X$ is the projection on the first factor. Then there is $f \in \mathrm{L}^0(X,\mu,Y)$ such that $(x, f(x)) \in A$ for almost all $x \in X$.*

Proof. The Jankov-von Neumann theorem grants us the existence of a $\sigma(\mathbf{\Sigma}^1_1)$-measurable map $f : X \to Y$ such that $(x, f(x)) \in A$ for all $x \in X$ (see [Kec95, 18.A]). The conclusion follows by combining the following two facts: every $\sigma(\mathbf{\Sigma}^1_1)$ set is Lebesgue measurable (see [Kec95, Thm. 29.7]), and every Lebesgue measurable function is equal to a Borel function up to measure zero. □

We can now prove the announced result.

Theorem 8.17 ([KLM15, Thm. 6])**.** *Let G be a Polish group with ample generics, then $\mathrm{L}^0(X,\mu,G)$ also has ample generics.*

Proof. Let $n \in \mathbb{N}$, consider the diagonal $\mathrm{L}^0(X,\mu,G)$-action on $\mathrm{L}^0(X,\mu,G)^n$ by conjugacy, we need to find a dense G_δ orbit for this action. Fix $(g_1,...,g_n) \in G^n$ with a dense G_δ-orbit, which we denote by O. The previous proposition ensures uses that $\mathrm{L}^0(X,\mu,O)$ is dense G_δ. It follows that $\mathrm{L}^0(X,\mu,O)^n$ is dense G_δ, as is any product of dense G_δ sets for the product topology.

So in order to finish the proof, it suffices to show that $\mathrm{L}^0(X,\mu,O)^n$ consists of a single $\mathrm{L}^0(X,\mu,G)$-orbit for the diagonal action. For each $i \in \{1,...,n\}$ denote by $\bar{g}_i$ the constant function equal to g_i, then $(\bar{g}_1,\dots,\bar{g}_n) \in \mathrm{L}^0(X,\mu,O)^n$. Now take $(f_1,...,f_n) \in \mathrm{L}^0(X,\mu,O)^n$. By assumption for all $x \in X$ there is $g \in G$ such that

$$\big(gf_1(x)g^{-1},\dots,gf_n(x)g^{-1}\big) = (g_1,\dots,g_n).$$

To conclude the proof, it suffices to show that we can choose this g in a measurable manner: we need a Borel function $g : X \to G$ such that for almost all $x \in X$,

$$\big(g(x)f_1(x)g^{-1}(x),\dots,g(x)f_n(x)g^{-1}(x)\big) = (g_1,\dots,g_n).$$

The fact that such a function exists is a direct consequence of the Jankov-von Neumann theorem as stated before (Theorem 8.16), applied to the Borel set

$$A = \left\{(x,g) \in X \times G\colon\ \big(gf_1(x)g^{-1},\dots,gf_n(x)g^{-1}\big) = (g_1,\dots,g_n).\right\}.$$

We conclude that the $\mathrm{L}^0(X,\mu,G)$-orbit of $(\bar{g}_1,\dots,\bar{g}_n)$ is equal to $\mathrm{L}^0(X,\mu,O)^n$. Since the latter is dense G_δ, this finishes the proof that $\mathrm{L}^0(X,\mu,G)$ has ample generics. □

Remark 8.18. Kwiatkowska and Malicki have proven that the converse also holds: if $\mathrm{L}^0(X,\mu,G)$ has ample generics then so does G, see [KM19, Cor. 3.2].

The main interest of our result is that $\mathrm{L}^0(X,\mu,G)$ is always contractible (in particular, connected) as soon as (X,μ) is standard: indeed we can assume $X=[0,1]$ endowed with the Lebesgue measure, and then the map $\Phi:[0,1]\times\mathrm{L}^0(X,\mu,G)\to\mathrm{L}^0(X,\mu,G)$ given by

$$\Phi(t,f)(x)=\begin{cases} f(x) & \text{if } x\geqslant t\\ 1_G & \text{if } x<t\end{cases}$$

is a homotopy from $\mathrm{id}_{\mathrm{L}^0(X,\mu,G)}$ to the constant map equal to the neutral element of $\mathrm{L}^0(X,\mu,G)$, thus witnessing that $\mathrm{L}^0(X,\mu,G)$ is contractible. Since G embeds into $\mathrm{L}^0(X,\mu,G)$ via constant maps, we have the following corollary, yielding examples of Polish groups with ample generics but far from being non-archimedean.

Corollary 8.19 ([KLM15, Cor. 8])**.** *Every Polish group with ample generics embeds in a contractible Polish group with ample generics.* □

8.2.3 Full groups of non-singular hyperfinite equivalence relations.

Our last class of examples comes from ergodic theory: with Kaïchouh we showed that *every full group of a type* III *ergodic hyperfinite equivalence relation has ample generics*. Using the reconstruction theorem of Fremlin [Fre02, 384D] and Krieger's classification of type III ergodic automorphisms up to orbit equivalence into types III_0, III_λ $(0<\lambda<1)$ and III_1 [Kri69], we get a continuum of connected[7] simple Polish groups with ample generics. Concrete examples are provided by the second item in Example 7.3. The description of the dense G_δ sets in $\mathrm{Hom}(\mathbb{F}_n,[\mathcal{R}])$ consisting of a single orbit are given by the following result.

Theorem 8.20. *Let $\mathcal{R}$ be a type* III *ergodic hyperfinite equivalence relation, let $n\geqslant 1$.*

Consider the subset $\mathcal{H}_n$ of $\mathrm{Hom}(\mathbb{F}_n,[\mathcal{R}])$ given by homomorphisms $\alpha:\mathbb{F}_n\to[\mathcal{R}]$ such that α has only finite orbits, and for every transitive $\mathbb{F}_n$-action β on a finite set, the set of $x\in X$ such that the restriction of α to the $\alpha(\Gamma)$-orbit of x is conjugate to β has positive measure.

Then $\mathcal{H}_n$ is dense G_δ and consists of a single conjugacy class. In particular, $[\mathcal{R}]$ has ample generics.

We won't give the proof and refer the reader to [KLM15, Sec. 4]. However we will now explain why the type II_∞ ergodic hyperfinite equivalence relation (as described in the last item of Example 7.3) has a full group with ample generics, a result that was left open by our paper. The ideas is very similar to our original proof, and the description of the dense G_δ diagonal conjugacy classes is very natural in view of the description of the dense G_δ diagonal conjugacy class for $\mathfrak{S}_\infty$ (see Theorem 8.3), noting that $\mathfrak{S}_\infty$ can be seen as the full group of the type I_∞ ergodic equivalence relation.

Theorem 8.21. *Let $\mathcal{R}$ be a type II_∞ hyperfinite ergodic equivalence relation, let μ be an invariant σ-finite measure.*

Consider the subset $\mathcal{H}_n$ of $\mathrm{Hom}(\mathbb{F}_n,[\mathcal{R}])$ given by homomorphisms $\alpha:\mathbb{F}_n\to[\mathcal{R}]$ such that α has only finite orbits, and for every transitive $\mathbb{F}_n$-action β on a finite set, the set of $x\in X$ such that the restriction of α to the $\alpha(\Gamma)$-orbit of x is conjugate to β has infinite measure.

Then $\mathcal{H}_n$ is dense G_δ and consists of a single conjugacy class. In particular, $[\mathcal{R}]$ has ample generics.

[7] Full groups are always connected by a result of Bezuglyi and Golodets [BG80, Thm. 2.3], in particular they are not non-archimedean.

Proof. Since there is only one hyperfinite type II_∞ ergodic equivalence relation up to isomorphism, we may as well assume that we are in the following variation of the situation described by the last item of Example 7.3. We take $X = \mathbb{N} \times \{0,1\}^{\mathbb{N}}$ endowed with the equivalence relation $\mathcal{R} = (\mathbb{N} \times \mathbb{N}) \times \mathcal{R}_0$ and the measure $\mu = \sum_{k\in\mathbb{N}} \delta_k \otimes \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)^{\otimes\mathbb{N}}$. As a side note which will be useful later on, observe that we can easily obtain an element $\tilde{\beta}$ of $\mathcal{H}_n$ from an action $\beta : \mathbb{F}_n \curvearrowright \mathbb{N}$ in $\mathcal{A}_n$ (as defined in Theorem 8.3) by defining $\tilde{\beta}(\gamma)k, x = (\beta(\gamma)k, x)$.

Observe $\mathcal{R}$ is hyperfinite, e.g. because it can be written as the increasing union of $\mathcal{R}_k$ where

$$((i,P),(j,Q)) \in \mathcal{R}_k \text{ iff } P \mathbin{\triangle} Q \subseteq \{0,...,k\} \text{ and } (i = j \text{ or} \max(i,j) \leqslant k).$$

We then have the following natural result, which is a direct consequence of [KT10, Lem. 4.7] (their proof works as well in the non-singular setup; we nevertheless give a simple proof for our restricted setup).

Lemma. The union $\bigcup_{k\in\mathbb{N}}[\mathcal{R}_k]$ is dense in $[\mathcal{R}]$.

Proof of the lemma. Take $T \in [\mathcal{R}]$, define $T_k \in [\mathcal{R}_k]$ by closing the finite segments obtained by intersecting the graph of T with $\mathcal{R}_k$, namely:

$$T_k(x) = \begin{cases} T(x) & \text{if } (x, T(x)) \in \mathcal{R}_k \\ T^{-j}(x) & \text{otherwise, where } j = \min\{i \geqslant 1 : (x, T^{-i}(x)) \notin \mathcal{R}_k\} - 1. \end{cases}$$

The finiteness of the equivalence classes of $\mathcal{R}_k$ and the fact that T is a bijection ensures us that T_k is well-defined and belongs to $[\mathcal{R}_k]$. Since $\mathcal{R} = \bigcup_k \mathcal{R}_k$, for all $x \in X$ we have $T_k(x) = T(x)$ for large enough k, so we conclude that $d_u(T_k, T) \to 0$. $\square_{\text{lemma}}$

We now move on to show that $\mathcal{H}_n$ is dense. To this end, the key point is to show the following analogue of the claims from the proofs of Theorem 8.3 and 8.12. It relies on the previous lemma and hence on hyperfiniteness.

Claim. The set of homomorphisms $\alpha : \mathbb{F}_n \to [\mathcal{R}]$ such that

- α has only finite orbits and
- the support of α has finite measure

is dense in $\mathrm{Hom}(\mathbb{F}_n, [\mathcal{R}])$.

Proof of the claim. Take any $\alpha : \mathbb{F}_n \to [\mathcal{R}]$, we need to approximate it by elements satisfying the conclusion of the claim. By the previous lemma we can approximate $\alpha(a_1), ..., \alpha(a_n)$ by elements of $\bigcup_k [\mathcal{R}_k]$, so we may as well assume that $\alpha(a_1), ..., \alpha(a_n) \in [\mathcal{R}_k]$, and thus that α has only finite orbits.

Now by σ-finiteness we way write X as an increasing union $X = \bigcup_k X_k$ where each X_k has finite measure. Then since α has finite orbits, for all $x \in X$ the whole $\alpha(\mathbb{F}_n)$-orbit of x is contained in X_k for k large enough. So if we let

$$A_k = \{x \in X_k \colon \alpha(\mathbb{F}_n)x \subseteq X_k\},$$

we have that $\bigcup_k A_k = X$ and each A_k is α-invariant. Denoting by α_k the action which is the identity outside of A_k and coincides with α on A_k, we clearly have $\alpha_k \to \alpha$ and α_k has the desired property. $\square_{\text{claim}}$

Now that the claim has been established, we can explain why $\mathcal{H}_n$ is dense. It suffices to approximate elements as in the claim by elements of $\mathcal{H}_n$. So take any α as in the claim. We also fix $\tilde{\beta} \in \mathcal{H}_n$ (such elements exist as explained in the end of the first paragraph of the proof). Let (B_k) be a decreasing sequence of Borel subsets of $X \setminus \operatorname{supp} \alpha$ with infinite measure and trivial intersection, and for each k let $\varphi_k \in [[\mathcal{R}]]$ with domain X and range B_k as provided by Lemma 7.15. Then let

$$\alpha_k(\gamma)x = \begin{cases} \alpha(\gamma)x & \text{if } x \in \operatorname{supp} \alpha; \\ \varphi_k \tilde{\beta}(\gamma)\varphi_k(x) & \text{if } x \in B_k; \\ x & \text{otherwise.} \end{cases}$$

Since $\tilde{\beta} \in \mathcal{H}_n$ we have $\alpha_k \in \mathcal{H}_n$. Moreover the actions α_k and α only differ on B_k. Since the sets B_k are decreasing and $\bigcap_k B_k = \emptyset$, we have $\nu(B_k) \to 0$ where ν is any finite measure equivalent to μ, and we conclude that $\alpha_k \to \alpha$ as wanted.

We are now left with proving that $\mathcal{H}_n$ consists of a single conjugacy class and that it is G_δ.

We first prove $\mathcal{H}_n$ is G_δ. We start by showing that the set $\mathcal{H}_n^{\text{fin}}$ of $\alpha : \mathbb{F}_n \to [\mathcal{R}]$ with only finite orbits is G_δ. To see this, recall from the proof of Theorem 8.3 that the space $\mathcal{A}_n^{\text{fin}}$ of $\mathbb{F}_n$-actions on $\mathbb{N}$ with only finite orbits is G_δ since we have the equality

$$\mathcal{A}_n^{\text{fin}} = \left(\bigcap_{x \in \mathbb{N}} \bigcup_{\beta \in \mathcal{B}} \mathcal{A}_{\beta,x} \right),$$

where $\mathcal{B}$ denotes the set of all transitive actions on finite sets of the form $\{1, .., k\}$ and for $\beta \in \mathcal{B}$, the set $\mathcal{A}_{\beta,x}$ is the open set of all actions on $\mathbb{N}$ whose restriction to the orbit of x is conjugate to β. Using Proposition 8.14 we obtain that $\mathrm{L}^0(X, \mu, \mathcal{A}_n^{\text{fin}})$ is also G_δ.

Now let (f_n) be a decomposition of $\mathcal{R}$ as in Definition 7.10, denote by

$$\Phi \colon [\mathcal{R}] \to \mathrm{L}^0(X, \mu, \mathfrak{S}_\infty)$$

the corresponding continuous embedding as provided by Proposition 8.14. Then by definition

$$\mathcal{H}_n^{\text{fin}} = \Phi^{-1}\left(\mathrm{L}^0(X, \mu, \mathcal{A}_n^{\text{fin}})\right),$$

so $\mathcal{H}_n^{\text{fin}}$ is G_δ as wanted.

In order to show that $\mathcal{H}_n$ is G_δ, it now suffices to show that for every $\beta \in \mathcal{B}$, the set $\mathcal{H}_n^\beta$ of α such that

$$\mu\left(\left\{x \in X \colon \alpha_{\restriction \alpha(\mathbb{F}_n)x} \text{ is conjugate to } \beta\right\}\right) = +\infty$$

is a G_δ set. Once this is done, the proof will be over since by definition

$$\mathcal{H}_n = \mathcal{H}_n^{\text{fin}} \cap \bigcap_{\beta \in \mathcal{B}} \mathcal{H}_n^\beta.$$

So to see this, observe that since our decomposition (f_n) satisfies $f_0 = \mathrm{id}_X$, we can rewrite $\mathcal{H}_n^\beta$ as the set of all α such that

$$\mu\left(\{x \in X \colon \Phi(\alpha)(x) \in \mathcal{A}_{\beta,0}\}\right) = +\infty$$

We now rely on a final lemma.

Lemma. Let U be an open subset of a Polish space Y, let μ be a σ-finite measure on a standard space X. Then the following set is G_δ in $\mathrm{L}^0(X, \mu, Y)$:

$$\mathcal{M}_\infty(U) \coloneqq \{f \in \mathrm{L}^0(X, \mu, Y) \colon \mu(f^{-1}(U)) = +\infty\}.$$

Proof of the lemma. Observe that $\mathcal{M}_\infty(U) = \bigcap_k \mathcal{M}_{>k}(U)$ where

$$\mathcal{M}_{>k}(U) \coloneqq \{f \in \mathrm{L}^0(X, \mu, Y) \colon \mu(f^{-1}(U)) > n\}$$

It thus suffices to show each $\mathcal{M}_{>k}(U)$ is open. Let us thus fix $f \in \mathcal{M}_{>k}(U)$, let $f_n \to f$ in measure and assume by contradiction that $f_n \notin \mathcal{M}_{<k}(U)$. Up to taking a subsequence we have that $f_n(x)$ converges to $f(x)$ for almost every x. In particular since U is open, for almost all $x \in f^{-1}(U)$, we also have $x \in f_n^{-1}(U)$ for n large enough. It follows that up to measure zero,

$$f^{-1}(U) \subseteq \bigcup_N \bigcap_{n \geqslant N} f_n^{-1}(U)$$

and since by assumption $\mu(f_n^{-1}(U)) \leqslant k$ for all n we conclude that $\mu(f^{-1}(U)) \leqslant k$, a contradiction. $\square_{\text{lemma}}$

Using the notation of the lemma, we can finally rewrite

$$\mathcal{H}_n^\beta = \Phi^{-1}(\mathcal{M}_\infty(\mathcal{A}_{\beta,0}))$$

and since $\mathcal{A}_{\beta,0}$ is open the desired conclusion that $\mathcal{H}_n^\beta$ (and hence $\mathcal{H}_n$) is G_δ follows.

We finally prove that $\mathcal{H}_n$ consists of a single conjugacy class. At this point, it might be a good idea for the reader to do the exercise from the proof of Theorem 8.3, namely to give an explicit proof that the set $\mathcal{A}_n$ from Theorem 8.3 consists of a single conjugacy class. Indeed, we are now going to do that *in a measurable manner* in our context.

To this end, it will be more convenient to work with finite index subgroups of $\mathbb{F}_n$ rather than our set $\mathcal{B}$ of transitive actions on finite sets. Let (Λ_i) be a sequence of elements of finite index subgroups of $\mathbb{F}_n$ such that

- Every subgroup of finite index is conjugate to some Λ_i;
- For all $i \neq j$ the subgroup Λ_i is not conjugate to Λ_j.

Now let $\alpha_1, \alpha_2 \in \mathcal{H}_n$. For $i \in \mathbb{N}$, denote by X_1^i the set of $x \in X$ such that there is y in the α_1-orbit of x with

$$\mathrm{Stab}_{\alpha_1}(y) = \Lambda_i.$$

Since $\alpha_1 \in \mathcal{H}_n$ and (Λ_i) is an injective enumeration of conjugacy classes of finite index subgroups, we have that $(X_1^i)_{i\in\mathbb{N}}$ is a partition of X consisting of infinite measure subsets. We analogously define a partition $(X_2^i)_{i\in\mathbb{N}}$ corresponding to the action α_2 by $x \in X_2^i$ iff there is y in the α_2-orbit of x such that $\mathrm{Stab}_{\alpha_2}(y) = \Lambda_i$.

Fix $j \in \{1, 2\}$, we first define the following Borel subset of X_j^i:

$$Y_j^i \coloneqq \{y \in X \colon \mathrm{Stab}_{\alpha_j}(y) = \Lambda_i\}$$

We can now take advantage of the finiteness of the α_j-orbits to (measurably) pick a point $z \in Y_j^i$ in each orbit contained in X_j^i. Let us give a detailed argument for the sake of completeness: one first fixes a Borel total order $\leqslant_X$ on X (obtained for instance by identifying X to $[0,1]$ and taking for $\leqslant_X$ the usual linear order on the reals). Here is then for $i \in \mathbb{N}$ the Borel subset of Y_j^i which intersects exactly once each orbit contained in X_j^i:

$$Z_j^i := \{z \in Y_j^i \colon \forall y \in Y_j^i \cap (\alpha_j(\mathbb{F}_n)z), z \leqslant_X y\}.$$

In plain words, Z_j^i is obtained from X_j^i by selecting in each orbit the minimum of the (finite!) set of elements with stabilizer Λ_i. The fact that that Z_j^i is Borel becomes clear once one notices that $z \in Z_j^i$ iff $z \in Y_j^i$ and for all $\gamma \in \mathbb{F}_n$ such that $\mathrm{Stab}_{\alpha_j}(z) = \Lambda_i$, one has $z \leqslant_X \alpha_j(\gamma)z$, allowing us to rewrite Z_j^i as

$$\begin{aligned} Z_j^i &= \bigcap_{\gamma \in \mathbb{F}_n} \{z \in Y_j^i \colon \alpha_j(\gamma)z \in Y_j^i \text{ and } z \leqslant_X \alpha_j(\gamma)z\} \\ &= \bigcap_{\gamma \in \mathbb{F}_n} \left(\alpha_j(\gamma)^{-1}(Y_j^i) \cap \{z \in Y_j^i : z \leqslant_X \alpha_j(\gamma z)\}\right). \end{aligned}$$

Now observe that if $F_i \Subset \mathbb{F}_n$ is a (necessarily finite) set of left coset representatives for Λ_i (i.e. $\mathbb{F}_n = \bigsqcup_{\gamma \in F_i} \gamma\Lambda_i$), then

$$X_j^i = \bigsqcup_{\gamma \in F_i} \alpha_j(\gamma) Z_j^i.$$

Since X_j^i has infinite measure, it follows that Z_j^i also has infinite measure.

By Lemma 7.15, for each $i \in \mathbb{N}$ we find $\varphi_i \in [[\mathcal{R}]]$ of domain Z_1^i and of range Z_2^i. We then define the conjugacy map $S \in [\mathcal{R}]$ by: for all $\gamma \in F_i$, for all $x \in \gamma Z_1^i$,

$$S(x) = \alpha_2(\gamma)\varphi_i\alpha_1(\gamma)^{-1}x.$$

Noting that the map S does not depend on the particular choice of coset representatives F_i, it is then straightforward to check that S conjugates α_1 to α_2. This finishes the proof that $\mathcal{H}_n$ consists of a single conjugacy class, so the theorem is proven. □

8.3 Quasi non-archimedianity

We will now discuss a feature which is shared by the various examples of ample generic groups we presented. This feature is a generalization of being non-archimedean that we introduced in a joint work with Tsachik Gelander, and we call it *quasi non-archimedean* [GLM17].

Let us start by repeating the definition of non-archimedianity: a topological group G is non-archimedean if its identity element admits a basis of neighborhoods consisting of open subgroups. Every open subgroup of a topological group has to be also closed because its complement is a disjoint union of its cosets which are open. It follows that every non-archimedean topological group is **totally disconnected**: all its connected components are trivial, or more concretely given any two distinct points, there is a clopen set that contains the first point but not the second. It is well-known consequence of van Dantzig's theorem that conversely, every *locally compact* Polish group is non-archimedean as soon as it is totally disconnected (see [Tao14, Rmk. 1.6.10] for a proof of van Dantzig's theorem). However, this is not true of Polish groups in general, as the following nice example shows.

Example 8.22. Recall from Definition 8.11 that if we endow $\mathbb{N}$ with the measure μ defined by $\mu(\{n\}) = \frac{1}{n}$, Tsankov's group G_μ of permutations $\sigma \in \mathfrak{S}_\infty$ whose support has finite measure is Polish for the topology induced by the left-invariant metric d_μ defined by

$$d_\mu(f, g) \coloneqq \mu(\{n \in \mathbb{N} \colon f(n) \neq g(n)\}).$$

Since its inclusion in the totally disconnected space $\mathfrak{S}_\infty$ is continuous, it has to be totally disconnected. However, using the fact that $\sum_n \frac{1}{n} = \infty$ but $\lim_n \frac{1}{n} = 0$, it is not hard to see that given any $\epsilon > 0$ and $k \in \mathbb{N}$ one can find a k-tuple $(\sigma_i)_{i=1}^k$ of finitely supported elements of G_μ with pairwise disjoint supports, each of measure belonging to the interval $[\frac{\epsilon}{2}, \epsilon)$. In particular, their product $\sigma_1 \cdots \sigma_k$ has a support of measure at least $\frac{k\epsilon}{2}$. It follows that given an open ball B of radius $\delta > 0$ centered at the identity, there is no open subgroup contained in B. So the topological group G_μ is far from being non-archimedean.

We now introduce quasi non-archimedeanity. As a motivation, observe that a topological group G is non-archimedean if and only if for every neighborhood of the identity U, there is a neighborhood of the identity V such that for every $n \in \mathbb{N}$ and every $g_1, ..., g_n \in V$, the group generated by $\{g_1, ..., g_n\}$ is contained in U (this is essentially the property we negated in the previous example).

Definition 8.23. A topological group is **quasi non-archimedean** (QNA) if for every neighborhood of the identity U and every $n \in \mathbb{N}$, there is a neighborhood of the identity V such that for and $g_1, ..., g_n \in V$, the group generated by $\{g_1, ..., g_n\}$ is contained in U.

As the reader can see, the difference with non-archimedean is that n is now given *before* having to chose V.

Example 8.24. Tsankov's group G_μ is quasi-non archimedean. Indeed, without loss of generality we can take U to be an open ball of d_μ-radius $\epsilon > 0$. Given $n \in \mathbb{N}$, we let V be the open ball of d_μ radius $\frac{\epsilon}{n}$. Let $\sigma_1, ..., \sigma_n \in V$, and let A be the union of their supports. By hypothesis $\mu(A) < \epsilon$. Now observe that every element of the group generated by $\sigma_1, ..., \sigma_n$ has support contained in A, in particular it belongs to U as wanted. However G_μ is not non-archimedean, as we saw in Example 8.22.

Theorem 8.25 (Gelander and the author, see [GLM17])**.** *Let G be a locally compact group, then G is quasi non-archimedean if and only if it is non-archimedean.*

We give the easy proof suggested by Cornulier and Caprace.

Proof. Suppose that G quasi non-archimedean, and assume by contradiction that G is not non-archimedean. By van Dantzig's theorem, the connected component of the identity G_0 of G is not trivial. It is then well-known that G_0 contains a one-parameter subgroup, i.e. there is a non-trivial continuous group homomorphism $\psi : \mathbb{R} \to G_0$ (see [MZ55, 4.7.2] in the metric case, [MZ55, 4.15] and [HM20, Lem. 7.41(ii)] for the general case).

Let us see why this contradicts that G is quasi-non archimedean even for $n = 1$. Since ψ is not trivial we find a neighborhood of the identity U in G which does not contain $\psi(\mathbb{R})$. Since G is quasi non-archimedean we should find (for $n = 1$) a neighborhood of the identity V such that for all $g \in V$, the group generated by g is contained in U. However, $\psi^{-1}(V)$ contains $(-\epsilon, \epsilon)$ for some $\epsilon > 0$, and since every real can be written as kt for some $t \in (-\epsilon, \epsilon)$ and $k \in \mathbb{N}$, we see that $\psi(\mathbb{R})$ should be contained in U, a contradiction. □

Remark 8.26. As noted in [GLM17], it is not hard to check that the class of quasi non-archimedean topological groups is closed under taking subgroups (with the induced topology), quotients by closed normal subgroups, and products.

8.4 Almost archimedeanity

Motivated by the above proof, we now introduce a property which we think is the correct opposite of quasi non-archimedean, rather than being *infinitesimally finitely generated*, which was studied in [GLM17] (we will come back to this other notion at the end of this section).

Definition 8.27. Let $n \in \mathbb{N}$. A topological group G is **almost n-archimedean** if for every neighborhood of the identity U, the union $\bigcup_{g_1,...,g_n \in U} \langle g_1, ..., g_n \rangle$ is dense in G.

A topological group is **almost archimedean** if it is almost n-archimedean for some $n \in \mathbb{N}$, and it is **infinitesimally generated** if it is generated by every neighborhood of the identity.

Observe that for every U open neighborhood of the identity in a topological group G, the union $\bigcup_{g_1,...,g_n \in U} \langle g_1, ..., g_n \rangle$ is open and contained in the group generated by U. In particular every almost archimedean topological group has to be infinitesimally generated. Also note that every connected group is automatically infinitesimally generated.

Example 8.28. Every connected locally compact group is almost 1-archimedean since the union of its one-parameter subgroups is dense (see [MZ55, 4.15]).

The following is a natural strengthening of [GLM17, Prop. 5.5].

Proposition 8.29. *Let G be an almost archimedean topological group, let H be a quasi non-archimedean topological group. Then every continuous group homomorphism $\pi : G \to H$ is trivial.*

Proof. We first show that $\pi^{-1}(U)$ is dense whenever U is an open neighborhood of the identity in H. Let n such that G is almost n-archimedean. Let U be an open neighborhood of the identity in H. By assumption there is a smaller open neighborhood V such that given any $h_1, ..., h_n \in V$, the group generated by $h_1, \ldots, h_n$ is contained in U. Now let $g_1, ..., g_n \in \pi^{-1}(V)$, then the group $\pi(\langle g_1, \ldots, g_n \rangle) = \langle \pi(g_1), \ldots, \pi(g_n) \rangle$ is contained in U. Since the union of all $\langle g_1, \ldots, g_n \rangle$ is dense in G, we have that $\pi^{-1}(U)$ is dense in G as wanted.

We can now finish the proof in a standard manner. Let now W be a neighborhood of the identity in H, let U an open neighborhood of the identity such that $U^2 \subseteq W$, we have that $\pi^{-1}(U)$ is dense and open so $\pi^{-1}(U)^2 = G$. Moreover $\pi^{-1}(U)^2 \subseteq \pi \in (U^2) \subseteq \pi^{-1}(W)$ so $\pi^{-1}(W) = G$. Since this holds for every neighborhood of the identity W and since H is Hausdorff we conclude that π is the trivial homomorphism. □

Question 2. Is there a Polish group which is almost 2-archimedean but not almost 1-archimedean?

We will now provide a notion which keeps being equivalent to connectedness for locally compact separable groups, but which is not for Polish groups. To the best of our knowledge, it first appeared in Kechris' monograph in order to rule out continuous homomorphisms from the real line to full groups of measure-preserving equivalence [Kec10], under the name *locally topologically finitely generated*.

Definition 8.30. Let $n \in \mathbb{N}$. A topological group G ha **infinitesimal rank** at most n (or is **infinitesimally n-generated**) if for every neighborhood of the identity U, there are $g_1, \ldots, g_n \in U$ which generated a dense subgroup of G. It is **infinitesimally finitely generated** if it has finite infinitesimal rank.

Observe that if G has infinitesimal rank at most n, then G is almost n-archimedean, in particular it is infinitesimally generated and cannot be quasi non-archimedean.

Example 8.31. For all $n \in \mathbb{N}$, the topological group $\mathbb{R}^n$ has infinitesimal rank $n+1$. Indeed, given a neighborhood U of 0, one first takes $K \in \mathbb{N}$ large enough so that $\frac{1}{K}e_1, ..., \frac{1}{K}e_n \in U$, where $(e_i)_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$. Denote by Γ the discrete subgroup generated by $(\frac{1}{K}e_1, ..., \frac{1}{K}e_n)$ and π the projection $\mathbb{R}^n \to \mathbb{R}^n/\Gamma$. By Kronecker's theorem, if we now take $f \in \mathbb{R}^n \cap U$ with irrational coordinates which are rationally independent, we have that $\pi(f)$ generates a dense subgroup of $\mathbb{R}^n/\Gamma$, which implies that the $n+1$-elements set $\{\frac{1}{K}e_1, ..., \frac{1}{K}e_n, f\}$ generates a dense subgroup of $\mathbb{R}^n$. This shows that the infinitesimal rank of $\mathbb{R}^n$ is at most $n+1$. To see the reverse inequality observe that given any n vectors, the group they generate is contained in the vector space they generate, so if we assume by contradiction that they generate a dense subgroup then they must form a basis of $\mathbb{R}^n$. In particular, the group they generate must be discrete in $\mathbb{R}^n$, a contradiction.

Remark 8.32. Note that $\mathbb{R}^{\mathbb{N}}$ is an example of a Polish group which is almost 1-archimedean without being infinitesimally finitely generated.

We have already observed that connected locally compact groups are almost 1-archimedean because the set of elements belonging to a 1-parameter subgroup is dense. For separable groups, using deep structural results on connected locally compact groups, the following strengthening was obtained in the aforementioned joint work with Gelander.

Theorem 8.33 ([GLM17])**.** *Let G be a connected separable locally compact group. Then the infinitesimal rank of G is finite. Furthermore,*

- *if G is compact nonabelian then its infinitesimal rank is* 2*;*
- *if G is compact abelian then its infinitesimal rank is* 1*.*

Our result for compact groups strengthens results of Halmos-Samelson in the abelian case [HS42], and Hoffman-Morris in the non-abelian case (they deal with the topological rank rather than the infinitesimal rank). We emphasize that we do not assume G to be Polish here; e.g. our result applies to the compact group $K^{[0,1]}$ as soon K is a compact separable group. Moreover, for Polish locally compact groups, our result is a direct consequence of stronger results such as the Schreier-Ulam theorem, see the discussion in the introduction of [GLM17].

8.5 Archimedes versus ample generics

We finish this section by going back to ample generics outside of the non-archimedean world as asked by Kechris and Rosendal.

We have already explained that, as proven by Malicki, Tsankov's groups have ample generics (Theorem 8.12), are not non-archimedean (Example 8.22), quasi non-archimedean (Example 8.24), and totally disconnected.

An argument analogue to that from Example 8.24 shows that full groups of non-singular equivalence relations are all quasi non-archimedean (see also [GLM17, Thm. 5.2]). These groups have ample generics as soon as the equivalence relation is hyperfinite and has type II_∞ (Theorem 8.21) or type III (Theorem 8.20), and they are connected [BG80, Thm. 2.3] hence not non-archimedean.

We have also seen that the class of groups with ample generics is stable under the groups of measurable maps construction (see Paragraph 8.2.2), yielding that the connected group $\mathrm{L}^0(X, \mu, G)$ has ample generics as soon as G has ample generics. Moreover, it is not hard to see that being quasi non-archimedean is also stable under this construction (see [GLM17, Prop. 5.4]).

So all the known examples of Polish groups with ample generics which are not non-archimedean are still quasi non-archimedean. Since non-archimedean groups are clearly quasi-non archimedean, we have the following very natural question.

Question 3. Must every Polish group with ample generics be quasi non-archimedean?

While we don't believe the answer to be positive, we don't have any candidate. We saw in the previous section a wealth of properties which negate quasi non-archimedeanity, so the following questions are probably a better start for someone who believes the answer to be positive.

Question 4. a) Can a Polish group with a dense set of elements belonging to 1-parameter subgroups have ample generics?

b) If not, can almost archimedean Polish groups have ample generics?

c) Can infinitesimally finitely generated Polish groups have ample generics?

Note that connected groups are infinitesimally generated and we have seen examples of connected groups with ample generics, so the adverb *finitely* is important in the last item of the above question.

9 Transitive actions of countable groups

9.1 The spaces of transitive actions over an infinite countable set

Given a countable group Γ, we will now focus on the subspace of transitive Γ-actions inside $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$, which its closely related to the space of infinite index subgroups of Γ.

Recall that given a subgroup $\Lambda \leqslant \Gamma$, we have a natural associated transitive action sometimes called the quasi-regular action $\alpha_\Lambda : \Gamma \curvearrowright \Gamma/\Lambda$, given by

$$\alpha_\Lambda(\gamma) g\Lambda = \gamma g \Lambda.$$

Actually, this action comes with a natural *root*, namely the coset Λ, whose stabilizer

$$\mathrm{Stab}_{\alpha_\Lambda}(\Lambda) \coloneqq \{\gamma \in \Gamma \colon \alpha_\Lambda(\gamma)\Lambda = \Lambda\}$$

is always equal to Λ since $\gamma\Lambda = \Lambda$ iff $\gamma \in \Lambda$. We thus make the following definition.

Definition 9.1. A **rooted transitive action** of the group Γ is a couple (α, x_0) where α is a transitive Γ-action on a set X and $x_0 \in X$.

Example 9.2. Given a subgroup $\Lambda \leqslant \Gamma$, we get the (quasi-regular) rooted action $(\alpha_\Lambda, \Lambda)$ associated to Λ. As explained before the definition, we then recover Λ as the stabilizer of the root.

Say that two pointed transitive rooted actions (α, x_0) and (β, y_0) on X and Y respectively are **isomorphic** if there is a bijection $f : X \to Y$ such that $f(x_0) = y_0$ and for all $x \in X$ and all $\gamma \in \Gamma$, $f(\alpha(\gamma)x) = \beta(\gamma)f(x)$. Observe that for a given rooted transitive action (α, x_0) on X, the map

$$\begin{aligned}\Gamma &\to X \\ \gamma &\mapsto \alpha(\gamma)x_0\end{aligned}$$

quotients down to an isomorphism of rooted actions between (α, x_0) and the quasi-regular rooted action associated to the α-stabilizer of x_0. It follows that two rooted transitive actions are isomorphic if and only if the respective stabilizers of their roots are equal. So subgroups and isomorphism classes of rooted transitive actions are two sides of the same coin.

Let us connect this to non rooted transitive actions. Recall that given any action $\alpha : \Gamma \curvearrowright X$ and any $x_0 \in X$, we have the following important formula:

$$\mathrm{Stab}_\alpha(\alpha(\gamma)x_0) = \gamma \,\mathrm{Stab}_\alpha(x_0)\gamma^{-1}. \tag{3}$$

Let us denote by $\mathrm{Sub}(\Gamma)$ the space of subgroups of Γ. By the above formula, if we associate to every transitive action the set of all its stabilizers, we get a conjugacy class in $\mathrm{Sub}(\Gamma)$, and two transitive actions are conjugate if and only if the corresponding conjugacy classes are equal in $\mathrm{Sub}(\Gamma)$. This motivates the study of the dynamical system $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$, where the action is given by

$$\gamma \cdot \Lambda \coloneqq \gamma\Lambda\gamma^{-1}.$$

Note that $\mathrm{Sub}(\Gamma)$ is a closed subspace of the compact space $\{0,1\}^\Gamma$ of subsets of Γ. In particular, $\mathrm{Sub}(\Gamma)$ is a compact Polish space.

We now have *two* models for the space of transitive actions of a countable group Γ over an infinite set: the subspace of $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ consisting of transitive actions on $\mathbb{N}$ and the space of infinite index subgroups of Γ. Our goal here is two show that theses models are essentially the same as far as descriptive set theory is concerned (see Theorem 9.9). Let us for now give these models some nicknames.

Definition 9.3. We denote by $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ the space of transitive Γ-actions on $\mathbb{N}$, and by $\mathrm{Sub}_{[\infty]}(\Gamma)$ the space of infinite index subgroups of Γ.

Before going further, let us connect the space of all Γ-actions on a countable set and the space of all subgroups of Γ via the stabilizer map.

Proposition 9.4 (see [GKM16, Lem. 2.5])**.** *Let Γ be a countable group, let $x_0 \in \mathbb{N}$. The map $\mathrm{Stab}_{x_0} : \alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty) \mapsto \mathrm{Stab}_\alpha(x_0)$ is continuous and open.*

The difficult part is to show that the stabilizer map is open. Since this proposition is fundamental in understanding the connection between our two models for the space of transitive actions, we provide a proof below (the hasty reader can jump directly to Theorem 9.9). It is useful to first restate the topology of $\mathrm{Sub}(\Gamma)$ in terms of pointed actions.

Definition 9.5. Given a finite symmetric subset $S \Subset \Gamma$, define the S-Schreier graph of an action α on a set X as the labeled graph[8] whose underlying set is X and such that for every $x \in X$ and $s \in S$, we have an s-labeled edge from x to $\alpha(s)x$ whose opposite edge is labeled s^{-1}.

[8]We follow Serre's conventions for graphs, which we recall at the beginning of Section 10.1 (see also Section 3.1 from Chapter I in [Ser80]).

By definition, the R**-ball** around a vertex v in a graph is the graph whose set of vertices is the set of all vertices connected to v by a path of length at most R, and whose edges are the edges which occur in paths of length at most R. The following lemma is well-known and key to a good understanding of the topology of $\mathrm{Sub}(\Gamma)$.

Lemma 9.6. *Let Γ be a discrete group. A family of basic neighborhoods of $\Lambda \in \mathrm{Sub}(\Gamma)$ is given by taking a radius $R > 0$, a finite symmetric subset $S \subseteq \Gamma$, and then the corresponding neighborhood of Λ is the set all Λ' such that the R-ball of the S-Schreier graph of $\Gamma \curvearrowright \Gamma/\Lambda'$ centered at Λ' is isomorphic to the R-ball of the S-Schreier graph of $\Gamma \curvearrowright \Gamma/\Lambda$ centered at Λ.*

Proof. The topology given by this neighborhood basis refines the topology on the space of subgroups because the intersection of Λ with a finite set S is detected in the radius 1 ball around Λ in the $S \cup S^{-1}$ Schreier graph of $\Gamma \curvearrowright \Gamma/\Lambda$. Indeed the intersection of Λ with S is equal to the set of labels $s \in S$ whose corresponding edge starting from Λ is a loop.

Conversely, fix $S \Subset \Gamma$ symmetric, and let $F = (S \cup \{1\})^R$. Let Λ' such that $\Lambda' \cap F^2 = \Lambda \cap F^2$, we claim that the map $F/\Lambda' \to F/\Lambda$ given by $\gamma\Lambda' \mapsto \gamma\Lambda$ is well-defined and bijective. Indeed, $\gamma\Lambda = \gamma'\Lambda$ iff $\gamma^{-1}\gamma' \in \Lambda$ and we have $\gamma^{-1}\gamma' \in F^2$ so the latter is equivalent to $\gamma^{-1}\gamma' \in \Lambda'$, which is itself equivalent to $\gamma\Lambda' = \gamma'\Lambda'$. Using a similar reasoning, one can check that this map actually induces a labeled graph isomorphism between the R-balls of the two Schreier graphs (recall that our definition of R-ball only keeps edges which belong to paths of length at most R). $\square$

Proof of Proposition 9.4. In order to see that Stab_x is continuous, first note that the topology on Sub_Γ is generated by finite intersections of open sets of the following form, where $\gamma \in \Gamma$:

$$\mathcal{I}(\gamma) = \{\Lambda \leqslant \Gamma \colon \gamma \in \Lambda\} \text{ and } \mathcal{O}(\gamma) = \{\Lambda \leqslant \Gamma \colon \gamma \notin \Lambda\} = \mathrm{Sub}(\Gamma) \setminus \mathcal{I}(\gamma).$$

It thus suffices to check that preimages by Stab_{x_0} of sets of the above form are open. But by definition,

$$\mathrm{Stab}_{x_0}^{-1}(\mathcal{I}(\gamma)) = \{\alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty) \colon \alpha(\gamma)x_0 = x_0\}$$

is clopen, so the preimage of the complement $\mathrm{Stab}_x^{-1}(\mathcal{O}(\gamma))$ is clopen as well, thus finishing the proof of the continuity.

In order to prove that Stab_{x_0} is open, we need to show that given any action α_0 and any basic neighborhood $\mathcal{U}$ of α_0, the direct image $\mathrm{Stab}_{x_0}(\mathcal{U})$ is a neighborhood of $\mathrm{Stab}_{x_0}(\alpha_0)$.

To this end, let α_0 be a fixed action of Γ on $\mathbb{N}$, by shrinking $\mathcal{U}$ we can assume there is $S \Subset \Gamma$ symmetric and $F \Subset \mathbb{N}$ containing x_0 such that

$$\mathcal{U} = \mathcal{U}_{S,F} = \{\alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty) \colon \forall (\gamma, y) \in S \times F, \alpha(\gamma)y = \alpha_0(\gamma)y\}.$$

Let now $Y = \alpha_0(\Gamma)x_0$, and $F_1 = F \cap Y$, $F_2 = F \setminus F_1$. Enlarging S if necessary (and thus shrinking $\mathcal{U}$ further), we may as well assume that every element $x \in F_1$ is of the form $\alpha(\gamma)x_0$ for some $\gamma \in S$.

We now define the neighborhood $\mathcal{V}$ of $\mathrm{Stab}_{x_0}(\alpha_0)$ which will be contained in $\mathrm{Stab}_{x_0}(\mathcal{U})$: $\mathcal{V}$ is the set consisting of all $\Lambda \leqslant \Gamma$ such that the ball of radius 2 of the S-Schreier graph of $\Gamma \curvearrowright \Gamma/\Lambda$ centered at Λ is isomorphic to that of α_0 centered at x_0.

Let us show that $\mathcal{V} \subseteq \mathrm{Stab}_{x_0}(\mathcal{U})$. Let $\Lambda \in \mathcal{V}$, let φ be a graph isomorphism from the radius 2 ball of the S-Schreier graph of $\Gamma \curvearrowright \Gamma/\Lambda$ centered at Λ to the radius 2 ball of the S-Schreier graph of α_0 centered at x_0, and extend φ arbitrarily to an injection $\tilde{\varphi} : \Gamma/\Lambda \to \mathbb{N}$ such that $\mathbb{N} \setminus \tilde{\varphi}(\Gamma/\Lambda)$ is infinite and contains $F_2 \cup \alpha_0(S)F_2$. Finally, let $\psi : \mathbb{N} \setminus Y \to \mathbb{N} \setminus \tilde{\varphi}(\Gamma/\Lambda)$ be an injection which is equal to the identity on $F_2 \cup \alpha_0(S)F_2$.

Let ρ denote the action of Γ on Γ/Λ, define an action α of Γ on $\mathbb{N}$ by

$$\alpha(\gamma)x = \begin{cases} \tilde{\varphi}\rho(\gamma)\tilde{\varphi}^{-1}(x) & \text{if } x \in \varphi(\Gamma/\Lambda), \\ \psi\alpha_0(\gamma)\psi^{-1}(x) & \text{if } x \in \psi(\mathbb{N} \setminus Y), \\ x \text{ otherwise.} \end{cases}$$

By construction the restriction of α to the α-orbit of x_0 is conjugate to ρ, so $\mathrm{Stab}_{x_0}(\alpha) = \Lambda$.

In order to finish the proof, we check that $\alpha \in \mathcal{U}$. Let $x \in F$ and $\gamma \in S$.

If $x \in F_1$, we have $x = \alpha_0(g)x_0$ for some $g \in S$, and hence both x and $\alpha_0(\gamma)x$ belong to the 2-ball centered at x_0 in the S-Schreier graph of α. It follows that $\varphi^{-1}(\alpha_0(\gamma)x) = \rho(\gamma)\varphi^{-1}(x)$, in other words $\alpha_0(\gamma)x = \varphi\rho(\gamma)\varphi^{-1}(x)$. Since $\tilde{\varphi}$ extends φ, we conclude that $\alpha(\gamma)x = \alpha_0(\gamma)x$.

If $x \in F_2$, we have $\psi(x) = x$, and $\alpha_0(\gamma)x \in \alpha_0(S)F_2$ so $\psi(\alpha_0(\gamma)x) = \alpha_0(\gamma)x$ as well. It follows that

$$\alpha(\gamma)x = \psi\alpha_0(\gamma)\psi^{-1}(x) = \psi\alpha_0(\gamma)x = \alpha_0(\gamma)x$$

as wanted, thus finishing the proof. □

With this in mind, we can go back to the space of transitive Γ-actions on $\mathbb{N}$ and the space of infinite index subgroups of Γ. Our goal is to show that these two topological spaces are essentially the same as far are descriptive set theory is concerned. First, we show that they both define G_δ subsets in their ambient topological spaces, hence they are both Polish.

Lemma 9.7. *The space* $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ *is* G_δ *in* $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$*, and the space* $\mathrm{Sub}_{[\infty]}(\Gamma)$ *is* G_δ *in* $\mathrm{Sub}(\Gamma)$*. In particular, both are Polish spaces. Moreover, for all* $x \in \mathbb{N}$ *the restriction of the stabilizer map to* $\mathrm{Stab}_x : \mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty) \to \mathrm{Sub}_{[\infty]}(\Gamma)$ *is continuous and open.*

Proof. Observe that $\alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ is transitive if and only if for all $x, y \in \mathbb{N}$ there is $\gamma \in \Gamma$ such that $\alpha(\gamma)x = y$. So if for $x, y \in \mathbb{N}$ we let $\mathcal{U}_{x,y}$ the open set of all α such that there is $\gamma \in \Gamma$ satisfying $\alpha(\gamma)x = y$, we have that

$$\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty) = \bigcap_{x,y \in \mathbb{N}} \mathcal{U}_{x,y},$$

thus showing that $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ is G_δ.

For the fact that $\mathrm{Sub}_{[\infty]}(\Gamma)$ is G_δ, observe that Λ has index a least N if and only if there are $\gamma_1, ..., \gamma_N \in \Gamma$ such that $\gamma_i\gamma_j^{-1} \notin \Lambda$ for all distinct $i, j \in \{1, \dots, N\}$. This shows that the set $\mathcal{V}_N$ of subgroups of index at least N is open, so $\mathrm{Sub}_{[\infty]} = \bigcap_{N \in \mathbb{N}} \mathcal{V}_N$ is G_δ as wanted. We conclude from Corollary 1.3 that both $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ and $\mathrm{Sub}_{[\infty]}(\Gamma)$ are Polish.

Let us finally establish that the restriction of the stabilizer map is continuous and open. Recall from Proposition 9.4 that $\mathrm{Stab}_x : \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty) \to \mathrm{Sub}(\Gamma)$ is continuous and open, in particular its restriction to $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ is continuous. We clearly have $\mathrm{Sub}_{[\infty]}(\Gamma) = \mathrm{Stab}_x(\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty))$. Since we also have

$$\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty) = \mathrm{Stab}_x^{-1}(\mathrm{Sub}_{[\infty]}(\Gamma)) = \mathrm{Stab}_x^{-1}(\mathrm{Stab}_x(\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)))$$

we conclude from Lemma 1.5 that the restriction of Stab_x to a map $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty) \to \mathrm{Sub}_{[\infty]}(\Gamma)$ is open as well. □

Remark 9.8. The fact that $\mathrm{Sub}_{[\infty]}(\Gamma)$ is G_δ can also be seen as a consequence of the fact that $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ is G_δ, by fixing $x \in \mathbb{N}$ and combining the following three facts with Corollary 1.6:

- the map Stab_x is continuous and open (Proposition 9.4);
- $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty) = \mathrm{Stab}_x^{-1}(\mathrm{Stab}_x(\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)))$;
- $\mathrm{Sub}_{[\infty]}(\Gamma) = \mathrm{Stab}_x\,(\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty))$.

Recall from the discussion at the beginning of this section that two transitive actions of Γ are conjugate if and only if the conjugacy class of their stabilizers are equal. In other words, the stabilizer map induces a bijection between the quotient spaces $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)/\mathfrak{S}_\infty$ and $\mathrm{Sub}_{[\infty]}(\Gamma)/\Gamma$. In particular, it induces a bijection between $\mathfrak{S}_\infty$-invariant subsets of $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ and Γ-invariant subsets of $\mathrm{Sub}_{[\infty]}(\Gamma)$, and the results we have seen have the following important consequence, which says that this bijection is very nice as far as descriptive set theory is concerned. Let us mention that this result is a somehow baby version of the inspiring results of Melleray and Tsankov on *category-preserving maps*, cf. appendix of [MT13]. Similar results have been obtained by Foreman, Rudolph and Weiss in the context of the space of measure-preserving actions of $\mathbb{Z}$ [Rud98], see also [For10].

Theorem 9.9. *Let P be a conjugacy-invariant property of transitive Γ-actions over a countable infinite set. Denote by $\mathcal{A}_P$ the set of $\alpha \in \mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ satisfying $\mathcal{P}$, and by $\mathcal{B}_P$ the set of $\Lambda \in \mathrm{Sub}_{[\infty]}(\Gamma)$ such that $\Gamma \curvearrowright \Gamma/\Lambda$ satisfies $\mathcal{P}$. The following hold:*

(1) $\mathcal{A}_P$ is open in $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ iff $\mathcal{B}_P$ is open in $\mathrm{Sub}_{[\infty]}(\Gamma)$.

(2) $\mathcal{A}_P$ is G_δ in $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ iff $\mathcal{B}_P$ is G_δ in $\mathrm{Sub}_{[\infty]}(\Gamma)$.

(3) $\mathcal{A}_P$ is dense in $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ iff $\mathcal{B}_P$ is dense in $\mathrm{Sub}_{[\infty]}(\Gamma)$

Proof. Fix $x \in \mathbb{N}$. Recall from Proposition 9.4 that the stabilizer map Stab_x is both continuous and open. By the conjugacy invariance of P, we both have $\mathcal{A}_P = \mathrm{Stab}_x^{-1}(\mathcal{B}_P)$ and $\mathcal{B}_P = \mathrm{Stab}_x(\mathcal{A}_P)$. The first equivalence (1) immediately follows.

Now if $\mathcal{A}_P$ is G_δ, then $\mathcal{B}_P = \mathrm{Stab}_x(\mathcal{A}_P)$ also is by Corollary 1.6, noting that $\mathcal{A}_P = \mathrm{Stab}_x^{-1}(\mathcal{B}_P) = \mathrm{Stab}_x^{-1}(\mathrm{Stab}_x(\mathcal{A}_P))$. The converse follows from the continuity of Stab_x and the equality $\mathcal{A}_P = \mathrm{Stab}_x^{-1}(\mathcal{B}_P)$, which establishes (2).

For the equivalence (3), if $\mathcal{A}_P$ is dense then since Stab_x is continuous surjective we get that $\mathcal{B}_P$ also is. Conversely, suppose that $\mathcal{A}_P$ is not dense, let U be a non-empty open set disjoint from $\mathcal{A}_P$. Since P is conjugacy invariant, $\mathrm{Stab}_x(U)$ is a non-empty open set disjoint from $\mathcal{B}_P$. So $\mathcal{B}_P$ is not dense, which finishes the proof. □

Here is an easy application which will be useful in the next section.

Corollary 9.10. *In the space $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$, actions with finitely generated stabilizers are dense.*

Proof. Every countable group $\Lambda = \{\lambda_n \colon n \in \mathbb{N}\}$ can be written as an increasing union of finitely generated subgroups $\Lambda_n = \langle \lambda_0, \dots, \lambda_n \rangle$. When Λ has infinite index, the Λ_n's must have infinite index as well, so the conjugacy invariant set of infinite index finitely generated subgroups is dense in $\mathrm{Sub}_{[\infty]}(\Gamma)$. The conclusion now follows from the last item of the above theorem. □

We end this section by characterizing when the G_δ set $\mathrm{Sub}_{[\infty]}(\Gamma)$ is actually closed.

Proposition 9.11. *Let Γ be a countable group.*

(1) If Γ is finitely generated, then $\mathrm{Sub}_{[\infty]}(\Gamma)$ is closed in $\mathrm{Sub}(\Gamma)$.

(2) If Γ is not finitely generated, then $\mathrm{Sub}_{[\infty]}(\Gamma)$ is dense non closed in $\mathrm{Sub}(\Gamma)$.

Proof. Observe that by Lemma 9.6, if Γ is generated by a finite set S, then finite index subgroups are isolated in $\mathrm{Sub}(\Gamma)$ because a sufficiently large ball in their S-Schreier graphs will completely determine the corresponding rooted transitive action up to isomorphism. In particular, finite index subgroups form an open set, which establishes the first item.

For the second item, suppose now that Γ is not finitely generated. Let Λ be a finite index subgroup of Γ, then Λ cannot be finitely generated because coset representatives of Γ/Λ along with generators of Λ form a generating set for Γ. If we enumerate $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$, we again have that the sequence of subgroups $(\langle \lambda_0, \ldots, \lambda_n \rangle)_{n\in\mathbb{N}}$ converges to Λ and consists of finitely generated (hence infinite index) subgroups. The density follows, and since $\mathrm{Sub}(\Gamma) \setminus \mathrm{Sub}_{[\infty]}(\Gamma)$ always contains Γ, the set $\mathrm{Sub}_{[\infty]}(\Gamma)$ cannot be closed. $\square$

Remark 9.12. If Γ is an infinite simple group then $\mathrm{Sub}(\Gamma) = \mathrm{Sub}_{[\infty]}(\Gamma) \sqcup \{\Gamma\}$ because every finite index subgroup Λ contains a finite index normal subgroup (obtained as the kernel of the action $\Gamma \curvearrowright \Gamma/\Lambda$). So $\mathrm{Sub}_{[\infty]}(\Gamma)$ is open when Γ is simple.

We will now review various conjugacy invariant properties of transitive actions which define G_δ subsets (in both models of the space of transitive actions by Theorem 9.9).

9.2 High transitivity

The following natural strengthening of transitivity was first considered for finite groups acting on finite sets.

Definition 9.13. Let $n \in \mathbb{N}$. An action of a countable group Γ on a set X of cardinality at least n is called **n-transitive** if for every pairwise distinct $x_1, \ldots, x_n \in X$, and pairwise distinct $y_1, \ldots, y_n$, there is $\gamma \in \Gamma$ such that for all $i \in \{1, \ldots, n\}$, we have $\gamma x_i = y_i$.

Observe that when X has cardinality n, the only group which admits a faithful n-transitive action on X is the symmetric group over n elements. But much more is true: for $n = 4$, the groups which admit a faithful 4-transitive action on a finite set are completely classified: these are some Matthieu groups, and then the symmetric groups over a set of cardinality $\geqslant 4$ and the alternating groups over a set of cardinality $\geqslant 6$, although the only known proofs of this appear to rely on the classification of finite simple groups. A more precise statement relies on the following notion.

Definition 9.14. The **transitivity degree** of a countable group Γ, denoted by $\mathrm{td}(\Gamma)$, is the supremum of the $n \in \mathbb{N}$ such that Γ admits a faithful n-transitive action.

The previously mentioned Matthieu groups with 4-transitive faithful actions actually have transitivity degree 4 or 5, so that for $n \geqslant 6$, the only finite groups of transitivity degree n are exactly the symmetric group $\mathfrak{S}_n$ and the alternating group $\mathfrak{A}_{n+2}$ (see [Cam81, Thm. 5.3]). We now switch to infinite countable groups, starting with a natural open question which to our knowledge was first asked by Hull and Osin.

Question 5. Is there a countable infinite group with finite transitivity degree $\geqslant 4$?

A natural source of groups with infinite transitivity degree are the *highly transitive* ones defined below. It is also unknown whether these are the only examples of countable groups with infinite transitivity degree.

Definition 9.15. An action of a countable group Γ on an infinite set X is called **highly transitive** when it is n-transitive for every $n \in \mathbb{N}$. A countable group is called **highly transitive** when it admits a faithful highly transitive action.

Remark 9.16. One can check that an action $\alpha : \Gamma \to S_\infty$ is highly transitive if and only if the image of α is dense in $\mathrm{Sym}(X)$.

Our first example is given by the group of finitely supported permutations $\mathfrak{S}_{(\infty)}$, closely followed by the group $\mathfrak{A}_{(\infty)}$ of finitely supported even permutations. But contrarily to what the situation for finite groups could suggest, there are many highly transitive countable groups!

A natural finitely generated example due to B.H. Neumann (see the introduction of [McD77]) is given by the group of all permutations σ of $\mathbb{Z}$ which are translations up to a finite set: there is $k \in \mathbb{N}$ and $F \Subset \mathbb{Z}$ such that for all $x \notin F$, we have $\sigma(x) = x+k$. Indeed, the natural action of this group on $\mathbb{Z}$ is highly transitive because the group contains all finitely supported permutations of $\mathbb{Z}$, and it is not hard to check that it is generated by $x \mapsto x+1$ and the transposition $(1 \quad 2)$.

More generally, any countable group admitting a faithful action on a countable set such that its image in $\mathfrak{S}(X)$ contains the group of finitely supported even permutations is highly transitive; such groups are called **partially finitary groups** by Le Boudec and Matte Bon, after having been first delineated in [HO16]. The following consequence of the Jordan-Wielandt theorem is due to Le Boudec an Matte Bon.

Theorem 9.17 ([LBMB22a, Prop. 2.4])**.** *Let Γ be a partially finitary group, as witnessed by a faithful action $\alpha : \Gamma \to \mathfrak{S}(X)$ whose image contains the group of finitely supported even permutations. Then α is the only 2-transitive Γ-action up to conjugacy.*

Let us also mention a nice characterization of partially finitary groups among highly transitive groups. Given a group Γ, we let $\mathbb{Z} = \langle t \rangle$ be the 1-generated free group and fix some $w \in \Gamma * \mathbb{Z}$. For every $g \in \Gamma$, the universal property of $\Gamma * \mathbb{Z}$ yields a unique homomorphism $\pi_g : \Gamma * \mathbb{Z}$ which is the identity on Γ and takes t to g, and we let $w(g) \coloneqq \pi_g(w)$. For instance if we fix some $\gamma_0 \in \Gamma$, one can consider $w = \gamma_0 t \gamma_0^{-1} t^{-1}$, and then $w(g) = 1$ iff g commutes with γ_0.

Definition 9.18. A group Γ satisfies a **mixed identity** if there is $w \in G * \mathbb{Z} \setminus \{1\}$ such that $w(g) = 1$ for *all* $g \in \Gamma$. If Γ satisfies no mixed identity, one says that Γ is **mixed identity free** (MIF).

Theorem 9.19 ([HO16, Thm. 5.9])**.** *Let Γ be a highly transitive group. Then Γ is partially finitary if and only if it satisfies a mixed identity.*

Sketch of proof. If Γ is partially finitary, let us identify Γ with its image in the symmetric group via its highly transitive action so that Γ contains all even permutations. Let us fix any four 3-cycles $c_1, c_2, c_3, c_4 \in \Gamma$ with disjoint supports, then if $c \in \Gamma$ is yet another 3-cycle, any conjugate of c must have disjoint support with at least one of the c_i's, so it must commute with at least one of the c_i's. So the iterated commutator

$$w = \Big[\Big[\big[[tct^{-1}, c_1], [tct^{-1}, c_2]\big], [tct^{-1}, c_3]\Big], [tct^{-1}, c_4]\Big]$$

is the desired mixed identity.

Suppose conversely that Γ is not partially finitary, and fix a highly transitive Γ-action. Using high transitivity and the proof of the simplicity of the infinite alternating group $\mathfrak{A}_{(\infty)}$, we see that all the elements of Γ must have infinite support because otherwise Γ would be partially finitary.

Now fix $w \in \Gamma * \mathbb{Z} \setminus \{1\}$. Conjugating w by a large enough power of t, we may and do assume that w begins and ends by $t^{\pm 1}$. Write

$$w = t^{k_{n+1}}\gamma_n t^{k_n} \cdots \gamma_1 t^{k_1}$$

where each k_i is nonzero, and each $\gamma_i \in \Gamma$ is non trivial.

Fix $y_0 \in X$, since $\gamma_1, \dots, \gamma_n$ have infinite support we can fix some elements $y_1, ..., y_n, y_{n+1} \in X$ such that the following elements:

$$y_0, y_1, \dots, y_n, y_{n+1}, \gamma_1 y_1, \gamma_2 y_2, \dots, \gamma_n y_n$$

are all distinct.

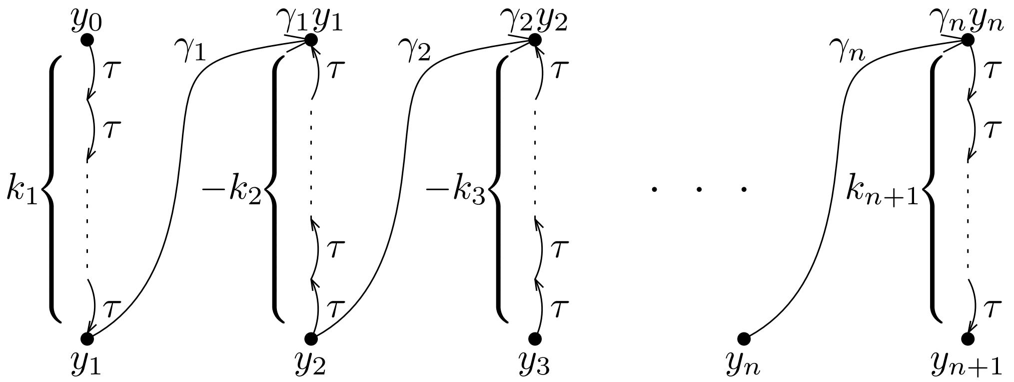

Figure 1: The construction of τ when $k_1 > 0$, $k_2 < 0$, $k_3 < 0$ and $k_{n+1} > 0$.

We can thus find a partial bijection τ of X of finite domain such that $\tau^{k_1}(y_0) = y_1$ and

$$\tau^{k_2}(\gamma_1 y_1) = y_2, \tau^{k_3}(\gamma_2 y_2) = y_3 \dots, \tau^{k_n}(\gamma_{n-1} y_{n-1}) = y_n, \tau^{k_{n+1}}(y_n) = y_{n+1}.$$

By high transitivity, there is $\gamma \in \Gamma$ which extends the restriction of τ to its support. By construction $w(\gamma)y_0 = y_{n+1} \neq y_0$ so $w(\gamma) \neq 1$ as wanted. $\square$

Remark 9.20. A more precise construction of τ yields that if $w \in \Gamma * \langle t \rangle$ is a mixed identity of length k satisfied by Γ and Γ is not partially finitary, then the transitivity degree of Γ is at most $k - 1$ (see [LBMB22b, Prop. A.1]). So groups satisfying a mixed identity do have infinite transitivity degree if and only if they are highly transitive.

Remark 9.21. A modification of the proof of the right to left implication can be used in the context of homogeneous actions on Urysohn spaces, showing for instance that *any* dense subgroup of the automorphism group of the random graph has to be MIF (see [EGLMM21, Sec. 6]).

A very large class of examples of non partially finitary groups coming with a natural highly transitive action is provided by topological full groups of actions of countable groups on the Cantor space with at least one infinite dense orbit: their action on any such orbit is highly transitive and faithful by density (see Example 11.10). This property is also true of their sufficiently large subgroups, so for instance the simple finitely generated groups considered by Matui in [Mat06] are highly transitive.

Remark 9.22. Juschenko and Monod showed later on that the simple finitely generated groups considered by Matui are moreover amenable [JM13]. Moreover Nekrashevych gave the optimal generalization of Matui's results by introducing the nicely behaved notion of alternating topological full group [Nek17].

Other than partially finitary groups and (subgroups of) topological full groups, we don't know any other countable groups coming with a natural highly transitive action. Nevertheless, there are many more highly transitive groups.

Indeed, free groups are highly transitive as was first proven by McDonough using a Baire category argument [McD77]. More generally, highly transitive free products can be characterized, as was discovered by Gunhouse and Hickin independently.

Theorem 9.23 ([Gun92, Hic92]). *Let Γ be a nontrivial free product, i.e. $\Gamma = \Gamma_1 * \Gamma_2$ with both $|\Gamma_1|, |\Gamma_2| \geqslant 2$. Then Γ is highly transitive iff one of its free factors Γ_i is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Remark 9.24. Gunhouse and Hickin actually show that when a nontrivial free product is not $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, it has a faithful highly transitive actions on $\mathbb{N}$ which is moreover **Jordan**: there is an infinite coinfinite subset $A \subseteq \mathbb{N}$ whose pointwise stabilizer acts highly transitively on the complement $\mathbb{N} \setminus A$. This definition of Jordan is much stronger than the one taken by Adeleke and Macpherson in their classification of non highly transitive Jordan infinite primitive permutation groups [AM96]. Note that the faithful highly transitive action of partially finitary groups is Jordan (any infinite coinfinite set witnesses this property!). It is also not hard to check that topological full groups satisfy a similar conclusion (all their actions on the orbits are Jordan).

Continuing our list of highly transitive groups, a wealth of examples such as surface groups [Kit12], outer automorphism groups of free groups of rank at least 4 [GG13] and hyperbolic groups with trivial finite radical [Cha12] were found to be highly transitive. All the above results on non partially finitary highly transitive groups were subsumed by the following result of Hull and Osin.

Theorem 9.25 ([HO16, Thm. 1.2]). *All acylindrically hyperbolic groups with trivial finite radical are highly transitive.*

In a somehow different direction, Fima, Moon and Stalder [FMS15] used Baire category techniques to obtain a large class of highly transitive groups acting on trees which intersects acylindrically hyperbolic groups (e.g. via free products, see [MO15, Cor. 2.2]) but is not contained in it as pointed out in Hull and Osin's paper [HO16, Cor. 5.12]. We finish this section by stating the optimal generalization of [FMS15] that we obtained with Fima, Moon and Stalder, deferring to Section 10 the needed definitions and examples, a sketch of its proof and a justification of its optimality, building on work of Le Boudec and Matte Bon on *non* highly transitive groups.

Theorem 9.26 ([FLMMS22, Thm. A]). *Let Γ be a countable group with a faithful action on a tree $\mathcal{T}$ which is minimal and of general type. Assume that the Γ-action on the boundary of $\mathcal{T}$ is topologically free. Then Γ is highly transitive.*

Remark 9.27. Before their aforementioned work with Fima [FMS15], Moon and Stalder, unaware of Gunhouse and Hickin's results, had obtained a completely different proof of Theorem 9.23 using Baire category methods [MS13]. As we will see, the proof of the above result relies on a generalization of their techniques, and we will show that in the case of free products, one can push these techniques a bit further to obtain a genericity result in the space of transitive actions.

9.3 High faithfulness, weak containment and totipotency

The following definition is a natural strengthening of faithfulness, somehow dual to high transitivity.

Definition 9.28. Let $\Gamma \curvearrowright X$ be any action. This action is called **highly faithful** if given any $\gamma_1, \cdots, \gamma_n \in \Gamma \setminus \{1\}$, there is $x \in X$ such that for all $i \in \{1, \ldots, n\}$, we have $\gamma_i x \neq x$.

Example 9.29. A nice example of a non highly faithful action is provided by highly transitive partially finitary group, since their only 2-transitive action (cf. Theorem 9.17) even has elements with disjoint support. In particular, they don't admit highly faithful highly transitive actions. The natural orbit actions of topological full groups are also never highly faithful for a similar reason, and we don't know wether topological full groups admit highly faithful highly transitive actions.

To our knowledge, the notion of high faithfulness appeared first in the work of de la Harpe [de 85] on C^*-simple groups under the name of strong faithfulness. Fima, Moon and Stalder introduced a slightly stronger notion which they called high faithfulness in [FMS15], but it agrees with the above notion for infinite groups, which is the only case of interest here as well as in their paper (see [FLMMS22, Prop. 2.5] for the equivalence of the two notions). In a joint unpublished work with Carderi and Gaboriau, we realized that the correct way to understand high faithfulness was in terms of the following definition, which is the natural analogue of Kechris' weak containment for measure-preserving actions [Kec10, Sec. 10].

Definition 9.30. An action $\alpha : \Gamma \curvearrowright X$ **weakly contains** an action $\beta : \Gamma \curvearrowright Y$ if given any $S \Subset \Gamma$ and any $F \Subset Y$, the graph induced by the S-Schreier graph of β on F is isomorphic to the graph induced by the S-Schreier graph of α on some finite subset $F' \Subset X$.

Note that given an action $\alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$, a neighborhood basis of α can be constructed by taking all $F \Subset \mathbb{N}$, $S \Subset \Gamma$ symmetric and then having as basic neighborhood of α the set of all actions β whose S-Schreier graph induce on F the same S-Schreier graph as β. We thus have the following result, analogous to [Kec10, Prop. 10.1].

Proposition 9.31. *An action $\alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ weakly contains $\beta \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ if and only if the closure of the $\mathfrak{S}_\infty$-orbit of α contains the closure of the $\mathfrak{S}_\infty$-orbit of β.* □

The next proposition is a routine consequence of the definition, and can also be directly deduced from the above proposition (see also [Kec10, Cor. 10.3] for its probability measure-preserving version).

Proposition 9.32. *The weak containment relation defines a G_δ subset of $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty) \times \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$, and for any Γ-action β on $\mathbb{N}$, the set of all actions $\alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ weakly contained in β is closed.* □

Finally, when we restrict to transitive actions, the description of the topology of $\mathrm{Sub}(\Gamma)$ given in Lemma 9.6 yields the following fact.

Proposition 9.33. *An action α weakly contains a transitive action β if and only if the closure of the set of stabilizers of α contains the set of the stabilizers of β.* □

Corollary 9.34. *An action of a countable group Γ is highly faithful if and only if it weakly contains the regular action $\Gamma \curvearrowright \Gamma$.* $\square$

Remark 9.35. Subgroups $\Lambda \leqslant \Gamma$ such that $\Gamma \curvearrowright \Gamma/\Lambda$ is not highly faithful are called **confined** and play an important role in Le Boudec and Matte Bon's remarkable classification of highly transitive actions of Higman-Thompson's groups V_d [LBMB22a, Thm. 4].

Remark 9.36. The reader acquainted with weak containment of unitary representations in the sense of Zimmer can note that if $\alpha : \Gamma \curvearrowright X$ weakly contains $\beta : \Gamma \curvearrowright Y$, then the unitary representations κ_α of Γ on $\ell^2(X)$ weakly contains the unitary representation κ_β of Γ on $\ell^2(Y)$ (this was actually used for high faithfulness by Burton and Kechris, see [BK20, Lem. F.1], and also very recently by Gerasimova and Monod under the name *unconfined*, see [GM23]). The converse is far from being true: for instance a nontrivial transitive action can never weakly contain the trivial action, while amenable groups are characterized as those whose regular unitary representation weakly contains the trivial representation.

Observe that given a fixed countable group Γ, there is a Γ-action on a countable set which weakly contains all Γ-actions. Indeed there are only countably many finite induced S-Schreier graphs up to isomorphism, so an arbitrary action is always weakly contained in a countable one. So if we take (α_n) dense in $\mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$, the action on $\mathbb{N} \times \mathbb{N} \simeq \mathbb{N}$ which we obtain by gluing all the α_n's together weakly contains all actions.

Using Proposition 9.31, we can reformulate this as the fact that the action $\mathfrak{S}_\infty \curvearrowright \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty)$ always has a dense orbit (see [Kec10, Thm. 10.7] for the probability measure preserving version; as was pointed out to us by Todor Tsankov, such a construction can also be adapted to show that $\mathrm{Hom}(\Gamma, [\mathcal{R}]))$ always contains a dense conjugacy class when $\mathcal{R}$ is an infinite type ergodic non-singular equivalence relation). The following definition was essentially introduced in [CGLM23] for invariant random subgroups; the version that we propose here is not exactly the same, as we discuss at the end of Section 12.

Definition 9.37. A transitive action of a countable group Γ on an infinite set is called **totipotent** when it weakly contains all transitive Γ-actions on infinite sets.

Using Theorem 9.9 and Proposition 9.31, we can reformulate totipotency as follows.

- An action $\alpha \in \mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ is totipotent when its conjugacy class is dense in $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$.
- A subgroup $\Lambda \leqslant \Gamma$ is **totipotent** when its conjugacy class is dense in $\mathrm{Sub}_{[\infty]}(\Gamma)$.

Example 9.38. Let us explain why $\mathbb{F}_2 = \langle a, b \rangle$ admits a totipotent action by explicitly building such an action (we will see some stronger results later on). Let (B_n) enumerate all the balls of $S = \{a, b, a^{-1}, b^{-1}\}$-Schreier graphs of transitive actions of $\mathbb{F}_2$ on infinite sets up to conjugacy, and for each n let α_n be a transitive $\mathbb{F}_2$-action admitting B_n as a ball in its S-Schreier graph. It suffices to construct a transitive $\mathbb{F}_2$-action whose S-Schreier graphs contains all the B_n's. Note that S-Schreier graphs of $\mathbb{F}_2$-actions are exactly S-labeled oriented graphs such that

(1) For every vertex x, there is exactly one s-labeled edge starting from x (in particular every vertex has outgoing degree $|S| = 4$)

(2) If an edge is labeled $s \in S$, then its opposite edge is labeled s^{-1}.

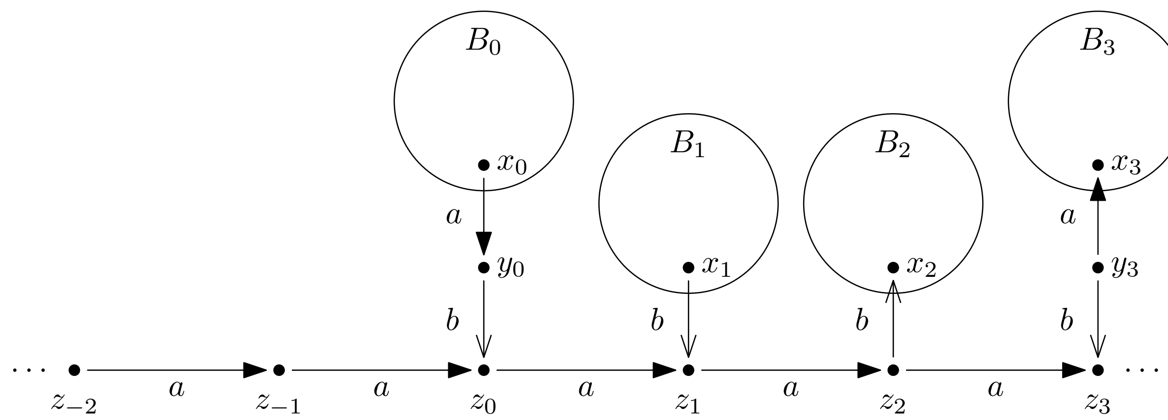

Figure 2: Gluing the B_n's together.

Our starting graph is the disjoint union of the balls B_n. We are going to attach these along a biinfinite a-labeled line. In the following construction, when we add an edge we will always implicitly add the corresponding opposite edge as required by Condition (2).

Since each α_n is transitive over an infinite set, for each n we find some $x_n \in B_n$ and $s_n \in S$ such that $\alpha_n(s_n)x_n \notin B_n$. Let $y_n = \alpha_n(s_n)x_n$, then we add to our graph the vertex y_n and the s_n-labeled edge from y_n to x_n. We now distinguish two cases:

- If $s_n \in \{b, b^{-1}\}$ let $z_n = y_n$.
- If $s_n \in \{a, a^{-1}\}$ let z_n be a new vertex different from all the previous ones and add a b-labeled edge from y_n to z_n.

Now let $(z_n)_{n<0}$ be a sequence of distinct additional vertices, all distinct from the previous ones. For each $n \in \mathbb{Z}$, we add an a-labeled edge from z_n to z_{n+1}.

Finally, we take care of Condition (1) as follows: we observe that the only infinite a or b-labeled line is the biinfinite line $(z_n)_{n\in\mathbb{Z}}$, in particular there is no infinite one-sided a or b-labeled line. So the points for which some a or b-labeled edge is missing belong to a (possibly empty) a or b-labeled finite segment which we simply close by adding an edge from its last vertex to its first vertex.

By construction, we have obtained a connected Schreier graph which contains a copy of every ball arising in the Schreier graph of every transitive $\mathbb{F}_2$-action over an infinite set: this is the Schreier graph of the desired totipotent $\mathbb{F}_2$-action.

Remark 9.39. Observe that no abelian group admits a totipotent action except the trivial group. It would be interesting to delineate the class of groups admitting a totipotent action more precisely, but we won't undertake this task here and only give positive results.

Totipotency admits an alternative definition when Γ is not finitely generated.

Proposition 9.40. *Let Γ be a non finitely generated group, then $\Lambda \leqslant \Gamma$ is totipotent if and only if its conjugacy class is dense in* $\mathrm{Sub}(\Gamma)$.

Proof. If Λ is totipotent, the density of $\mathrm{Sub}_{[\infty]}(\Gamma)$ in $\mathrm{Sub}(\Gamma)$ (see item (2) in Proposition 9.11) yields that its conjugacy class is dense in the whole $\mathrm{Sub}(\Gamma)$. Conversely, let $\Lambda \leqslant \Gamma$ have a dense conjugacy class in $\mathrm{Sub}(\Gamma)$. Then Λ cannot have finite index because otherwise its conjugacy class would be finite[9]. So Λ has infinite index and in particular its conjugacy class is dense in $\mathrm{Sub}_{[\infty]}(\Gamma)$: Λ is totipotent. $\square$

[9]Note that $\mathrm{Sub}(\Gamma)$ is infinite as soon as Γ is infinite, e.g. because the map $\gamma \mapsto \langle\gamma\rangle$ is finite-to-one.

Say that an action group on a Polish space by homeomorphisms is **topologically transitive** when it admits a dense orbit. We have the following well-known result.

Proposition 9.41. *Let a group G act by homeomorphisms on a Polish space X. Then the following are equivalent*

(i) the G-action is topologically transitive;

(ii) the set of $x \in X$ whose orbit is dense is a dense G_δ subset of X;

(iii) for all $U, V \subseteq X$ nonempty open, there is $g \in G$ such that $g \cdot U \cap V \neq \emptyset$.

Proof. We show the chain of implication (i)⇒(iii)⇒(ii)⇒(i).

To see the first implication (i)⇒(iii), take x with a dense orbit, then find $g, h \in G$ such that $g \cdot x \in U$ and $h \cdot x \in V$. It follows that $hg^{-1} \cdot (g \cdot x) \in V$ so $hg^{-1} \cdot U \cap V \neq \emptyset$.

The implication (iii)⇒(ii) holds because if we let (U_n) be a countable basis of nonempty open sets for the topology of X, condition (iii) implies that each $G \cdot U_n$ is dense in X, and since it is open we get that $\bigcap_{n\in\mathbb{N}} G \cdot U_n$ is dense G_δ. By construction $\bigcap_{n\in\mathbb{N}} G \cdot U_n$ is the set of all points with a dense orbit, so (ii) holds as wanted.

The implication (ii)⇒(i) is a direct consequence of the Baire category theorem, which finishes the proof of the proposition. □

By definition a transitive Γ-action over an infinite set is totipotent iff its conjugacy class of stabilizers is dense in the space of infinite index subgroups of Γ, so by the above proposition the set of totipotent actions is G_δ, and it is dense if and only if it is not empty.

It is an instructive exercise to understand how condition (iii) in Proposition 9.41 simplifies the proof of the existence of a totipotent $\mathbb{F}_2$-action: it suffices to show that given two balls in two Schreier graphs of transitive $\mathbb{F}_2$-actions on infinite sets, one can find a third transitive $\mathbb{F}_2$-actions on an infinite set which contains a copy of both previous balls. Using this approach, we will now identify a large class of locally finite groups admitting totipotent actions.

Definition 9.42. A locally finite group Γ is called **productive** if given any finite subgroup $G \leqslant \Gamma$, there is $\gamma \in \Gamma$ such that the subgroups G and $\gamma G \gamma^{-1}$ intersect trivially and commute.

In other words, Γ is productive if given any finite subgroup $G \leqslant \Gamma$, there is $\gamma \in \Gamma$ such that the group $\langle G \cup \gamma G \gamma^{-1} \rangle$ naturally decomposes as a direct product $G \times \gamma G \gamma^{-1}$.

Example 9.43. The group $\mathfrak{S}_{(\infty)}$ of finitely supported permutations of $\mathbb{N}$ is productive. Indeed, every finite subgroup G is contained in the group of permutations supported on a finite subset F of $\mathbb{N}$, and we can then find $\sigma \in \mathfrak{S}_{(\infty)}$ such that $\sigma(F) \cap F = \emptyset$, so that G commutes with $\sigma G \sigma^{-1}$ and $G \cap \sigma G \sigma^{-1} = \{\mathrm{id}_{\mathbb{N}}\}$. The same argument shows that the group $\mathfrak{A}_{(\infty)}$ of finitely supported permutations of $\mathbb{N}$ of even signature is also productive.

Proposition 9.44. *Every productive locally finite infinite group Γ admits a totipotent action.*

Proof. We have to show that the Γ-action on $\mathrm{Sub}_{[\infty]}(\Gamma)$ is topologically transitive. We thus apply condition (iii) from Proposition 9.41.

Let U and V be two nonempty open subsets of $\mathrm{Sub}_{[\infty]}(\Gamma)$. Since Γ is locally finite, up to shrinking down U and V we may find a finite group $G \leqslant \Gamma$ and subgroups $G_U, G_V \leqslant G$ such that

$$U = \{\Lambda \in \mathrm{Sub}_{[\infty]}(\Gamma) \colon \Lambda \cap G = G_U\} \text{ and } V = \{\Lambda \in \mathrm{Sub}_{[\infty]}(\Gamma) \colon \Lambda \cap G = G_V\}.$$

Since Γ is productive, we find $\gamma \in \Gamma$ such that $\gamma G \gamma^{-1} \cap G = \{1_\Gamma\}$ and $\gamma G \gamma^{-1}$ commutes with G. We then define

$$\Lambda = \left\langle G_V, \gamma G_U \gamma^{-1} \right\rangle.$$

Observe that Λ is naturally isomorphic to the direct product $G_V \times \gamma G_U \gamma^{-1} \leqslant G \times \gamma G \gamma^{-1}$. Moreover $\Lambda \cap G = G_V$ and $\gamma^{-1}\Lambda\gamma \cap G = G_U$, so $\gamma^{-1} \cdot U \cap V \neq \emptyset$ as wanted. $\square$

Remark 9.45. If a locally finite infinite group Γ admits a totipotent action, then its space of subgroups has no isolated points. Indeed every infinite element of $\mathrm{Sub}(\Gamma)$ is the union of its finite subgroups, in particular the space of finite subgroups is dense so no infinite subgroup is isolated. Moreover no finite subgroup can be totipotent, because for every $k \in \mathbb{N}$ the set of subgroups of cardinality at most k is closed. So if $\Lambda \leqslant \Gamma$ is totipotent, every finite subgroup is a nontrivial limit of conjugates of Λ, and we conclude that $\mathrm{Sub}(\Gamma)$ has no isolated point.

Another locally finite group to which the above result applies is the group $\mathfrak{S}_{2^\infty}$ of dyadic permutations, and we now recall its definition.

Definition 9.46. Consider for every $n \in \mathbb{N}$ the group $\mathfrak{S}_{2^n} \coloneqq \mathfrak{S}(\{0,1\}^n)$, and embed $\mathfrak{S}_{2^n}$ into $\mathfrak{S}_{2^{n+1}}$ by making it act on the first n coordinate on $\{0,1\}^{n+1} = \{0,1\}^n \times \{0,1\}$. The corresponding direct limit over $n \in \mathbb{N}$ is the **group of dyadic permutations** $\mathfrak{S}_{2^\infty}$.

Proposition 9.47. *The group $\mathfrak{S}_{2^\infty}$ is productive, in particular it has a totipotent action.*

Proof. Consider the permutation σ of $\{0,1\}^{2n}$ which takes $(x_i)_{i=0}^{2n-1}$ to $(x_{n+i \bmod 2^n})_{i=0}^{2n-1}$. Observe that it conjugates $\mathfrak{S}_{2^n}$ acting on the first n coordinates to $\mathfrak{S}_{2^n}$ acting on the last n coordinates on $\{0,1\}^{2n}$. These two actions commute and the corresponding subgroups of $\mathfrak{S}_{2^{2n}}$ intersect trivially, so $\mathfrak{S}_{2^\infty}$ is productive since the first action is the one given by the embedding $\mathfrak{S}_{2^n} \hookrightarrow \mathfrak{S}_{2^{2n}}$ through the direct limit which defines $\mathfrak{S}_{2^\infty}$. The existence of a totipotent action then follows from Proposition 9.44. $\square$

The above proposition could be extended by either replacing 2 by some varying integers, or also by considering alternating groups instead of symmetric groups. Such locally finite groups arise naturally in the study of topological full groups of minimal $\mathbb{Z}$-actions (see e.g. Section 4 in [JM13]), and $\mathfrak{S}_{2^\infty}$ arises more specifically from the 2-odometer. Remarkably, the minimal conjugacy closed invariant subsets of the space of subgroups (also known as URS) of these groups are classified when one only works with alternating groups, see [TT18, Thm. 10.3].

To conclude this section, let us mention part of the very recent work of Azuelos and Gaboriau, who have obtained a wide class of groups admitting a totipotent action, including many hyperbolic groups and many groups acting on trees [AG23]. We will cite in Section 10.5 a particular case of their results for groups acting on trees, and explain why it implies $\Gamma_1 * \Gamma_2$ admits a totipotent action as soon as $|\Gamma_1| \geqslant 3$ and $|\Gamma_2| \geqslant 2$.

10 High transitivity for groups acting on trees

10.1 Statement of the main result

In this section, we state the result obtained with Fima, Moon and Stalder which characterizes high transitivity for a large class of groups acting on trees [FLMMS22]. We start by giving the necessary definitions, starting with a definition of graphs which allows for multiple edges. Although such edges do not appear in the statement of our main result, they are unavoidable in its proof through the Bass-Serre graphs of actions that will appear in the next section.

A graph $\mathcal{G}$ is given by a vertex set $V(\mathcal{G})$ and a set of edges $E(\mathcal{G})$ endowed with the following structural maps:

- A **source map** $\mathsf{s} : E(\mathcal{G}) \to V(\mathcal{G})$ and **target map** $\mathsf{t} : E(\mathcal{G}) \to V(\mathcal{G})$;
- A fixed-point free involution $E(\mathcal{G}) \to E(\mathcal{G})$ which we denote by $\bar{\cdot} : e \mapsto \bar{e}$ and call the **antipode map**.

These structural maps are related by the following equation: for all edge $e \in \mathcal{E}(\mathcal{G})$, we have

$$\mathsf{s}(\bar{e}) = \mathsf{t}(e) \text{ and } \mathsf{t}(\bar{e}) = \mathsf{s}(e).$$

We say that two vertices $v_1, v_2 \in V(\mathcal{G})$ are **adjacent** or **connected by an edge** if there is $e \in E(\mathcal{G})$ such that $\mathsf{s}(e) = v_1$ and $\mathsf{t}(e) = v_2$.

A **path** of length $n \geqslant 1$ is a finite sequence of edges $(e_i)_{i=1}^n$ such that for all $i \in \{1, \dots, n-1\}$ $\mathsf{t}(e_i) = \mathsf{s}(e_{i+1})$. A path $(e_i)_{i=1}^n$ is called **reduced** if for all $i \in \{1, \dots, n-1\}$ we moreover have $e_{i+1} \neq \bar{e}_i$. Viewing edges as paths of length 1, we may extend the definition of s and t to the set of paths by putting

$$\mathsf{s}((e_i)_{i=1}^n) = \mathsf{s}(e_1) \text{ and } \mathsf{t}((e_i)_{i=1}^n) = \mathsf{t}(e_n).$$

We also say that $(e_i)_{i=1}^n$ is a path **from** $\mathsf{s}(e_1)$ **to** $\mathsf{t}(e_n)$. We also extend the definition of the antipode map to paths by letting $\overline{(e_i)_{i=1}^n} = (\bar{e}_{n-1-i})_{i=1}^n$. Finally, a **biinfinite reduced path** is a biinfinite sequence $(e_i)_{i\in\mathbb{Z}}$ of edges such that for all $i \in \mathbb{Z}$, $\mathsf{t}(e_i) = \mathsf{s}(e_{i+1})$ and $e_i \neq \bar{e}_{i+1}$.

Say that two vertices are **in the same connected component** if there is a path from one to the other. It is straightforward to see that being in the same connected component is an equivalence relation. Its equivalence classes are called the connected components of the graph. A graph is **connected** when it has only one connected component.

Finally a **cycle** is a reduced path whose target and source coincide.

Definition 10.1. A **forest** is a graph with no cycles, and a **tree** is a connected forest.

A nonempty tree is called **pruned** when it has no degree one vertices. Note that every pruned tree is infinite.

A **subgraph** of a graph is given by subsets both of the vertex and edge sets which are stable under the structural maps. Any subset V of the vertex set of a graph **induces** a subgraph whose vertex set is W and whose edge set is the set of edges whose source and target belong to W. Similarly, any subset E of the edge set induces a subgraph whose edge set is $E \cup \bar{E}$ and whose vertex set is the set of targets and sources of elements of E.

Definition 10.2. Given an edge e, its associated **half-graph** is the subgraph induced by the set of edges f such that there is a reduced path starting by e, not using $\bar{e}$, and whose last edge is equal to f. We denote it by $\mathcal{H}(e)$.

Note that the half-graph $\mathcal{H}(e)$ always contains the edge e. When $\mathcal{H}(e)$ is actually a tree (which is automatic if the ambient graph is a tree), we call it a **half-tree** and we say that e is a **treeing edge**.

We now start the hypothesis that we will have to make on our group actions on trees so as to characterize high transitivity. The first one is minimality.

Definition 10.3. A group action on a tree is called **minimal** when there is no non-trivial invariant subtrees.

Here is an easy structural consequence of admitting a minimal action, or equivalently having the whole automorphism group acting minimally.

Lemma 10.4. *Let $\mathcal{T}$ be a tree with at least two edges endowed with a minimal action of a group Γ. Then $\mathcal{T}$ is pruned.*

Proof. Suppose $\mathcal{T}$ is not pruned. Then its degree one vertices can be pruned off, thus obtaining a subtree $\mathcal{S} \subsetneq \mathcal{T}$. Since the tree $\mathcal{T}$ has at least two edges $\mathcal{S}$ is not empty, and it is Γ-invariant by construction, contradicting minimality. $\square$

Our next hypothesis is the most important one, and it relies on the fact that every tree automorphism α has to be of one of the following forms, obtained by considering the minimum distance between v and $\alpha(v)$:

- α is **elliptic** if it fixes a vertex (the minimum distance is 0).
- α is an **inversion** if it flips two adjacent vertices (the minimum distance is 1 and attained by the two vertices which are flipped)
- α is **hyperbolic** if there is an α-invariant biinfinite reduced path $(e_i)_{i\in\mathbb{Z}}$ onto which α acts by translation (meaning that there is a fixed $k \in \mathbb{Z} \setminus \{0\}$ such that $\alpha(e_i) = e_{i+k}$ for all $i \in \mathbb{Z}$; the elements sent at minimum distance from themselves are then precisely the elements of this axis, and this minimum distance is equal to $|k|$).

Definition 10.5. A group action on a tree is called **of general type** if there are two group elements acting hyperbolically and transversely (the intersection of their axes is finite and nonempty).

Example 10.6. The action of the free group $\mathbb{F}_2 = \langle a, b\rangle$ on its Cayley graph is of general type because a and b act hyperbolically, and their respective axes intersect only at the identity element.

We finally state the key property that will allow us to characterize high transitivity.

Definition 10.7. A Γ-action on a pruned tree $\mathcal{T}$ is called **topologically free on the boundary** if whenever $\gamma \in \Gamma \setminus \{1\}$ and $\mathcal{H}$ is a half-tree, there is a half-tree $\mathcal{H}' \subseteq \mathcal{H}$ such that $\gamma\mathcal{H}'$ is disjoint from $\mathcal{H}'$.

Remark 10.8. The terminology comes from the fact that this is equivalent to the action on the boundary $\partial\mathcal{T}$ of the tree (endowed with the natural Hausdorff topology associated to half-trees) being topologically free in the usual sense. For a minimal action of general type on a tree, topological freeness is actually equivalent to the action being highly faithful in the sense of Definition 9.28, see [BIO20, Prop. 3.8]. Also note that every tree endowed with a faithful minimal action of an infinite group must be pruned by Lemma 10.4, so the above definition does apply in the following theorem.

Here is our main result. Stated as such, it relies crucially on the work of Le Boudec and Matte Bon [LBMB22b]. Recall that $\mathrm{td}(\Gamma)$ is the transitivity degree of Γ, see Definition 9.14, and that MIF means mixed identity free, see Definition 9.18.

Theorem 10.9 ([FLMMS22, Thm. B]). *Suppose we are given a countable infinite group Γ with a faithful minimal Γ-action of general type on a tree $\mathcal{T}$. Then the following are equivalent:*

(i) Γ *admits a highly faithful highly transitive action;*

(ii) Γ *is highly transitive;*

(iii) $\mathrm{td}(\Gamma) \geqslant 4$*;*

(iv) Γ *is MIF;*

(v) the Γ*-action on the boundary of* $\mathcal{T}$ *is topologically free.*

Let us now state the key theorem which enables us to prove the above result.

Theorem 10.10 ([FLMMS22, Thm. A]). *Let Γ be a countable group with a minimal action of general type on a tree $\mathcal{T}$ which is moreover topologically free. Then Γ admits a highly faithful highly transitive action.*

Granting this result and the following lemma, we will prove theorem 10.9.

Lemma 10.11 (see also [FLMMS22, Lem. 2.5]). *Let $\Gamma \curvearrowright X$ be a highly faithful transitive action over an infinite set. Then every non trivial element of Γ has infinite support.*

Proof. Suppose by contradiction that $\gamma_0 \in \Gamma$ has finite support F. Since Γ is acting transitively on the infinite set X, by Neumann's lemma there is some $\gamma \in \Gamma$ such that $\gamma F \cap F = \emptyset$. But then γ_0 and $\gamma\gamma_0\gamma^{-1}$ have disjoint supports, thus contradicting high faithfulness. □

Proof of Theorem 10.9. The implication (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is clear.

Two important implications come from the work of Le Boudec and Matte Bon: (iv) $\Longrightarrow$ (v) is Proposition 3.7 in [LBMB22b], and the implication (iii) $\Longrightarrow$ (v) is Corollary 1.5 from the same paper.

The implication (v) $\Longrightarrow$ (i) is our main result, namely Theorem 10.10. So (i), (ii), (iii) and (v) are all equivalent, and since (iv) implies (v), we only need to show that one of them implies (iv).

We thus finish by showing that (i) implies (iv): observe that if $\Gamma \curvearrowright X$ is highly faithful, then every non trivial element of Γ has infinite support by the above lemma. In particular, Γ is not partially finitary so by Theorem 9.19 it is MIF as wanted. □

Our proof of theorem 10.10 relies crucially on Bass-Serre theory. Standard arguments reduce it to the case where Γ is either an amalgamated product or an HNN extension whose action on its Bass-Serre tree is of general type and topologically free (see Section 7.1 in [FLMMS22]). In this exposition, we will explain in details the proof only for free products, pointing out briefly the differences with the case of amalgamated free products (see Remark 10.27). We won't mention HNN extensions further. The construction that we present for free products originates in the work of Moon and Stalder [MS13], although they only use it for free products of finite groups in their paper.

10.2 A bit of Bass-Serre theory

Convention. In this section and in the two next ones, actions on graphs will always be *left* actions, while actions on sets are *right* actions.

Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product. One associates to any right Γ-action on a set X a (bipartite) **Bass-Serre graph** denoted by $\mathbf{BS}(X \curvearrowleft \Gamma)$ defined as follows:

- The set of vertices is the set of Γ_1 orbits union a disjoint copy of the set of Γ_2-orbits:

$$V(\mathbf{BS}(X \curvearrowleft \Gamma)) = X/\Gamma_1 \sqcup X/\Gamma_2$$

- We put an edge between a Γ_1-orbit and a Γ_2 orbit for every common element they have. More precisely, let

$$E(\mathbf{BS}(X \curvearrowleft \Gamma)) = X \times \{-,+\}.$$

  The source and antipode maps are then completely determined by the following assignment: for all $x \in X$, let $\mathsf{s}(x,+) = x \cdot \Gamma_1$, let $\mathsf{t}(x,+) = x \cdot \Gamma_2$, and let $\overline{(x,+)} = (x,-)$ (so $\mathsf{s}(x,-) = x \cdot \Gamma_2$, $\mathsf{t}(x,-) = x \cdot \Gamma_1$ and $\overline{(x,-)} = (x,+)$).

Observe that this graph is connected if and only if the action is transitive, and that its connected components can naturally be identified to Γ-orbits. Also note that by the definition of edges and the fact that our graph is bipartite, any path must consist of edges of alternating signs. Here is a key observation.

Lemma 10.12. *Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product. If we have a free Γ-action $X \curvearrowleft \Gamma$, then the associated Bass-Serre graph $\mathbf{BS}(X \curvearrowleft \Gamma)$ is a forest.*

Proof. Suppose for instance $(e_1, ..., e_n)$ is a cycle with target and source $x\Gamma_1$. Then since the graph is bipartite n is even. Without loss of generality assume that the first edge is positive: then $e_1 = (x_1, +)$, $e_2 = (x_2, -)$, ... $e_n = (x_n, -)$. By definition for $i = 2, ..., n$ we have γ_i such that $x_{i+1} = x_i\gamma_i$, where for i even we have $\gamma_i \in \Gamma_1$, for i odd we have $\gamma_i \in \Gamma_2$ and in both cases $\gamma_i \neq 1$ because the cycle is a reduced path. Let $\gamma = \gamma_1 \cdots \gamma_{n-1}$, then γ is a non-trivial reduced word in $\Gamma_1 * \Gamma_2$ whose last letter γ_{n-1} belongs to γ_2. By construction $x_1\gamma$ belongs to the same Γ_1-orbit as x_1, so we find $\gamma_n \in \Gamma_1$ such that $x_1\gamma\gamma_n = x_1$. But since γ was a non-trivial reduced word ending with an element of Γ_2, $\gamma\gamma_n$ is a non-trivial element of γ fixing a point, contradicting the freeness of the Γ-action. The argument for cycles with target and source $x\Gamma_2$ is the same, switching the roles of Γ_1 and Γ_2. □

Definition 10.13. Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product. The Bass-Serre tree $\mathbf{BS}(\Gamma \curvearrowleft \Gamma)$ obtained from the (free!) right action by right translation $\Gamma \curvearrowleft \Gamma$ is called the **Bass-Serre tree** of the free product decomposition of $\Gamma = \Gamma_1 * \Gamma_2$. Since we will be working with a fixed free product $\Gamma = \Gamma_1 * \Gamma_2$, we will simply call this tree the Bass-Serre tree of Γ.

The left action of $\Gamma = \Gamma_1 * \Gamma_2$ by left multiplication on itself commutes with the action by right multiplication, so it induces a Γ action by automorphisms on its Bass-Serre tree. More generally, whenever we have two right actions $X \curvearrowleft \Gamma$, $Y \curvearrowleft \Gamma$, any Γ-equivariant map $\pi : X \to Y$ induces a graph homomorphism $\pi_* : \mathbf{BS}(X \curvearrowleft \Gamma) \to \mathbf{BS}(Y \curvearrowleft \Gamma)$ which is surjective if π is. This homomorphism is actually always *star surjective* in the following sense.

Definition 10.14. Given a graph $\mathcal{G}$, the **star** of a vertex $v \in V(\mathcal{G})$ is the set of edges whose source is equal to v

$$\mathrm{star}(v) = \{e \in E(\mathcal{G}) \colon \mathsf{s}(e) = v\}.$$

Let $\pi : \mathcal{G} \to \mathcal{H}$ be a graph homomorphism, then π is **star surjective** if it is surjective and for every $v \in V(\mathcal{G})$, $\pi(\mathrm{star}(v)) = \mathrm{star}(\pi(v))$.

Here is a straightforward but important consequence of star surjectivity.

Lemma 10.15. *Let $\pi : \mathcal{G} \to \mathcal{H}$ be a surjective graph homomorphism, assume in addition that π is star surjective. Then any reduced path $(f_1, \ldots, f_n)$ in $\mathcal{H}$ can be lifted to a reduced path $(e_1, \ldots, e_n)$ in $\mathcal{G}$, meaning that for all $i \in \{1, \ldots, n\}$ we have $\pi(e_i) = f_i$. Moreover, lifts can be extended: if the reduced path $(f_1, \ldots, f_{n+m})$ extends $(f_1, \ldots, f_n)$, then there are $e_{n+1}, ..., e_{n+m}$ such that the reduced path $(e_1, \ldots, e_{n+m})$ is a lift of $(f_1, \ldots, f_{n+m})$.* □

A subtree $\mathcal{T}$ of a graph $\mathcal{G}$ is called **spanning** when $V(\mathcal{T}) = V(\mathcal{G})$. Spanning subtrees always exist, see e.g. [Ser80, Prop. 11 in I.2.3]. We have the following basic lemma, which is an important tool for developing Bass-Serre theory (see also [Ser80, Prop. 14 in I.3.1]).

Lemma 10.16. *Let $\pi : \mathcal{G} \to \mathcal{H}$ be a surjective graph homomorphism, assume in addition that π is star surjective. Then given a subtree $\mathcal{T} \subseteq \mathcal{H}$, there is a graph homomorphism $j : \mathcal{T} \to \mathcal{G}$ which lifts $\mathcal{T}$: for every vertex $v \in V(\mathcal{T})$ and every edge $e \in E(\mathcal{T})$,*

$$\pi(j(v)) = v \text{ and } \pi(j(e)) = e.$$

Proof. Observe that any lift of a proper subtree $j' : \mathcal{T}' \subseteq \mathcal{T} \to \mathcal{G}$ can be extended to an additional vertex connected to $\mathcal{T}'$ inside $\mathcal{T}$, using star surjectivity. The conclusion then follows by a direct application of Zorn's lemma. □

Given a group Λ acting on a tree $\mathcal{T}$, Bass-Serre theory offers a natural presentation of Λ in terms of some of its edge and vertex stabilizers, along with some additional individual elements of Λ. This presentation is a description of Λ as the *fundamental group of a graph of groups*. We now give this presentation in details in the general case, and then explain how the picture is much simpler in our setup of free products.

Building a presentation from a group acting on a tree. Let us start with a group Λ acting on a tree $\mathcal{T}$ without inversions. We can then form the quotient graph $\mathcal{G} = \Lambda\backslash\mathcal{T}$ and denote by $\pi : \mathcal{T} \to \mathcal{G}$ the quotient map, which is surjective and star surjective. Fix a spanning subtree $\mathcal{S} \subseteq \mathcal{G}$ and a lift $j : \mathcal{S} \to \mathcal{T}$ as provided by Lemma 10.16.

We also fix an *orientation* of $\mathcal{G}$, i.e. a subset $A \subseteq E(\mathcal{G})$ such that for every edge $e \in E(\mathcal{G})$, either $e \in A$ or $\bar{e} \in A$ and these two cases are mutually exclusive.

We can start describing the vertex and edge stabilizers appearing in the presentation of Λ (we will need some additional edge stabilizers eventually):

- For every $v \in V(\mathcal{G})$, let $\Lambda_v = \mathrm{Stab}_\Lambda(j(v))$
- For every $e \in E(\mathcal{S}) \cap A$, let $\Lambda_e = \mathrm{Stab}_\Lambda(j(e))$

We then extend j to $E(\mathcal{G})$ by first letting, for $e \in A$, $j(e)$ be a lift of e such that

$$j(\mathsf{s}(e)) = \mathsf{s}(j(e)).$$

Such a lift exists by local surjectivity. Then for $e \in B$ we let $j(e) = \overline{j(\bar{e})}$. We extend our list of edge stabilizers to all the positively oriented edges of $\mathcal{G}$:

- for $e \in (E(\mathcal{G}) \setminus E(\mathcal{S})) \cap A$, let $\Lambda_e = \mathrm{Stab}_\Lambda(j(e))$

Observe that j is still a section of π, but it is not a graph homomorphism anymore because it does not commute with the target map of edges in A (and the source map of edges in B). Nevertheless, by the definition of the quotient map, for every edge $e \in A$, there is *some* edge in the Λ-orbit of $j(e)$ whose target is equal to $t(j(e))$, so the following definition makes sense:

- For every $e \in A \cap (E(\mathcal{G}) \setminus E(\mathcal{S}))$, we fix $\lambda_e \in \Lambda$ such that $\lambda_e j(\mathsf{t}(e)) = \mathsf{t}(j(e))$.

We then have $\lambda_e \mathrm{Stab}_\Lambda(j(\mathsf{t}(e)))\lambda_e^{-1} = \mathrm{Stab}_\Lambda(\mathsf{t}(j(e)))$, in other words

$$\lambda_e \Lambda_{\mathsf{t}(e)} \lambda_e^{-1} = \mathrm{Stab}_\Lambda(\mathsf{t}(j(e))) \geqslant \mathrm{Stab}_\Lambda(j(e)) = \Lambda_e. \tag{4}$$

We finally define various embeddings which will allow us to give the group presentation we seek:

- For $e \in A$, let ι_e^{s} be the natural inclusion of Λ_e in $\Lambda_{\mathsf{s}(e)}$ given by the fact that
$$\Lambda_e = \mathrm{Stab}_\Lambda(j(e)) \leqslant \mathrm{Stab}_\Lambda(\mathsf{s}(j(e))) = \mathrm{Stab}_\Lambda(j(\mathsf{s}(e))) = \Lambda_{\mathsf{s}(e)}.$$

- For $e \in A \cap E(\mathcal{S})$, let ι_e^{t} be the natural inclusion of Λ_e in $\Lambda_{\mathsf{t}(e)}$ given by the fact that
$$\Lambda_e = \mathrm{Stab}_\Lambda(j(e)) \leqslant \mathrm{Stab}_\Lambda(\mathsf{t}(j(e))) = \mathrm{Stab}_\Lambda(j(\mathsf{t}(e))) = \Lambda_{\mathsf{t}(e)}.$$

- For $e \in A \setminus E(\mathcal{S})$, let ι_e^{t} be the embedding of Λ_e into $\Lambda_{\mathsf{t}(e)}$ provided by Equation (4): for all $\gamma \in \Lambda_e$, let
$$\iota_e^{\mathsf{t}}(\gamma) = \lambda_e^{-1}\gamma\lambda_e = \lambda_e^{-1}\iota_e^{\mathsf{s}}(\gamma)\lambda_e^{-1}.$$

Theorem 10.17 (see [Ser80, Thm. 13 in I.5.3])**.** *The group Λ is actually given by the following presentation: it is generated by the groups $(\Lambda_v)_{v \in V(\mathcal{G})}$ and the additional generators $(\lambda_e)_{e \in A \setminus E(\mathcal{S})}$, subject to the following relations*

- *(amalgamation relations) for all $e \in A \cap E(\mathcal{S})$ and all $\gamma \in \Lambda_e$:*
$$\iota_e^{\mathsf{t}}(\gamma) = \iota_e^{\mathsf{s}}(\gamma).$$

- *(HNN relations) for all $e \in A \setminus E(\mathcal{S})$ and all $\gamma \in \Lambda_e$:*
$$\iota_e^{\mathsf{t}}(\gamma) = \lambda_e^{-1}\iota_e^{\mathsf{s}}(\gamma)\lambda_e.$$

Example 10.18. If Λ is acting freely on $\mathcal{T}$, both edge and vertex stabilizers are trivial, so the amalgamation and HNN relations are trivial, and we deduce that Λ is freely generated by $(\lambda_e)_{e \in A \setminus E(\mathcal{S})}$, thus recovering the characterization of free groups as groups freely acting on trees.

Example 10.19. Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product, let $\mathcal{T}$ be its Bass-Serre tree. By construction the Γ-action on $\mathcal{T}$ is free at the level of edges, and has two orbits at the level of vertices, namely Γ/Γ_1 and Γ/Γ_2. The stabilizer of the coset Γ_1 is Γ_1, and the stabilizer of the coset Γ_2 is Γ_2, so stabilizers of vertices are conjugates of either Γ_1 or Γ_2. The quotient graph is then a single edge, and we can take as a lift the edge 1_Γ whose vertex stabilizers are Γ_1 and Γ_2. Since the action on edges is free, again the amalgamation and HNN relations trivialize and we recover the free product decomposition of Γ.

More generally, take $\Lambda \leqslant \Gamma$, then again edge stabilizers are trivial so we conclude from the above theorem that Λ is naturally isomorphic to a free product of a free group over $|A \setminus E(\mathcal{S})|$ generators with subgroups of either Γ_1 or Γ_2 (Kurosh's theorem). In particular, if Λ is finitely generated, there are only finitely many non trivial Λ_v, and the set $E(\mathcal{G}) \setminus E(\mathcal{S})$ is finite (in other words, $\mathcal{G}$ has a finitely generated fundamental group).

Remark 10.20. There are actually two natural trees that we can make $\mathbb{F}_2 = \langle a, b\rangle$ act on. The first is the standard Cayley graph, and then the quotient graph is a bouquet of two circles, obtained by doing two trivial HNN extensions of the trivial group via generators a and b. The second is associated to the free product decomposition $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$, and it is not locally finite. Here vertex stabilizers are isomorphic to $\mathbb{Z}$, the action on edges is free, and the quotient graph is as above an edge whose two extremities are decorated with copies of $\mathbb{Z}$.

Both trees can be used in order to build highly transitive actions of $\mathbb{F}_2$, but do note that in this text, we only present a full proof for free products, and hence for $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ we are working with the second tree, which is not locally finite.

We end this section with an important well-known lemma towards proving our high transitivity result for free products.

Lemma 10.21. *Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product with $|\Gamma_1| \geqslant 3$ and $|\Gamma_2| \geqslant 2$, let $\mathcal{T}$ be the associated Bass-Serre tree. Then $\Gamma \curvearrowright \mathcal{T}$ is minimal, topologically free on the boundary and of general type.*

Proof. Recall that the Γ-action on $\mathcal{T}$ is transitive on the edges, hence minimal. There are only two vertex orbits, corresponding to Γ_1 or Γ_2 cosets, so up to conjugacy there are two types of elliptic elements: those of Γ_1 and those of Γ_2. But by freeness of the action on the edges and the fact that both Γ_1 and Γ_2 have cardinality at least 2, it is clear that these elliptic elements act topologically freely on the boundary.

The fact that the hyperbolic elements of Γ also act topologically freely follows from the fact that $\mathcal{T}$ has no isolated points in its boundary: given any half-tree $\mathcal{H}$, one can find two disjoint half-trees $\mathcal{H}_1$ and $\mathcal{H}_2$ contained in $\mathcal{H}$. (the latter property is a direct consequence of the fact that the vertices corresponding to Γ_1 cosets have degree $|\Gamma_1| = 3$ and every edge touches such a vertex). Now given any hyperbolic automorphism α of $\mathcal{T}$, if $\mathcal{H}$ is any half-tree, we find two smaller disjoint half trees $\mathcal{H}_1$, $\mathcal{H}_2$. Then one of them, say $\mathcal{H}_1$, is disjoint from the axis of α, and hence $\alpha(\mathcal{H}_1)$ is disjoint from $\mathcal{H}_1$ as wanted.

Finally, let us show that the action is of general type. Observe that given $\gamma_1 \in \Gamma_1 \setminus \{1\}$ and $\gamma_2 \in \Gamma_2 \setminus \{1\}$, the element $\gamma_1\gamma_2$ is hyperbolic and its axis can be described as follows: if W be the set of all finite words (of length $\geqslant 1$) of the form $\gamma_1\gamma_2\gamma_1\gamma_2\dots$ then the set of edges of the axis of $\gamma_1\gamma_2$ is $W \sqcup \{1\} \sqcup W^{-1}$. With that observation in hand, we take distinct $\gamma_1, \gamma_1' \in \Gamma_1 \in \Gamma_1 \setminus \{1\}$ and $\gamma_2 \in \Gamma_2 \setminus \{1\}$, and then the hyperbolic tree automorphisms $\gamma_1\gamma_2$ and $\gamma_1'\gamma_2$ are transverse, thus showing that the action is of general type. $\square$

10.3 Finitely generated subgroups of free products and half-trees

Our proof of the genericity of high transitivity for free products will use crucially the fact that finitely generated groups of infinite index form a dense subset of the set of infinite index subgroups. This allows us to bypass some constructions from [FLMMS22], using

Bass-Serre theory as described in the previous section, noting that Bass-Serre graphs of finitely generated subgroups are essentially isomorphic to the whole Bass-Serre graph, except on a finite part, which is the content of the following proposition. I am very grateful to Damien Gaboriau for explaining this to me a certain number of times.

Proposition 10.22. *Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product, let $\mathcal{T}$ be its Bass-Serre tree, and let $\Lambda \leqslant \Gamma$ be an infinite index finitely generated subgroup, let $\mathcal{G} = \Lambda\backslash\mathcal{T}$ and denote by $\pi : \mathcal{T} \to \mathcal{G}$ the quotient map. There is a finite connected set $K \subseteq E(\mathcal{G})$ whose complement induces a forest $\mathcal{F}$ with infinitely many vertices, and such that for all $v \in V(\mathcal{F})$ and all $w \in V(\mathcal{T})$ such that $\pi(w) = v$, the map π is* ***star-bijective*** *at w, meaning that it induces a bijection* $\mathrm{star}(w) \to \mathrm{star}(v)$.

Proof. We apply Theorem 10.17: we fix an orientation A of $\Lambda\backslash\mathcal{T}$, a spanning tree $\mathcal{S} \subseteq \Lambda\backslash\mathcal{T}$, a lift $j : \mathcal{S} \to \mathcal{T}$ extended to all the edges of A so that $j(\mathsf{s}(e)) = \mathsf{s}(j(e))$ for all $e \in A$. Finally, for all $e \in A \setminus E(\mathcal{S})$, we fix some $\lambda_e \in \Lambda$ such that $\lambda_e j(\mathsf{t}(e)) = \mathsf{t}(j(e))$. Since the Λ-action on $\mathcal{T}$ is free at the level of edges, Theorem 10.17 yields that Λ decomposes naturally as a free product as follows:

$$\Lambda = \left(\mathop{*}_{v \in V(\Lambda\backslash\mathcal{T})} \mathrm{Stab}_{\Lambda\curvearrowright\mathcal{T}}(j(v)) \right) * \mathbb{F}_{A\setminus E(\mathcal{S})},$$

where $\mathbb{F}_{A\setminus E(\mathcal{S})}$ is a free group freely generated by $(\lambda_e)_{e\in A\setminus E(\mathcal{S})}$. Since Λ is finitely generated, there are only finitely many $v \in V(\Lambda\backslash\mathcal{T})$ such that $\mathrm{Stab}_{\Lambda\curvearrowright\mathcal{T}}(j(v)) \neq \{1\}$, and the set $A \setminus E(\mathcal{S})$ is finite. Moreover, since Λ has infinite index, the edge set of $\Lambda\backslash\mathcal{T}$ is infinite. It follows that the vertex set of $\Lambda\backslash\mathcal{T}$ is infinite (otherwise $A \setminus E(\mathcal{S})$ would have to be infinite since a tree has exactly one more vertex than positively oriented edges).

Let us define K as the finite connected set of edges spanned by A along with the vertices $v \in V(\Lambda\backslash\mathcal{T})$ such that $\mathrm{Stab}_{\Lambda\curvearrowright\mathcal{T}}(j(v)) \neq \{1\}$. Since all the vertices from A belong to K and $\mathcal{S}$ is a spanning tree such that $E(\mathcal{G}) = E(\mathcal{S}) \sqcup A$, the complement of K induces a forest. Finally, if we fix any vertex $v \in V(\mathcal{G}) \setminus K$, we have $\mathrm{Stab}_{\Lambda\curvearrowright\mathcal{T}}(j(v)) = \{1\}$, so for every $w \in V(\mathcal{T})$ such that $\pi(w) = v$, we also have $\mathrm{Stab}_{\Lambda\curvearrowright\mathcal{T}}(w) = \{1\}$. By definition of the quotient map, this implies that for all $w \in V(\mathcal{T})$ such that $\pi(w) = v$, π induces a bijection $\mathrm{star}(w) \to \mathrm{star}(v)$ as wanted. $\square$

In order to use the above proposition, we need a better understanding of half-trees. We first reproduce a useful lemma from [FLMMS22]. Recall that an edge is called a *treeing edge* when its associated half-graph is a tree.

Lemma 10.23. *Let $\mathcal{G}$ be a connected graph admitting a treeing edge, and let ω be a reduced path in $\mathcal{G}$. Then ω can be extended to a reduced path ω' whose last edge is a treeing edge.*

Proof. See [FLMMS22, Lem. 2.17]. $\square$

Proposition 10.22 will be used together with the following lemma.

Lemma 10.24. *Let $\mathcal{T}$ be a pruned nonempty tree, let $\pi : \mathcal{T} \to \mathcal{G}$ be a star-surjective graph homomorphism. Suppose that there is a finite set $K \subseteq E(\mathcal{G})$ whose complement induces a forest $\mathcal{F}$ with infinitely many vertices, and such that for all $v \in V(\mathcal{F})$, all $w \in V(\mathcal{T})$ such that $\pi(w) = v$ the map π induces a bijection* $\mathrm{star}(w) \to \mathrm{star}(v)$. *The following hold:*

(i) If $\mathcal{H}$ is a half-tree in $\mathcal{T}$ such that $\pi(\mathcal{H})$ is contained in the forest $\mathcal{F}$, then $\pi(\mathcal{H})$ is a half-tree and the restriction of π to $\mathcal{H}$ is a graph isomorphism $\mathcal{H} \to \pi(\mathcal{H})$.

(ii) Given any half-tree $\mathcal{H}$ in $\mathcal{T}$, there is a smaller half-tree $\mathcal{H}' \subseteq \mathcal{H}$ such that $\pi(\mathcal{H}')$ is contained in $\mathcal{F}$.

Proof. We start by proving item (i). Let $\mathcal{H} = \mathcal{H}(e)$ be a half-tree in $\mathcal{T}$ such that $\pi(\mathcal{H})$ is contained in $\mathcal{F}$. Clearly $\pi(\mathcal{H})$ is a subtree since it is connected and contained in a forest. Moreover, any reduced path in $\mathcal{H}$ has to be sent by π to a reduced path in $\mathcal{F}$ since for all $v \in V(\mathcal{F})$, and all $w \in V(\mathcal{T})$ such that $\pi(w) = v$ the map π induces a bijection $\mathrm{star}(w) \to \mathrm{star}(v)$. It follows that $\pi(\mathcal{H}(e)) = \mathcal{H}(\pi(e))$, in particular $\pi(\mathcal{H})$ is a half-tree as wanted. Moreover, using star-bijectivity again, we obtain that π is an injection at the level of reduced paths starting by e, and hence the restriction of π to $\mathcal{H}$ is a graph isomorphism $\mathcal{H} \to \pi(\mathcal{H})$.

We now prove item (ii). Again let $\mathcal{H} = \mathcal{H}(e)$ be a half-tree in $\mathcal{T}$, let $f = \pi(e)$, let $v = \mathsf{t}(e)$. By the above lemma, there is a reduced path $(f, f_2, \ldots, f_n)$ in $\mathcal{G}$ starting by f whose last edge f_n is a treeing edge, meaning that the half-graph $\mathcal{H}(f_n)$ is actually half-tree in $\mathcal{G}$. We now distinguish two cases.

- If the half-tree $\mathcal{H}(f_n)$ is infinite, it contains a half-tree contained in $\mathcal{F}$ since K is finite. We can thus extend our reduced path to $(f, f_2, \ldots, f_{n+m})$ so that $\mathcal{H}(f_{n+m})$ is a half-tree contained in $\mathcal{F}$. We then lift this reduced path to $(e, e_2, ..., e_{n+m})$, and then the half-tree $\mathcal{H}(e_{n+m})$ is sent by π to the half-tree $\mathcal{H}(f_{n+m})$, which is contained in $\mathcal{F}$ as wanted.

- Otherwise $\mathcal{H}(f_n)$ is finite so it must contain a degree one vertex v. Let $(f_n, ..., f_{n+m})$ be the unique geodesic path in $\mathcal{H}(f_n)$ from $\mathsf{s}(f_n)$ to v. Let $(e, e_2, \ldots, e_{n+m})$ be a lift of the reduced path $(f, f_2, \ldots, f_{n+m})$. Since $\mathcal{T}$ is pruned, we may find $e' \neq \bar{e}_{n+m}$ such that $\mathsf{s}(e') = \mathsf{t}(e_{n+m})$. Since $v = \mathsf{t}(f_{n+m})$ has degree one, we have $\pi(e') = \bar{f}_{n+m}$, and so we may lift the reduced path $(\bar{f}_{n+m}, \bar{f}_{n+m-1}, \ldots, \bar{f}_n)$ to a reduced path of the form $(e', e_{n+m+1}, \ldots, e_{n+2m})$. It follows that the path
$$(e, e_2, \ldots, e_{n+m}, e', e_{n+m+1}, \ldots, e_{n+2m})$$
is reduced with $\pi(e_{n+2m}) = \bar{f}_n$. Then $\mathcal{H}(\bar{f}_n)$ is infinite since $\mathcal{H}(f_n)$ is finite and the vertex set of $\mathcal{F}$ is infinite. By the first case, we can further extend the lift $(e, e_2, \ldots, e_{n+m}, e', e_{n+m+1}, \ldots, e_{n+2m})$ to a reduced path whose last edge e'' satisfies $\pi(\mathcal{H}(e'')) \subseteq \mathcal{F}$. Then the half-tree $\mathcal{H}(e'')$ is contained in $\mathcal{H}$ and $\pi(\mathcal{H}(e''))$ is contained in $\mathcal{F}$ as wanted.

Since we found the desired half-tree in both cases, the proof is finished. □

10.4 Proof of high transitivity for free products

Convention. Our convention that actions on trees are left actions while actions on sets are right actions still holds. Since here actions on sets will be seen as group homomorphisms $\Gamma \to \mathfrak{S}_\infty$, we write composition in $\mathfrak{S}_\infty$ in the reverse direction: if $\sigma, \tau \in \mathfrak{S}_\infty$ and $x \in \mathbb{N}$ we write $x\sigma\tau$ for the element obtained by applying first σ to x and then τ to $x\sigma$.

The argument uses Baire category techniques, and we will be working directly in the Polish space of all transitive Γ-actions over an infinite set, which is an important difference with [MS13]. Here is the main statement.

Theorem 10.25. *Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product with $|\Gamma_1| \geqslant 3$ and $|\Gamma_2| \geqslant 2$. Then the space of highly transitive actions of Γ on $\mathbb{N}$ is dense G_δ in the space $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, \mathfrak{S}_\infty)$ of transitive Γ-actions on $\mathbb{N}$.*

The following lemma justifies the importance of topological freeness while encompassing two key claims from [FLMMS22], as we will see later. It also shows that topological freeness implies high faithfulness of the action on the tree in a strong sense, which is not hard to see with the usual definition of topological freeness as well.

Lemma 10.26. *Let $\Gamma \curvearrowright \mathcal{T}$ be an action on a pruned tree which is topologically free on the boundary. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ be pairwise distinct. Then given any half-tree $\mathcal{H}$, there is another half-tree $\mathcal{H}' \subseteq \mathcal{H}$ such that the half trees $\gamma_1\mathcal{H}', \ldots, \gamma_n\mathcal{H}'$ are pairwise disjoint.*

Proof. We prove the result by induction on $n \geqslant 1$. For $n = 1$, there is nothing to prove. Assuming the result holds for some $n \geqslant 1$, we take $\gamma_1, \ldots, \gamma_{n+1} \in \Gamma$ pairwise distinct and fix a half-tree $\mathcal{H} \subseteq \mathcal{T}$. By our inductive hypothesis, there is some half-tree $\mathcal{H}_0 \subseteq \mathcal{H}$ such that the half-trees $\gamma_1\mathcal{H}_0, \ldots, \gamma_n\mathcal{H}_0$ are pairwise disjoint. Applying the definition of topological freeness to the non-trivial elements $\gamma_i^{-1}\gamma_{n+1}$ successively for $i \in \{1, ..., n\}$ we find $\mathcal{H}_i \subseteq \mathcal{H}_{i-1}$ such that $\gamma_i^{-1}\gamma_{n+1}\mathcal{H}_i$ is disjoint from $\mathcal{H}_i$. It follows that by letting $\mathcal{H}' = \mathcal{H}_n \subseteq \mathcal{H}_0$, we have that each $\gamma_i\mathcal{H}'$ is disjoint from $\gamma_{n+1}\mathcal{H}'$. Since $\mathcal{H}' \subseteq \mathcal{H}_0$ all whose translates by $\gamma_1, \ldots, \gamma_n$ are disjoint, the induction is proven and the lemma follows. □

Proof of Theorem 10.25. We first spell out the open sets which are used to show that the space of highly transitive actions is dense G_δ. For all $n \geqslant 1$, denote by $\mathbb{N}^{(n)}$ the set of n-tuples consisting of pairwise distinct integers. Given $(x_1, \ldots x_n, y_1, \ldots, y_n) \in \mathbb{N}^{(2n)}$, consider the following open set

$$\mathcal{U}_{x_1,\ldots,x_n \to y_1,\ldots,y_n} = \{\alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty) \colon \exists \gamma \in \Gamma \text{ s.t. } \forall i \in \{1, \ldots, n\}, y_i = x_i\alpha(\gamma)\}.$$

Then if we denote

$$\mathcal{HT} = \{\alpha \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty) \colon \alpha \text{ is highly transitive}\},$$

we claim that

$$\mathcal{HT} = \bigcap_{n\geqslant 1} \bigcap_{(x_1,\ldots,x_n,y_1,\ldots,y_n)\in\mathbb{N}^{(2n)}} \mathcal{U}_{x_1,\ldots,x_n \to y_1,\ldots,y_n}. \tag{5}$$

Indeed, the right-to-left inclusion is clear, and conversely if α belongs to right-hand term and if $(x_1, \ldots, x_n) \in \mathbb{N}^{(n)}$ and $(y_1, \ldots, y_n) \in \mathbb{N}^{(n)}$, we can pick $(z_1, \ldots, z_n) \in \mathbb{N}^{(n)}$ distinct from all the x_i's and y_i's. Since $\alpha \in \mathcal{U}_{x_1,\ldots,x_n \to z_1,\ldots,z_n}$ we find $\gamma_1 \in \Gamma$ such that $z_i = x_i\alpha(\gamma_1)$ for all $i \in \{1, \ldots, n\}$, and since $\alpha \in \mathcal{U}_{z_1,\ldots,z_n \to y_1,\ldots,y_n}$ we find $\gamma_2 \in \Gamma$ such that $y_i = z_i\alpha(\gamma_2)$ for all $i \in \{1, \ldots, n\}$. We thus have $y_i = x_i\alpha(\gamma_1\gamma_2)$ for all $i \in \{1, \ldots, n\}$, thus proving Equation (5).

By the Baire category theorem, it now suffices to prove that each of the open sets in Equation (5) is dense. Fix $(x_1, \ldots x_n, x_{n+1}, \ldots, x_{2n}) \in \mathbb{N}^{(2n)}$, let us show that the open set $\mathcal{U}_{x_1,\ldots,x_n \to x_{n+1},\ldots,x_{2n}}$ is dense.

We thus fix a transitive action α and a neighborhood $\mathcal{U}$ of α. By Corollary 9.10, we may as well take α with finitely generated stabilizers. Since Γ is generated by $\Gamma_1 \cup \Gamma_2$, we can then find a finite set $F \subseteq \mathbb{N}$ such that $\mathcal{U}$ contains the closed set

$$\mathcal{C} = \{\beta \in \mathrm{Hom}(\Gamma, \mathfrak{S}_\infty) \colon \forall x \in F, \forall \gamma \in \Gamma_1 \cup \Gamma_2, x\beta(\gamma) = x\alpha(\gamma)\}.$$

We will now construct $\beta \in \mathcal{C} \cap \mathcal{U}_{x_1,\ldots,x_n \to x_{n+1},\ldots,x_{2n}}$, thus showing the desired density.

Let $\mathcal{T}$ be the Bass-Serre tree of $\Gamma = \Gamma_1 * \Gamma_2$. Let us fix a basepoint $x_0 \in \mathbb{N}$, the map π^{x_0} : $\gamma \mapsto x\alpha(\gamma)$ is a Γ-equivariant surjection from the Γ-action on itself by right translation to the action α. It thus induces a graph homomorphism between the corresponding Bass-Serre graphs $\pi_*^{x_0} : \mathbf{BS}(\Gamma \curvearrowleft \Gamma) \to \mathbf{BS}(\mathbb{N} \overset{\alpha}{\curvearrowleft} \Gamma)$. Let $\Lambda = \mathrm{Stab}_\Gamma^\alpha(x_0)$. Then π^{x_0} descends to an isomorphism of actions between $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ and $\mathbb{N} \overset{\alpha}{\curvearrowleft} \Gamma$. But the Bass-Serre graph of $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ naturally identifies to $\Lambda\backslash\mathcal{T}$, so π^{x_0} yields an isomorphism $\tilde{\pi}^{x_0} : \Lambda\backslash\mathcal{T} \xrightarrow{\cong} \mathbf{BS}(\mathbb{N} \overset{\alpha}{\curvearrowleft} \Gamma)$. Finally, if we denote by $\pi : \mathcal{T} \to \Lambda\backslash\mathcal{T}$ the quotient map, the following diagram commutes, where the vertical arrows are graph isomorphisms.

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\pi"] \arrow[d, "\mathrm{id}"] & \Lambda\backslash\mathcal{T} \arrow[d, "\tilde{\pi}^{x_0}"] \\
\mathbf{BS}(\Gamma \curvearrowleft \Gamma) \arrow[r, "\pi_*^{x_0}"] & \mathbf{BS}(\mathbb{N} \overset{\alpha}{\curvearrowleft} \Gamma)
\end{tikzcd} \tag{6}$$

We now apply Proposition 10.22: we have a finite connected set $K \subseteq E(\Lambda\backslash\mathcal{T})$ whose complement induces a forest $\mathcal{F}$ with infinitely many vertices, and such that for all $v \in V(\mathcal{F})$ and all $w \in V(\mathcal{T})$ such that $\pi(w) = v$, the map π induces a bijection $\mathrm{star}(w) \to \mathrm{star}(v)$. We can thus find a larger finite connected subgraph $\mathcal{B}$ in $\Lambda\backslash\mathcal{T}$ which contains all the edges of the form $(\tilde{\pi}^{x_0})^{-1}(x, \pm)$ for $x \in F \cup \{x_1, \dots, x_{2n}\}$, all the edges in $A \setminus E(\mathcal{S})$, and such that π is star-bijective at the pullback of every vertex $v \in V(\Lambda\backslash\mathcal{T}) \setminus V(\mathcal{B})$. This implies that the Γ_1 right action on $\Lambda g\Gamma_1$ is free for every vertex $\Lambda g\Gamma_1 \notin V(\mathcal{B})$, and similarly that the Γ_2 right action on $\Lambda g\Gamma_2$ is free for every vertex $\Lambda g\Gamma_2 \notin V(\mathcal{B})$.

For $i \in \{1, \dots, 2n\}$, fix some $\gamma_i \in \Gamma$ such that

$$x_0\alpha(\gamma_i) = x_i.$$

Using Lemma 10.24 $2n$ times, we find a half-tree $\mathcal{H}$ in $\mathcal{T}$ such that for all $i \in \{1, \dots, 2n\}$, $\pi(\gamma_i\mathcal{H})$ is a half-tree in $\Lambda\backslash\mathcal{T}$ contained in $\mathcal{F}$.

Let us then fix a spanning subtree $\mathcal{S} \subseteq \Lambda\backslash\mathcal{T}$ containing $\mathcal{F}$ and a lift $j : \mathcal{S} \to \mathcal{T}$. Replacing each γ_i by another coset representative in $\Lambda\backslash\Gamma$, we may further assume that $\gamma_i\mathcal{H}$ is contained in $j(\mathcal{S})$. Since the x_i's are pairwise distinct, the γ_i's are pairwise distinct. By Lemma 10.26 and since the Γ-action on $\mathcal{T}$ is topologically free (see Lemma 10.21), there is a half-tree $\mathcal{H}' \subseteq \mathcal{H}$ such that the $\gamma_i\mathcal{H}'$ for $i \in \{1, \dots, 2n\}$ are pairwise disjoint. Since they are all contained in $j(\mathcal{S})$, we also have that the half-trees $\pi(\gamma_i\mathcal{H}')$ for $i \in \{1, \dots, 2n\}$ are pairwise disjoint in $\Lambda\backslash\mathcal{T}$. By taking a smaller half-tree, we can even assume $\mathcal{H}'$ is at even distance from the vertex Γ_1.

We now fix a geodesic reduced path $p = (e_1, \dots, e_{2k})$ in $\mathcal{T}$ with starting vertex Γ_1 such that $\mathcal{H}' = \mathcal{H}(e_{2k})$. Consider the connected subgraph $\mathcal{G}_0$ of $\mathbf{BS}(\mathbb{N} \overset{\alpha}{\curvearrowleft} \Gamma)$ defined as the union of $\tilde{\pi}^{x_0}(\mathcal{B})$ with the paths $\pi_*^{x_0}(\gamma_i p)$, where $i \in \{1, \dots, 2n\}$ (to see that it is connected, note that the first vertex of the path $\pi_*^{x_0}(\gamma_i p)$ is the $\alpha(\Gamma_1)$-orbit of $x_0\alpha(\gamma_i) = x_i$, which belongs to $\tilde{\pi}^{x_0}(\mathcal{B})$). We are now going to build the desired action β.

Let X_1 denote the union of the vertices of $\mathcal{G}_0$ which are $\alpha(\Gamma_1)$-orbits, and X_2 denote the reunion of the vertices of $\mathcal{G}_0$ which are $\alpha(\Gamma_2)$-orbits. We start by defining partially β as follows:

$$\begin{aligned}
\forall \gamma \in \Gamma_1, \forall x \in X_1, \quad x\beta(\gamma) = x\alpha(\gamma) \\
\forall \gamma \in \Gamma_2, \forall x \in X_2, \quad x\beta(\gamma) = x\alpha(\gamma).
\end{aligned}$$

Let $e_{2k} = (g_{2k}, -)$ be the last edge of the path p, which points towards the half-tree $\mathcal{H}'$. Then $x_i\alpha(g_{2k}) = \pi^{x_0}(\gamma_i g_{2k})$, so $x_i\alpha(g_{2k})$ is an edge whose target $x_i\alpha(g_{2k}\Gamma_2)$ has degree one in $\mathcal{G}_0$, in other words $x_i\alpha(g_{2k})$ is the only element of $x_i\alpha(g_{2k}\Gamma_2)\cap X_1$. Fix $\kappa_2 \in \Gamma_2 \setminus \{1\}$, then $x_i\alpha(g_{2k}\kappa_2) \notin X_1$ since Γ_2 is freely acting on the Γ_2-orbit of $x_i\alpha(g_{2k})$.

Let us also fix some non trivial $\kappa_1 \in \Gamma_1$. We now extend $\beta_{\restriction\Gamma_1}$ a bit further so as to have $\beta \in \mathcal{U}_{x_1,\dots,x_n\to x_{n+1},\dots,x_{2n}}$: for $i \in \{1,\dots,n\}$, we require

$$x_i\alpha(g_{2k}\kappa_2)\beta(\kappa_1) = x_{n+i}\alpha(g_{2k}\kappa_2). \tag{7}$$

This can be done for instance by adding n copies of Γ_1 onto which $\beta(\Gamma_1)$ acts on the right, identifying in the i'th copy the element 1_{Γ_1} to $x_i\alpha(g_{2k}\kappa_2)$ and the element κ_1 to $x_{n+i}\alpha(g_{2k}\kappa_2)$ so as to have the desired identity (7). Note that the restriction of β to Γ_1 is now defined on a larger set $\tilde{X}_1 \supseteq X_1$, and since $\mathbb{N}\setminus X_1$ is infinite we may as well assume $\tilde{X}_1 \subseteq \mathbb{N}$ and $\mathbb{N}\setminus(\tilde{X}_1\cup X_2)$ is infinite. We then extend arbitrarily β to a transitive Γ-action on $\mathbb{N}$, e.g. by first filling $\mathbb{N}$ with infinitely many new free Γ_1 orbits, and then adding infinitely many new free Γ_2 orbits, making sure that the Γ_2-orbits connect the Γ_1 orbits together and partition $\mathbb{N}$ as well[10].

It is clear that $\beta \in \mathcal{C}$. Let us check that $\beta \in \mathcal{U}_{x_1,\dots,x_n\to x_{n+1},\dots,x_{2n}}$. Our main claim is that

$$x_i\alpha(g_{2k}\kappa_2) = x_i\beta(g_{2k}\kappa_2).$$

To see this, we first need to spell out precisely what the path $p = (e_1,\dots,e_{2k})$ connecting the vertex Γ_1 to the half tree $\mathcal{H}'$ looks like. Since p is a path starting by Γ_1, we have $e_1 = (g_1,+)$ for some $g_1 \in \Gamma_1$, and then $e_j = (g_j,\epsilon_j)$ for some $g_j \in \Gamma$, where $\epsilon_j = +$ if j is odd and $\epsilon_j = -$ if j is even, for $j \in \{2,\dots,2k\}$. Since the starting vertex is actually Γ_1, we have $g_1 \in \Gamma_1$. Furthermore, for $j \in \{1,\dots,2k-1\}$ we can write $g_{j+1} = g_jh_j$ where for j even we have $h_j \in \Gamma_2$, while for i odd we have $h_i \in \Gamma_1$. Since the path is reduced, no h_i can be equal to 1. We thus have $g_{2k} = g_1h_1\dots h_{2n-1}$ with $g_1 \in \Gamma_1$, and then for i odd $h_i \in \Gamma_2\setminus\{1\}$, for i even $h_i \in \Gamma_1\setminus\{1\}$.

Let us now take $i \in \{1,\dots,2n\}$. By construction the path $\pi_*^{x_0}(\gamma_ip)$ is contained in $\mathcal{G}_0$. We show by induction on $j \in \{1,\dots,2k\}$ that $x_i\alpha(g_j) = x_i\beta(g_j)$. For $j = 1$, this follows from the fact that $x_i \in X_1$. Assuming the result holds for some $j < 2k$, we write $g_{j+1} = g_jh_j$ and note that $(x_i\alpha(g_j),\epsilon_j)$ is equal to $\pi_*^{x_0}(\gamma_ie_j) \in E(\mathcal{G}_0)$. If j is odd, $\epsilon_j = +$ so the target of $(x_i\alpha(g_j),\epsilon_j)$ is $x_i\alpha(g_j)\alpha(\Gamma_1)$, in particular $x_i\alpha(g_j) \in X_1$ and hence

$$x_i\alpha(g_j)\alpha(h_j) = x_i\alpha(g_j)\beta(h_j) = x_i\beta(g_j)\beta(h_j),$$

so $x_i\alpha(g_{j+1}) = x_i\alpha(g_{j+1})$ as wanted. The case j is even is similar and left to the reader. This finishes the proof of the induction, in particular $x_i\alpha(g_{2k}) = x_i\beta(g_{2k})$, and since $x_i\alpha(g_{2k}) \in X_2$ we finally conclude

$$x_i\alpha(g_{2k}\kappa_2) = x_i\beta(g_{2k}\kappa_2)$$

as wanted.

Now take $i \in \{1,\dots,n\}$, we have by the previous equality and the definition of β

$$\begin{aligned} x_i\beta(g_{2k}\kappa_2\kappa_1) &= x_i\alpha(g_{2k}\kappa_2)\beta(\kappa_1) \\ &= x_{n+i}\alpha(g_{2k}\kappa_2) \\ &= x_{n+i}\beta(g_{2k}\kappa_2). \end{aligned}$$

[10]A much more precise version of this argument is made in [FLMMS22] where the *free globalization of a pre-action of* Γ is constructed.

So the element $\gamma \coloneqq g_{2k}\kappa_2\kappa_1\kappa_2^{-1}g_{2k}^{-1}$ satisfies

$$x_i\beta(\gamma) = x_{n+i}$$

for all $i \in \{1, \ldots, n\}$, thus finishing the proof. $\square$

Remark 10.27. Let us now briefly point out the adaptations one needs to make for the case of amalgamated products. Let Σ be a common subgroup of two countable groups Γ_1 and Γ_2, consider the amalgamated product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. First, the definition of the edges of the Bass-Serre graph of a Γ-action changes as follows: the set of vertices remains the set of Γ_1 and Γ_2 orbits, but we put one edge between such orbits for every Σ-orbit contained in both. In this way, the Bass-Serre graph of a free transitive Γ-action still coincides with the Bass-Serre tree of Γ. But the main difference is that the genericity result holds in a smaller setup: we need to restrict to actions where Σ acts freely so as to be able to make the modifications needed for high transitivity (Equation (7)): we cannot connect a Γ_1-orbit where Σ acts freely to a Γ_2-orbit where it does not. In [FLMMS22, Sec. 5 and 6], we actually work with transitive actions where both Γ_1 and Γ_2 act freely with infinitely many orbits, but this may not be necessary. It would be interesting to understand for which amalgamated products Theorem 10.25 holds. In an unpublished result with Carderi and Gaboriau, we show that the generic transitive action of a surface group of genus $\geqslant 2$ is highly transitive, using full groups. This approach will be presented for free groups instead of surface groups in the next chapter.

10.5 Consequences on the space of subgroups of free products

We now derive consequences of the previous results for the space of transitive actions of free products, or equivalently for their space of infinite index subgroups. We will use the following recent result of Azuelos and Gaboriau in a restricted form which is sufficient for our purposes[11].

Theorem 10.28 ([AG23, Thm. 5.12])**.** *Let Γ be a countable group endowed with a faithful minimal action of general type on a tree $\mathcal{T}$. Suppose that there are two edges $e_1, e_2 \in E(\mathcal{T})$ such that $\mathrm{Stab}_\Gamma(e_1) \cap \mathrm{Stab}_\Gamma(e_2)$ is finite. Consider the space $\mathrm{Sub}_{|\cdot\backslash\mathcal{T}|\infty}$ of all subgroups $\Lambda \leqslant \Gamma$ such that the edge set of $\Lambda\backslash\mathcal{T}$ is infinite. Then the action of Γ by conjugacy on $\mathrm{Sub}_{|\cdot\backslash\mathcal{T}|\infty}$ is topologically transitive.*

Corollary 10.29. *Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product with $|\Gamma_1| \geqslant 3$ and $|\Gamma_2| \geqslant 2$. Then its space of infinite index subgroups $\mathrm{Sub}_{[\infty]}(\Gamma)$ the action of Γ by conjugacy on $\mathrm{Sub}_{[\infty]}(\Gamma)$ has a dense orbit: Γ admits a totipotent action.*

Proof. Consider the Γ-action on its Bass-Serre tree as described in Example 10.19. Since Γ is acting freely on edges, the previous theorem applies: the space $\mathrm{Sub}_{|\cdot\backslash\mathcal{T}|\infty}$ of all subgroups $\Lambda \leqslant \Gamma$ such that the edge set of $\Lambda\backslash\mathcal{T}$ is infinite has no isolated points and the action by conjugacy on it is topologically transitive. However, since the Γ-action on edges is free and consists of one orbit up to flipping edges, we have that

$$\mathrm{Sub}_{|\cdot\backslash\mathcal{T}|\infty} = \mathrm{Sub}_{[\infty]}(\Gamma).$$

The result follows. $\square$

[11] Their result are stronger in two directions: they allow for a finite kernel and actually describe when the action is *highly topologically transitive* (see Remark 12.6 for more on this last notion).

Theorem 10.30. *Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product with $|\Gamma_1| \geqslant 3$ and $|\Gamma_2| \geqslant 2$. Then the set of all infinite index subgroups Λ of Γ such that the action $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ is both totipotent and highly transitive is dense G_δ in the space* $\mathrm{Sub}_{[\infty]}(\Gamma)$ *of infinite index subgroups of* Γ.

Proof. The fact that totipotent subgroups form a G_δ set which is dense as soon as it is not empty was already noticed right after Proposition 9.41. So the previous corollary yields that the space of totipotent subgroups is dense G_δ in $\mathrm{Sub}_{[\infty]}(\Gamma)$. Theorem 10.25 can then be combined with Theorem 9.9 to obtain that the conjugacy invariant space of infinite index subgroups $\Lambda \leqslant \Gamma$ such that $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ is highly transitive forms a dense G_δ set in $\mathrm{Sub}_{[\infty]}(\Gamma)$. Being the intersection of two dense G_δ sets in the Polish space $\mathrm{Sub}_{[\infty]}(\Gamma)$, the desired set of infinite index subgroups Λ of Γ such that the action $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ is both totipotent and highly transitive is dense G_δ in $\mathrm{Sub}_{[\infty]}(\Gamma)$ by the Baire category space. $\square$

We end this section by mentioning a consequence on amenable actions of free products. First recall that a (non necessarily transitive) action $\Gamma \curvearrowright X$ is **amenable** if for every $\epsilon > 0$ and $S \Subset \Gamma$, one can find an (S, ϵ)-invariant finite set, namely $F \Subset X$ such that for all $\gamma \in F$,

$$\frac{|F \triangle \gamma F|}{|F|} < \epsilon.$$

A subgroup $\Lambda \leqslant \Gamma$ is **coamenable** if the transitive action $\Gamma \curvearrowright \Gamma/\Lambda$ is amenable. By definition a group Γ is amenable if its action by left translation on itself is amenable, which can easily be shown to be equivalent to the fact that every *free* Γ-action is amenable.

As first noticed by van Douwen for free groups [van90], some non amenable groups admit nevertheless faithful amenable actions[12]. Following Glasner and Monod, we denote by $\mathcal{A}$ the class of countable groups admitting a faithful amenable action [GM07].

Glasner and Monod have obtained a characterization of free products belonging to the class $\mathcal{A}$, i.e. admitting a faithful amenable action [GM07].

Since any action which weakly contains an amenable action has to be amenable[13], we conclude from the previous theorem that if a free product admits an amenable transitive action on an infinite set, then the space of infinite index subgroups Λ such that $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ is totipotent, highly transitive and amenable is comeager.

We also note that since totipotent actions are automatically (highly) faithful, a free product $\Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geqslant 3$ and $|\Gamma_2| \geqslant 2$ is in $\mathcal{A}$ if and only if it admits an amenable transitive action on an infinite set. A version of this argument was very recently used by Azuelos and Gaboriau, allowing them to obtain many new examples of countable groups in $\mathcal{A}$, see [AG23, Sec. 8].

Example 10.31. As per Example 9.38 (or as a very special case of Corollary 10.29), the free group on two generators $\mathbb{F}_2 = \langle a, b\rangle$ admits a totipotent action. Moreover, any surjective group homomorphism $\mathbb{F}_2 \to \mathbb{Z}$ (e.g. the one obtained by sending a to 1 and b to 0) yields an action of $\mathbb{F}_2$ on $\mathbb{Z}$ by translation which is amenable since $\mathbb{Z}$ is. So any totipotent $\mathbb{F}_2$ action is amenable, and since such actions are moreover highly faithful, we conclude that $\mathbb{F}_2$ belongs to the class $\mathcal{A}$. The same argument shows that for every $n \geqslant 2$, the free group $\mathbb{F}_n$ is in $\mathcal{A}$.

[12] Van Douwen's action is actually much more than faithful: every nontrivial group element fixes only finitely many points.

[13] More generally, one can show that if an action α weakly contains a family of actions $\Gamma \curvearrowright X_i$ such that the disjoint union of these actions $\Gamma \curvearrowright \bigsqcup_i X_i$ is amenable, then α is amenable.

III Non-free actions in the measurable context

11 Consequences of density in full groups

We now begin our study of countable dense subgroups of full groups (see Section 7 for the definition of the latter). The results we present here were obtained jointly with Carderi and Gaboriau in an unpublished work, and generalize earlier results of Eisenmann-Glasner [EG16] and the author [LM18a]. They will be applied in the next section to free groups. For simplicity we will stick to the type II_1 case, and we may as well assume that our type II_1 equivalence relation $\mathcal{R}$ on the standard probability space (X,μ) is p.m.p. (probability measure preserving), meaning that the probability measure μ is preserved by the elements of the full group of $\mathcal{R}$, or equivalently that $\mathcal{R}$ comes from an action of a countable group which preserves the measure μ.

This allows us to make use of the following well-known construction: given $A \subseteq X$ and $T \in \mathrm{Aut}(X,\mu)$, the Poincaré recurrence theorem implies that for almost all $x \in A$ there is $n \geqslant 1$ such that $T^n(x) \in A$. Defining for $x \in X$

$$\tau_{T,A}(x) = \begin{cases} \min\{n \geqslant 1 \colon T^n(x) \in A\} & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases}$$

the transformation **induced** by T on A, denoted by T_A, is given by $T_A(x) = T^{\tau_{T,A}(x)}(x)$. It is a measure-preserving transformation, and if $T \in [\mathcal{R}]$ then $T_A \in [\mathcal{R}]$ as well.

Our first result shows that density in the full group is inherited by point stabilizers, as opposed to what happens for the infinite symmetric group (although point stabilizers of a countable dense subgroup of $\mathfrak{S}_\infty$ are themselves highly transitive). This can be seen as a manifestation of the fact that the measure μ does not see singletons.

Note that we make a slight abuse of notation in the statement: when we write $\Gamma \leqslant [\mathcal{R}]$, we are actually fixing some Borel Γ-action on X which induces the inclusion of Γ in the full group of $[\mathcal{R}]$ (the whole full group $[\mathcal{R}]$ does not act on X since its elements are only defined up to measure zero).

Theorem 11.1. *Let $\mathcal{R}$ be a p.m.p. equivalence relation, let $\Gamma \leqslant [\mathcal{R}]$ be a countable dense subgroup. Then there is a full measure Borel subset $X_0 \subseteq X$ such that for all $x_1, ..., x_n \in X_0$, the subgroup $\mathrm{Stab}_\Gamma(x_1) \cap \cdots \cap \mathrm{Stab}_\Gamma(x_n)$ is still dense in $[\mathcal{R}]$.*

The proof of this theorem relies on the following key lemma.

Lemma 11.2. *Let $\mathcal{R}$ be a p.m.p. equivalence relation, suppose $\Gamma \leqslant [\mathcal{R}]$ is a countable dense subgroup and let $T \in [\mathcal{R}]$. Fix $A \subseteq X$ Borel, then there is a full measure Borel subset $X_A \subseteq X$ such that whenever $x_1, \dots, x_k \in A \cap X_A$, there is $\gamma \in \Gamma$ such that $\gamma x_i = x_i$ for all $i \in \{1, \dots, k\}$ and*

$$d_u(\gamma, T) < 3\mu(A).$$

Proof. Observe that the induced transformation $T_{X\setminus A}$ satisfies $T_{X\setminus A}(x) = T(x)$ as soon as $T(x) \notin A$ and $x \notin A$. Since $\mu(T^{-1}(A)) = \mu(A)$, it follows that $d_u(T, T_{X\setminus A}) \leqslant 2\mu(A)$. By density, for every $n \in \mathbb{N}$ we can pick $\gamma_n \in \Gamma$ such that $d_u(\gamma_n, T_{X\setminus A}) < 2^{-n}$. By definition of the uniform distance and the Borel-Cantelli lemma, we have a full measure Borel subset $X_A \subseteq X$ such that for all $x \in X_A$, there is $N \in \mathbb{N}$ such that $\gamma_n(x) = T_{X\setminus A}(x)$ for all $n \geqslant N$. So if we now pick $x_1, \dots, x_k \in A \cap X_A$, for large enough $n \in \mathbb{N}$ we have

$\gamma_n(x_i) = T_{X\setminus A}(x_i) = x_i$, and if we furthermore took n so large that $2^{-n} < \mu(A)$, we have by the triangle inequality

$$d_u(\gamma_n, T) \leqslant d_u(\gamma_n, T_{X\setminus A}) + d_u(T_{X\setminus A}, T) \leqslant 2^{-n} + 2\mu(A) < 3\mu(A),$$

so we can take $\gamma \coloneqq \gamma_n$ and the proof is finished. □

Proof of Theorem 11.1. Let $\mathcal{A}$ be a countable algebra of Borel subsets of X such that for every $n \in \mathbb{N}$ there is a partition of X into n elements of $\mathcal{A}$ of measure $\frac{1}{n}$.

The above lemma grants us for every $A \in \mathcal{A}$ a full measure Borel subset $X_A \subseteq X$ such that for every $T \in [\mathcal{R}]$, every $x_1, \dots, x_k \in A \cap X_A$, there is $\gamma \in \Gamma$ such that $\gamma x_i = x_i$ for all $i \in \{1, \dots, k\}$ and

$$d_u(\gamma, T) < 3\mu(A).$$

If we let $X_0 \coloneqq \bigcap_{A\in\mathcal{A}} X_A$, we thus have that for every $A \in \mathcal{A}$, every $T \in [\mathcal{R}]$, every $x_1, \dots, x_k \in A \cap X_0$, there is $\gamma \in \Gamma$ such that $\gamma x_i = x_i$ for all $i \in \{1, \dots, k\}$ and

$$d_u(\gamma, T) < 3\mu(A).$$

Let us now show the desired density: let $x_1, \dots, x_k \in X_0$ and $\epsilon > 0$. Let $n \geqslant 1$ such that $\frac{3k}{n} < \epsilon$, then by our assumption on $\mathcal{A}$ we have a partition $(A_1, \dots, A_n) \in \mathcal{A}^n$ of X into sets of measure $\frac{1}{n}$. Taking the union of those A_j which contain some x_i, we obtain a set $A \in \mathcal{A}$ of measure at most $\frac{k}{n}$ containing all the x_i's. By construction there is $\gamma \in \Gamma$ such that $\gamma x_i = x_i$ for all $i \in \{1, \dots, k\}$ and

$$d_u(\gamma, T) < 3\mu(A) < \epsilon,$$

which concludes the proof that the subgroup $\mathrm{Stab}_\Gamma(x_1) \cap \dots \cap \mathrm{Stab}_\Gamma(x_n)$ is dense in the full group $[\mathcal{R}]$. □

Now that we know that density is inherited by finite intersections of stabilizer subgroups, we move to consequences of density for the action of the group on a full measure subset of the *set* X, and then on the ergodic properties of the action of the group on (X, μ).

11.1 Density, permutationally full actions and the fullness degree

This section is devoted to a new property for actions on sets which we call permutational fullness, and which as we will see puts under the same umbrella *high transitivity on every orbit* and *total non freeness*. Given any set X, let us denote by $\mathrm{Sym}(X)$ its group of bijections, so that by definition $\mathfrak{S}_\infty = \mathrm{Sym}(\mathbb{N})$.

Definition 11.3. A Γ-action on a set X is called **permutationally full** if given a finite subset $F \Subset X$ and $\sigma \in \mathrm{Sym}(F)$ such that for all $x \in F$, $\sigma(x)$ is in the ρ-orbit of x, there is $\gamma \in \Gamma$ such that for every $x \in F$,

$$\rho(\gamma)x = \sigma(x).$$

Note that a transitive action over an infinite set is highly transitive if and only if it is permutationally full. Just as high transitivity can be restated as density in the permutation group $\mathfrak{S}_\infty$ (see Remark 9.16), permutational fullness can be restated as density in the the *permutational full group*.

Definition 11.4. The **permutational full group** of a Γ-action ρ on a set X is the group of permutations $\sigma \in \mathrm{Sym}(X)$ such that $\sigma(x) \in \rho(\Gamma)x$ for all $x \in X$. We denote it by $[\rho]$.

Not that $[\rho]$ is a closed subgroup of $\mathrm{Sym}(X)$, equipped with the topology of pointwise convergence where we view X as a discrete set.

Proposition 11.5. *Let ρ be a Γ-action on a set X with $|X| \geqslant 3$. The following are equivalent:*

(i) The action ρ is permutationally full;

(ii) The restriction of ρ to any finite union of orbits is permutationally full.

(iii) Given finitely many distinct ρ-orbits $O_1, ..., O_k$ and finite subsets $F_1 \Subset O_1, \dots, F_k \Subset O_k$, for every permutations $\sigma_1 \in \mathrm{Sym}(F_1), ..., \sigma_k \in \mathrm{Sym}(F_k)$, there is $\gamma \in \Gamma$ such that for every $i \in \{1, ..., k\}$ and every $x \in F_i$, we have $\rho(\gamma)x = \sigma_i(x)$.

(iv) The closure of $\rho(\Gamma)$ in $\mathrm{Sym}(X)$ is equal to the permutational full group of ρ.

Proof. The equivalence between (i) and (ii) is a direct consequence of the definition of permutational fullness. Also, (iii) is a reformulation of (i) by cutting F according to the partition of X into ρ-orbits, so the two conditions (iii) and (i) are also equivalent.

Since every permutation σ as in the definition of permutational fullness extends to an element of $[\rho]$ by letting it act trivially outside of F we have that (iv) implies (i). The converse follows from the fact that if we are given some $\tau \in [\rho]$ and $F \Subset X$, the restriction $\rho_{\restriction F}$ can be extended to a permutation σ of a larger finite set $F' \supseteq F$ such that for all $x \in F'$, $\sigma(x)$ is in the ρ-orbit of x. □

Straightforward examples of a permutationally full action are provided by highly transitive actions. The point of the above definition is that it provides a stronger notion because if an action is permutationally full, its restriction to any infinite orbit is highly transitive. Another nice consequence of permutational fullness is total non freeness.

Definition 11.6 (Vershik, [Ver12, Def.-Th. 1, sect. 2.3])**.** Let Γ be a countable group. A Γ-action on a set X is called ***totally non free*** when the stabilizer map $x \mapsto \mathrm{Stab}_\Gamma(x)$ is injective.

Proposition 11.7. *Every permutationally full action all whose orbits have cardinality at least* 3 *is totally non free.*

Proof. Suppose $\Gamma \curvearrowright X$ is permutationally full. Let $x \neq y$. Since all orbits have cardinality at least 3, we find z in the same ρ-orbit as y, distinct from both x and y. Consider the permutation $\sigma \in \mathrm{Sym}(\{x, y, z\})$ defined by $\sigma(x) = x$, $\sigma(y) = z$ and $\sigma(z) = y$. Then any $\gamma \in \Gamma$ such that $\rho(\gamma)$ extends σ witnesses that $\mathrm{Stab}_\Gamma(x) \neq \mathrm{Stab}_\Gamma(y)$. □

Remark 11.8. In particular, when all the orbits have cardinality at least 3, a permutationally full action has all its restrictions two distinct orbits non isomorphic (otherwise they would have some common stabilizer). The above argument shows that even when there are orbits of cardinality 2, the Γ-actions restricted to two distinct orbits of cardinality 2 cannot be isomorphic.

In view of the fact that every permutationally full action with infinite orbits is highly transitive onto every orbit, the following result generalizes the result of Eisenmann-Glasner that dense subgroups act highly transitively on orbits [EG16, Prop. 1.19]. On the other hand, Proposition 11.7 yields that it also generalizes the fact that after removing a null set, dense subgroups of full groups act totally non freely on the underlying space, as was first proved by the author in [LM18a, Prop. 1.10].

Theorem 11.9. *Let Γ be a dense subgroup of $[\mathcal{R}]$, where $\mathcal{R}$ is a p.m.p. equivalence relation. Then there is a conull $\mathcal{R}$-invariant Borel subset $X_0 \subseteq X$ such that the restriction of the Γ-action to X_0 is permutationally full and its orbits are equal to the $\mathcal{R}$-classes.*

Proof. We start by fixing a countable Γ-invariant subalgebra $\mathcal{A}$ of the Borel subsets of X which separates points. Let G be a countable group generating the equivalence relation $\mathcal{R}$ (for instance, one could take $G = \Gamma$, but we use a different name for clarity since G is only used to "name" finite subsets of orbits).

For every n-tuple of points $(x_1, \cdots, x_n)$ with pairwise distinct orbits and every n-tuple of finite subsets $F_1 \Subset Gx_1, \dots, F_n \Subset Gx_n$, there exists an n-tuple $(K_1, \cdots, K_n)$ of finite subsets of G such that the maps

$$\begin{array}{rccc} \varphi_i: & K_i & \to & F_i \\ & k & \mapsto & kx_i \end{array}$$

are bijections. For $i \in \{1, \dots, n\}$, let $\sigma_i \in \mathrm{Sym}(F_i)$. Let $\sigma'_i = \varphi_i^{-1}\sigma_i\varphi_i \in \mathrm{Sym}(K_i)$. We are after an element γ of Γ such that

$$\gamma k x_i = \sigma_i(kx_i) = \sigma'_i(k)x_i$$

for all $i \in \{1, \cdots, n\}$ and $k \in K_i$.

We claim there is an n-tuple $(A_1, \cdots, A_n)$ of elements of the algebra $\mathcal{A}$ such that $x_i \in A_i$ and the sets $(kA_i)_{i\in\{1,\dots,n\},k\in K_i}$ are pairwise disjoint. Indeed, since the points $(kx_i)_{i\in\{1,\dots,n\},k\in K_i}$ are pairwise distinct we can find a disjoint family $(A_{i,k})_{i\in\{1,\dots,n\},k\in F_i}$ of elements of the separating algebra $\mathcal{A}$ such that $kx_i \in A_{i,k}$. Then $A_i := \bigcap_{k\in K_i} f^{-1}A_{i,k}$ is as wanted.

Consider now any n-tuple $(K_i, \sigma'_i, A_i)_{i=1,\cdots,n}$ of finite subsets $K_i \Subset G$, $\sigma'_i \in \mathrm{Sym}(K_i)$ and $A_i \in \mathcal{A}$ such that the sets $(kA_i)_{i\in\{1,\dots,n\},k\in K_i}$ are pairwise disjoint. We introduce the element $T = T_{(K_i,\sigma'_i,A_i)_{i=1,\cdots,n}} \in [\mathcal{R}]$ defined as follows:

$$T(x) = \begin{cases} \sigma'_i(k)k^{-1}x & \text{if } x \in kA_i \text{ for some } i \in \{1, ..., n\} \text{ and } k \in K_i \\ x & \text{otherwise.} \end{cases}$$

It satisfies $T(k(x_i)) = \sigma'_i(k)x_i$ for every $x_i \in A_i, k \in F_i$. By density of Γ in the full group $[\mathcal{R}]$, there is a sequence $(\gamma_m)_m$ of elements of Γ such that $d_u(\gamma_m, T) \to 0$, and we may as well assume $d_u(\gamma_m, T) < 2^{-m}$. By the Borel-Cantelli lemma, for almost all $x \in X$, we have $\gamma_m(x) = T(x)$ for all but finitely many $m \in \mathbb{N}$. This provides us a conull set $Z_{(K_i,\sigma'_i,A_i)_{i=1,\cdots,n}}$ restricted to which, if we pick finitely many elements $x_i \in A_i$, for m large enough, we do have $\gamma_m k x_i = \sigma_i(kx_i)$ as wanted. Taking the intersection of the countably many conull sets $Z_{(K_i,\sigma'_i,A_i)_{i=1,\cdots,n}}$ and further intersecting with its G-translates so as to make it $\mathcal{R}$-invariant, we arrive at the desired set X_0. □

Question 6. Does the converse hold: if $\Gamma \curvearrowright (X, \mu)$ is a p.m.p. action which is permutationally full as an action on the set X, must Γ be dense in the full group of $\mathcal{R}_\Gamma$?

An easy example of a permutationally full action of a countable group on a set with infinitely many orbits is given by taking a highly transitive action of a group Γ on a set X and then making $\bigoplus_{\mathbb{N}} \Gamma$ act on $X \times \mathbb{N}$. But note that the action is not faithful when restricted to one of its orbits.

Example 11.10. A nice example of a permutationally full action which is faithful on every orbit is provided by topological dynamics: given a minimal action of a countable group on the Cantor space by homeomorphisms, by using arguments similar to those of Theorem 11.9 it can be shown that the action of its alternating topological full group on the Cantor space is permutationally full, and it is faithful on every orbit by minimality. It is however not highly faithful since there are (many!) nontrivial group elements with disjoint supports.

Since we are only considering countable infinite groups, every action which is faithful on every orbit has only infinite orbits.

Definition 11.11. Let Γ be a countable infinite group. Let $\mathcal{PF}$ denote the class of all permutationally full actions (ρ, X) of Γ on arbitrary sets X which are faithful onto every orbit. The ***fullness degree*** is the maximal cardinality of the set of orbits $\rho(\Gamma)\backslash X$ where $(\rho, X) \in \mathcal{PF}$.

Remark 11.12. By Proposition (11.7), if $(\rho, X) \in \mathcal{PF}$ then the stabilizer map embeds X in $\mathrm{Sub}(\Gamma)$, which has cardinality at most $2^{\aleph_0}$. It follows that the fullness degree of a countable group is well-defined and at most $2^{\aleph_0}$.

Theorem 11.9 has the following remarkable consequence.

Corollary 11.13. *Suppose Γ is a dense subgroup of the full group of an ergodic p.m.p. equivalence relation on (X, μ), then the fullness degree of Γ is equal to $2^{\aleph_0}$.*

Proof. Note that by ergodicity and faithfulness of the Γ-action on (X, μ), the Γ-action on almost every orbit is faithful: for each $\gamma \in \Gamma \setminus \{1\}$, the support of γ has positive measure, and hence must intersect almost every orbit. So we may as well assume the Γ-action is faithful on each Γ-orbit, and then the result follows from the previous theorem. $\square$

Remark 11.14. Any partially finitary group (e.g. the group of finitely supported permutations) has only one highly transitive action up to conjugacy, namely the action on $\mathbb{N}$ which makes it partially finitary (see Theorem 9.17). So the fullness degree of partially finitary groups is 1 in view of Remark 11.8. In particular such groups cannot be dense subgroups of full groups by the above corollary.

Observe that a group is highly transitive if and only if its fullness degree is at least 1, so non highly transitive groups have fullness degree 0. The above remark provides an example of a of groups with fullness degree 1, a dense countable subgroups of full groups have fullness degree $2^{\aleph_0}$. The following question is particularly appealing to us.

Question 7. Is there a countable group whose fullness degree does not belong to $\{0, 1, 2^{\aleph_0}\}$?

11.2 Ergodic theoretic consequences of density

In this section we list the ergodic theoretic features of actions yielding dense subgroups in full groups of ergodic p.m.p. equivalence relations. They all rely on the fact that full groups of ergodic p.m.p. equivalence relations are dense in $\mathrm{Aut}(X,\mu)$ for the weak topology.

The key fact that we use is the irreducibility of the Koopman unitary representation of $\kappa : \mathrm{Aut}(X,\mu) \to \mathcal{U}(\mathrm{L}^2_0(X,\mu))$, where

$$\mathrm{L}^2_0(X,\mu) = \left\{ f \in \mathrm{L}^2(X,\mu) \colon \int_X f = 0 \right\} = (\mathbb{C}1_X)^\perp$$

and $\kappa(T)f(x) = f(T^{-1}x)$. Let us give an easy proof of the irreducibility, which is due to Glasner-Furstenberg-Weiss [Gla03, Thm. 5.14] and Bekka-de la Harpe (unpublished).

Theorem 11.15. *The Koopman representation* $\kappa : \mathrm{Aut}(X,\mu) \to \mathcal{U}(\mathrm{L}^2_0(X,\mu))$ *is irreducible.*

Proof. Let $f \in \mathrm{L}^2_0(X,\mu)$ be a non zero function, and let $\mathcal{K}$ be the the Hilbert space spanned by $\mathrm{Aut}(X,\mu)\cdot f \oplus \mathbb{C}1_X$. We have to show that $\mathcal{K} = \mathrm{L}^2(X,\mu)$. Because $\int_X f = 0$, we have $\mu(\{x \in X : f(x) > 0\}) > 0$, and we let $\alpha = \mu(\{x \in X : f(x) > 0\})$. By density of step functions in $\mathcal{H}$, it suffices to show that $\mathcal{K}$ contains all the characteristic functions of Borel subsets of X. Because $\mathrm{Aut}(X,\mu)$ acts transitively on sets of equal measure, and because the sum of the characteristic functions of two disjoint sets is the characteristic function of their union, it suffices to show that for every $\epsilon \in (0,\alpha)$, there exists $A \subseteq X$ such that $\mu(A) = \epsilon$ and $\mathcal{K}$ contains the characteristic function of A.

To this end, we fix $\epsilon \in (0,\alpha)$ and $A \subseteq \{x \in X : f(x) > 0\}$ of measure ϵ. Let $T \in \mathrm{Aut}(X,\mu)$ whose ergodic components are A and $X\setminus A$. We then apply von Neumann's mean ergodic theorem to T and f, which yields that the function

$$\tilde{f} = \frac{\int_A f}{\mu(A)}\chi_A + \frac{\int_{X\setminus A} f}{1-\mu(A)}\chi_{X\setminus A}$$

arises as a limit of Cesaro averages of f, and thus belongs to $\mathcal{K}$. By subtracting $\frac{\int_{X\setminus A} f}{1-\mu(A)}\cdot 1_X$ to it and renormalizing, we find that χ_A belongs to $\mathcal{K}$, which ends the proof (note that $\int_{X\setminus A} f < 0$, which guarantees that $\tilde{f}$ takes two distinct values). $\square$

Since the Koopman representation is continuous (when endowing $\mathrm{Aut}(X,\mu)$ with the weak topology and $\mathcal{U}(\mathrm{L}^2_0(X,\mu))$ with the strong topology), we get the following corollary[14].

Corollary 11.16. *Let* $\rho : \Gamma \to \mathrm{Aut}(X,\mu)$ *be a p.m.p. action with dense image in* $\mathrm{Aut}(X,\mu)$*. Then the Koopman representation of* ρ *on* $\mathrm{L}^2_0(X,\mu)$ *is irreducible, and as a consequence* ρ *has no nontrivial factor.*

Proof. If the Koopman representation of ρ were reducible, let $\mathcal{K} \subseteq \mathrm{L}^2_0(X,\mu)$ be a proper $\kappa(\Gamma)$-invariant nontrivial subspace, then by density and continuity $\mathcal{K}$ is also $\kappa(\mathrm{Aut}(X,\mu))$ invariant, thus contradicting the above theorem.

Now observe that if $\pi : (X,\mu) \to (Y,\nu)$ is a factor map of ρ onto some other p.m.p. action ρ', then $\mathrm{L}^2_0(Y,\nu)$ embeds into $\mathrm{L}^2_0(X,\mu)$ via $f \mapsto f\circ\pi$ as an invariant subspace. So either (Y,ν) is a point or π induces an isomorphism $\mathrm{L}^2_0(Y,\nu) \to \mathrm{L}^2_0(X,\mu)$, which means that π is an isomorphism of actions. $\square$

[14] We are grateful to Todor Tsankov for pointing out the absence of nontrivial factors, and its relationship with weak mixing (see Remark 11.18).

Say that a p.m.p. action $\Gamma \curvearrowright (X,\mu)$ is **weakly mixing** if for all $\epsilon > 0$, whenever $(A_1,\dots,A_n)$ and $(B_1,\dots,B_n)$ are finite partitions of X, there is $\gamma \in \Gamma$ such that for all $i,j \in \{1,\dots,n\}$,

$$|\mu(A_i \cap \gamma B_j) - \mu(A_i)\mu(B_j)| < \epsilon.$$

Proposition 11.17. *Let $\rho : \Gamma \to \mathrm{Aut}(X,\mu)$ be a p.m.p. action with dense image in $\mathrm{Aut}(X,\mu)$. Then ρ is weakly mixing.*

Proof. Let $(A_1,...,A_n)$ and $(B_1,...,B_n)$ be partitions of X, let $\epsilon > 0$. For each $i \in \{1,...,n\}$, we find a partition $(C_{i,j})_{j=1}^n$ of A_i such that for every $j \in \{1,...,n\}$, we have $\mu(C_{i,j}) = \mu(A_i)\mu(B_j)$. For $j \in \{1,...,n\}$, we then define $C_j = \bigsqcup_{i=1}^n C_{i,j}$. Observe that $\mu(C_j) = \mu(B_j)$ and for every $i \in \{1,...,n\}$ we have $\mu(A_i \cap C_j) = \mu(A_i)\mu(B_j)$.

Let us now fix $T \in \mathrm{Aut}(X,\mu)$ such that $T(B_j) = C_j$ for every $j \in \{1,...,n\}$. Observe that for every $i,j \in \{1,...,n\}$, such a T satisfies

$$\mu(A_i \cap T(B_j)) = \mu(A_i)\mu(B_j).$$

So any $\gamma \in \Gamma$ sufficiently close to T in the weak topology will satisfy

$$|\mu(A_i \cap \rho(\gamma)B_j) - \mu(A_i)\mu(B_j)| < \epsilon$$

as wanted. □

Remark 11.18. For details on the definitions and facts which follow, see [KL16, Sec. 2.2]. A p.m.p. action is **compact** when its image in $\mathrm{Aut}(X,\mu)$ has compact closure. Every p.m.p. action has a maximal compact factor, and weakly mixing actions are exactly those whose maximal compact factor is trivial. In view of the absence of nontrivial factors for p.m.p. actions with dense image, the above proposition can also be obtained as a consequence of the fact that $\mathrm{Aut}(X,\mu)$ is not compact.

The following proposition will enable us to apply the above results to dense subgroups of full groups.

Proposition 11.19 ([Kec10, Prop. 3.1])**.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, then its full group $[\mathcal{R}]$ is dense in $\mathrm{Aut}(X,\mu)$ for the weak topology.*

Proof. By definition of the weak topology, a basic neighborhood $\mathcal{O}$ of $T \in \mathrm{Aut}(X,\mu)$ is given by some $\epsilon > 0$ and $(A_1,\dots,A_n)$ partition of X as the set

$$\mathcal{O} = \{U \in \mathrm{Aut}(X,\mu) \colon \forall i \in \{1,\dots,n\}, \mu(U(A_i) \bigtriangleup T(A_i)) < \epsilon\}.$$

By Lemma 7.15, for every $i \in \{1,...,n\}$ there is some $\varphi_i \in [[\mathcal{R}]]$ with domain A_i and range $T(A_i)$. The p.m.p. transformation $U \coloneqq \bigsqcup_{i=1}^n \varphi_i$ then belongs to $[\mathcal{R}]$ and to our fixed neighborhood $\mathcal{O}$ of T since it actually takes exactly each A_i to $T(A_i)$. □

Putting these results together, we arrive at the following conclusion.

Theorem 11.20. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation on (X,μ), let $\Gamma \leqslant [\mathcal{R}]$ be dense. Then the Γ-action on (X,μ) is weakly mixing and has no nontrivial factor. Moreover, there is a full measure Borel subset $X_0 \subseteq X$ such that for any $x_1,...,x_n \in X_0$, the action of the subgroup*

$$\bigcap_{i=1}^n \mathrm{Stab}_\Gamma(x_i)$$

on (X,μ) is weakly mixing and has no nontrivial factor. In particular, the Γ action on (X,μ) is metrically k-transitive for all $k \geqslant 1$ in the sense of Vershik [Ver12, Def. 8].

Proof. Since the uniform topology refines the weak topology and the full group $[\mathcal{R}]$ is dense in $\mathrm{Aut}(X,\mu)$ by the previous proposition, Γ is dense in $\mathrm{Aut}(X,\mu)$. So by Proposition 11.17, the Γ-action is weakly mixing, and by Corollary 11.16 it has no nontrivial factor. Finally, Theorem 11.1 yields a full measure set $X_0 \subseteq X$ such that the finite intersections of stabilizers of elements of X_0 are dense as well, and hence satisfy the same conclusion as Γ itself. $\square$

Remark 11.21. Let us remark that weak mixing seems to be the strongest mixing property that dense subgroups of full groups have. For instance, they are not mildly mixing[15]: any injective sequence (γ_n) approximating the identity[16] satisfies $\lim_{n\to+\infty}\mu(\gamma_n A \triangle A) = 0$ for all $A \subseteq X$, thus contradicting mild mixing.

11.3 Density and invariant random subgroups

We now exhibit the strong connection between the measure-preserving action associated to a dense subgroup of a full group and invariant random subgroups.

Recall that an **invariant random subgroup** (or IRS) of a countable group Γ is a probability measure on $\mathrm{Sub}(\Gamma)$ which is preserved by the conjugacy action.

Given any p.m.p. action ρ of Γ on (X,μ), the stabilizer map $\mathrm{Stab}: X \to \mathrm{Sub}(\Gamma)$ is equivariant if we endow X with the action ρ and $\mathrm{Sub}(\Gamma)$ with the conjugacy action. It follows that the probability measure $\mathrm{Stab}_*\mu$ is an invariant random subgroup, which we call the **IRS associated to** the p.m.p. action ρ. Observe that any two isomorphic p.m.p. actions must have the same IRS.

Remark 11.22. Recalling from Section 9.1 that subgroups are the same thing as isomorphism classes rooted transitive actions, we could equivalently define an *invariant random rooted transitive action* as a Borel probability measure on the space of isomorphism classes of rooted transitive actions which is invariant under the natural rerooting action of Γ on this space: $\gamma \cdot [\alpha, x_0] = [\alpha, \alpha(\gamma)x_0]$. Given a Γ-action ρ on a set X, the stabilizer map becomes the map which takes $x \in X$ to the (isomorphism class of the) restriction of ρ to the orbit of x, rooted at x. The invariant random rooted transitive action associated to a p.m.p. Γ-action ρ on (X,μ) is thus the pushforward of μ by the map $x \mapsto [\rho_{\restriction \rho(\Gamma)x}, x]$.

The prototypical example of an IRS is that of the Dirac measure δ_N on a normal subgroup $N \lhd \Gamma$, and IRS are often presented as generalizations of normal subgroup. Also note that the Bernoulli shift $\Gamma/N \curvearrowright [0,1]^{\Gamma/N}$, viewed as a p.m.p. Γ-action, has associated IRS δ_N. A random version of this construction, due to Abért, Glasner and Virag, yields that every IRS actually comes from a p.m.p. action [AGV16, Prop. 12].

The analogy between normal subgroups and IRS's is mostly motivated by the fact that rigidity results on normal subgroups actually tend to generalize to IRS's (see [SZ94] for the first result in this direction, although the terminology IRS had not been coined at the time). However, our results here provide IRS's which are far from being normal in many ways.

A good source of interesting (and far from normal) IRS's is provided by totally non free p.m.p. actions. In this context, total non freeness means that the stabilizer map becomes injective when restricted to a full measure set, which is actually Vershik's original

[15]By definition a p.m.p. action is mildly mixing if $\liminf_{\gamma\to\infty}\mu(A \triangle \gamma A) > 0$ for all Borel $A \subseteq X$ such that $\mu(A) > 0$, see e.g. [Sch84].

[16]Such a sequence exists because there are no isolated points in $\mathrm{Aut}(X,\mu)$, which is actually a contractible space (see [Kec10, Thm. 2.8]).

definition from [Ver12]. In particular, any totally non free p.m.p. action on (X,μ) is isomorphic to the action by conjugacy on $\mathrm{Sub}(\Gamma)$ endowed with the corresponding IRS $\mathrm{Stab}_* \mu$.

As already noted by Vershik, any two totally non free p.m.p. actions are isomorphic iff they have the same IRS[17]. Moreover, the IRS $\mathrm{Stab}_* \mu$ of any totally non free action is **self-normalizing**: for $\mathrm{Stab}_* \mu$ almost every $\Lambda \leqslant \Gamma$, we have that the normalizer of Λ is equal to Λ (indeed the stabilizer subgroup is self-normalizing iff the restriction of the stabilizer map to the corresponding orbit is injective).

We can now restate the properties of dense subgroups of full groups in terms of the corresponding IRS's.

Theorem 11.23. *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation on (X,μ), let $\rho : \Gamma \to [\mathcal{R}]$ be a faithful p.m.p. action with dense image in $[\mathcal{R}]$. Denote by ν the corresponding IRS. Then there is a ν-full measure subset $S_0 \subseteq \mathrm{Sub}(\Gamma)$ such that for all $n \geqslant 0$, for all $\Lambda_1, ..., \Lambda_n \in S_0$, the intersection $\bigcap_{i=1}^n \Lambda_i$ (taken to be equal to Γ when $n = 0$) satisfies the following two properties*

- *it acts in a weakly mixing manner on $(\mathrm{Sub}(\Gamma), \nu)$, and this action is permutationally full up to a null set.*
- *for ν-almost all $\Lambda \leqslant \Gamma$, the action*

$$\bigcap_{i=1}^n \Lambda_i \curvearrowright \Gamma/\Lambda$$

is highly transitive and faithful.

Proof. By Theorem 11.9, there is a full measure $\rho(\Gamma)$-invariant subset $X_0 \subseteq X$ restricted to which ρ is permutationally full with orbits equal to the $\mathcal{R}$-classes. Being permutationally full, the action is in particular totally non free by Proposition 11.7. It follows that ρ is isomorphic to the conjugacy action of its IRS ν via the stabilizer map.

We then apply Theorem 11.1 and find a full measure subset $X_0 \subseteq X$ such that for all $x_1, \dots, x_n \in X_0$, the subgroup $\bigcap_{i=1}^n \mathrm{Stab}_\Gamma^\rho(x_i)$ is still dense in $[\mathcal{R}]$. Letting $S_0 = \mathrm{Stab}_\Gamma^\rho(X_0)$, we obtain the desired full measure subset of $\mathrm{Sub}(\Gamma)$.

Indeed, for all $\Lambda_1, ..., \Lambda_n \in S_0$, the subgroup $\bigcap_{i=1}^n \Lambda_i$ is dense in $[\mathcal{R}]$, so by Theorem 11.20 its action is weakly mixing, and by Theorem 11.9 it is permutationally full after restricting it to a suitable full measure set, and its orbits are equal to the $\mathcal{R}$-classes. In particular, for almost all x, the ρ-action of $\bigcap_{i=1}^n \Lambda_i$ on $[x]_\mathcal{R} = \rho(\Gamma)x$ is highly transitive, and faithful by ergodicity and faithfulness of ρ. Pushing this forward via the stabilizer map, we obtain the desired statement. □

Remark 11.24. Let ν be the IRS of a ρ-action as in the above theorem. The second item implies in particular that given any $\Lambda_1, ..., \Lambda_n$ in the full measure set S_0, their intersection $\bigcap_{i=1}^n \Lambda_i$ is infinite. Taking $\Lambda_1, \dots, \Lambda_n$ in the same conjugacy class, we deduce that ν-almost every $\Lambda \leqslant \Gamma$ is n-step s-normal in the sense of Bader, Furman and Sauer [BFS14].

[17] In particular, the conjugacy relation on the space of totally non-free p.m.p. actions is smooth in the sense of invariant descriptive set theory.

12 Cost and totipotent dense actions of free groups

We start by giving Levitt's original definition of cost, which is very useful in practice. First recall that given a p.m.p. equivalence relation $\mathcal{R}$ on (X, μ), its pseudo full group as defined in Definition 7.14 is denoted by $[[\mathcal{R}]]$. A **graphing** of $\mathcal{R}$ is a countable subset $\Phi \subseteq [[\mathcal{R}]]$, and its **cost** is

$$\mathrm{Cost}(\Phi) = \sum_{\varphi \in \Phi} \mu(\mathrm{dom}\, \varphi) = \sum_{\varphi \in \Phi} \mu(\mathrm{rng}\, \varphi),$$

the last equality being a consequence of the fact that $\mathcal{R}$ preserves μ. A graphing Φ is **generating** if up to a null set, $\mathcal{R}$ is the smallest equivalence relation whose pseudo full group contains Φ.

Definition 12.1 (Levitt [Lev95])**.** Given a p.m.p. equivalence relation $\mathcal{R}$, its **cost** is the infimum of the costs of its generating graphings.

Two remarks are in order. First, every p.m.p. equivalence relation comes from a p.m.p. action of a countable group Γ, say ρ, and then the graphing $\rho(\Gamma)$ witnesses that the infimum in the above definition is above a nonempty set. Second, when $\mathcal{R}$ comes from an action ρ of a *finitely generated* group Γ, then the cost of $\mathcal{R}$ is bounded above by the minimal number of generators of Γ, since any finite generating set of Γ becomes generating for $\mathcal{R}$ via the action map ρ.

A founding result of Gaboriau is that p.m.p. equivalence actions coming from *free* actions of $\mathbb{F}_n$ have cost n [Gab00]. Our work is motivated by the following easier consequence of the theory of cost.

Theorem 12.2 (Gaboriau)**.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, let $n \geqslant 2$. Then* $\mathrm{Cost}(\mathcal{R}) < n$ *if and only if there is a* non-free *action of the free group $\mathbb{F}_n$ inducing the equivalence relation $\mathcal{R}$.*

The main result of our PhD thesis can be seen as a strengthening of the above theorem: under the same assumptions, $\mathrm{Cost}(\mathcal{R}) < n$ iff there is an action ρ of the free group $\mathbb{F}_n$ such that $\rho(\mathbb{F}_n)$ is dense in $[\mathcal{R}]$ [LM14b, LM14a]. In [LM18a], we obtained a much stronger conclusion by achieving as well the following two properties:

1. amenability of the action on almost every orbit.
2. high faithfulness of the action on almost every orbit.

In particular we could make the action ρ faithful, thus getting the following reinforcement of the above theorem:

Theorem 12.3 ([LM18a])**.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, let $n \geqslant 2$. Then* $\mathrm{Cost}(\mathcal{R}) < n$ *if and only if there is a dense free group on n generators in $[\mathcal{R}]$ whose action on almost every orbit is amenable and highly faithful.*

Noting that dense subgroups act totally non-freely (which as we saw in the last section is actually a consequence of permutational fullness), this has the following nice consequence, reinforcing a result of Bowen which is the statement below without the moreover part (see [Bow15, Cor. 5.4]).

Corollary 12.4. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, let $n \geqslant 2$. Then* $\mathrm{Cost}(\mathcal{R}) < n$ *if and only if there is an IRS ν of $\mathbb{F}_n$ such that $\mathcal{R}$ is isomorphic to the equivalence relation relation generated by the* faithful *conjugacy action of $\mathbb{F}_n$ on* $(\mathrm{Sub}(\mathbb{F}_n), \nu)$. *Moreover, such an IRS can actually be taken to be supported on co-amenable unconfined subgroups of $\mathbb{F}_n$.*

We have seen that for finitely generated free groups of rank $n \geqslant 2$, high faithfulness and amenability (or in the space of subgroups, unconfinedness and co-amenability) are implied by totipotency. So the following result, obtained with Carderi and Gaboriau, implies the previous ones, yielding dense subgroups of full groups whose action onto almost every orbit can imitate any transitive $\mathbb{F}_n$-action over an infinite set.

Theorem 12.5 (Carderi, Gaboriau and the author [CGLM23])**.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, let $n \geqslant 2$. Then* $\mathrm{Cost}(\mathcal{R}) < n$ *if and only if there is a dense free group on n generators in $[\mathcal{R}]$ whose action on almost every orbit is totipotent.*

Here totipotency can be restated as the fact that the support of the associated IRS is as large as possible. In contrast, the construction the author had for Theorem 12.3 produced IRS's whose support was contained in the space of rooted actions where the first generator acts freely. We sketch very briefly the proof of Theorem 12.5 below, referring the reader to [CGLM23] for details:

- A key new idea compared to our previous approach is to first take $Y \subseteq X$ of sufficiently large measure so that the restriction of $\mathcal{R}$ is still of cost less than n.
- We then construct an $\mathbb{F}_n$-action ρ_Y with dense image in the full group of $[\mathcal{R}_{\restriction Y}]$, in a way which is sufficiently flexible so that we can perturb the last $n-1$ generators and retain density.
- Towards getting totipotency, as in Example 9.38 we enumerate all the possible balls of Schreier graphs of transitive $\mathbb{F}_n$-actions over infinite sets up to isomorphism as $(\mathcal{B}_k)_{k\in\mathbb{N}}$.
- We partition $X \setminus Y$ into countably many sets C_k, and find partial isomorphisms φ_1,...,φ_n such that on each C_k, the graph generated by φ_1,...,φ_n is almost surely finite and isomorphic to $\mathcal{B}_k$.
- Writing $\mathbb{F}_n = \langle a_1, \dots, a_n \rangle$, we finally patch together the φ_i's with a perturbed version of the $\rho_Y(a_i)$ so as to obtain an action ρ of $\mathbb{F}_n$ on X with the desired properties. The patching has to be done so that the orbits of ρ are equal to the $\mathcal{R}$-classes, but this is the only constraint towards proving density thanks to the flexibility of ρ_Y and a general lemma allowing to upgrade density "in the full group of the subset Y" to density in the whole full group, see [CGLM23, Cor. 3.4].

Remark 12.6. Since density in the full group implies weak mixing of the corresponding IRS ν (Theorem 11.23), and weak mixing implies topological transitivity of the diagonal action on $(\operatorname{supp} \nu)^k$ for every k, we conclude from the above theorem that the $\mathbb{F}_n$-action on its space of infinite index subgroups is topologically k-transitive for every k, i.e. highly topologically transitive in the terminology of Azuelos and Gaboriau [AG23] (this conclusion also follows directly from their work). Also note that for $k = 1$, since the subgroups we get are both almost surely totipotent and co-highly transitive (by the second item of Theorem 11.23), we also get another proof that the generic transitive $\mathbb{F}_n$-action over an infinite set is highly transitive, a special case of Theorem 10.25.

The fact that the IRS's supports are as large as possible in the above theorem was advertised a bit differently in our paper. We noted that any dense subgroup Γ of the full group of an ergodic p.m.p. equivalence relation has an atomless IRS associated to its action. This IRS thus has to be supported on the *perfect kernel* of $\mathrm{Sub}(\Gamma)$, namely the largest closed subset of $\mathrm{Sub}(\Gamma)$ without isolated points for the induced topology. We thus called totipotent an ergodic IRS with support equal the perfect kernel of $\mathrm{Sub}(\Gamma)$, noting that by ergodicity, having support equal to the perfect kernel is equivalent to having almost every conjugacy class dense in the perfect kernel (using the well-known Proposition 2.3 from [CGLM23]).

We propose here to change terminology and rather call subgroups $\Lambda \leqslant \Gamma$ whose orbit is dense in the perfect kernel of $\mathrm{Sub}(\Gamma)$ **pluripotent**[18].

One could then call a not necessarily ergodic IRS pluripotent when the corresponding random subgroup is almost surely pluripotent. Since the complement of the perfect kernel is countable, any subgroup outside the perfect kernel but in the support of some IRS must have finite conjugacy class, in particular its conjugacy class cannot be dense. So pluripotency is the strongest notion one can hope for. Nevertheless, when the perfect kernel contains the space of infinite index subgroups (which is the case for $\mathbb{F}_n$), the random subgroup is automatically almost surely totipotent, and thus we can still call the IRS totipotent. The following natural question was suggested by Andreas Thom. I don't know the answer even when Γ has a countable space of subgroups, and hence trivial perfect kernel.

Question 8. Is there a finitely generated group Γ whose space of subgroups has a perfect kernel properly contained in $\mathrm{Sub}_{[\infty]}(\Gamma)$, but which admits a totipotent subgroup?

Remark 12.7. It follows from our work with Carderi, Gaboriau and Stalder that the perfect kernel can be a proper nonempty subset of the space of infinite index subgroups for some Baumslag-Solitar groups, but the phenotype partition prevents the existence of pluripotent (in particular, totipotent) subgroups, see [CGLMS23].

On the other hand, fundamental groups of closed hyperbolic 3-manifold also have their perfect kernel properly contained in their space of infinite infinite subgroups, and they do admit pluripotent subgroups by the work of Azuelos and Gaboriau [AG23, Thm. 6.2]. However, as explained in their paper, the infinite index subgroups which are outside the perfect kernel are *virtual fibers*, so their normalizer has finite index. In other words, their conjugacy class is finite, in particular they cannot be totipotent. I am very grateful to Penelope Azuelos for pointing this out to me.

Finally, given a measure-preserving ergodic equivalence relation $\mathcal{R}$ of cost $< n$, the space $\mathrm{Hom}_{\mathrm{gen}}(\mathbb{F}_n, [\mathcal{R}])$ of all group homomorphisms $\rho : \mathbb{F}_n \to [\mathcal{R}]$ such that $\rho(\mathbb{F}_n)$ generates $\mathcal{R}$ is a Polish space which is a natural ergodic analogue of the space of transitive $\mathbb{F}_n$-actions over an infinite set. The first part of the following question was also asked in [CGLM23].

Question 9. Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, let Γ be a finitely generated non-abelian free group. Is the set of $\rho \in \mathrm{Hom}_{\mathrm{gen}}(\Gamma, [\mathcal{R}])$ such that $\rho(\Gamma)$ is dense in $[\mathcal{R}]$ a dense G_δ set? What about totipotency of ρ on almost every orbit?

[18]According to Wikipedia, pluripotency is the second strongest notion of potency for cells, right after totipotency. The reader who comes up with weaker notions for subgroups will be glad to learn that there are also (in decreasing order of potency) multipotency, oligopotency, and finally unipotency, which would have provided a good name for a normal subgroup if the notion of unipotent group did not already exist!

References

[AG23] Pénélope Azuelos and Damien Gaboriau. Perfect kernel and dynamics: From Bass-Serre theory to hyperbolic groups, 2023. arXiv:2308.05954, doi:10.48550/arXiv.2308.05954.

[AGV16] Miklós Abért, Yair Glasner, and Bálint Virág. The Measurable Kesten Theorem. *The Annals of Probability*, 44(3):1601–1646, 2016. arXiv:24735838.

[AM96] S. A. Adeleke and Dugald Macpherson. Classification of Infinite Primitive Jordan Permutation Groups. *Proceedings of the London Mathematical Society*, s3-72(1):63–123, 1996. doi:10.1112/plms/s3-72.1.63.

[Ban32] Stefan Banach. *Théorie des opérations linéaires.* Warsaw, 1932.

[BFS14] Uri Bader, Alex Furman, and Roman Sauer. Weak Notions of Normality and Vanishing up to Rank in L2-Cohomology. *International Mathematics Research Notices*, 2014(12):3177–3189, 2014. doi:10.1093/imrn/rnt029.

[BG80] S. I. Bezuglyi and V. Ya. Golodets. Topological properties of complete groups of automorphisms of a measure space. *Siberian Mathematical Journal*, 21(2):147–155, 1980. doi:10.1007/BF00968260.

[BIO20] Rasmus Sylvester Bryder, Nikolay A. Ivanov, and Tron Omland. C^*-simplicity of HNN extensions and groups acting on trees. *Annales de l'Institut Fourier*, 70(4):1497–1543, 2020. doi:10.5802/aif.3378.

[BK96] Howard Becker and Alexander S. Kechris. *The Descriptive Set Theory of Polish Group Actions.* London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996. doi:10.1017/CBO9780511735264.

[BK20] Peter J. Burton and Alexander S. Kechris. Weak containment of measure-preserving group actions. *Ergodic Theory and Dynamical Systems*, 40(10):2681–2733, 2020. doi:10.1017/etds.2019.26.

[Bow15] Lewis Bowen. Invariant random subgroups of the free group. *Groups, Geometry, and Dynamics*, 9(3):891–916, 2015. doi:10.4171/GGD/331.

[BYBM13] Itaï Ben Yaacov, Alexander Berenstein, and Julien Melleray. Polish topometric groups. *Transactions of the American Mathematical Society*, 365(7):3877–3897, 2013. doi:10.1090/S0002-9947-2013-05773-X.

[Cam81] Peter J. Cameron. Finite Permutation Groups and Finite Simple Groups. *Bulletin of the London Mathematical Society*, 13(1):1–22, 1981. doi:10.1112/blms/13.1.1.

[CGLM23] Alessandro Carderi, Damien Gaboriau, and François Le Maître. On dense totipotent free subgroups in full groups. *Geometry & Topology*, 27(6):2297–2318, 2023. doi:10.2140/gt.2023.27.2297.

[CGLMS23] Alessandro Carderi, Damien Gaboriau, François Le Maître, and Yves Stalder. On the space of subgroups of Baumslag-Solitar groups II: The Cantor-Bendixson rank. *In preparation*, 2023.

[Cha12] Vladimir Vladimirovich Chaynikov. *Properties of Hyperbolic Groups: Free Normal Subgroups, Quasiconvex Subgroups and Actions of Maximal Growth*. PhD thesis, Vanderbilt University, 2012.

[CLM16] Alessandro Carderi and François Le Maître. More Polish full groups. *Topology and its Applications*, 202:80–105, 2016. doi:10.1016/j.topol.2015.12.065.

[CLM18] Alessandro Carderi and François Le Maître. Orbit full groups for locally compact groups. *Transactions of the American Mathematical Society*, 370(4):2321–2349, 2018. doi:10.1090/tran/6985.

[Coh13] Donald L. Cohn. *Measure Theory: Second Edition*. Birkhäuser Advanced Texts Basler Lehrbücher. Springer, New York, NY, 2013. doi:10.1007/978-1-4614-6956-8.

[de 85] Pierre de la Harpe. Reduced C*-algebras of discrete groups which are simple with a unique trace. In Huzihiro Araki, Calvin C. Moore, Şerban-Valentin Stratila, and Dan-Virgil Voiculescu, editors, *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Mathematics, pages 230–253, Berlin, Heidelberg, 1985. Springer. doi:10.1007/BFb0074887.

[Dij05] Jan J. Dijkstra. On Homeomorphism Groups and the Compact-Open Topology. *The American Mathematical Monthly*, 112(10):910–912, 2005. arXiv:30037630, doi:10.2307/30037630.

[EG16] Amichai Eisenmann and Yair Glasner. Generic IRS in free groups, after Bowen. *Proceedings of the American Mathematical Society*, 144(10):4231–4246, 2016. doi:10.1090/proc/13020.

[EGLMM21] Mahmood Etedadialiabadi, Su Gao, François Le Maître, and Julien Melleray. Dense locally finite subgroups of automorphism groups of ultraextensive spaces. *Advances in Mathematics*, 391:107966, 2021. doi:10.1016/j.aim.2021.107966.

[FLMMS22] Pierre Fima, François Le Maître, Soyoung Moon, and Yves Stalder. A characterization of high transitivity for groups acting on trees. *Discrete Analysis*, 2022. doi:10.19086/da.37645.

[FMS15] Pierre Fima, Soyoung Moon, and Yves Stalder. Highly transitive actions of groups acting on trees. *Proceedings of the American Mathematical Society*, 143(12):5083–5095, 2015. doi:10.1090/proc/12659.

[For10] Matthew Foreman. Models for measure preserving transformations. *Topology and its Applications*, 157(8):1404–1414, 2010. doi:10.1016/j.topol.2009.06.021.

[Fre02] D. H. Fremlin. *Measure Theory Vol. 3: Measure Algebras*. Torres Fremlin, Colchester, 2002.

[Gab00] Damien Gaboriau. Coût des relations d'équivalence et des groupes. *Inventiones mathematicae*, 139(1):41–98, 2000. doi:10.1007/s002229900019.

[Gao09] Su Gao. *Invariant Descriptive Set Theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[GG13] Shelly Garion and Yair Glasner. Highly transitive actions of $\mathrm{Out}(F_n)$. *Groups, Geometry, and Dynamics*, 7(2):357–376, 2013. doi:10.4171/ggd/185.

[GKM16] Yair Glasner, Daniel Kitroser, and Julien Melleray. From isolated subgroups to generic permutation representations. *Journal of the London Mathematical Society*, 94(3):688–708, 2016. doi:10.1112/jlms/jdw054.

[Gla03] Eli Glasner. *Ergodic Theory via Joinings*, volume 101 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 2003. doi:10.1090/surv/101.

[GLM17] Tsachik Gelander and François Le Maître. Infinitesimal topological generators and quasi non-archimedean topological groups. *Topology and its Applications*, 218:97–113, 2017. doi:10.1016/j.topol.2016.12.019.

[GM07] Y. Glasner and N. Monod. Amenable actions, free products and a fixed point property. *Bulletin of the London Mathematical Society*, 39(1):138–150, 2007. doi:10.1112/blms/bdl011.

[GM23] Maria Gerasimova and Nicolas Monod. A family of exotic group C*-algebras, 2023. arXiv:2305.01990, doi:10.48550/arXiv.2305.01990.

[Gun92] Steven V. Gunhouse. Highly transitive representations of free products on the natural numbers. *Archiv der Mathematik*, 58(5):435–443, 1992. doi:10.1007/BF01190113.

[HHLS93] Wilfrid Hodges, Ian Hodkinson, Daniel Lascar, and Saharon Shelah. The Small Index Property for ω-Stable ω-Categorical Structures and for the Random Graph. *Journal of the London Mathematical Society*, s2-48(2):204–218, 1993. doi:10.1112/jlms/s2-48.2.204.

[Hic92] K. K. Hickin. Highly Transitive Jordan Representations of Free Products. *Journal of the London Mathematical Society*, s2-46(1):81–91, 1992. doi:10.1112/jlms/s2-46.1.81.

[HM20] Karl H. Hofmann and Sidney A. Morris. *The Structure of Compact Groups: A Primer for the Student – A Handbook for the Expert*. De Gruyter, 2020. doi:10.1515/9783110695991.

[HO16] Michael Hull and Denis Osin. Transitivity degrees of countable groups and acylindrical hyperbolicity. *Israel Journal of Mathematics*, 216(1):307–353, 2016. doi:10.1007/s11856-016-1411-9.

[HS42] P. R. Halmos and H. Samelson. On Monothetic Groups. *Proceedings of the National Academy of Sciences of the United States of America*, 28(6):254–258, 1942. doi:10.1073/pnas.28.6.254.

[IT65] A. Ionescu Tulcea. On the Category of Certain Classes of Transformations in Ergodic Theory. *Transactions of the American Mathematical Society*, 114(1):261–279, 1965. arXiv:1994001, doi:10.2307/1994001.

[JM13] Kate Juschenko and Nicolas Monod. Cantor systems, piecewise translations and simple amenable groups. *Annals of Mathematics*, 178(2):775–787, 2013. doi:10.4007/annals.2013.178.2.7.

[Kal85] Robert R. Kallman. Uniqueness results for groups of measure preserving transformations. *Proceedings of the American Mathematical Society*, 95(1):87–90, 1985. doi:10.2307/2045579.

[Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. doi:10.1007/978-1-4612-4190-4.

[Kec10] Alexander S. Kechris. *Global Aspects of Ergodic Group Actions*, volume 160 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010. doi:10.1090/surv/160.

[Kit12] Daniel Kitroser. Highly-transitive actions of surface groups. *Proceedings of the American Mathematical Society*, 140(10):3365–3375, 2012. doi:10.1090/S0002-9939-2012-11195-5.

[KL16] David Kerr and Hanfeng Li. *Ergodic Theory: Independence and Dichotomies*. Springer Monographs in Mathematics. Springer International Publishing, Cham, 2016. doi:10.1007/978-3-319-49847-8.

[KLM15] Adriane Kaïchouh and François Le Maître. Connected Polish groups with ample generics. *Bulletin of the London Mathematical Society*, 47(6):996–1009, 2015. doi:10.1112/blms/bdv078.

[KM19] Aleksandra Kwiatkowska and Maciej Malicki. Automorphism groups of countable structures and groups of measurable functions. *Israel Journal of Mathematics*, 230(1):335–360, 2019. doi:10.1007/s11856-018-1825-7.

[KR07] Alexander S. Kechris and Christian Rosendal. Turbulence, amalgamation, and generic automorphisms of homogeneous structures. *Proceedings of the London Mathematical Society*, 94(2):302–350, 2007. doi:10.1112/plms/pdl007.

[Kri69] Wolfgang Krieger. On non-singular transformations of a measure space. I, II. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 11(2):83–117, 1969. doi:10.1007/BF00531811.

[KT10] John Kittrell and Todor Tsankov. Topological properties of full groups. *Ergodic Theory and Dynamical Systems*, 30(2):525–545, 2010. doi:10.1017/S0143385709000078.

[LBMB22a] Adrien Le Boudec and Nicolás Matte Bon. Confined subgroups and high transitivity. *Annales Henri Lebesgue*, 5:491–522, 2022. doi:10.5802/ahl.128.

[LBMB22b] Adrien Le Boudec and Nicolás Matte Bon. Triple transitivity and non-free actions in dimension one. *Journal of the London Mathematical Society*, 105(2):884–908, 2022. doi:10.1112/jlms.12521.

[Lev95] Gilbert Levitt. On the cost of generating an equivalence relation. *Ergodic Theory and Dynamical Systems*, 15(6):1173–1181, 1995. doi:10.1017/S0143385700009846.

[LM14a] François Le Maître. The number of topological generators for full groups of ergodic equivalence relations. *Inventiones mathematicae*, 198(2):261–268, 2014. doi:10.1007/s00222-014-0503-6.

[LM14b] François Le Maître. *Sur les groupes pleins préservant une mesure de probabilité*. PhD thesis, ENS Lyon, 2014.

[LM18a] François Le Maître. Highly faithful actions and dense free subgroups in full groups. *Groups, Geometry, and Dynamics*, 12(1):207–230, 2018. doi:10.4171/GGD/446.

[LM18b] François Le Maître. On a measurable analogue of small topological full groups. *Advances in Mathematics*, 332:235–286, 2018. arXiv:1608.07399, doi:10.1016/j.aim.2018.05.008.

[LM21] François Le Maître. On a measurable analogue of small topological full groups II. *Annales de l'Institut Fourier*, 71(5):1885–1927, 2021. doi:10.5802/aif.3443.

[LM22] François Le Maître. Polish topologies on groups of non-singular transformations. *Journal of Logic and Analysis*, 14, 2022. doi:10.4115/jla.2022.14.4.

[LMS21] François Le Maître and Konstantin Slutsky. L^1 full groups of flows. *arXiv:2108.09009 [math]*, 2021. arXiv:2108.09009.

[Mal16a] Maciej Malicki. Consequences of the existence of ample generics and automorphism groups of homogeneous metric structures. *The Journal of Symbolic Logic*, 81(3):876–886, 2016. doi:10.1017/jsl.2015.73.

[Mal16b] Maciej Malicki. An example of a non non-archimedean Polish group with ample generics. *Proceedings of the American Mathematical Society*, 144(8):3579–3581, 2016. doi:10.1090/proc/13017.

[Mat06] Hiroki Matui. Some remarks on topological full groups of Cantor minimal systems. *International Journal of Mathematics*, 17(02):231–251, 2006. doi:10.1142/S0129167X06003448.

[McD77] T. P. McDonough. A permutation representation of a free group. *The Quarterly Journal of Mathematics*, 28(3):353–356, 1977. doi:10.1093/qmath/28.3.353.

[MO15] Ashot Minasyan and Denis Osin. Acylindrical hyperbolicity of groups acting on trees. *Mathematische Annalen*, 362(3):1055–1105, 2015. doi:10.1007/s00208-014-1138-z.

[MS13] Soyoung Moon and Yves Stalder. Highly transitive actions of free products. *Algebraic & Geometric Topology*, 13(1):589–607, 2013. doi:10.2140/agt.2013.13.589.

[MT13] Julien Melleray and Todor Tsankov. Generic representations of abelian groups and extreme amenability. *Israel Journal of Mathematics*, 198(1):129–167, 2013. doi:10.1007/s11856-013-0036-5.

[MZ55] Deane Montgomery and Leo Zippin. *Topological Transformation Groups*. Interscience Publishers, 1955.

[Nek17] V. Nekrashevych. Simple groups of dynamical origin. *Ergodic Theory and Dynamical Systems*, pages 1–26, 2017. doi:10.1017/etds.2017.47.

[Rud98] Daniel J. Rudolph. Residuality orbit equivalence. In *Topological Dynamics and Applications*, volume 215 of *Contemporary Mathematics*, University of Minnesota, 1998. Amer. Math. Soc., Providence, RI.

[Sab19] Marcin Sabok. Automatic continuity for isometry groups. *Journal of the Institute of Mathematics of Jussieu*, 18(3):561–590, 2019. doi:10.1017/S1474748017000135.

[Sch84] Klaus Schmidt. Asymptotic Properties of Unitary Representations and Mixing. *Proceedings of the London Mathematical Society*, s3-48(3):445–460, 1984. doi:10.1112/plms/s3-48.3.445.

[Ser80] Jean-Pierre Serre. *Trees*. Springer-Verlag, Berlin Heidelberg, 1980. doi:10.1007/978-3-642-61856-7.

[SZ94] Garrett Stuck and Robert J. Zimmer. Stabilizers for Ergodic Actions of Higher Rank Semisimple Groups. *Annals of Mathematics*, 139(3):723–747, 1994. arXiv:2118577, doi:10.2307/2118577.

[Tao14] Terence Tao. *Hilbert's Fifth Problem and Related Topics*, volume 153 of *Grad. Stud. Math.* American Mathematical Society (AMS), Providence, RI, 2014.

[Tsa06] Todor Tsankov. Compactifications of $\mathbb{N}$ and Polishable subgroups of S_∞. *Fundamenta Mathematicae*, 189(3):269–284, 2006. doi:10.4064/fm189-3-4.

[TT18] Simon Thomas and Robin Tucker-Drob. Invariant random subgroups of inductive limits of finite alternating groups. *Journal of Algebra*, 503:474–533, 2018. doi:10.1016/j.jalgebra.2018.02.012.

[van90] Eric K. van Douwen. Measures invariant under actions of F2. *Topology and its Applications*, 34(1):53–68, 1990. doi:10.1016/0166-8641(90)90089-K.

[Ver12] A. M. Vershik. Totally nonfree actions and the infinite symmetric group. *Moscow Mathematical Journal*, 12(1):193–212, 216, 2012. doi:10.17323/1609-4514-2012-12-1-193-212.