# $L^1$ FULL GROUPS OF FLOWS

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ABSTRACT. We introduce the concept of an $L^1$ full group associated with a measure-preserving action of a Polish normed group on a standard probability space. Such groups are shown to carry a natural separable complete metric, and are thus Polish. Our construction generalizes $L^1$ full groups of actions of discrete groups, which have been studied recently by the first author.

We show that under minor assumptions on the actions, topological derived subgroups of $L^1$ full groups are topologically simple and — when the acting group is locally compact and amenable — are whirly amenable and generically two-generated.

For measure-preserving actions of the real line (also known as measure-preserving flows), the topological derived subgroup of an $L^1$ full groups is shown to coincide with the kernel of the index map, which implies that $L^1$ full groups of free measure-preserving flows are topologically finitely generated if and only if the flow admits finitely many ergodic components. The latter is in a striking contrast to the case of $\mathbb{Z}$-actions, where the number of topological generators is controlled by the entropy of the action.

## 1. INTRODUCTION

Full groups were introduced by H. Dye [Dye59] in the framework of measure-preserving actions of countable groups as measurable analogues of unitary groups of von Neumann algebras, by mimicking the fact that the latter are stable under countable cutting and pasting of partial isometries. These Polish groups have since been recognized as important invariants as they encode the induced partition of the space into orbits. A similar viewpoint applies in the setup of minimal homeomorphisms on the Cantor space [GPS99], where likewise the full groups are responsible for the orbit equivalence class of the action.

Full groups are defined to consist of transformations which act by a permutation on each orbit. When the action is free, one can associate with an element $h$ of the full group a cocycle defined by the equation $h(x) = \rho_h(x)\cdot x$. From the point of view of topological dynamics, it is natural to consider the subgroup of those $h$ for which the cocycle map is continuous, which is the defining condition for the so-called topological full groups. The latter has a much tighter control of the action, and encodes minimal homeomorphisms of the Cantor space up to flip-conjugacy (see [GPS99]).

A celebrated result of H. Dye states that all ergodic $\mathbb{Z}$-actions produce the same partition up to isomorphism, and hence the associated full groups are all isomorphic. The first named author has been motivated by the above to seek for the analog of topological full groups in the context of ergodic theory, which was achieved in [LM18] by imposing integrability conditions on the cocycle. In particular, he introduced $L^1$ full groups of measure-preserving ergodic transformations, and showed based on the result of R. M. Belinskaja [Bel68] that they also determine the action up to flip-conjugacy. Unlike in the context of Cantor dynamics, these $L^1$ full groups are uncountable, but they carry a natural Polish topology.

In this paper, we widen the concept of an $L^1$ full group and associate such an object with any measure-preserving Borel action of a Polish normed group (the reader may consult Appendix A for a concise reminder about group norms). Quasi-isometric compatible norms will result in the same $L^1$ full groups, so actions of Polish boundedly generated groups have canonical $L^1$ full groups associated with them based on to the work of C. Rosendal [Ros18]. Our study also parallels the generalization of the full group construction introduced by A. Carderi and the first named author in [CLM16], where full groups were defined for Borel measure-preserving actions of Polish groups.

1.1. **Main results.** Let $G$ be a Polish group with a compatible norm $\|\cdot\|$ and consider a Borel measure-preserving action $G \curvearrowright X$ on a standard probability space $(X,\mu)$. The group action defines an orbit equivalence relation $\mathcal{R}_G$ by declaring points $x_1, x_2 \in X$ equivalent whenever $G\cdot x_1 = G\cdot x_2$. The norm induces a metric onto each $\mathcal{R}_G$-class via

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$D(x_1,x_2)=\inf_{g\in G}\{\|g\| : gx_1=x_2\}$. Following [CLM16], a **full group** of the action is denoted by $[\mathscr{R}_G]$ and is defined as the collection of all measure-preserving $T\in \mathrm{Aut}(X,\mu)$ that satisfy $x\mathscr{R}_G Tx$ for all $x\in X$. The $\mathrm{L}^1$ **full group** $[G\curvearrowright X]_1$ is given by those $T\in[\mathscr{R}_G]$ for which the map $X\ni x\mapsto D(x,Tx)$ is integrable. This defines a subgroup of $[\mathscr{R}_G]$, and we show in Theorem 2.9 that these groups are Polish in the topology of the norm $\|T\|=\int_X D(x,Tx)\,d\mu$. The strategy of establishing this statement is analogous to that of [CLM18], where the Polish topology for full groups $[\mathscr{R}_G]$ was defined.

It is a general and well-known phenomenon in the study of all kinds and variants of full groups that their structure is usually best understood through the derived subgroups. Our setup is no exception.

**Theorem 1.** *The topological derived group of any aperiodic* $\mathrm{L}^1$ *full group is equal to the closed subgroup generated by involutions.*

The argument needed for Theorem 1 is quite robust. We extract the idea used in [LM18], isolate the class of finitely full groups, and show that under mild assumptions on the action, Theorem 1 holds for such groups. Since the method here is based on the prior work, we provide these arguments in Appendix C and in Corollary C.11 in particular. Alongside we mention Corollary C.17 which implies that $\mathrm{L}^1$ full groups of ergodic actions are topologically simple.

For the rest of our results we narrow down the generality of the acting groups, and consider *locally compact* Polish normed groups. In Section 3 we show that if $H<G$ is a dense subgroup of locally compact Polish normed group $G$ then $[H\curvearrowright X]_1$ is dense in $[G\curvearrowright X]_1$. In fact, we prove a considerably stronger statement by showing that for each $T\in[G\curvearrowright X]$ and $\epsilon>0$ there is $S\in[H\curvearrowright X]$ such that $\operatorname{ess\,sup}_{x\in X} D(Tx,Sx)<\epsilon$.

Recall that a topological group is **amenable** if all of its continuous actions on compact spaces preserve some Radon probability measure, and that it is **whirly amenable** if it is amenable and moreover every invariant Radon measure is supported on the set of fixed points. The following is a combination of Theorem 4.6 and Corollary 4.8.

**Theorem 2.** *Let $G\curvearrowright X$ be a measure-preserving action of a locally compact Polish normed group. Consider the following three statements:*

*(1) $G$ is amenable;*
*(2) the topological derived group $D([G\curvearrowright X]_1)$ is whirly amenable.*
*(3) the* $\mathrm{L}^1$ *full group $[G\curvearrowright X]_1$ is amenable;*

*The implications* $(1)\implies(2)\implies(3)$ *always hold. If $G$ is unimodular and the action is free, then the three statements above are all equivalent.*

When the acting group is amenable and orbits of the action are uncountable, we are able to compute the topological rank of the derived $\mathrm{L}^1$ full groups — that is, the minimal number of elements needed to generate a dense subgroup. Theorem 4.17 contains a stronger version of the following.

**Theorem 3.** *Let $G\curvearrowright X$ be a measure-preserving action of an amenable locally compact Polish normed group on a standard probability space $(X,\mu)$. If all orbits of the action are uncountable, then the topological rank of the derived* $\mathrm{L}^1$ *full group $D([G\curvearrowright X]_1)$ is equal to* 2.

It is instructive to contrast the situation with the actions of finitely generated groups, where finiteness of the topological rank of the derived $\mathrm{L}^1$ full group is equivalent to finiteness of the Rokhlin entropy of the action [LM19].

Our most refined understanding of $\mathrm{L}^1$ full groups is achieved for free actions of $\mathbb{R}$, which are known as **flows**. All the results we described so far are valid for all compatible norms on the acting group. When it comes to the actions of $\mathbb{R}$, however, we consider only the standard Euclidean norm on it. Just like the actions of $\mathbb{Z}$, flows give rise to an important homomorphism, known as the **index map**. Assuming the flow is ergodic, the index map can be described most easily as $[\mathbb{R}\curvearrowright X]_1\ni T\mapsto\int_X|\rho_T(x)|\,d\mu$, where $\rho_T$ is the cocycle of $T$. Section 5 is devoted to the analysis of the index map for general $\mathbb{R}$-flows.

The most technically challenging result of our work is summarized in Theorem 9.1, which identifies the derived $\mathrm{L}^1$ full group of a flow with the kernel of the index map, and describes the abelianization of $[\mathbb{R}\curvearrowright X]_1$.

**Theorem 4.** *Let $\mathscr{F}$ be a measure-preserving flow on $(X,\mu)$. The kernel of the index map is equal to the derived* $\mathrm{L}^1$ *full group of the flow, and the topological abelianization of $[\mathscr{F}]_1$ is $\mathbb{R}$.*

The statement of Theorem 4 parallels the known results for $\mathbb{Z}$-actions from [LM18]. The structure of its proof, however, has an important difference. We rely crucially on the fact that each element of the full group acts in a measure-preserving manner on each orbit. This allows us to use Hopf's decomposition (described in Appendix B.5) in order to separate any given element $T \in [\mathbb{R} \curvearrowright X]_1$ into two parts — recurrent and dissipative. If the acting group were discrete, the recurrent part would reduce to periodic orbits only. This is not at all the case for non-discrete groups, hence we need a new machinery to understand non-periodic recurrent transformations. To cope with this, we introduce the concept of an **intermitted** transformation, which plays the central role in Section 7, and which we hope will find other applications.

Finally, Theorems 3 and 4 allow us to obtain estimates for the topological rank of the whole $L^1$ full groups of flows, which is the content of Proposition 9.3.

**Theorem 5.** *Let $\mathscr{F}$ be a free measure-preserving flow on a standard probability space $(X,\mu)$. The topological rank* $\mathrm{rk}([\mathscr{F}]_1)$ *is finite if and only if the flow has finitely many ergodic components. Moreover, if $\mathscr{F}$ has exactly $n$ ergodic components then*

$$n+1 \leq \mathrm{rk}([\mathscr{F}]_1) \leq n+3.$$

In particular, the topological rank of the $L^1$ full group of an ergodic flow is equal to either 2, 3 or 4. We conjecture that it is always equal to 2, and more generally that the topological rank of the $L^1$ full group of any measure-preserving flow is equal to $n+1$ where $n$ is the number of ergodic components.

1.2. **Preliminaries.** Our work belongs to the field of ergodic theory, which means that all the constructions are defined and results are proven up to null sets. On a number of occasions, we allow ourselves to deviate from the pedantic accuracy and write "for all $x\ldots$" when we really ought to say "for almost all $x\ldots$", etc. The only part where certain care needs to be exercised in this regard appears in Section 2. We define $L^1$ full groups for *Borel* measure-preserving actions of Polish normed groups, and we need a genuine action on the space $X$ for these to make sense just as in [CLM16]. Boolean actions (also called near actions) of general Polish groups do not admit realizations in general [GTW05], and even when they do, it could happen that different realizations yield different full groups. This subtlety disappears once we move our attention to locally compact group actions, which is the case for Section 3 and onwards. All boolean actions of locally compact second countable groups admit Borel realizations which are all conjugate up to measure zero (and hence have the same full group), so null sets can be neglected just as they always are in ergodic theory.

By a *standard probability space* we mean the unique (up to isomorphism) separable atomless measure space $(X,\mu)$ with $\mu(X)=1$, i.e., the unit interval $[0,1]$ with the Lebesgue measure. A few times in Section 4 and Appendix B.1 we refer to a *standard Lebesgue space*, by which we mean a separable finite measure space, $\mu(X)<\infty$, thus in contrast to the notion of the standard probability space allowing atoms and omitting the normalization requirement.

Any group action $G \curvearrowright X$ induces the orbit equivalence relation $\mathscr{R}_{G\curvearrowright X}$, where two points $x, y \in X$ are $\mathscr{R}_{G\curvearrowright X}$-equivalent whenever $G\cdot x = G\cdot y$. We will usually write this equivalence relation simply as $\mathscr{R}_G$ for brevity. For the actions $\mathbb{Z} \curvearrowright X$ generated by an automorphism $T \in \mathrm{Aut}(X,\mu)$, we denote the corresponding orbit equivalence relation by $\mathscr{R}_T$. We shall encounter various equivalence relations throughout this article. An equivalence class of a point $x \in X$ under the relation $\mathscr{R}$ is denoted by $[x]_{\mathscr{R}}$ and the *saturation* of a set $A \subseteq X$ is denoted by $[A]_{\mathscr{R}}$ and is defined to be the union of $\mathscr{R}$-equivalence classes of the elements of $A$: $[A]_{\mathscr{R}} = \bigcup_{x\in A}[x]_{\mathscr{R}}$. In particular, $[x]_{\mathscr{R}_T}$ is the orbit of $x$ under the action of $T$. The reader may notice that the notation for a saturation $[A]_{\mathscr{R}}$ resembles that for the full group of an action $[G \curvearrowright X]$ (see Section 2). Both notations are standard, and we hope that confusion will not arise, as it applies to objects of different nature — sets and actions, respectively.

Consider a measure-preserving action of a locally compact Polish (equivalently, second-countable) group $G$ on a standard Lebesgue space $(X,\mu)$. A *complete section* for the action is a measurable set $\mathscr{C} \subseteq X$ that intersects almost every orbit, i.e., $\mu(X \setminus G\cdot\mathscr{C}) = 0$. A *cross section* is a complete section $\mathscr{C} \subseteq X$ such that for some non-empty neighborhood of the identity $U \subseteq G$ we have $Uc \cap Uc' = \emptyset$ whenever $c, c' \in \mathscr{C}$ are distinct. When the need to mention such a neighborhood $U$ explicitly arises, we say that $\mathscr{C}$ is a *$U$-lacunary* cross section.

With any cross section $\mathscr{C}$ one associates a decomposition of the phase space known as the Voronoi tessellation. Slightly more generally, Appendix B.2 defines the concept of a tessellation over a cross section, which corresponds to a set $\mathscr{W} \subseteq \mathscr{C} \times X$ for which the fibers $\mathscr{W}_c = \{x \in X : (c,x) \in \mathscr{W}\}$, $c \in \mathscr{C}$, partition the phase space. Every tessellation $\mathscr{W}$ gives rise to an equivalence relation $\mathscr{R}_{\mathscr{W}}$, where points $x, y \in X$ are deemed equivalent whenever they belong to the same fiber $\mathscr{W}_c$, and to the projection map $\pi_{\mathscr{W}} : X \to \mathscr{C}$ that associates with each $x \in X$ the unique $c \in \mathscr{C}$ which fiber $\mathscr{W}_c$ the point $x$ belongs to, and is therefore defined by the condition $(\pi_{\mathscr{W}}(x), x) \in \mathscr{W}$ for all $x \in X$.

When the action $G \curvearrowright X$ is free, each orbit $G\cdot x$ can be identified with the acting group. Such a correspondence $g \mapsto gx$ depends on the choice of the anchor point $x$ within the orbit, but suffices to transfer structures invariant under right translations from the group $G$ onto the orbits of the action. For instance, if the acting group is locally compact, then a right-invariant Haar measure $\lambda$ can be pushed onto orbits by setting $\lambda_x(A) = \{g \in G : gx \in A\}$ as discussed in Appendix B.4. Freeness of the action $G \curvearrowright X$ gives rise to the *cocycle map* $\rho : \mathscr{R}_{G\curvearrowright X} \to G$ which is well-defined by the condition $\rho(x, y) \cdot x = y$. Elements of the full group $[G \curvearrowright X]$ are characterized as measure-preserving transformations $T \in \mathrm{Aut}(X,\mu)$ such that $(T(x), x) \in \mathscr{R}_{G\curvearrowright X}$ for all $x \in X$. With each $T \in [G \curvearrowright X]$ one may therefore associate the map $\rho_T : X \to G$, also known as the *cocycle map*, and defined by $\rho_T(x) = \rho(x, Tx)$. Both the context and the notation will clarify which cocycle map is being referred to.

All these concepts appear prominently in the sections which deal with free measure-preserving flows, that is actions of $\mathbb{R}$ on the standard probability space. We use the additive notation for such actions: $\mathbb{R} \times X \ni (x, r) \mapsto x + r \in X$. The group $\mathbb{R}$ carries a natural linear order which is invariant under the group operation and can therefore be transferred onto orbits. More specifically, given a free measure-preserving flow $\mathbb{R} \curvearrowright X$ we use the notation $x < y$ whenever $x$ and $y$ belong to the same orbit and $y = x + r$ for some $r > 0$. Every cross section $\mathscr{C}$ of a free flow intersects each orbit in a bi-infinite fashion — each $c \in \mathscr{C}$ has a unique successor and a unique predecessor in the order of the orbit. One therefore has a bijection $\sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, called the *the first return map* or *the induced map*, which sends $c \in \mathscr{C}$ to the next element of the cross section within the same orbit. We also make use of the gap function that measures the lengths of intervals of the cross section, i.e. $\mathrm{gap}_{\mathscr{C}}(c) = \rho(c, \sigma_{\mathscr{C}}(c))$.

There is also a canonical tessellation associated with a cross section $\mathscr{C}$ which partitions each orbit into intervals between adjacent points of $\mathscr{C}$ and is given by $W_{\mathscr{C}} = \{(c, x) \in \mathscr{C} \times X : c \le x < \sigma_{\mathscr{C}}(c)\}$. The associated equivalence relation $\mathscr{R}_{W_{\mathscr{C}}}$ is denoted simply by $\mathscr{R}_{\mathscr{C}}$ and groups points $(x, y) \in \mathscr{R}_{\mathbb{R}\curvearrowright X}$ which belong to the same interval of the tessellation, $\pi_{\mathscr{C}}(x) = \pi_{\mathscr{C}}(y)$. The $\mathscr{R}_{\mathscr{C}}$-equivalence class of $x \in X$ is equal to $[x]_{\mathscr{R}_{\mathscr{C}}} = \pi_{\mathscr{C}}(x) + \big[0, \mathrm{gap}_{\mathscr{C}}(\pi_{\mathscr{C}}(x))\big)$.

Often enough we need to restrict sets and functions to an $\mathscr{R}_{\mathscr{C}}$-class. Since such a need arises very frequently, especially in Section 8, we adopt the following shorthand notations. Given a set $A \subseteq X$ and a point $c \in \mathscr{C}$ the intersection $A \cap [c]_{\mathscr{R}_{\mathscr{C}}}$ is denoted simply by $A(c)$. Likewise, $\lambda_c^{W_{\mathscr{C}}}(A)$ stands for $\lambda(\{r \in \mathbb{R} : c + r \in A \cap [c]_{\mathscr{R}_{\mathscr{C}}}\})$ and corresponds to the Lebesgue measure of the set $A \cap [c]_{\mathscr{R}_{\mathscr{C}}}$. Moreover, $\lambda_c^{W_{\mathscr{C}}}(A)$ will usually be shortened to $\lambda_c^{\mathscr{C}}(A)$, when the tessellation is clear from the context.

## 2. $\mathrm{L}^1$ Full Groups of Polish Group Actions

We begin by defining the key notion of interest for our work, namely the $\mathrm{L}^1$ full groups of measure-preserving actions of Polish normed groups on a standard probability space. Admittedly, the overall focus will be on actions of locally compact groups, and $\mathbb{R}$-flows in particular. Nonetheless, the concept of an $\mathrm{L}^1$ full group can be introduced for all Polish normed group actions, and in this section we therefore keep this level of generality.

### 2.1. $\mathrm{L}^1$ spaces with values in metric spaces.

By a *Polish metric space* we mean a separable complete metric space.

**Definition 2.1.** Let $(X,\mu)$ be a standard probability space, let $(Y, d_Y)$ be a Polish metric space, and let $\hat{e} : X \to Y$ be a measurable function. We define the $\hat{e}$-pointed $\mathrm{L}^1$ space $\mathrm{L}^1_{\hat{e}}(X, Y)$ as the space of measurable functions $f : X \to Y$ such that $\int_X d_Y\big(\hat{e}(x), f(x)\big)\, d\mu(x) < +\infty$. The space $\mathrm{L}^1_{\hat{e}}(X, Y)$ is naturally equipped with the metric

$$\tilde{d}_Y(f_1, f_2) = \int_X d_Y(f_1(x), f_2(x))\, d\mu(x),$$

which is finite by the triangle inequality using the function $\hat{e}$ as the middle point.

**Proposition 2.2.** *Let $(X,\mu)$ be a standard probability space, $(Y, d_Y)$ be a Polish metric space, and $\hat{e} : X \to Y$ be a measurable function. The space $(\mathrm{L}^1_{\hat{e}}(X, Y), \tilde{d}_Y)$ is a Polish metric space.*

*Proof.* The argument follows closely the classical proof that $(\mathrm{L}^1(X,\mathbb{R}), \tilde{d}_{\mathbb{R}})$ is a Polish metric space. To check completeness, let us pick a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^1_{\hat{e}}(X, Y)$. Without loss of generality we may assume that $\tilde{d}_Y(f_n, f_{n+1}) < 2^{-n}$, $n \in \mathbb{N}$. Consider the sets $A_n = \{x \in X : d_Y(f_n(x), f_{n+1}(x)) \ge 1/n^2\}$, $n \ge 1$. Chebyshev's inequality shows that $\mu(A_n) \le n^2 2^{-n}$, whence $\sum_n \mu(A_n) < \infty$. The Borel-Cantelli lemma implies that $(f_n(x))_{n\in\mathbb{N}}$ is pointwise Cauchy for almost every $x \in X$. Since $(Y, d_Y)$ is complete, the pointwise limit of $(f_n)_{n\in\mathbb{N}}$ exists; let us denote it by $f : X \to Y$. Define functions $h_n, h : X \to \mathbb{R}^{\ge 0}$ by

$$h_n(x) = \sum_{i<n} d_Y(f_i(x), f_{i+1}(x)), \quad h(x) = \sum_{i\in\mathbb{N}} d_Y(f_i(x), f_{i+1}(x)) = \lim_{n\to\infty} h_n(x),$$

and note that $h \in L^1(X,\mathbb{R})$ by Fatou's lemma. Finally, we conclude that

$$\begin{aligned}\tilde{d}_Y(f_n, f) = \int_X d_Y(f_n(x), f(x))\, d\mu(x) &\le \int_X \sum_{k=n}^{\infty} d_Y(f_k, f_{k+1}(x))\, d\mu(x)\\ &= \int_X (h(x) - h_n(x))\, d\mu(x) \to 0,\end{aligned}$$

where the last convergence follows from Lebesgue's dominated convergence theorem.

To verify separability, pick a countable dense set $D \subseteq Y$ and use Chebychev's inequality to note that the subspace of maps taking values in $D$ is $\tilde{d}_Y$-dense (a stronger statement holds, as this subspace is dense, in fact, in the *sup* metric). Moreover, the subspace of functions that takes only finitely many values (all of which are element of $D$) is still dense. Finally, one uses a dense countable subalgebra of the measure algebra on $X$ and only considers the functions which are measurable with respect to this countable subalgebra. The resulting countable collection is dense in $L^1_{\hat{e}}(X, \tilde{d}_Y)$. □

The group of measure-preserving automorphisms $\mathrm{Aut}(X,\mu)$ has a natural action by composition on $L^1_{\hat{e}}(X, Y)$, i.e., $(T \cdot f)(x) = f(T^{-1}x)$. Every automorphism acts by an isometry.

**Proposition 2.3.** *Let $(X,\mu)$ be a standard probability space, $(Y, d_Y)$ be a Polish metric space, and $\hat{e} : X \to Y$ be a measurable function. The action $\mathrm{Aut}(X,\mu) \curvearrowright L^1_{\hat{e}}(X, Y)$ is continuous.*

*Proof.* The argument mirrors the one in [CLM16, Prop. 2.9(1)]. Given sequences $T_n \to T$ and $f_n \to f$ we need to shows that $T_n \cdot f_n \to T \cdot f$. Since the action is by isometries,

$$\tilde{d}_Y(T_n \cdot f_n, T \cdot f) = \tilde{d}_Y(f_n, T_n^{-1} T \cdot f) \le \tilde{d}_Y(f_n, f) + \tilde{d}_Y(f, T_n^{-1} T \cdot f).$$

It therefore suffices to show that for any $f \in L^1_{\hat{e}}(X, Y)$ and any convergent sequence of automorphisms $T_n \to T$ one has $\tilde{d}_Y(f, T_n^{-1} \circ T \cdot f) \to 0$ as $n \to \infty$. The latter is enough to check for functions that take only finitely many values as those are dense in $L^1_{\hat{e}}(X, Y)$. If $f$ is such a step function over sets $A_1, \ldots, A_m$, then convergence $T_n \to T$ implies $\mu(T_n(A_i) \triangle A_i) \to 0$ for all $1 \le i \le m$. The latter easily yields $\tilde{d}_Y(f, T_n^{-1} T \cdot f) \to 0$. □

When $Y$ is a Polish group, there is a natural choice of the function $\hat{e}$, namely the constant function $\hat{e}(x) = e$ that is equal to the identity of the group. We therefore will simplify the notation in this case and write $L^1(X, Y)$, omitting the subscript $\hat{e}$.

Recall that a *Polish normed group* is a Polish group together with a compatible norm on it (see Appendix A). In particular, if $(G, \|\cdot\|)$ is a Polish normed group, we have a canonical choice of a complete metric on $G$ given by $d_G(u, v) = (\|u^{-1}v\| + \|vu^{-1}\|)/2$. The corresponding space $L^1(X, G)$ is Polish by Proposition 2.2.

**Proposition 2.4.** *Let $(G, \|\cdot\|)$ be a Polish normed group, and let $G \curvearrowright X$ be a Borel measure-preserving action on a standard probability space. The space $L^1(X, G)$ is a Polish normed group under the pointwise operations, $(f \cdot g)(x) = f(x)g(x)$, $f^{-1}(x) = f(x)^{-1}$, and the norm $\|f\|_1^{L^1(X,G)} = \int_X \|f(x)\|\, d\mu(x)$.*

2.2. $L^1$ **full groups of Polish normed group actions.** Let $(G, \|\cdot\|)$ be a Polish normed group, and let $G \curvearrowright X$ be a measure-preserving Borel action on a standard probability space $(X, \mu)$. Let also $\mathscr{R}_G \subseteq X \times X$ denote the equivalence relation induced by this action. The norm induces a metric on each $\mathscr{R}_G$-equivalence class via

$$D(x, y) = \inf_{u \in G} \{\|u\| : ux = y\} \text{ for } (x, y) \in \mathscr{R}_G. \tag{1}$$

Properties of the metric are straightforward except, possibly, for the implication $D(x, y) = 0 \implies x = y$. To justify the latter let $u_n \in G$, $n \in \mathbb{N}$, be a sequence such that $u_n \to e$ and $u_n x = y$. Elements $u_n^{-1} u_0$, $n \in \mathbb{N}$, belong to the stabilizer of $x$. By Miller's Theorem [Mil77], stabilizers of all points are closed, whence $u_0 = \lim_n u_n^{-1} u_0$ fixes $x$. Thus $u_0 x = x$, and $x = y$ as claimed.

A. Carderi and the second-named author introduced in [CLM16] full groups of Borel measure-preserving Polish group actions on standard probability spaces. Given such an action $G \curvearrowright X$, they define the full group of the action $[G \curvearrowright X]$ to consist of those measure-preserving transformations $T \in \mathrm{Aut}(X, \mu)$ that preserve the equivalence relation $\mathscr{R}_G$. They showed that full groups are Polish with respect to a natural topology of convergence in measure.

Suppose that the acting group $G$ is furthermore endowed with a compatible norm, which therefore induces a metric $D$ on the equivalence classes of $\mathscr{R}_G$. We define a subgroup of $[G \curvearrowright X]$ that consists of those automorphisms $T$ for which the map $x \mapsto D(x, Tx)$ is integrable. Such a subgroup, we argue in this subsection, also carries a natural Polish topology.

**Definition 2.5.** Let $G \curvearrowright X$ be a Borel measure-preserving action of a Polish normed group $(G, \|\cdot\|)$ on a standard probability space $X$; let $D : \mathscr{R}_G \to \mathbb{R}^{\geq 0}$ be the associated metric on the orbits of the action. The $\mathrm{L}^1$ norm of an automorphism $T \in [G \curvearrowright X]$ is denoted by $\|T\|_1$ and is defined by the integral $\|T\|_1 = \int_X D(x, Tx)\, d\mu(x)$. In general, many elements of the full group will have an infinite norm, and the $\mathrm{L}^1$ **full group** of the action consists of the automorphisms for which the norm is finite: $[G \curvearrowright X]_1 = \{T \in [G \curvearrowright X] : \|T\|_1 < \infty\}$.

Elements of $[G \curvearrowright X]_1$ form a group under the composition, as can readily be verified using the triangle inequality for $D$ and the fact that transformations are measure-preserving. Likewise, it is straightforward to check that $\|\cdot\|$ is indeed a norm on $[G \curvearrowright X]_1$. Our goal is to prove that the topology of the norm $\|\cdot\|_1$ on $[G \curvearrowright X]_1$ is a Polish topology. Following the approach taken in [CLM16], we provide a different definition of the $\mathrm{L}^1$ full group, where Polishness of the topology will be readily obtainable, and then argue that the two constructions are isometrically isomorphic. Note that our construction encompasses that from [CLM16] since the latter corresponds to the case when $G$ is equipped with a compatible *bounded* norm.

We recall some basic facts from [CLM16]. $\mathrm{L}^0(X, G)$ denotes the space of measurable functions $f : X \to G$; this space is Polish with respect to the topology of convergence in measure. One can endow the space $X$ with a Polish topology so that the evaluation map $\Phi : \mathrm{L}^0(X, G) \to \mathrm{L}^0(X, X)$ given by $\Phi(f)(x) = f(x) \cdot x$ becomes continuous.

*Remark* 2.6. In [CLM16] the continuity of $\Phi$ is obtained by appealing to the remarkable but difficult result from Becker and Kechris which states that every Borel flow has a continuous model [BK96, Thm. 5.2.1]. Let us point out that one can also derive this from the much easier fact that every Borel $G$-flow on $X$ can be embedded into a continuous Polish $G$-flow on some $X'$ (see [BK96, Thm. 2.6]). Indeed we can then endow $X'$ with the pushforward measure and work with it instead, since we have an isomorphism between $X$ and a full measure subset of $X'$.

Let the set $\mathrm{PF} \subseteq \mathrm{L}^0(X, G)$ be the preimage of $\mathrm{Aut}(X, \mu)$ under the above-defined map $\Phi$:

$$\mathrm{PF} = \{f \in \mathrm{L}^0(X, G) : \Phi(f) \in \mathrm{Aut}(X, \mu)\}.$$

Since $\mathrm{Aut}(X, \mu)$ is a $G_\delta$ subset of $\mathrm{L}^0(X, X)$ (cf. [CLM16, Prop. 2.9]), PF is $G_\delta$ in $\mathrm{L}^0(X, G)$, hence Polish in the induced topology. The group operations can be pulled from $\mathrm{Aut}(X, \mu)$ onto PF (cf. [CLM16, p. 91]) as follows: for $f, g \in \mathrm{PF}$ and $x \in X$ define the multiplication via $(f * g)(x) = f(\Phi(g)(x))g(x)$ and the inverse[1] by $\mathrm{inv}(f)(x) = f(\Phi(f)^{-1}(x))^{-1}$. These operations turn PF into a Polish group and $\Phi : \mathrm{PF} \to \mathrm{Aut}(X, \mu)$ into a continuous homomorphism.

The space $\mathrm{L}^1(X, G)$ of integrable functions admits a natural inclusion $\iota : \mathrm{L}^1(X, G) \hookrightarrow \mathrm{L}^0(X, G)$. This inclusion is continuous, as can be seen by noting that the equivalent metric $\rho_G = \min\{1, d_G\}$ on $G$ generates the convergence in measure topology on $\mathrm{L}^0(X, G)$ (see [CLM16, Prop. 2.7]), and $\tilde{d}_G(f, g) \geq \tilde{\rho}_G(f, g)$ for all $f, g \in \mathrm{L}^1(X, G)$. Set $\mathrm{PF}^1 = \iota^{-1}(\mathrm{PF})$, which we endow with the topology induced form $\mathrm{L}^1(X, G)$. Since $\mathrm{L}^1(X, G)$ is a subset of $\mathrm{L}^0(X, G)$, we may omit the inclusion map $\iota$ when convenient.

**Proposition 2.7.** $\mathrm{PF}^1$ *is a Polish group with the multiplication* $(f, g) \mapsto (f * g)$ *and the inverse* $f \mapsto \mathrm{inv}(f)$. *The function* $f \mapsto \|f\|_1^{\mathrm{L}^1(X,G)}$ *is a compatible group norm on* $\mathrm{PF}^1$ *and* $\Phi \circ \iota \restriction_{\mathrm{PF}^1} : \mathrm{PF}^1 \to \mathrm{Aut}(X, \mu)$ *is a continuous homomorphism.*

*Proof.* First of all, we need to show that these operations are well-defined in the sense that functions $f * g$ and $\mathrm{inv}(f)$ belong to $\mathrm{L}^1(X, G)$ whenever so do their arguments. To this end observe that for $f, g \in \mathrm{PF}^1$

$$\|f * g\|_1^{\mathrm{L}^1(X,G)} = \int_X \|f(\Phi(g)(x))g(x)\|\, d\mu(x) \leq \int_X \|f(\Phi(g)(x))\|\, d\mu(x) + \int_X \|g(x)\|\, d\mu(x).$$

Now note that since $\Phi(g)$ is measure-preserving, we have $\int_X \|f(\Phi(g)(x))\|\, d\mu(x) = \int_X \|f(x)\|\, d\mu(x)$, so

$$\|f * g\|_1^{\mathrm{L}^1(X,G)} \leq \int_X \|f(x)\|\, d\mu(x) + \int_X \|g(x)\|\, d\mu(x) = \|f\|_1^{\mathrm{L}^1(X,G)} + \|g\|_1^{\mathrm{L}^1(X,G)}.$$

In particular, $f * g \in \mathrm{L}^1(X, G)$, and thus $\mathrm{PF}^1$ is closed under the multiplication. Similarly, $\Phi(f) \in \mathrm{Aut}(X, \mu)$ implies

$$\|\mathrm{inv}(f)\|_1^{\mathrm{L}^1(X,G)} = \int_X \|f(\Phi(f)^{-1}(x))^{-1}\|\, d\mu(x) = \int_X \|f(x)^{-1}\|\, d\mu(x) = \|f\|_1^{\mathrm{L}^1(X,G)}.$$

Thus $\mathrm{PF}^1$ is also closed under the inverse. Since these operations define a group structure on PF, it follows that $\mathrm{PF}^1$ is an (abstract) subgroup of PF. Note that we have also established that $\|\cdot\|^{\mathrm{L}^1(X,G)}$ is a group norm on $\mathrm{PF}^1$.

[1] The symbol $f^{-1}$ has already been used in the definition of the pointwise inverse on all of $\mathrm{L}^1(X, G)$. We introduce a different operation here, hence the slightly unusual choice of the symbol to denote the inverse operation.

The multiplication and the operation of taking the inverse are continuous in the topology of $\mathrm{L}^1(X,G)$, which is a consequence of the continuity of $\Phi\circ\iota$, Proposition 2.3 and Proposition 2.4. Since $\mathrm{PF}^1$ is a $G_\delta$ subset of $\mathrm{L}^1(X,G)$, we conclude that it is a Polish group in the topology induced by the norm $\|\cdot\|_1^{\mathrm{L}^1(X,G)}$. □

Let $K \trianglelefteq \mathrm{PF}^1$ denote the kernel of $\Phi\circ\iota\restriction_{\mathrm{PF}^1}$, and let $\|\cdot\|_1^{\mathrm{PF}^1/K}$ denote the quotient norm induced by $\|\cdot\|_1^{\mathrm{L}^1(X,G)}$. The factor group $(\mathrm{PF}^1/K, \|\cdot\|_1^{\mathrm{PF}^1/K})$ is evidently a Polish group, and it turns out to be isometrically isomorphic to the $\mathrm{L}^1$ full group introduced in Definition 2.5 as we will now see. Let $\tilde\Phi : \mathrm{PF}^1/K \to \mathrm{Aut}(X,\mu)$ denote the homomorphism induced by $\Phi\circ\iota\restriction_{\mathrm{PF}^1}$ onto the factor group.

**Proposition 2.8.** *The homomorphism* $\tilde\Phi : \mathrm{PF}^1/K \to \mathrm{Aut}(X,\mu)$ *establishes an isometric isomorphism between the normed groups* $(\mathrm{PF}^1/K, \|\cdot\|_1^{\mathrm{PF}^1/K})$ *and* $([G \curvearrowright X]_1, \|\cdot\|_1)$.

*Proof.* We begin by showing that for any $gK \in \mathrm{PF}^1/K$ one has $\left\|gK\right\|_1^{\mathrm{PF}/K} = \left\|\tilde\Phi(gK)\right\|_1$. By the definition of the quotient norm $\left\|gK\right\|_1^{\mathrm{PF}^1/\mathrm{K}} = \inf_{k\in K}\int_X \left\|g(x)k(x)\right\| d\mu(x)$. For any fixed $k \in K$ we have $g(x)k(x)\cdot x = g(x)\cdot x$, and therefore $D(x, g(x)\cdot x) \le \left\|g(x)k(x)\right\|$ for almost every $x \in X$. This readily implies the inequality $\left\|\tilde\Phi(gK)\right\|_1 \le \left\|gK\right\|_1^{\mathrm{PF}^1/K}$. For the other direction, let $\epsilon > 0$ and consider the set

$$\{(x,u) \in X\times G : g(x)\cdot x = u\cdot x \text{ and } \|u\| \le D(x, g(x)\cdot x)+\epsilon\}.$$

Using Jankov-von Neumann uniformization theorem, one may pick a measurable map $g_0 : X \to G$ that satisfies $g_0(x)\cdot x = g(x)\cdot x$ and $\left\|g_0(x)\right\| \le D(x, g(x)\cdot x)+\epsilon$ for almost all $x \in X$. Since $x \mapsto g(x)^{-1}g_0(x) \in K$, we have

$$\left\|\tilde\Phi(gK)\right\|_1 = \int_X D(x, g(x)\cdot x)\, d\mu(x) \ge \int_X \left\|g(x)g(x)^{-1}g_0(x)\right\| d\mu(x) - \epsilon \ge \left\|gK\right\|_1^{\mathrm{PF}^1/K} - \epsilon.$$

As $\epsilon$ is an arbitrary positive real, we conclude that $\left\|gK\right\|_1^{\mathrm{PF}^1/K} = \left\|\tilde\Phi(gK)\right\|_1$.

It remains to check that $\tilde\Phi$ is surjective. For an automorphism $T \in [G \curvearrowright X]_1$ consider the set

$$\{(x,u) \in X\times G : Tx = u\cdot x \text{ and } \|u\| \le D(x, Tx)+1\}.$$

Applying the Jankov-von Neumann uniformization theorem once again we get a map $g \in \mathrm{L}^0(X,G)$ such that $\Phi(g) = T$ and $\left\|g(x)\right\| \le D(x, Tx)+1$. The latter inequality together with the assumption that $T \in [G \curvearrowright X]_1$ easily imply that $g \in \mathrm{L}^1(X,G)$ and thus $gK \in \mathrm{PF}^1/K$ is the preimage of $T$ under $\tilde\Phi$. □

Results of this section can be summarized as follows.

**Theorem 2.9.** *Let $G \curvearrowright X$ be a Borel measure-preserving action of a Polish normed group $(G, \|\cdot\|)$ on a standard probability space. The $\mathrm{L}^1$ full group $[G \curvearrowright X]_1$ is a Polish normed group relative to the norm $\|T\|_1 = \int_X D(x, Tx)\, d\mu(x)$.*

*Remark* 2.10. When the acting group is finitely generated equipped with the word length metric with respect to the finite generating set, it can be shown that the right-invariant metric induced by the norm on the $\mathrm{L}^1$ full group is complete (see Prop. 3.4, Prop. 3.5 and the remark thereafter in [LM18] for a more general statement). Nevertheless, $\mathrm{L}^1$ full groups are not CLI in general, for instance if $G = \mathbb{R}$ is acting by rotation on the circle, the $\mathrm{L}^1$ full group of the action is $\mathrm{Aut}(\mathbb{S}^1, \lambda)$ itself, which is not CLI.

Let us point out a possible more general framework for the above theorem. Given a standard probability space $(X,\mu)$, consider a metric $D$ on $X$ which is allowed to take the value $+\infty$ and which is Borel (an example being given by equation (1) which actually makes sense for all $x, y \in X$). Note that the relation $D(x,y) < +\infty$ is an equivalence relation, and define the $\mathrm{L}^1$ full group of $D$ as the group of all $T \in \mathrm{Aut}(X,\mu)$ such that $\int_X D(x, T(x))d\mu(x) < +\infty$.

**Question 2.11.** *Assume that for every $x \in X$, the metric $D$ restricts to a complete separable metric on the equivalence class $\{y \in X\colon D(x,y) < +\infty\}$, is it true that the $\mathrm{L}^1$ full group of $D$ is Polish ?*

2.3. $\mathrm{L}^1$ **full groups of boundedly generated Polish group actions.** We would now like to explain how one can make sense of the $\mathrm{L}^1$ full group of any measure-preserving action of a *boundedly generated* Polish group without fixing a norm ahead of time, using the fact that such groups come with a canonical (up to quasi-isometry) norm as identified by C. Rosendal (see [Ros18] for more on this).

**Definition 2.12** (Rosendal)**.** Let $G$ be a Polish group. A subset $A \subseteq G$ is **coarsely bounded** if for every continuous isometric action of $G$ on a metric space $(M, d_M)$, the set $A\cdot m$ is bounded for all $m \in M$, i.e., there is $K > 0$ such that $d(a_1\cdot m, a_2\cdot m) \le K$ for all $a_1, a_2 \in A$. A Polish group $G$ is **boundedly generated** if it is generated by a coarsely bounded set.

Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a Polish group $G$ are **quasi-isometric** if there exists $C>0$ such that for all $g\in G$,

$$\frac{1}{C}\|g\|-C\leq\|g\|'\leq C\|g\|+C.$$

**Lemma 2.13.** *Let $\|\cdot\|$ and $\|\cdot\|'$ be two quasi-isometric compatible norms on a Polish group $G$, and let $G\curvearrowright(X,\mu)$ be a measure-preserving action. The* $\mathrm{L}^1$ *full groups associated with the norms $\|\cdot\|$ and $\|\cdot\|'$ are equal* as topological groups.

*Proof.* The quasi-isometry condition implies that a function $f:X\to G$ satisfies $\int_X\|f(x)\|\,d\mu<+\infty$ if and only if $\int_X\|f(x)\|'\,d\mu<+\infty$. In particular, the $\mathrm{L}^1$ full groups associated to both norms are equal as abstract groups. Moreover, the topologies of the two norms induce the same Borel structure on $[G\curvearrowright X]_1$, and therefore have to coincide (see [BK96, Sec. 1.6]). □

A compatible norm $\|\cdot\|$ on $G$ is called *maximal* if for all compatible norms $\|\cdot\|'$ there is $C>0$ such that

$$\|g\|'\leq C\|g\|+C\quad\text{for all } g\in G.$$

**Theorem 2.14** (Rosendal)**.** *Every boundedly generated Polish group admits a maximal compatible norm.*

Since all maximal norms are quasi-isometric, the theorem above and Lemma 2.13 allow to introduce the following definition.

**Definition 2.15.** Let $G$ be a boundedly generated Polish group, and let $G\curvearrowright(X,\mu)$ be a measure-preserving action. The $\mathrm{L}^1$ **full group** $[G\curvearrowright X]_1$ of this action is the $\mathrm{L}^1$ full group associated with the normed group $(G,\|\cdot\|)$, where $\|\cdot\|$ is a maximal compatible norm on $G$.

When $G=\mathbb{R}$, we recover the definition of $\mathrm{L}^1$ full groups of flows as given in the introduction.

2.4. **Stability under the first return map.** Let $T\in\mathrm{Aut}(X,\mu)$ be a measure-preserving transformation. For a given measurable set $A$ consider the induced map $T_A$, defined as the identity on $X\setminus A$, and as $T^n(x)$ for $x\in A$ where $n\geq 1$ is the smallest integer such that $T^n(x)\in A$. By Poincaré recurrence theorem, such a map yields a well-defined measure-preserving transformation.

**Proposition 2.16.** *Let $G\curvearrowright X$ be a Borel measure-preserving action of a Polish normed group $(G,\|\cdot\|)$. For any element $T\in[G\curvearrowright X]_1$ and any measurable set $A\subseteq X$ the induced transformation $T_A$ belongs to $[G\curvearrowright X]_1$ and moreover $\|T_A\|_1\leq\|T\|_1$.*

*Proof.* For $n\geq1$ let $A_n$ be the set of elements of $A$ whose return time is equal to $n$; note that $X=\bigsqcup_{n\geq1}\bigsqcup_{i=0}^{n-1}T^i(A_n)$. Let as before $D:\mathscr{R}_G\to\mathbb{R}^{\geq0}$ be the metric induced by the group norm $\|\cdot\|$ on the orbits of the action. To estimate the value of $\|T_A\|_1$ observe that

$$\begin{aligned}\|T_A\|_1&=\int_X D(x,T_Ax)\,d\mu(x)=\sum_{n=1}^{\infty}\int_{A_n}D(x,T_Ax)\,d\mu(x)=\sum_{n=1}^{\infty}\int_{A_n}D(x,T^nx)\,d\mu(x)\\ \because\text{ triangle inequality}&\leq\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_{A_n}D(T^ix,T^{i+1}x)\,d\mu(x)=\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_{T^i(A_n)}D(x,Tx)\,d(\mu\circ T^{-i})(x)\\ \because\ T\text{ preserves }\mu&=\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_{T^i(A_n)}D(x,Tx)\,d\mu(x)=\int_X D(x,Tx)\,d\mu(x)=\|T\|_1.\end{aligned}$$

Thus $T_A\in[G\curvearrowright X]_1$ and $\|T_A\|_1\leq\|T\|_1$ as claimed. □

## 3. Full groups of locally compact group actions

In this section we narrow down the generality of the narrative and focus on actions of *locally compact* Polish groups, or equivalently of locally compact second-countable groups. Such restrictions enlarge our toolbox in a number of ways. For instance, all locally compact Polish group actions admit cross sections to which the so-called Voronoi tessellations can be associated. A concise reminder of the needed facts can be found in Appendices B.2 and B.1.

Throughout this section we fix a measure-preserving action $G\curvearrowright X$ of a locally compact Polish group on a standard probability space $(X,\mu)$, a compatible norm $\|\cdot\|$ on $G$ and a dense subset $H\subseteq G$. As before, $D:\mathscr{R}_G\to\mathbb{R}_+$ denotes the family of metrics on the orbits induced by the norm.

**Definition 3.1.** A measure-preserving transformation $T : A \to B$ between two measurable sets $A, B \subseteq X$ is said to be $H$-**decomposable** if there exist a measurable partition $A = \bigsqcup_{k\in\mathbb{N}} A_k$ and elements $h_k \in H$ such that $T\restriction_{A_k} = h_k$ for all $k \in \mathbb{N}$.

The purpose of this section is to prove that any element of a full group can be approximated by an element that piecewise acts by elements in a given dense set. More precisely, the following result will be established.

**Theorem 3.2.** *Let $G \curvearrowright X$ be a measure-preserving action of a locally compact Polish group, let $\|\cdot\|$ be a compatible norm on $G$ with the associated metric on the orbits $D : \mathscr{R}_G \to \mathbb{R}_+$, and let $H \subseteq G$ be a dense set. For any $T \in [G \curvearrowright X]$ and any $\epsilon > 0$ there exists an $H$-decomposable transformation $S \in [G \curvearrowright X]$ such that* $\operatorname{ess\,sup}_{x\in X} D(Tx, Sx) < \epsilon$.

Our main motivation for proving this result was the following immediate corollary.

**Corollary 3.3.** *Let $G \curvearrowright X$ be a measure-preserving action of a locally compact Polish group, let $\|\cdot\|$ be a compatible norm on $G$, and let $H \subseteq G$ be a dense subgroup. Then $[H \curvearrowright X]_1$ is dense in $[G \curvearrowright X]_1$.*

*Remark* 3.4. Theorem 3.2 improves upon the conclusion of [CLM18, Thm. 2.1], which shows that $[H \curvearrowright X]$ is dense in $[G \curvearrowright X]$ whenever $H$ is a dense subgroup of $G$. While the proof presented below establishes density in a much stronger topology through more elementary means, we note that methods of [CLM18] apply to all suitable actions of Polish groups (as already mentioned in Theorem 2.3 therein), whereas our approach crucially uses local compactness of the acting group to deduce existence of various cross sections.

We begin by setting up the context for the argument. First of all, note that we may (and do) assume without loss of generality that the norm $\|\cdot\|$ is proper. This follows from the simple observation that if $\|\cdot\|'$ and $\|\cdot\|$ are arbitrary compatible norms, then for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|g\| < \delta$ implies $\|g\|' < \epsilon$, and therefore also $D(x, y) < \delta \implies D'(x, y) < \epsilon$ for all $(x, y) \in \mathscr{R}_G$. In particular, if Theorem 3.2 is proved for one compatible norm it is automatically valid for all of them. Another harmless simplification is that $H$ can be assumed countable, as we can always shrink it while retaining density in $G$.

Let $\mathscr{C}$ be a cross section for a measure-preserving action $G \curvearrowright X$ and let $\mathscr{W}$ be a tessellation over $\mathscr{C}$ (see Appendix B.2). Let $\nu_{\mathscr{W}}$ be the push-forward measure $(\pi_{\mathscr{W}})_*\mu$ on the cross section and $(\mu_c)_{c\in\mathscr{C}}$ be the disintegration of $\mu$ over $(\pi_{\mathscr{W}}, \nu_{\mathscr{W}})$ (Theorem B.1).

**Definition 3.5.** Two Borel sets $A, B \subseteq X$ are said to be

- $\mathscr{W}$**-proportionate** if the equivalence $\mu_c(A) = 0 \iff \mu_c(B) = 0$ holds for $\nu_{\mathscr{W}}$-almost all $c \in \mathscr{C}$;
- $\mathscr{W}$**-equimeasurable** if $\mu_c(A) = \mu_c(B)$ for $\nu_{\mathscr{W}}$-almost all $c \in \mathscr{C}$.

A measure-preserving map $T : A \to B$ is $\mathscr{W}$**-coherent** if $\mu$-almost surely one has $\pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(Tx)$.

We begin by stating the following elementary observation.

**Lemma 3.6.** *Let $\mathscr{W}$ be an $N$-lacunary tessellation for an open neighborhood of the identity $N \subseteq G$. If $A, B \subseteq N \cdot \mathscr{C}$ are $\mathscr{W}$-proportionate Borel sets then $\mu(B \setminus N \cdot \pi_{\mathscr{W}}(A)) = 0$.*

For the proof of the next lemma we need the notion of a suitable action, introduced by H. Becker [Bec13, Def. 1.2.7]. A measure-preserving action $G \curvearrowright X$ is **suitable** if for all Borel sets $A, B \subseteq X$ one of the two options holds:

(1) for any open neighborhood of the identity $M \subseteq G$ there exists $g \in M$ such that $\mu(gA \cap B) > 0$;
(2) there exist Borel sets $A' \subseteq A$, $B' \subseteq B$ such that $\mu(A \setminus A') = 0 = \mu(B \setminus B')$ and an open neighborhood of the identity $M \subseteq G$ such that $M \cdot A' \cap B' = \varnothing$.

All measure-preserving actions of locally compact Polish groups are suitable (see [Bec13, Thm.1.2.9]).

**Lemma 3.7.** *Suppose $\mathscr{W}$ is an $N$-lacunary tessellation for some open neighborhood of the identity $N \subseteq G$. For all non-negligible $\mathscr{W}$-proportionate Borel sets $A, B \subseteq N \cdot \mathscr{C}$ there exists an open set $U \subseteq N^2$ such that $\mu(gA \cap B) > 0$ for all $g \in U$.*

*Proof.* Let $H_1 = H \cap N$ be a dense set in $N$ and put $A_1 = H_1 \cdot A$. We apply the dichotomy in the definition of a suitable action to the sets $A_1$ and $B$ and show that item (2) cannot hold.

Indeed, suppose there exist $A_1' \subseteq A_1$ and $B' \subseteq B$ such that $\mu(A_1 \setminus A_1') = 0 = \mu(B \setminus B')$ and an open neighborhood of the identity $M \subseteq G$ such that $M \cdot A_1' \cap B' = \varnothing$. Set $A' = \bigcap_k (h_k^{-1} A_1' \cap A)$, where $(h_k)_{k\in\mathbb{N}}$ is an enumeration of $H_1$, and note that $\mu(A \setminus A') = 0$ and $MH_1 \cdot A' \cap B' = \varnothing$ simply because $H_1 \cdot A' \subseteq A_1'$. Since $H_1$ is dense in $N$, we have $N \subseteq MH_1$

and thus $N \cdot A' \cap B' = \varnothing$. The letter, however, contradicts the conclusion of Lemma 3.6 applied to non-negligible $\mathscr{W}$-proportionate sets $A'$ and $B'$.

We are left with the alternative of the item (1), and so there has to exist some $g \in N$ such that $\mu(gA_1 \cap B) > 0$. Since $A_1 = H_1 \cdot A$, there exist $h \in H_1$ such that $\mu(ghA \cap B) > 0$. It remains to note that $\mu(g' A \cap B) > 0$ is an open condition, since the action of $G$ on the measure algebra is continuous. □

**Lemma 3.8.** *Suppose $\mathscr{W}$ is an $N$-lacunary tessellation for an open neighborhood of the identity $N \subseteq G$. For any non-empty open $V \subseteq N$ and for any non-negligible Borel set $A \subseteq X$ there exists $h \in H$ such that*

$$\mu(\{x \in A : hx \in V \cdot \mathscr{C} \textit{ and } \pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(hx)\}) > 0.$$

*Proof.* Let $\zeta : X \to \mathscr{W}$ be the Borel bijection $\zeta(x) = (\pi_{\mathscr{W}}(x), x)$ and consider the push-forward measure $\zeta_*\mu$, which for $Z \subseteq \mathscr{W}$ can be expressed as $\zeta_*\mu(Z) = \int_{\mathscr{C}} \mu_c(Z_c)\, d\nu_{\mathscr{W}}(c)$. Let $(h_n)_{n\in\mathbb{N}}$ be an enumeration of $H$ and set $W_n = \{(c,x) \in \mathscr{W} : \pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(h_n x)$ and $h_n x \in V \cdot \mathscr{C}\}$. We claim that $\bigcup_n W_n = \mathscr{W}$. Indeed, for each $(c,x) \in \mathscr{W}$ the set of $g \in G$ such that $gx \in V \cdot c$ is non-empty and open, hence there is $h_n \in H$ such that $h_n x \in V \cdot c$.

Since $0 < \mu(A) = \zeta_*\mu(\zeta(A))$ by assumption, there exists $W_n$ such that $\zeta_*\mu(\zeta(A) \cap W_n) > 0$, which translates into

$$\mu(\{x \in A : h_n x \in V \cdot \mathscr{C} \text{ and } \pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(h_n x)\}) > 0.$$ □

**Lemma 3.9.** *For all non-negligible $\mathscr{W}$-proportionate Borel sets $A, B \subseteq X$ there exists $h \in H$ such that*

$$\mu(\{x \in A : hx \in B \textit{ and } \pi_{\mathscr{C}}(x) = \pi_{\mathscr{C}}(hx)\}) > 0.$$

*Proof.* We are going to reduce the setup of this lemma to that of Lemma 3.7. Let $V \subseteq G$ be so small that $\mathscr{W}$ is $V^3$-lacunary. Apply Lemma 3.8 to find $h_1 \in H$ such that for

$$A' = \{x \in A : h_1 x \in V \cdot \mathscr{C} \text{ and } \pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(h_1 x)\}$$

one has $\mu(A') > 0$. Set $A_1 = h_1 A'$, $B_1 = \pi^{-1}(\{c \in \mathscr{C} : \mu_c(A_1) > 0\}) \cap B$ and note that $A_1$ and $B_1$ are non-negligible $\mathscr{W}$-proportionate sets; moreover $A_1 \subseteq V \cdot \mathscr{C}$.

Repeat the same step for the set $B_1$ to find $h_2 \in H$ such that for $B_1' = \{x \in B_1 : h_2 x \in V \cdot \mathscr{C}$ and $\pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(h_2 x)\}$ we have $\mu(B_1') > 0$. Set $B_2 = h_2 B_1'$, $A_2 = A_1 \cap \pi^{-1}(\{c \in \mathscr{C} : \mu_c(B_2) > 0\})$. Once again, sets $A_2$ and $B_2$ are non-negligible, $\mathscr{W}$-proportionate and are both contained in $V \cdot \mathscr{C}$. Lemma 3.7 applies, and yields an open set $U \subseteq V^2$ such that $\mu(gA_2 \cap B_2) > 0$ for all $g \in U$. Note that since $U \subseteq V^2$ and $\mathscr{W}$ is $V^3$-lacunary, $\pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(gx)$ holds for all $x \in V \cdot \mathscr{C}$ and $g \in U$. We conclude that $\mu(h_2^{-1} g h_1 A \cap B) > 0$ for all $g \in U$ and any $h \in h_2^{-1} U h_1 \cap H$ satisfies the conclusions of the lemma. □

**Lemma 3.10.** *For all $\mathscr{W}$-equimeasurable Borel sets $A, B \subseteq X$ there exists a $\mathscr{W}$-coherent $H$-decomposable measure-preserving bijection $T : A \to B$.*

*Proof.* Let $(h_n)_{n\in\mathbb{N}}$ be an enumeration of $H$. Consider the set

$$A_0 = \{x \in A : \pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(h_0 x) \text{ and } h_0 x \in B\},$$

and let $B_0 = h_0 A_0$. Note that the sets $A \setminus A_0$ and $B \setminus B_0$ are $\mathscr{W}$-equimeasurable, so we may continue in the same fashion and construct sets $A_k$ such that

$$A_k = \Big\{x \in A \setminus \bigsqcup_{i<k} A_i : \pi_{\mathscr{W}}(x) = \pi_{\mathscr{W}}(h_k x) \text{ and } h_k x \in B \setminus \bigsqcup_{i<k} B_i\Big\}.$$

We define $T : \bigsqcup_{k\in\mathbb{N}} A_k \to \bigsqcup_{k\in\mathbb{N}} B_k$ by the condition $Tx = h_k x$ for $x \in A_k$ and note that

$$\mu(A \setminus \bigsqcup_{k\in\mathbb{N}} A_k) = 0 = \mu(B \setminus \bigsqcup_{k\in\mathbb{N}} B_k)$$

by Lemma 3.9. □

**Lemma 3.11.** *Suppose $\mathscr{W}$ is a cocompact tessellation and let $A, B \subseteq X$ be $\mathscr{W}$-equimeasurable Borel sets. For any $\epsilon > 0$ and any $\mathscr{W}$-coherent measure-preserving $T : A \to B$ there exists is a $\mathscr{W}$-coherent $H$-decomposable $\tilde{T} : A \to B$ such that* $\operatorname{ess\,sup}_{x\in A} D(Tx, \tilde{T}x) < \epsilon$.

*Proof.* Let $\mathscr{V}$ be a $K'$-cocompact tessellation over some cross section $\mathscr{C}'$ such that the diameter of each region in $\mathscr{V}$ is less than $\epsilon$. Suppose $\mathscr{W}$ is $K$-cocompact. By Lemma B.9, we can find a finite partition of $\mathscr{C}' = \bigsqcup_{i\leq n} \mathscr{C}_i'$ such that each $\mathscr{C}_i'$ is $K'K^2K'$-lacunary, which guarantees that for each $i$ every $\mathscr{W}_c$ intersects at most one class $\mathscr{V}_c$, $c \in \mathscr{C}_i'$. For each $i, j < n$ set $A_{(i,j)} = \{x \in A : \pi_{\mathcal{V}}(x) \in \mathcal{C}'_i, \pi_{\mathcal{V}}(Tx) \in \mathcal{C}'_j\}$ and $B_{(i,j)} = TA_{(i,j)}$. We re-enumerate sets $A_{(i,j)}$ and $B_{(i,j)}$ as a sequence $A_k$, $B_k$, $k \le n^2$ and note that for all $x, y \in A_k$ one has

$$\pi_{\mathcal{C}}(x) = \pi_{\mathcal{C}}(y) \implies \big(\pi_{\mathcal{C}'}(x) = \pi_{\mathcal{C}'}(y) \text{ and } \pi_{\mathcal{C}'}(Tx) = \pi_{\mathcal{C}'}(Ty)\big).$$

Moreover, sets $A_k$ and $T(A_k)$ are $\mathcal{V}$-equimeasurable, so Lemma 3.10 yields $\mathcal{V}$-coherent $H$-decomposable measure-preserving maps $T_k : A_k \to T(A_k)$. Set $\tilde{T} : A \to B$ by the condition $\tilde{T}x = T_k x$ whenever $x \in A_k$. It is easy to check that $\tilde{T}$ is as claimed. □

*Proof of Theorem 3.2.* Fix a cocompact cross section $\mathcal{C}$, and let $U_n \subseteq G$ be the open ball of radius $n$. For all $n \in \mathbb{N}$ select based on Lemma B.9 a finite sequence of cocompact cross sections $\mathcal{C}^n_1, \dots, \mathcal{C}^n_{k_n}$ such that each $\mathcal{C}^n_i$ is $U_n$-lacunary and $\mathcal{C} = \bigcup_{i=1}^{k_n} \mathcal{C}^n_i$. Re-enumerate cross sections $\mathcal{C}^n_i$, $n \in \mathbb{N}$, $1 \le i \le k_n$, into a sequence $(\mathcal{C}_k)_{k=0}^{\infty}$ and let $\mathcal{V}_k$ be the Voronoi tessellation over $\mathcal{C}_k$.

Let $A_0 = \{x \in X : \pi_{\mathcal{V}_0}(x) = \pi_{\mathcal{V}_0}(Tx)\}$, and use Lemma 3.11 to find an $H$-decomposable measure-preserving map $T_0 : A_0 \to T(A_0)$ such that $\operatorname{ess\,sup}_{x \in A_0} D(T_0 x, Tx) < \epsilon$. Set $A_k = \{x \in X : \pi_{\mathcal{V}_k}(x) = \pi_{\mathcal{V}_k}(Tx) \text{ and } x \notin \bigsqcup_{l<k} A_l\}$ and observe that $A_k$, $k \in \mathbb{N}$, form a partition of $X$. Find transformations $T_k : A_k \to T(A_k)$ by repeated applications of Lemma 3.11 applied to the tessellations $\mathcal{V}_k$. The element $S \in [G \curvearrowright X]$ defined by $Sx = T_k x$ satsifies the conclusion of the theorem. □

## 4. Derived $\mathrm{L}^1$ full groups of actions of locally compact amenable groups

We impose one further restriction on the acting group, and consider a second countable *amenable* normed group $(G, \|\cdot\|)$. As before, we fix a measure-preserving action $G \curvearrowright X$ on a standard probability space $(X, \mu)$, and let $D : \mathcal{R}_G \to \mathbb{R}_+$ denote the family of metrics induced onto the orbits by the norm.

4.1. **Dense chain of subgroups.** An equivalence relation $\mathcal{R} \subseteq \mathcal{R}_G$ is said to be **uniformly bounded** if there is $M > 0$ and $X' \subseteq X$ such that $\mu(X \setminus X') = 0$ and $\sup_{(x_1,x_2) \in \mathcal{R}'} D(x_1, x_2) \le M$, where $\mathcal{R}' = \mathcal{R} \cap X' \times X'$.

**Lemma 4.1.** *Let $(G, \|\cdot\|)$ be a locally compact amenable Polish normed group acting on a standard probability space $(X, \mu)$. There exists a sequence of cross sections $\mathcal{C}_n$ and tessellations $\mathcal{W}_n$ over $\mathcal{C}_n$ such that for all $n \in \mathbb{N}$*

*(1) $\mathcal{R}_{\mathcal{W}_n} \subseteq \mathcal{R}_{\mathcal{W}_{n+1}}$ and $\bigcup_{k \in \mathbb{N}} \mathcal{R}_{\mathcal{W}_k} = \mathcal{R}_G$ (up to a null set);*
*(2) $\mathcal{R}_{\mathcal{W}_n}$ is uniformly bounded.*

*Proof.* Let $\mathcal{C}$ be a cocompact cross section, $\mathcal{V}_{\mathcal{C}}$ be the Voronoi tessellation over $\mathcal{C}$, $\pi_{\mathcal{V}_{\mathcal{C}}} : X \to \mathcal{C}$ be the associated reduction, and $\nu = (\pi_{\mathcal{V}_{\mathcal{C}}})_* \mu$ be the push-forward measure on $\mathcal{C}$. Recall that $\mathcal{R}_{\mathcal{V}_{\mathcal{C}}}$ is uniformly bounded, since $\mathcal{C}$ is cocompact. Let $E$ be the equivalence relation obtained by restricting $\mathcal{R}_G$ onto $\mathcal{C}$. By a theorem of A. Connes, J. Feldman, and B. Weiss [CFW81], $E$ is hyperfinite on an invariant set of $\nu_{\mathcal{C}}$-full measure. Throwing away a $G$-invariant null set, we may write $E = \bigcup_{n \ge 1} E_n$ where $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of Borel equivalence relations with finite classes. For $m, n \in \mathbb{N}$ define $A_{n,m}$ to be the set of points in the cross section whose $E_n$-class is bounded in diameter by $m$:

$$A_{n,m} = \big\{c \in \mathcal{C} : D(c_1, c_2) \le m \text{ for all } c_1, c_2 \in \mathcal{C} \text{ such that } c_i E_n c\big\}.$$

Note that the sets $A_{n,m}$ are $E_n$-invariant, nested, and $\bigcup_m A_{n,m} = \mathcal{C}$ for every $n \in \mathbb{N}$. Pick $m_n$ so large as to ensure $\nu(\mathcal{C} \setminus A_{n,m_n}) < 2^{-n}$ and let $B_n = \bigcap_{k \ge n} A_{k,m_k}$. The sets $B_n$ are $E_n$-invariant, increasing, and $\lim_n \nu(B_n) = \nu(\mathcal{C})$. Define equivalence relations $F_n$ on $\mathcal{C}$ by setting $c_1 F_n c_2$ whenever $c_1 = c_2$ or $c_1, c_2 \in B_n$ and $c_1 E_n c_2$. Note that $D(c_1, c_2) \le m_n$ whenever $c_1 F_n c_2$. Let $\mathcal{C}_n \subseteq \mathcal{C}$ be a Borel transversal for $F_n$ and define $\mathcal{W}_n = \{(c, x) \in \mathcal{C}_n \times X : c F_n \pi_{\mathcal{V}_{\mathcal{C}}}(x)\}$. It is straightforward to check that $\mathcal{W}_n$ is a tessellation over $\mathcal{C}_n$ and equivalence relations $\mathcal{R}_{\mathcal{W}_n}$ satisfy the conclusions of the lemma. □

The equivalence relations $\mathcal{R}_{\mathcal{W}_n}$ produced by Lemma 4.1 give rise to a nested chain of groups $[\mathcal{R}_{\mathcal{W}_0}] \le [\mathcal{R}_{\mathcal{W}_1}] \le \cdots$. Some basic facts about such groups can be found in Appendix B.2. The following lemma establishes that such a chain is dense in the *derived* $\mathrm{L}^1$ full group.

**Lemma 4.2.** *Let $(G, \|\cdot\|)$ be a locally compact amenable Polish normed group acting on a standard probability space $(X, \mu)$ and let $(\mathcal{R}_n)_{n \in \mathbb{N}}$ be a sequence of equivalence relations as in Lemma 4.1. If the action is aperiodic, then the union $\bigcup_n [\mathcal{R}_n]$ is contained in the derived $\mathrm{L}^1$ full group $D([\mathcal{R}_G]_1)$ and is dense therein.*

*Proof.* By definition, $[\mathcal{R}_n]$ is a subgroup of $[\mathcal{R}_G]$. Since equivalence relations $\mathcal{R}_n$ are uniformly bounded, we actually have $[\mathcal{R}_n] \leq [G \curvearrowright X]_1$, and the topology induced by the $\mathrm{L}^1$ metric on $[\mathcal{R}_n]$ coincides with the topology induced from $[\mathcal{R}_G]$. Moreover, in view of Proposition B.7, $[\mathcal{R}_n]$ is topologically generated by periodic transformations, so we actually have $[\mathcal{R}_n] \leq D([G \curvearrowright X]_1)$ as a consequence of Lemma C.9 and Corollary C.11.

It remains to verify that the union $\bigcup_n [\mathcal{R}_n]$ is dense in $D([G \curvearrowright X]_1)$. To this end, recall that by Corollary C.11 the derived $\mathrm{L}^1$ full group $D([G \curvearrowright X]_1)$ is topologically generated by involutions. So let $U \in D([G \curvearrowright X]_1)$ be an involution and set $X_n = \{x \in X : x\mathcal{R}_n Ux\}$, $n \in \mathbb{N}$. Note that $X_n$ is $U$-invariant since $U$ is an involution. Moreover, $\mu(X_n) \to 1$ as $\bigcup_n \mathcal{R}_n = \mathcal{R}_G$, and thus the induced transformations $U_{X_n} \in [\mathcal{R}_n]$ converge to $U$ in the topology of $[G \curvearrowright X]_1$. We conclude that $\bigcup_n [\mathcal{R}_n]$ is dense in the derived $\mathrm{L}^1$ full group. □

**Corollary 4.3.** *Let $(G, \|\cdot\|)$ be a locally compact amenable Polish normed group acting on a standard probability space $(X,\mu)$. Suppose that almost every orbit of the action is uncountable. There exists a chain $H_0 \leq H_1 \leq \cdots \leq D([G \curvearrowright X]_1)$ of closed subgroups such that $\bigcup_n H_n$ is dense in $D([G \curvearrowright X]_1)$, and each $H_n$ is isomorphic to $\mathrm{L}^0(Y_n, \nu_n, \mathrm{Aut}([0,1],\lambda))$ for some standard Lebesgue space $(Y_n, \nu_n)$. If moreover each orbit of the action has measure zero, then one can assume that all $(Y_n,\nu_n)$ are atomless and each $H_n$ is isomorphic to $\mathrm{L}^0([0,1],\lambda,\mathrm{Aut}([0,1],\lambda))$.*

*Proof.* Apply Lemmas 4.1 and 4.2 to get a dense chain of subgroups $[\mathcal{R}_0] \leq [\mathcal{R}_1] \leq \cdots \leq D([G \curvearrowright X]_1)$ and use Corollary B.14 to deduce that each $[\mathcal{R}_n]$ has the desired form. □

4.2. **Whirly amenability.** Lemma 4.2 is a powerful tool to deduce various dynamical properties of derived $\mathrm{L}^1$ full groups. Recall that a Polish group $G$ is said to be **whirly amenable** if it is amenable and for any continuous action of $G$ on a compact space any invariant measure is supported on the set of fixed points of the action. In particular, each such action has to have some fixed points, so whirly amenable groups are extremely amenable.

**Proposition 4.4.** *Let $\mathcal{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space $(X,\mu)$. If $\mu$ is atomless, then the full group $[\mathcal{R}]$ is whirly amenable.*

*Proof.* In view of Proposition B.6, the full group $[\mathcal{R}]$ is isomorphic to

$$\mathrm{L}^0([0,1],\lambda,\mathrm{Aut}([0,1],\lambda))^{\epsilon_0} \times \mathrm{Aut}([0,1],\lambda)^{\kappa_0} \times \prod_{n\geq 1} \mathrm{L}^0([0,1],\lambda,\mathfrak{S}_n)^{\epsilon_0},$$

where $\mathfrak{S}_n$ is the group of permutations of an $n$-element set, and $\epsilon_n \in \{0,1\}$, $\kappa_0 \leq \aleph_0$. Since a product of whirly amenable groups is whirly amenable, it suffices to show that the groups appearing in the decomposition above, namely $\mathrm{L}^0([0,1],\lambda,\mathrm{Aut}([0,1],\lambda))$, $\mathrm{Aut}([0,1],\lambda)$, and $\mathrm{L}^0([0,1],\lambda,\mathfrak{S}_n)$, $n \geq 1$, are whirly amenable.

The group $\mathrm{Aut}([0,1],\lambda)$ is whirly amenable by [GP02] (it is, in fact, a so-called Levy group). Finally, we apply a theorem of V. Pestov and F. M. Schneider [PS17], which asserts that a group $\mathrm{L}^0([0,1],\lambda,G)$ is whirly amenable if and only if $G$ is amenable. This readily implies whirly amenability of $\mathrm{L}^0([0,1],\lambda,\mathfrak{S}_n)$ and $\mathrm{L}^0([0,1],\lambda,\mathrm{Aut}([0,1],\lambda))$. □

*Remark* 4.5. The assumption of $\mu$ being atomless cannot be omitted in the proposition above. Indeed, $[\mathcal{R}]$ will factor onto $\mathfrak{S}_n$ for some $n \geq 2$, as long as an $\mathcal{R}$-class contains at least 2 atoms of $\mu$ of the same measure. However, if all $\mu$-atoms within each $\mathcal{R}$-class have distinct measures, then the restriction of $[\mathcal{R}]$ onto the atomic part of $X$ is trivial, which suffices to conclude the whirly amenability of the group $[\mathcal{R}]$.

**Theorem 4.6.** *Let $G \curvearrowright X$ be a measure-preserving action of an amenable locally compact Polish normed group on a standard probability space $(X,\mu)$. If the action is aperiodic, then the derived $\mathrm{L}^1$ full group $D([G \curvearrowright X]_1)$ is whirly amenable. In particular, $[G \curvearrowright X]_1$ is amenable.*

*Proof.* Lemma 4.2 shows that $D([G \curvearrowright X]_1)$ has an increasing dense chain of subgroups $H_n$ of the form $[\mathcal{R}_n]$, where $\mathcal{R}_n$ are smooth measurable equivalence relations on $X$. Proposition 4.4 applies and shows that groups $H_n$ are whirly amenable. The latter is sufficient to conclude whirly amenability of $D([G \curvearrowright X]_1)$, as any invariant measure for the action of the derived group is also invariant for the induced $H_n$ actions, hence it has to be supported on the intersection of fixed points of all $H_n$, which coincides with the set of fixed points for the action of $D([G \curvearrowright X]_1)$.

The fact that $[G \curvearrowright X]_1$ is amenable now follows from the fact that every abelian group is amenable, and every amenable extension of an amenable group must itself be amenable (for instance,see [BdlHV08, Prop. G.2.2]). □

*Remark* 4.7. Note that in general $[G \curvearrowright X]_1$ is not extremely amenable. For flows it factors onto $\mathbb{R}$ via the index map (see Section 5) and $\mathbb{R}$ admits continuous actions on compact spaces without fixed points, so $[\mathbb{R} \curvearrowright X]_1$ is not extremely amenable (in particular, it is not whirly amenable).

**Corollary 4.8.** *Let $G \curvearrowright X$ be a free measure-preserving action of a unimodular locally compact Polish group on a standard probability space $(X,\mu)$. The following are equivalent:*

*(i)* *$G$ is amenable.*
*(ii)* *$[G \curvearrowright X]_1$ is amenable.*
*(iii)* *The derived $\mathrm{L}^1$ full group $D([G \curvearrowright X]_1)$ is amenable.*
*(iv)* *The derived $\mathrm{L}^1$ full group $D([G \curvearrowright X]_1)$ is extremely amenable.*
*(v)* *The derived $\mathrm{L}^1$ full group $D([G \curvearrowright X]_1)$ is whirly amenable.*

*Proof.* We established the implication (i) $\Longrightarrow$ (v) in Theorem 4.6. The chain of implications (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) is straightforward, and (iii) $\Longrightarrow$ (ii) follows from the stability of amenability under group extensions, which was already discussed in Theorem 4.6.

For the last implication (ii) $\Longrightarrow$ (i), first recall that the orbit full group of the action is generated by involutions (as is any full group). It follows that the orbit full group is topologically generated by involutions whose cocycles are integrable (actually, one can even ask that the cocycles are bounded). In particular, the $\mathrm{L}^1$ full group $[G \curvearrowright X]_1$ is dense in the orbit full group, and so assuming (ii) we conclude that the orbit full group $[G \curvearrowright X]$ is amenable. The amenability of $G$ then follows from [CLM18, Thm. 5.1]. □

*Remark* 4.9. We have to require unimodularity in order to be able to apply [CLM18, Thm. 5.1]. It seems likely that the unimodularity hypothesis can be dropped in this result, but we do not pursue this question further.

4.3. **Topological generators.** We now concern ourselves with the question of determining the topological rank of derived full groups. Our approach will be based on the dense chain of subgroups established in Corollary 4.3, and the first step is to study the topological rank of the group $\mathrm{L}^0([0,1],\mathrm{Aut}([0,1]))$.

Let $(Y,\nu)$ and $(Z,\lambda)$ be standard Lebesgue spaces. Consider the product space $Y \times Z$ equipped with the product measure $\nu \times \lambda$ and let $\mathscr{R}$ be the product of the discrete equivalence relation on $Y$ and the anti-discrete on $Z$; in other words, $(y_1,z_1)\mathscr{R}(y_2,z_2)$ if and only if $y_1 = y_2$. As discussed in Appendix B.1, the following two groups are one and the same:

(1) the full group $[\mathscr{R}]$;
(2) the topological group $\mathrm{L}^0(Y,\nu,\mathrm{Aut}(Z,\lambda))$.

Suppose that $(Z,\lambda)$ is atomless. Pick a hyperfinite ergodic equivalence relation $E$ on $Z$ so that $\mathrm{APER}(Z) \cap [E]$ is dense in $\mathrm{Aut}(Z,\lambda)$. Set $\mathscr{R}_0 = \mathrm{id}_Y \times E$ to be the equivalence relation on $Y \times Z$ given by $(y_1,z_1)\mathscr{R}_0(y_2,z_2)$ whenever $y_1 = y_2$ and $z_1 E z_2$. A standard application of the Jankov-von Neumann uniformization theorem yields the following lemma.

**Lemma 4.10.** $\mathrm{APER}(Y \times Z) \cap [\mathscr{R}_0]$ *is dense in* $[\mathscr{R}] \simeq \mathrm{L}^0(Y,\nu,\mathrm{Aut}([0,1],\lambda))$.

Our first goal is to establish that the topological rank of $[\mathscr{R}]$ is 2. We do so by first verifying this under the assumption that $(Y,\nu)$ is atomless, and then deducing the general case.

We say that a topological group $G$ is **generically $k$-generated**, $k \in \mathbb{N}$, if the set of $k$-tuples $(g_1,\dots,g_k) \in G^k$ that generate a dense subgroup of $G$ is dense in $G^k$. Note that the set of such tuples is always a $G_\delta$ set, so if $G$ is generically $k$-generated, then a comeager set of $k$-tuples generates a dense subgroup of $G$.

**Proposition 4.11.** *Suppose that $(Y,\nu)$ is atomless. The group $[\mathscr{R}]$ is generically 2-generated.*

*Proof.* By Theorem 5.1 from [LM16] the set of pairs $(S,T) \in (\mathrm{APER}(Y \times Z) \cap [\mathscr{R}_0]) \times [\mathscr{R}_0]$ such that $\overline{\langle S,T\rangle} = [\mathscr{R}_0]$ is dense $G_\delta$ in $(\mathrm{APER} \cap [\mathscr{R}_0]) \times [\mathscr{R}_0]$ for the uniform topology. In view of Lemma 4.10, this implies that $[\mathscr{R}]$ is generically 2-generated. □

**Lemma 4.12.** *For all topological groups $G$ and $H$ one has*

$$\mathrm{rk}(G \times H) \geq \max\{\mathrm{rk}(G),\mathrm{rk}(H)\}.$$

*If $G \times H$ is generically $k$-generated, then so are $G$ and $H$ as well.*

*Proof.* The inequality on ranks is immediate from the trivial observation that if $\langle (g_1,h_1),\dots,(g_k,h_k)\rangle$ is dense in $G \times H$, then $\langle g_1,\dots,g_k\rangle$ is dense in $G$ and $\langle h_1,\dots,h_k\rangle$ is dense in $H$.

Suppose $G \times H$ is generically $k$-generated, pick an open set $U \subseteq G^k$ and note that $U \times H^k$ naturally corresponds to an open subset of $(G \times H)^k$ via the isomorphism $(G \times H)^k \simeq G^k \times H^k$. Since $G \times H$ is generically $k$-generated, there is a tuple $(g_i,h_i)_{i=1}^k \in (G \times H)^k$ that generates a dense subgroup and $(g_i,h_i)_{i=1}^k \in U \times H^k$. We conclude that $(g_i)_{i=1}^k \in U$ generates a dense subgroup of $G$ and the lemma follows. □

**Lemma 4.13.** *For any separable topological group $G$*

$$\mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G)) = \mathrm{rk}\big(\mathrm{L}^0([0,1],\lambda,G)\times G^{\mathbb{N}}\big).$$

*If* $\mathrm{L}^0([0,1],\lambda,G)$ *is generically $k$-generated for some $k\in\mathbb{N}$, then so is* $\mathrm{L}^0([0,1],\lambda,G)\times G^{\mathbb{N}}$.

*Proof.* In view of Lemma 4.12, $\mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G)) \le \mathrm{rk}\big(\mathrm{L}^0([0,1],\lambda,G)\times G^{\mathbb{N}}\big)$, and since the group $G$ is separable, we only need to consider the case when $\mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G))$ is finite.

It is notationally convenient to shrink the interval and work with the group $\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$ instead as it can naturally be viewed as a closed subgroup of $\mathrm{L}^0([0,1],\lambda,G)$ via the identification $f\times(g_i)_{i\in\mathbb{N}}\mapsto\zeta$, where

$$\zeta(t)=\begin{cases} f(t) & \text{if } 0\le t<1/2,\\ g_i & \text{if } 1-2^{-i-1}\le t<1-2^{-i-2} \text{ for } i\in\mathbb{N}.\end{cases}$$

Pick families $(\xi_l)_{l\in\mathbb{N}}$ dense in $\mathrm{L}^0([0,1/2],\lambda,G)$, and $(h_m)_{m\in\mathbb{N}}$ dense in $G$.

Let us call a function $\alpha:\mathbb{N}\to\mathbb{N}$ a multi-index if $\alpha(i)=0$ for all but finitely many $i\in\mathbb{N}$. We use $\mathbb{N}^{<\mathbb{N}}$ to denote the set of all multi-indices. Given $\alpha\in\mathbb{N}^{<\mathbb{N}}$, let $h_\alpha=(h_{\alpha(i)})_{i\in\mathbb{N}}$ be an element of $G^{\mathbb{N}}$. Note that $\{h_\alpha:\alpha\in\mathbb{N}^{<\mathbb{N}}\}$ is dense in $G^{\mathbb{N}}$ and thus $\{\xi_l\times h_\alpha : l\in\mathbb{N},\alpha\in\mathbb{N}^{<\mathbb{N}}\}$ is a dense family in $\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$.

Pick a tuple $f_1,\dots,f_k\in\mathrm{L}^0([0,1],\lambda,G)$ that generates a dense subgroup. For each pair $(l,\alpha)\in\mathbb{N}\times\mathbb{N}^{<\mathbb{N}}$, there exists a sequence of reduced words $(w_n^{l,\alpha})_{n\in\mathbb{N}}$ in the free group on $k$ generators such that $w_n^{l,\alpha}(f_1,\dots,f_k)$ converges to $\xi_l\times h_\alpha$ in measure. By passing to a subsequence, we may assume that $w_n^{l,\alpha}(f_1,\dots,f_k)\to\xi_l\times h_\alpha$ pointwise almost surely. In other words, the set $P_{l,\alpha}=\big\{t\in[0,1]: w_n^{l,\alpha}(f_1,\dots,f_k)(t)\to(\xi_l\times h_\alpha)(t)\big\}$ has Lebesgue measure 1 for all $(l,\alpha)\in\mathbb{N}\times\mathbb{N}^{<\mathbb{N}}$, and hence so does $P=\bigcap_{l\in\mathbb{N}}\bigcap_{\alpha\in\mathbb{N}^{<\mathbb{N}}}P_{l,\alpha}$.

Pick some $t_j\in P\cap[1-2^{-j-1},1-2^{-j-2})$, $j\in\mathbb{N}$, and set

$$\tilde f_i(t)=\begin{cases} f_i(t) & \text{for } 0\le t<1/2,\\ f_i(t_j) & \text{for } 1-2^{-j-1}\le t<1-2^{-j-2} \text{ for } j\in\mathbb{N}.\end{cases}$$

Elements $\tilde f_i$ naturally belong to $\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$, and we claim that they generate a dense subgroup therein, thus witnessing $\mathrm{rk}(\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}})\le k$. To this end note that for any $\xi_l\times h_\alpha\in\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$, we have $w_n^{l,\alpha}(f_1,\dots,f_k)\to\xi_l\times h_\alpha$ pointwise almost surely. In particular, $w_n^{l,\alpha}(f_1,\dots,f_k)\restriction_{[0,1/2]}\to\xi_l\times h_\alpha\restriction_{[0,1/2]}$ in measure and for each $j\in\mathbb{N}$,

$$w_n^{l,\alpha}(f_1,\dots,f_k)(t_j)\to(\xi_l\times h_\alpha)(t_j)=h_{\alpha(j)}$$

by choosing $t_j\in P$. We conclude that $w_n^{l,\alpha}(\tilde f_1,\dots,\tilde f_k)\to\xi_l\times h_\alpha$ in $\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$, and therefore

$$\mathrm{rk}(\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}})\le k.$$

Finally, suppose that $\mathrm{L}^0([0,1],\lambda,G)$ is generically $k$-generated. Let $U_i\subseteq\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$, $1\le i\le k$, be open sets and pick any open $V_i\subseteq\mathrm{L}^0([0,1],\lambda,G)$, $1\le i\le k$, such that $V_i\cap\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}=U_i$. Since $\mathrm{L}^0([0,1],\lambda,G)$ is assumed to be generically $k$-generated, there is a tuple $(f_1,\dots,f_k)$ generating a dense subgroup in $\mathrm{L}^0([0,1],\lambda,G)$ such that $f_i\in V_i$ for each $i$. Running the above construction, we get a tuple $(\tilde f_1,\dots,\tilde f_k)\in\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$ such that $\tilde f_i\in U_i$, $1\le i\le k$, whence $\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$ is generically $k$-generated. □

Lemma 4.13 remains valid if we take the product with a finite power of $G$, which follows from Lemma 4.12.

**Corollary 4.14.** *For any separable topological group $G$ and any $m\in\mathbb{N}$ one has*

$$\mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G)) = \mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G))\times G^m.$$

*If* $\mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G))$ *is generically $k$-generated for some $k\in\mathbb{N}$, then so is* $\mathrm{L}^0([0,1],\lambda,G)\times G^m$.

We may now strengthen Proposition 4.11 by dropping the assumption on $(Y,\nu)$ being atomless.

**Proposition 4.15.** *Let $(Y,\nu)$ be a standard Lebesgue space and $(Z,\lambda)$ be a standard probability space. The topological group* $\mathrm{L}^0(Y,\nu,\mathrm{Aut}(Z,\lambda))$ *is generically 2-generated.*

*Proof.* Let $Y_a$ be the set of atoms of $Y$, put $Y_0=Y\setminus Y_a$ and $\nu_0=\nu\restriction_{Y_0}$. The group $\mathrm{L}^0(Y,\nu,\mathrm{Aut}(Z,\lambda))$ is naturally isomorphic to

$$\mathrm{L}^0(Y_0,\nu_0,\mathrm{Aut}(Z,\lambda))\times\mathrm{Aut}(Z,\lambda)^{|Y_a|}.$$

An application of Proposition 4.11 together with Lemma 4.13 or Corollary 4.14 (depending on whether $Y_a$ is infinite or not) finishes the proof. □

**Proposition 4.16.** *Let $G$ be a Polish group and let $H_0 \le H_1 \le \cdots \le G$ be a dense chain of Polish subgroups, $\overline{\bigcup_n H_n} = G$. If each $H_n$ is generically $k$-generated, then $G$ is generically $k$-generated.*

*Proof.* We need to show that for any open $U \subseteq G^k$ and any open $V \subseteq G$ there is a tuple $(g_1, \ldots, g_k) \in U$ such that $\langle g_1, \ldots, g_k \rangle \cap V \neq \varnothing$. Since groups $H_n$ are nested and $\bigcup_n H_n$ is dense in $G$, there is $n$ so large that $U \cap H_n^k \neq \varnothing$ and $V \cap H_n \neq \varnothing$. It remains to use the fact that $H_n$ is generically $k$-generated to find the required tuple. □

**Theorem 4.17.** *Let $G \curvearrowright X$ be a measure-preserving action of an amenable locally compact Polish normed group on a standard probability space $(X, \mu)$. If almost every orbit of the action is uncountable, then the derived* $\mathrm{L}^1$ *full group $D([G \curvearrowright X]_1)$ is generically* 2*-generated.*

*Proof.* In view of Corollary 4.3, there is a chain of subgroups $H_0 \le H_1 \le \cdots \le D([G \curvearrowright X]_1)$, $\overline{\bigcup_n H_n} = D([G \curvearrowright X]_1)$, where each $H_n$ is isomorphic to $\mathrm{L}^0(Y_n, \nu_n, \mathrm{Aut}([0,1], \lambda))$ for some standard Lebesgue space $(Y_n, \nu_n)$. By Proposition 4.15, every $H_n$ is generically 2-generated and we may apply Proposition 4.16. □

**Corollary 4.18.** *Let $G \curvearrowright X$ be a measure-preserving action of an amenable locally compact Polish normed group on a standard probability space $(X, \mu)$. If almost every orbit of the action is uncountable, then the derived* $\mathrm{L}^1$ *full group $D([G \curvearrowright X]_1)$ has topological rank* 2.

*Proof.* Theorem 4.17 implies that the topological rank is at most two. To see that it is actually equal to 2, simply note that $D([G \curvearrowright X]_1)$ is not abelian (e.g. by the proof of Proposition C.7). □

The assumption for orbits to be uncountable is essential, and Corollary 4.18 is in a striking difference with the dynamical interpretation of the topological rank of derived $\mathrm{L}^1$ full groups for discrete groups actions. As shown in [LM19, Thm. 4.3], an aperiodic action of a finitely generated group $\Gamma \curvearrowright X$ has finite Rokhlin entropy if and only if the topological rank of $D([\Gamma \curvearrowright X]_1)$ is finite.

## 5. INDEX MAP

We now turn out attention to the so-called *index map*, which is an important tool in the study of full groups of measure-preserving flows. This map is a continuous homomorphism from the $\mathrm{L}^1$ full group of the flow $\mathcal{F}$ to the additive group of reals, which can be thought of measuring the average shift distance. When the flow is ergodic, such averages are the same across orbits. By taking the ergodic decomposition of $(X, \mu, \mathcal{F})$, we can adopt a slightly more general vantage point and view the index map $\mathcal{I}$ as a homomorphism into the $\mathrm{L}^1$ space of functions on the space of invariant measures $(\mathcal{E}, p)$: $\mathcal{I} : [\mathcal{F}]_1 \to \mathrm{L}^1(\mathcal{E}, p)$.

Understanding the kernel of the index map is the task of fundamental importance. We will subsequently identify $\ker \mathcal{I}$ with the derived topological subgroup of $[\mathcal{F}]_1$. This will allow us to describe abelianizations of $\mathrm{L}^1$ full groups of flows and estimate the number of their topological generators.

It has already been mentioned that any element $T$ of a full group of a flow induces Lebesgue measure-preserving transformations on orbits (Appendix B.4). When $T$, moreover, belongs to the $\mathrm{L}^1$ full group, these transformations are special—they leave half-lines invariant up to a set of finite measure. Such transformations form the so-called *commensurating group*. Let us therefore begin with a more formal treatment of this group, which, by the way, has already appeared in the literature before, for instance in [RS98].

5.1. **Self commensurating automorphisms of a subset.** Consider an infinite measure space $(Z, \lambda)$. We say that two measurable sets $A, B \subseteq Z$ are *commensurate* if the measure of their symmetric difference is finite, $\lambda(A \triangle B) < \infty$. The relation of being commensurate is an equivalence relation, and all sets of finite measure fall into a single class. Note also that if $A$ and $B$ are both commensurate to some $C$, then so is the intersection $A \cap B$; in other words, all equivalence classes of commensurability are closed under finite intersections.

Let $\mathfrak{C}(B)$ denote the collection of all measurable $A \subseteq Z$ that are commensurate to $B$. Fix some $Y \subseteq Z$ and consider the semigroup of measure-preserving transformations between elements of $\mathfrak{C}(Y)$. More precisely, let $\mathrm{Iso}^\star(Y, \lambda)$ be the collection of measure-preserving maps $T : A \to B$ between sets $A, B \in \mathfrak{C}(Y)$, which we call the *self commensurating semigroup* of $(Y, \lambda)$.

We use the notation $\operatorname{dom} T = A$ and $\operatorname{rng} T = B$ to refer to the domain and the range of $T$, respectively. As usual, we identify two maps that differ on a null set. Since classes of commensurability are closed under finite intersections, the set $\mathrm{Iso}^\star(Y, \lambda)$ forms a semigroup under the composition.

This semigroup carries a natural equivalence relation: $T \sim S$ whenever the transformations disagree on a set of finite measure, $\lambda(\{x : Tx \neq Sx\}) < \infty$. This equivalence is, moreover, a congruence, i.e., if $T_1 \sim S_1$ and $T_2 \sim S_2$, then $T_1 \circ T_2 \sim S_1 \circ S_2$. One may therefore push the semigroup structure from $\mathrm{Iso}^\star(Y,\lambda)$ onto the set of equivalence classes, which we denote by $\mathrm{Aut}^\star(Y,\lambda)$. An important observation is that $\mathrm{Aut}^\star(Y,\lambda)$ is a group. Indeed, the identity corresponds to the map $x \mapsto x$ on $Y$, and for a representative $T \in \mathrm{Iso}^\star(Y,\lambda)$, its inverse inside $\mathrm{Aut}^\star(Y,\lambda)$ is, naturally, given by $T^{-1} : \mathrm{rng}\, T \to \mathrm{dom}\, T$. We call $\mathrm{Aut}^\star(Y,\lambda)$ the *self commensurating automorphism group* of $Y$.

The self commensurating semigroup admits an important homomorphism $\mathscr{I} : \mathrm{Iso}^\star(Y,\lambda) \to \mathbb{R}$, called the *index map*, and defined by $\mathscr{I}(T) = \lambda(\mathrm{dom}\, T \setminus \mathrm{rng}\, T) - \lambda(\mathrm{rng}\, T \setminus \mathrm{dom}\, T)$.

**Lemma 5.1.** *The index map satisfies the following properties for all $T \in \mathrm{Iso}^\star(Y,\lambda)$:*

*(1) if $A \in \mathfrak{C}(Y)$ is such that $\mathrm{dom}\, T \subseteq A$ and $\mathrm{rng}\, T \subseteq A$, then $\mathscr{I}(T) = \lambda(A \setminus \mathrm{rng}\, T) - \lambda(A \setminus \mathrm{dom}\, T)$;*
*(2) if $T' \in \mathrm{Iso}^\star(Y,\lambda)$ is a restriction of $T'$, i.e., $T' = T\restriction_{\mathrm{dom}\, T'}$, then $\mathscr{I}(T') = \mathscr{I}(T)$.*

*Proof.* (1) If $A \subseteq Z$ is commensurate to $Y$ and $\mathrm{dom}\, T \subseteq A$, $\mathrm{rng}\, T \subseteq A$, then

$$\begin{aligned}\mathscr{I}(T) &= \lambda(\mathrm{dom}\, T \setminus \mathrm{rng}\, T) - \lambda(\mathrm{rng}\, T \setminus \mathrm{dom}\, T)\\ &= \lambda(A \setminus \mathrm{rng}\, T) - \lambda(A \setminus (\mathrm{dom}\, T \cup \mathrm{rng}\, T)) - (\lambda(A \setminus \mathrm{dom}\, T) - \lambda(A \setminus (\mathrm{dom}\, T \cup \mathrm{rng}\, T)))\\ &= \lambda(A \setminus \mathrm{rng}\, T) - \lambda(A \setminus \mathrm{dom}\, T).\end{aligned}$$

(2) If $T' \in \mathrm{Iso}^\star(Y,\lambda)$ is a restriction of $T$, then $T(\mathrm{dom}\, T \setminus \mathrm{dom}\, T') = \mathrm{rng}\, T \setminus \mathrm{rng}\, T'$. Thus for any $A \in \mathfrak{C}(Y)$ containing both $\mathrm{dom}\, T$ and $\mathrm{rng}\, T$ item (1) implies

$$\begin{aligned}\mathscr{I}(T) &= \lambda(A \setminus \mathrm{dom}\, T) - \lambda(A \setminus \mathrm{rng}\, T)\\ &= \lambda(A \setminus \mathrm{dom}\, T') - \lambda(\mathrm{dom}\, T \setminus \mathrm{dom}\, T') - (\lambda(B \setminus \mathrm{rng}\, T') - \lambda(\mathrm{rng}\, T \setminus \mathrm{rng}\, T'))\\ &= \lambda(A \setminus \mathrm{dom}\, T') - \lambda(A \setminus \mathrm{rng}\, T') = \mathscr{I}(T'),\end{aligned}$$

which proves (2). □

**Proposition 5.2.** *The index map $\mathscr{I} : \mathrm{Iso}^\star(Y,\lambda) \to \mathbb{R}$ is a homomorphism. Moreover, if $T, S \in \mathrm{Iso}^\star(Y,\lambda)$ are equivalent, $T \sim S$, then $\mathscr{I}(T) = \mathscr{I}(S)$.*

*Proof.* In view of Lemma 5.1(2), to check that $\mathscr{I}(T_1 \circ T_2) = \mathscr{I}(T_1) + \mathscr{I}(T_2)$ we may pass to restrictions of these transformations and assume that $\mathrm{rng}\, T_2 = \mathrm{dom}\, T_1$. Pick a set $A \in \mathfrak{C}(Y)$ large enough to contain the domains and ranges of $T_1$ and $T_2$; by Lemma 5.1(1)

$$\begin{aligned}\mathscr{I}(T_1 \circ T_2) &= \lambda(A \setminus \mathrm{rng}\, T_1) - \lambda(A \setminus \mathrm{dom}\, T_2)\\ &= \lambda(A \setminus \mathrm{rng}\, T_1) - \lambda(A \setminus \mathrm{dom}\, T_1) + \lambda(A \setminus \mathrm{rng}\, T_2) - \lambda(A \setminus \mathrm{dom}\, T_2)\\ &= \mathscr{I}(T_1) + \mathscr{I}(T_2).\end{aligned}$$

For the moreover part, suppose that $T, S \in \mathrm{Iso}^\star_Y(Y,\lambda)$ are equivalent. Let $U$ be the restriction of $T$ and $S$ onto the set $\{x : Tx = Sx\}$. Using Lemma 5.1(2) once again, we get $\mathscr{I}(T) = \mathscr{I}(U) = \mathscr{I}(S)$, hence the index map is invariant under the equivalence relation $\sim$. □

The proposition above implies that the index map respects the relation $\sim$, and hence gives rise to a map from $\mathrm{Aut}^\star(Y,\lambda)$ to the reals.

**Corollary 5.3.** *The index map factors to a group homomorphism $\mathscr{I} : \mathrm{Aut}^\star(Y,\lambda) \to \mathbb{R}$.*

5.2. **The commensurating automorphism group.** Let us again consider an infinite measure space $(Z,\lambda)$ and $Y \subseteq Z$ a measurable subset. We now define the *commensurating automorphism group of $Y$ in $Z$* as the group of all measure-preserving transformations $T \in \mathrm{Aut}(Z,\lambda)$ such that $\lambda(Y \triangle T(Y)) < \infty$. We denote this group by $\mathrm{Aut}_Y(Z,\lambda)$.

Every $T \in \mathrm{Aut}_Y(Z,\lambda)$ naturally gives rise to an element of $\mathrm{Aut}^\star(Y,\lambda)$ by considering its restriction $T\restriction_Y$. The following lemma shows that in this case we may use any other set $A$ commensurate to $Y$ instead without changing the corresponding element of the commensurating group.

**Lemma 5.4.** *Let $T \in \mathrm{Aut}(Z,\lambda)$ be a measure-preserving automorphism. If $T\restriction_A \in \mathrm{Iso}^\star(Y,\lambda)$ for some $A \in \mathfrak{C}(Y)$, then $T\restriction_B \in \mathrm{Iso}^\star(Y,\lambda)$ and $T\restriction_B \sim T\restriction_A$ for all $B \in \mathfrak{C}(Y)$.*

*Proof.* Since commensuration is an equivalence relation and $A$ is commensurate to $Y$, the assumption $T\restriction_A \in \mathrm{Iso}^\star(Y,\lambda)$ is equivalent to $\lambda(A \triangle T(A)) < \infty$. Moreover, given $B \in \mathfrak{C}(Y)$, we only need to show that $\lambda(B \triangle T(B))$ is finite in order to conclude that $T\restriction_B \in \mathrm{Iso}^\star(Y,\lambda)$. So we compute

$$\begin{aligned}\lambda(B\triangle T(B)) &= \lambda(B\setminus T(B))+\lambda(T(B)\setminus B)\\ &\leq \lambda(A\setminus T(A))+\lambda(B\setminus A)+\lambda(T(A\setminus B))+\lambda(T(A)\setminus A)+\lambda(A\setminus B)+\lambda(T(B\setminus A))\\ &= \lambda(A\triangle T(A))+2\lambda(A\triangle B)<\infty.\end{aligned}$$

Thus the measure $\lambda(B\triangle T(B))$ is finite, hence $T\restriction_B \in \mathrm{Iso}^\star(Y,\lambda)$ for all $B\in\mathfrak{C}(Y)$. Finally, $T\restriction_A \sim T\restriction_B$, since these transformations agree on $A\cap B$. □

To summarize, if $T\restriction_A \in \mathrm{Iso}^\star(Y,\lambda)$ for some $A\in\mathfrak{C}(Y)$, then all restrictions $T\restriction_B$, $B\in\mathfrak{C}(Y)$, are pairwise equivalent, hence correspond to the same element $T\restriction_Y \in \mathrm{Aut}^\star(Y,\lambda)$. According to Proposition 5.2, the index $\mathscr{I}(T\restriction_Y)$ of this element can be computed as $\mathscr{I}(T\restriction_Y)=\lambda(B\setminus T(B))-\lambda(B\setminus T^{-1}(B))$ for any $B\in\mathfrak{C}(Y)$.

5.3. **Index map on $\mathrm{L}^1$ full groups of $\mathbb{R}$-flows.** Let $\mathscr{F}=\mathbb{R}\curvearrowright X$ be a free measure-preserving Borel flow, let $[\mathscr{F}]_1$ be the associated $\mathrm{L}^1$ full group, where we endow $\mathbb{R}$ with the standard Euclidean norm, and let $T\in[\mathscr{F}]_1$. Recall that the cocycle of $T$ is denoted by $\rho_T : X\to\mathbb{R}$ and defined by the equality $T(x)=x+\rho_T(x)$ for all $x\in X$. We are going to argue that on every orbit $T$ induces a measure-preserving transformation that belongs to the commensurate group of $\mathbb{R}^{\geq 0}$, when the orbit is identified with the real line.

Consider the function $f:\mathscr{R}_{\mathscr{F}}\to\{-1,1\}$ defined by

$$f(x,y)=\begin{cases}1 & \text{if } x<y<T(x),\\ -1 & \text{if } T(x)<y<x,\\ 0 & \text{otherwise.}\end{cases}$$

One can think of $f$ as a "charge function" that spreads charge $+1$ over each interval $(x,T(x))$ and $-1$ over $(T(x),x)$. Note that $\int_{\mathbb{R}} f(x,x+r)\,d\lambda(r)=\rho_T(x)$. Since $T$ belongs to the $\mathrm{L}^1$ full group, its cocycle is integrable, which means that $f$ is $M$-integrable (see Appendix B.4). We apply the mass transport principle, which shows that

$$\int_X\int_{\mathbb{R}} f(x,x+r)\,d\lambda(r)d\mu(x)=\int_X\int_{\mathbb{R}} f(x+r,x)\,d\lambda(r)d\mu(x).$$

Let $T_x\in\mathrm{Aut}(\mathbb{R},\lambda)$ denote the transformation induced by $T$ onto the orbit of $x$ obtained by identifying the origin of the real line with $x$. The following two quantities are therefore finite:

$$\int_{\mathbb{R}} |f(x+r,x)|\,d\lambda(r)=\lambda\big(\mathbb{R}^{\geq 0}\setminus T_x(\mathbb{R}^{\geq 0})\big)+\lambda\big(T_x(\mathbb{R}^{\geq 0})\setminus\mathbb{R}^{\geq 0}\big),$$

$$\int_{\mathbb{R}} f(x+r,x)\,d\lambda(r)=\lambda\big(\mathbb{R}^{\geq 0}\setminus T_x(\mathbb{R}^{\geq 0})\big)-\lambda\big(T_x(\mathbb{R}^{\geq 0})\setminus\mathbb{R}^{\geq 0}\big).$$

In particular, $T_x\restriction_{\mathbb{R}^{\geq 0}}$ belongs to the commensurating group of $\mathbb{R}^{\geq 0}$. The second quantity, on the other hand, is equal to the index of $T_x\restriction_{\mathbb{R}^{\geq 0}}$. By Subsection 5.2, $\mathscr{I}(T_x\restriction_{\mathbb{R}^{\geq 0}})=\mathscr{I}(T_y\restriction_{\mathbb{R}^{\geq 0}})$ whenever $x\mathscr{R}_{\mathscr{F}}y$. For any $T\in[\mathscr{F}]_1$, we therefore have an orbit invariant measurable map $h_T : X\to\mathbb{R}$ given by $h_T(x)=\int_{\mathbb{R}} f(x+r,x)\,d\lambda(r)$. Note that for any $\mathscr{F}$-invariant set $Y\subseteq X$ we have

$$\int_Y \rho_T(x)\,d\mu=\int_Y h_T(x)\,d\mu. \tag{2}$$

Let $(\mathscr{E},p)$, $X\ni x\mapsto\nu_x\in\mathscr{E}$, be the ergodic decomposition of $(X,\mu,\mathscr{F})$ (see Appendix B.3). Since the map $h_T$ is $\mathscr{R}_{\mathscr{F}}$-invariant, it produces a map $\tilde{h}_T:\mathscr{E}\to\mathbb{R}$ via $\tilde{h}(\nu)=h(x)$ for any $x$ such that $\nu=\nu_x$ or, equivalently, via

$$\tilde{h}_T(\nu)=\int_X\int_{\mathbb{R}} f(x+r,x)\,d\lambda(r)d\nu(x).$$

Note also that

$$\int_X h_T(x)\,d\mu=\int_X\int_{\mathbb{R}} f(x+r,x)\,d\lambda(r)d\mu(x)=\int_{\mathscr{E}}\tilde{h}_T(\nu)\,dp(\nu),$$

thus $\tilde{h}_T\in\mathrm{L}^1(\mathscr{E},\mathbb{R})$. We can now define the index map as a function $\mathscr{I}:[\mathscr{F}]_1\to\mathrm{L}^1(\mathscr{E},\mathbb{R})$.

**Definition 5.5.** Let $\mathscr{F}=\mathbb{R}\curvearrowright X$ be a free measure-preserving flow on a standard probability space $(X,\mu)$; let also $(\mathscr{E},p)$ be the space of $\mathscr{F}$-invariant ergodic probability measures, where $p$ is the disintegration of $\mu$. The index map is the function $\mathscr{I}:[\mathscr{F}]_1\to\mathrm{L}^1(\mathscr{E},\mathbb{R})$ given by $\mathscr{I}(T)(\nu)=\tilde{h}_T(\nu)=\int_X\int_{\mathbb{R}} f(x+r,x)\,d\lambda(r)d\nu(x)$.

**Proposition 5.6.** *For any free measure-preserving flow $\mathscr{F}=\mathbb{R}\curvearrowright X$ the index map $\mathscr{I}:[\mathscr{F}]_1\to\mathrm{L}^1(\mathscr{E},\mathbb{R})$ is a continuous and surjective homomorphism.*

*Proof.* The index map is a homomorphism, since, as we have discussed earlier, $h_T(x)$ is equal to the index of $T_x\restriction_{\mathbb{R}^{\geq 0}}$. Continuity follows from the fact that $\mathscr{I}$ is a Borel homomorphism between Polish groups. To see surjectivity pick any $\tilde{h}\in \mathrm{L}^1(\mathscr{E},\mathbb{R})$, view it as map $h: X\to\mathbb{R}$ via the identification $h(x)=\tilde{h}(v_x)$. Define the automorphism $T\in \mathrm{Aut}(X,\mu)$ by $T(x)=x+h(x)$. It is straightforward to check that $T\in[\mathscr{F}]_1$ and $\mathscr{I}(T)=h$. □

The factor group $[\mathscr{F}]_1/\ker\mathscr{I}$ naturally inherits the factor metric given by $\|T\ker\mathscr{I}\|_1=\inf_{S\in\ker\mathscr{I}}\|TS\|_1$. In view of Proposition 5.6, the index map induces an isomorphism between $[\mathscr{F}]_1/\ker\mathscr{I}$ and $\mathrm{L}^1(\mathscr{E},\mathbb{R})$. We argue that this isomorphism is, in fact, an isometry.

**Proposition 5.7.** *The index map $\mathscr{I}$ induces an isometric isomorphism between $[\mathscr{F}]_1/\ker\mathscr{I}$ and $\mathrm{L}^1(\mathscr{E},\mathbb{R})$, where the former is endowed with the factor metric and the letter bears the usual $\mathrm{L}^1$ norm.*

*Proof.* Since $\int_X |h_T(x)|\,d\mu=\int_{\mathscr{E}}|\tilde{h}_T(v)|\,dp$ for all $T\in[\mathscr{F}]_1$, it suffices to show that for all $T\in[\mathscr{F}]_1$

$$\inf_{S\in\ker\mathscr{I}}\|TS\|_1=\int_X |h_T(x)|\,d\mu.$$

Let $T\in[\mathscr{F}]_1$. We first show that the inequality $\inf_{S\in\ker\mathscr{I}}\|TS\|_1\geq\int_X|h_T(x)|\,d\mu(x)$ holds. Pick $S\in\ker\mathscr{I}$, we have

$$\|TS\|_1=\int_X|\rho_{TS}(x)|\,d\mu\geq\left|\int_X\rho_{TS}(x)\right|\,d\mu,$$

so by the cocycle identity $\|TS\|_1\geq\left|\int_X\rho_T(S(x))+\rho_S(x)\,d\mu\right|$. By assumption $\int_X\rho_S(x)\,d\mu=0$, and since $S$ is measure-preserving $\int_X\rho_T(S(x))\,d\mu=\int_X\rho_T(x)\,d\mu=\int_X h_T(x)\,d\mu$. We conclude that

$$\|TS\|_1\geq\left|\int_X h_T(x)\,d\mu\right|,$$

thus obtaining the inequality

$$\inf_{S\in\ker\mathscr{I}}\|TS\|_1\geq\int_X|h_T(x)|\,d\mu.$$

For the other direction consider a transformation $T'$ defined by $T'(x)=x+h_T(x)$; note that $T'\in[\mathscr{F}]_1$, $\rho_{T'}(x)=h_{T'}(x)=h_T(x)$ for all $x\in X$, and $T^{-1}T'\in\ker\mathscr{I}$. Therefore

$$\inf_{S\in\ker\mathscr{I}}\|TS\|_1\leq\left\|TT^{-1}T'\right\|_1=\left\|T'\right\|_1=\int_X|h_{T'}(x)|\,d\mu=\int_X|h_T(x)|\,d\mu,$$

and the desired equality of norms follows. □

Using a similar reasoning, we get the following characterization of the $\mathrm{L}^1$ full group and the index map, where for all $T\in[\mathscr{R}_{\mathscr{F}}]$ we let $r_T$ be the measure-preserving transformation of $(\mathscr{R}_{\mathscr{F}},M)$ given by $r_T(x,y)=(x,T(y))$, where $\mathscr{R}_{\mathscr{F}}$ is the associated equivalence relation (cf. Appendix B.4).

**Proposition 5.8.** *Let $\mathscr{F}=\mathbb{R}\curvearrowright X$ be a free measure-preserving $\mathbb{R}$-flow. Consider the set $\mathscr{R}^{\geq0}=\{(x,y)\in\mathscr{R}_{\mathscr{F}}: x\geq y\}$. Then for every $T\in[\mathbb{R}\curvearrowright X]$, we have*

$$\|T\|_1=M\left(\mathscr{R}^{\geq0}\,\triangle\,r_T(\mathscr{R}^{\geq0})\right).$$

*In particular, the $\mathrm{L}^1$ full group of $\mathscr{F}$ can be seen as the commensurating group of $\mathscr{R}^{\geq0}$ inside the full group of $\mathscr{R}$. Moreover, in the ergodic case, the index of $T$ as defined above is equal to its index as a commensurating transformation of the set $\mathscr{R}^{\geq0}$.*

*Proof.* Through the identification $(x,t)\mapsto(x,x+t)$, the measure-preserving transformation $r_T$ is acting on $X\times\mathbb{R}$ as $\mathrm{id_X}\times T_x$, and the set $\mathscr{R}^{\geq0}$ becomes $X\times\mathbb{R}^{\geq0}$. We then have

$$\begin{aligned}M(\mathscr{R}^{\geq0}\,\triangle\,r_T(\mathscr{R}^{\geq0}))&=\int_X\lambda(\mathbb{R}^{\geq0}\,\triangle\,(T_x(\mathbb{R}^{\geq0})))\,d\mu(x)\\&=\int_X|c_T(x)|\,d\mu(x)\end{aligned}$$

by the mass transport principle, which yields the conclusion since by definition $\|T\|_1=\int_X|c_T(x)|\,d\mu(x)$. The moreover part follows from a similar computation. □

*Remark* 5.9. One could use the fact that the commensurating automorphism group of $\mathscr{R}^{\geq0}$ is a Polish group in order to get another proof that $\mathrm{L}^1$ full groups of measure-preserving $\mathbb{R}$-flows are themselves Polish, using the fact that the full group of $\mathscr{R}$ embeds via $T\mapsto r_T$ in the group of measure-preserving transformations of $(\mathscr{R},M)$.

## 6. Bounded Elements of the Full Group which are Ergodic on each Orbit

The purpose of this section is to contrast some of the differences in the dynamics of the elements of full groups of $\mathbb{Z}$-actions and those arising from $\mathbb{R}$-flows. Let $S \in [\mathbb{Z} \curvearrowright X]$ be an element of the full group of a measure-preserving aperiodic transformation and let $\rho_{S^k} : X \to \mathbb{Z}$ be the cocycle associated with $S^k$ for $k \in \mathbb{Z}$. Since $\mathbb{Z}$ is a discrete group, the conservative part in the Hopf's decomposition for $S$ (see Appendix B.5) reduces to the set of periodic orbits. In particular, an aperiodic $S \in [\mathbb{Z} \curvearrowright X]$ has to be dissipative, hence $|\rho_{S^k}(x)| \to \infty$ as $k \to \infty$. When $S$ belongs to the $\mathrm{L}^1$ full group of the action, a theorem of R. M. Belinskaja [Bel68, Thm. 3.2] strengthens this conclusion and asserts that for almost all $x$ in the dissipative component of $S$ either $\rho_{S^k}(x) \to +\infty$ or $\rho_{S^k}(x) \to -\infty$. This property is an important tool in studying elements of $\mathrm{L}^1$ full group of $\mathbb{Z}$ actions.

Given an arbitrary free measure-preserving flow $\mathbb{R} \curvearrowright X$, we build an example of an aperiodic $S \in [\mathbb{R} \curvearrowright X]_1$ for which the signs in $\{\rho_{S^k}(x) : k \in \mathbb{N}\}$ keep alternating indefinitely for almost all $x \in X$. In fact, we present a transformation that acts ergodically on each orbit of the flow (in particular, it is conservative and globally ergodic provided so is the flow). Moreover, we ensure it has a uniformly bounded cocycle. Our argument uses a variant of the well-known cutting and stacking construction adapted for infinite measure spaces. Additional technical difficulties arise for the necessity to work across all orbits of the flow simultaneously. The transformation will arise as a limit of special partial maps we call *castles*, which we now define.

The **pseudo full group** of the flow is the set of injective Borel maps $\varphi : \mathrm{dom}\,\varphi \to \mathrm{rng}\,\varphi$ between Borel sets $\mathrm{dom}\,\varphi \subseteq X$, $\mathrm{rng}\,\varphi \subseteq X$ for which there exists a countable Borel partition $(A_n)_{n\in\mathbb{N}}$ of the domain $\mathrm{dom}\,\varphi$ and a countable family of reals $(t_n)_{n\in\mathbb{N}}$ such that $\varphi(x) = x + t_n$ for every $x \in A_n$. Such maps are measure-preserving isomorphisms between $(\mathrm{dom}\,\varphi, \mu\restriction_{\mathrm{dom}\,\varphi})$ and $(\mathrm{rng}\,\varphi, \mu\restriction_{\mathrm{rng}\,\varphi})$. The **support** of $\varphi$ is the set

$$\mathrm{supp}\,\varphi = \{x \in \mathrm{dom}\,\varphi : \varphi(x) \neq x\} \cup \{x \in \mathrm{rng}\,\varphi : \varphi^{-1}(x) \neq x\}.$$

Given $\varphi$ in the pseudo full group and a Borel set $A \subseteq X$, we let $\varphi(A) = \{\varphi(x) \colon x \in A \cap \mathrm{dom}\,\varphi\}$; in particular, $\varphi(A) = \varnothing$ if $A$ is disjoint from $\mathrm{dom}\,\varphi$. A **castle** is an element $\varphi$ of the pseudo full group of the flow such that for $B = \mathrm{dom}\,\varphi \setminus \mathrm{rng}\,\varphi$ the sequence $(\varphi^k(B))_{k\in\mathbb{N}}$ consists of pairwise disjoint subsets which cover its support. Since $\varphi$ is measure-preserving, for almost every $x \in B$ there is $k \in \mathbb{N}$ such that $\varphi^k(x) \notin \mathrm{dom}\,\varphi$. It follows that $\varphi^{-1}$ is also a castle. The set $B$ is called the **basis** of the castle, and the basis of its inverse $C$ is called its **ceiling**, which is equal to $\mathrm{rng}\,\varphi \setminus \mathrm{dom}\,\varphi$. Observe that if two castles have disjoint supports, then their union is also a castle. We denote by $\vec{\varphi} : B \to C$ the element of the pseudo full group which takes every element of the basis of $\varphi$ to the corresponding element of the ceiling.

*Remark* 6.1. Equivalently, one could define a castle as an element $\varphi$ of the pseudo full group which induces a *graphing* consisting of finite segments only (see [KM04, Sec. 17] for the definition of a graphing). It induces a partial order $\leq_\varphi$ defined by $x \leq_\varphi y$ if and only if there is $k \in \mathbb{N}$ such that $y = \varphi^k(x)$. The basis of the castle is the set of minimal elements, while the ceiling is the set of maximal ones. Finally, $\vec{\varphi}$ is the map which takes a minimal element to the unique maximal element above it.

**Theorem 6.2.** *Let $\mathbb{R} \curvearrowright X$ be a free measure-preserving flow. There exists $S \in [\mathbb{R} \curvearrowright X]$ that acts ergodically on every orbit of the flow and whose cocycle is bounded by* 4. *Moreover, the signs in $\{\rho_{S^k}(x) : k \in \mathbb{N}\}$ keep alternating indefinitely for almost all $x \in X$.*

*Proof.* Fix a free measure-preserving flow $\mathbb{R} \curvearrowright X$, and let $\mathscr{C} \subset X$ be a cross section. Since $\mathscr{C}$ is lacunary, for any $c \in \mathscr{C}$ there exists $\min\{r > 0 : c + r \in \mathscr{C}\}$; we denote this value by $\mathrm{gap}_{\mathscr{C}}(c)$. This gives the first return map $\sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ via $\sigma_{\mathscr{C}}(c) = c + \mathrm{gap}_{\mathscr{C}}(c)$, which is Borel. There is also a natural bijective correspondence between $X$ and the set $\{(c, r) \in \mathscr{C} \times \mathbb{R}^{\geq 0} : c \in \mathscr{C}, 0 \leq r < \mathrm{gap}_{\mathscr{C}}(c)\}$. Let $\lambda_c^{\mathscr{C}}$ be the "Lebesgue measure" on $c + [0, \mathrm{gap}_{\mathscr{C}}(c))$ given by

$$\lambda_c^{\mathscr{C}}(A) = \lambda(\{r \in \mathbb{R} : 0 \leq r < \mathrm{gap}_{\mathscr{C}}(c), c + r \in A\}).$$

The measure $\mu$ on $X$ can be disintegrated as $\mu(A) = \int_{\mathscr{C}} \lambda_c^{\mathscr{C}}(A)\, d\nu(c)$ for some finite (but not necessarily probability) measure $\nu$ on $\mathscr{C}$ (see, for instance, [Slu17, Sec. 4] and Appendix B.1).

Let $(\mathscr{C}_n)_{n\in\mathbb{N}}$ be a vanishing sequence of markers—a sequence of nested cross sections $\mathscr{C}_1 \supset \mathscr{C}_2 \supset \mathscr{C}_3 \cdots$ with the empty intersection: $\bigcap_{n\in\mathbb{N}} \mathscr{C}_n = \varnothing$. We may arrange $\mathscr{C}_1$ to be such that $\mathrm{gap}_{\mathscr{C}_1}(c) \in (2,3)$ for all $c \in \mathscr{C}_1$. Put $\mathscr{C}_0 = \{c + k \colon c \in \mathscr{C}_1, k \in \{0,1,2\}\}$ and $Y = \mathscr{C}_1 + [0,2)$. Note that $\mu(X \setminus Y) \leq \frac{1}{3}$. Our first goal is to define an element $\varphi$ of the pseudo full group with domain and range equal to $Y$ such that for almost every $x \in Y$, the action of $\varphi$ on the intersection of the orbit of $x$ with $Y$ is ergodic, and which has cocycle bounded by 3. It will then be easy to modify $\varphi$ to an element of the full group whose action on each orbit of the flow is ergodic at the cost of increasing the cocycle bound to 4.

Our first transformation $\varphi$ will arise as the limit of a sequence of castles $(\varphi_n)_{n\in\mathbb{N}}$, with each $\varphi_n$ belonging to the pseudo full group of $\mathscr{R}_{\mathscr{C}_n}$. We also use another family of castles $(\psi_n)_{n\in\mathbb{N}}$ which allows us to extend $\varphi_n$ by "going back" from its ceiling to its basis while keeping the cocycle bound (this is our main adjustment compared to the usual cutting and stacking procedure). Both sequences of castles will have their cocycles bounded by 3. Here are the basic constraints that these sequences have to satisfy:

(1) for all $n \geq 1$, $Y = \operatorname{supp}\varphi_n \sqcup \operatorname{supp}\psi_n$;
(2) for all $n \geq 1$, $\varphi_{n+1}$ extends $\varphi_n$;
(3) $\mu(\operatorname{supp}\psi_n)$ tends to 0 as $n$ tends to $+\infty$.

Bases and ceilings of $(\varphi_n)$ and $(\psi_n)$ will satisfy additional constraints which will enable us to make the induction work and ensure ergodicity on each orbit of the flow. In order to specify these constraints properly, we introduce the following notation.

Each orbit of the flow comes with the linear order $<$ inherited from $\mathbb{R}$ via $x < y$ if and only if $y = x + t$ for some $t > 0$. Set $\kappa_{\mathscr{C}_n}(x)$ to be the minimum of the intersection of $\mathscr{C}_n$ with the cone $\{y \in X : y \geq x\}$.

Let $\mathscr{D}_1 := \mathscr{C}_1 + 2 \subseteq \mathscr{C}_0$ and $\mathscr{D}_n$ be the set of those $x \in \mathscr{D}_1$ which are maximal in $\kappa_{\mathscr{C}_n}^{-1}(c)$ among points of $\mathscr{D}_1$ for some $c \in \mathscr{C}_n$; in other words, $\mathscr{D}_n = \{x \in \mathscr{D}_1 : (x, \kappa_{\mathscr{C}_n}(x)) \cap \mathscr{C}_0 = \varnothing\}$. Note that by construction the distance between $x$ and $\kappa_{\mathscr{C}_n}(x)$ is less than 1 for each $x \in \mathscr{D}_n$. Let $\iota_n$ be the map $\mathscr{C}_n \to \mathscr{D}_n$ which assigns to $c \in \mathscr{C}_n$ the $<$-least element of $\mathscr{D}_n$ which is greater than $c$.

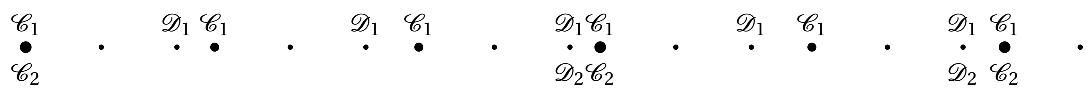


FIGURE 1. An example of cross sections $\mathscr{C}_0$ (all points), $\mathscr{C}_1$ (dots of size $\bullet$ and above), $\mathscr{C}_2$ (marked as $\bullet$) and $\mathscr{D}_1, \mathscr{D}_2$.

The bases and ceilings of $\varphi_n$ and $\psi_n$ are as follows.

- the basis of $\varphi_n$ is $A_n := \mathscr{C}_n + \left[0, \frac{1}{2^n}\right)$;
- the ceiling of $\varphi_n$ is $B_n := \mathscr{D}_n + \left[-\frac{1}{2} - \frac{1}{2^n}, -\frac{1}{2}\right)$;
- the basis of $\psi_n$ is $C_n := \mathscr{D}_n + \left[-\frac{1}{2}, -\frac{1}{2} + \frac{1}{2^n}\right)$;
- the ceiling of $\psi_n$ is $D_n := \mathscr{C}_n + \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}\right)$.

Furthermore, we impose two **translation conditions**, which help us to preserve the above concrete definitions of the bases and ceilings at the inductive step when we construct $\varphi_{n+1}$ and $\psi_{n+1}$:

- $\vec{\varphi}_n(c + t) = \iota_n(c) + t - \frac{1}{2} - \frac{1}{2^n}$ for all $c \in \mathscr{C}_n$ and all $t \in \left[0, \frac{1}{2^n}\right)$.
- $\vec{\psi}_n(d + t) = \iota_n^{-1}(d) + t + 1$ for all $d \in \mathscr{D}_n$ and all $t \in \left[-\frac{1}{2}, -\frac{1}{2} + \frac{1}{2^n}\right)$.

The first step of the construction consists of the castle $\varphi_1 : x \mapsto x + 1$, which has the basis $A_1 = \mathscr{C}_1 + [0, \frac{1}{2})$ and ceiling $B_1 = \mathscr{D}_1 + [-1, -\frac{1}{2})$, and the castle $\psi_1 : x \mapsto x - 1$ defined for $x \in C_1$ with ceiling $D_1 = \mathscr{C}_1 + [\frac{1}{2}, 1)$.

We now concentrate on the induction step: suppose $\varphi_n$ and $\psi_n$ have been built for some $n \geq 1$, let us construct $\varphi_{n+1}$ and $\psi_{n+1}$.

The strategy is to split the basis of $\varphi_n$ and $\psi_n$ into two equal intervals and "interleave" the "two halves" of $\varphi_n$ with "one half" of $\psi_n$ followed by "gluing" adjacent ceilings and basis within the same $\mathscr{C}_{n+1}$ segment (see Figure 2). To this end, we introduce two intermediate castles $\tilde{\varphi}_n$ and $\tilde{\psi}_n$ which will ensure that $\varphi_{n+1}$ "wiggles" more than $\varphi_n$, yielding ergodicity of the final transformation.

Define two new half measure subsets of the bases $A_n$ and $C_n$ respectively:

- $A_n^1 := \mathscr{C}_n + \left[0, \frac{1}{2^{n+1}}\right)$;
- $C_n^0 := \mathscr{D}_n + \left[-\frac{1}{2} + \frac{1}{2^{n+1}}, -\frac{1}{2} + \frac{1}{2^n}\right)$;

and let $B_n^0 \coloneqq \vec{\varphi}_n(A_n^1) = \mathscr{D}_n + \left[-\frac{1}{2} - \frac{1}{2^n}, -\frac{1}{2} - \frac{1}{2^{n+1}}\right)$, and $D_n^0 \coloneqq \vec{\psi}_n(C_n^0) = \mathscr{C}_n + \left[\frac{1}{2} + \frac{1}{2^{n+1}}, \frac{1}{2} + \frac{1}{2^n}\right)$, where the two equalities are consequences of the translation conditions. Let $E_n$ be the $\psi_n$-saturation of $C_n^0$, and note that the restriction of $\psi_n$ to $E_n$ is a castle with support $E_n$, whose basis is $C_n^0$ and whose ceiling is $D_n^0$. Finally, let $A_n^0 \coloneqq A_n \setminus A_n^1 = \mathscr{C}_n + \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$.

We define the partial measure-preserving transformation $\xi_n : B_n^0 \sqcup D_n^0 \to C_n^0 \sqcup A_n^0$ to be used for "gluing together" $\varphi_n$ and the restriction of $\psi_n$ to $E_n$:

- $\xi_n(b) = b + \frac{3}{2^{n+1}} \in C_n^0$ for all $b \in B_n^0$ and
- $\xi_n(d) = d - \frac{1}{2} \in A_n^0$ for all $d \in D_n^0$.

Set $\tilde{\varphi}_n \coloneqq \varphi_n \sqcup \xi_n \sqcup \psi_{n\restriction E_n}$, whereas $\tilde{\psi}_n$ is simply the restriction of $\psi_n$ onto the complement of $E_n$. Observe that $\tilde{\varphi}_n$ has basis $A_n^1$ and ceiling $B_n^1 \coloneqq B_n \setminus B_n^0 = \mathscr{D}_n + \left[-\frac{1}{2} - \frac{1}{2^{n+1}}, -\frac{1}{2}\right)$, while $\tilde{\psi}_n$ has basis $C_n^1 \coloneqq C_n \setminus C_n^0 = \mathscr{D}_n + \left[-\frac{1}{2}, -\frac{1}{2} + \frac{1}{2^{n+1}}\right)$ and ceiling $D_n^1 \coloneqq D_n \setminus D_n^0 = \mathscr{C}_n + \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2^{n+1}}\right)$. We continue to have $Y = \operatorname{supp} \tilde{\varphi}_n \sqcup \operatorname{supp} \tilde{\psi}_n$, but the support of $\tilde{\psi}_n$ is half the support of $\psi_n$, and $\mu(\operatorname{supp} \tilde{\psi}_n) = \frac{1}{2}\mu(\operatorname{supp} \psi_n)$.

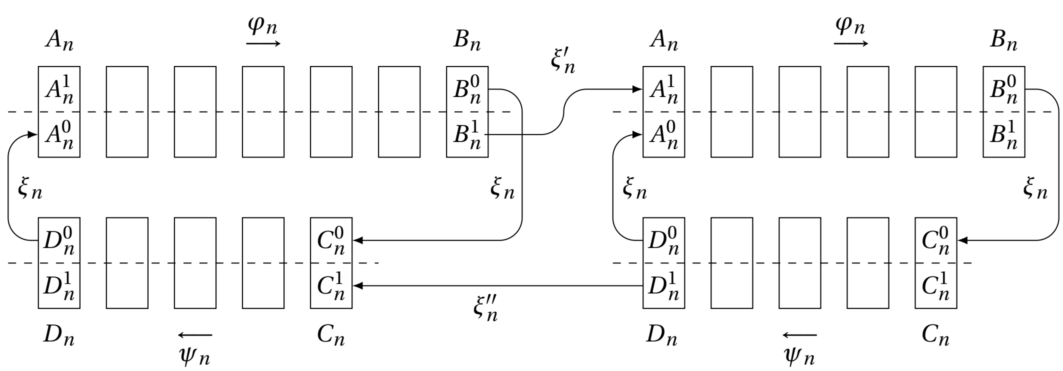


FIGURE 2. Inductive step.

The ceiling of $\tilde{\varphi}_n$ is equal to $B_n^1 = \mathscr{D}_n + \left[-\frac{1}{2} - \frac{1}{2^{n+1}}, -\frac{1}{2}\right)$, whereas we need the ceiling of $\varphi_{n+1}$ to be equal to $B_{n+1} = \mathscr{D}_{n+1} + \left[-\frac{1}{2} - \frac{1}{2^{n+1}}, -\frac{1}{2}\right)$. We obtain the required $\varphi_{n+1}$ and $\psi_{n+1}$ out of $\tilde{\varphi}_n$ and $\tilde{\psi}_n$ respectively by "passing through each element of $\mathscr{C}_n \setminus \mathscr{C}_{n+1}$".

Note that $\mathscr{D}_{n+1}$ is equal to the set of $d \in \mathscr{D}_n$ such that $\kappa_{\mathscr{C}_n}(d) \in \mathscr{C}_{n+1}$. Each $x \in B_n^1 \setminus B_{n+1}$ can be written uniquely as $x = d + t$ where $d \in \mathscr{D}_n \setminus \mathscr{D}_{n+1}$ and $t \in \left[-\frac{1}{2} - \frac{1}{2^{n+1}}, -\frac{1}{2}\right)$. Set

$$\xi_n'(x) = \kappa_{\mathscr{C}_n}(d) + t + \frac{1}{2} + \frac{1}{2^{n+1}},$$

and note that $\xi_n'(x)$ belongs to $(\mathscr{C}_n \setminus \mathscr{C}_{n+1}) + \left[0, \frac{1}{2^{n+1}}\right) = A_n^1 \setminus A_{n+1}$, hence $\xi_n'$ is a measure-preserving bijection from $B_n^1 \setminus B_{n+1}$ onto $A_n^1 \setminus A_{n+1}$.

The transformation $\varphi_{n+1}$ is set to be $\tilde{\varphi}_n \sqcup \xi_n'$, and we claim that it is a castle with basis $A_{n+1}$ and ceiling $B_{n+1}$. This amounts to showing that for all $x \in A_{n+1}$, there is $k \in \mathbb{N}$ such that $\varphi_{n+1}^k(x)$ is not defined. Pick $x \in A_{n+1}$ and write it as $c_0 + t$ for some $c_0 \in \mathscr{C}_{n+1}$ and $t \in [0, \frac{1}{2^{n+1}})$. Let $c_1$ be the successor of $c_0$ in $\mathscr{C}_n$, which we suppose not to be an element of $\mathscr{C}_{n+1}$. By the construction of $\tilde{\varphi}_n$ and $\xi_n'$, there is $k \in \mathbb{N}$ such that $\xi_n'(\tilde{\varphi}_n^k(x)) \in c' + [0, \frac{1}{2^{n+1}})$, which means that $\varphi_{n+1}^{k+1}(x) \in c' + [0, \frac{1}{2^{n+1}})$. Iterating this argument, we eventually find $k_0, p \in \mathbb{N}$ such that $\varphi_{n+1}^{k_0}(x) \in c_p + [0, \frac{1}{2^{n+1}})$ for some $c_p \in \mathscr{C}_n$ such that the successor $c_{p+1}$ of $c_p$ in $\mathscr{C}_n$ belongs to $\mathscr{C}_{n+1}$. By the definition of $\tilde{\varphi}_n$ we must have some $l \in \mathbb{N}$ such that $\varphi_{n+1}^{k_0+l}(x) = \tilde{\varphi}_n^l(\varphi_{n+1}^{k_0}(x)) \in B_{n+1}$, whereas $\varphi_{n+1}^{k_0+l+1}(x)$ is not defined, thus $\varphi_{n+1}$ is indeed a castle.

Extension $\psi_{n+1}$ of $\tilde{\psi}_n$ is defined similarly by connecting adjacent segments of $D_n^1$ and $C_n^1$ by a translation. More specifically, each $x \in D_n^1 \setminus D_{n+1}$ can be written uniquely as $x = c + t$ for some $c \in \mathscr{C}_n \setminus \mathscr{C}_{n+1}$ and $t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2^{n+1}})$. The restriction of $\kappa_{\mathscr{C}_n}$ to $\mathscr{D}_n$ is a bijection $\mathscr{D}_n \to \mathscr{C}_n$, we denote its inverse by $p_n$ and let $\xi_n''(x) = p_n(c) + t - 1$. The map $\psi_{n+1} \coloneqq \tilde{\psi}_n \sqcup \xi_n''$ can be checked to be a castle with basis $C_{n+1}$ and ceiling $D_{n+1}$ as desired. It also follows that the translation conditions continue to be satisfied by both of $\varphi_{n+1}$ and $\psi_{n+1}$.

Transformations $\varphi_n$ extend each other, so $\varphi = \bigcup_n \varphi_n$ is an element of the pseudo full group supported on $Y = \operatorname{supp}\varphi_n \sqcup \operatorname{supp}\psi_n$. Note also that $\mu(\operatorname{supp}\psi_{n+1}) = \mu(\operatorname{supp}\psi_n)/2$, and therefore $\operatorname{dom}\varphi = Y = \operatorname{rng}\varphi$. We claim that $\varphi$, seen as a measure-preserving transformation of $Y$, induces an ergodic measure-preserving transformation on $(y+\mathbb{R})\cap Y$ for almost all $y \in Y$, where $y+\mathbb{R}$ is endowed with the Lebesgue measure. This follows from the fact that $\varphi$ induces a *rank-one* transformation of the infinite measure space $(y+\mathbb{R})\cap Y$: for all Borel $A \subseteq (y+\mathbb{R})\cap Y$ of finite Lebesgue measure and all $\epsilon > 0$, there are $B \subseteq (y+\mathbb{R})\cap Y$, $k \in \mathbb{N}$, and a subset $F \subseteq \{0,\dots,k\}$ such that $B, \varphi(B), \dots, \varphi^k(B)$ are pairwise disjoint and

$$\lambda(A \triangle (\bigsqcup_{f\in F} \varphi^f(B))) < \epsilon.$$

Indeed, at each step $n$ for every $c \in \mathscr{C}_n$, the iterates of $c+[0,\frac{1}{2^n})$ by the restriction of $\varphi_n$ to the interval $[c, \iota_n(c))$ are disjoint "intervals of size $2^{-n}$", i.e., sets of the form $t+[0,\frac{1}{2^n})$, and these iterates cover a proportion $1-\frac{1}{2^n}$ of $[c,\iota_n(c))$ (the rest of this interval being $[c,\iota_n(c))\cap \operatorname{supp}\psi_n$).

It remains to extend $\varphi$ supported on $Y$ to a measure-preserving transformation $S$ with $\operatorname{supp} S = X$. Let $Z = X\setminus Y$ be the leftover set, $Z = \{c+t : c \in \mathscr{C}_1 : 2 \le t < \operatorname{gap}_{\mathscr{C}_1}(c)\}$, and put $Z' = \{c+t : c\in\mathscr{C}_1,\ 2-\operatorname{gap}_{\mathscr{C}_1}(c) \le t < 2\}$. Figure 3 illustrates an interval between $c \in \mathscr{C}_1$ and $c' = \sigma_{\mathscr{C}_1}(c)$. Within this gap, $Z$ corresponds to $[c+2, c+2+\operatorname{gap}_{\mathscr{C}_1}(c))$, and $Z'$ is an interval of the exact same length adjacent to it on the left. Note that $Z' \subseteq Y$ by construction. Let $\eta : Z' \to Z$ be the natural translation map, $\eta(x) = x + \operatorname{gap}_{\mathscr{C}_1}(c)$ for all $x \in Z'$ satisfying $x \in c+[0,\operatorname{gap}_{\mathscr{C}_1}(c))$. Observe that $\eta$ is a measure-preserving bijection and its cocycle is bounded by 1.

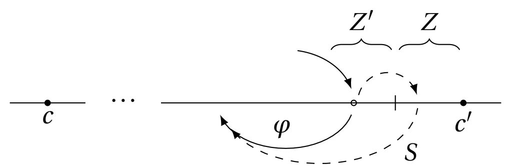


FIGURE 3. Construction of the transformation $S$.

We now rewire the orbits of $\varphi$ and define $S : X \to X$ as follows (see Figure 3):

$$S(x) = \begin{cases} \varphi(x) & \text{if } x \notin Z\cup Z';\\ \eta(x) & \text{if } x \in Z';\\ \varphi(\eta^{-1}(x)) & \text{if } x\in Z.\end{cases}$$

It is straightforward to verify that $S$ is a free measure-preserving transformation, and the distance $D(x,Sx) \le 4$ for all $x \in X$, because $|\rho_\varphi(x)| \le 3$ and $|\rho_\eta(x)| \le 1$ for all $x$ in their domains. Note that the transformation induced by $S$ on $Y$ is equal to $\varphi$, so since the latter is ergodic on every orbit of the flow intersected with $Y$ and since $X = Y \sqcup Z$, it follows that $S$ is ergodic on every orbit of the flow and satisfies the conclusion of the theorem. □

*Remark* 6.3. The bound 4 in the formulation of Theorem 6.2 is of no significance as by rescaling the flow it can be replaced with any $\epsilon > 0$.

## 7. CONSERVATIVE AND INTERMITTED TRANSFORMATIONS

Interesting dynamics of conservative transformations is present only in the non-discrete case, as it reduces to periodicity for countable group actions. Section 6 provides an illustrative construction of a conservative automorphism, and shows that they exist in $\mathrm{L}^1$ full groups of all free flows. The present section is devoted to the study of such elements. The central role is played by the concept of an intermitted transformation, which is related to the notion of induced transformation. Using this tool we show that all conservative elements of $[\mathbb{R}\curvearrowright X]_1$ can be approximated by periodic automorphisms, and hence belong to the derived subgroup of $[\mathbb{R}\curvearrowright X]_1$; see Corollary 7.8.

Throughout the section we fix a free measure-preserving flow $\mathbb{R}\curvearrowright X$ on a standard Lebesgue space $(X,\mu)$. Given a cross section $\mathscr{C} \subset X$, recall that we defined an equivalence relation by $\mathscr{R}_\mathscr{C}$ by declaring $x\mathscr{R}_\mathscr{C} y$ whenever there is $c \in \mathscr{C}$ such that both $x$ and $y$ belong to the gap between $c$ and $\sigma_\mathscr{C}(c)$. More formally, $x\mathscr{R}_\mathscr{C} y$ if there is $c\in\mathscr{C}$ such that $\rho(c,x)\ge 0$, $\rho(c,y)\ge 0$ and $\rho(x,\sigma_\mathscr{C}(c)) > 0$, $\rho(x,\sigma_\mathscr{C}(c)) > 0$. Such an equivalence relation is smooth.

7.1. **Intermitted transformations.** Let $T \in [\mathbb{R} \curvearrowright X]$ be a conservative transformation, and let $\mathscr{C}$ be a cross section of the flow. Under the action of $T$, almost every point returns to its $\mathscr{R}_{\mathscr{C}}$-class infinitely often, which suggests the idea of the first return map.

**Definition 7.1.** The **intermitted transformation** $T_{\mathscr{R}_{\mathscr{C}}} : X \to X$ is defined by

$$T_{\mathscr{R}_{\mathscr{C}}} x = T^{n(x)} x, \quad \text{where } n(x) = \min\{n \geq 1 : x \mathscr{R}_{\mathscr{C}} T^{n(x)} x\}.$$

The map $T_{\mathscr{R}_{\mathscr{C}}}$ is well-defined, since $T$ is conservative, and it preserves the measure $\mu$, since $T_{\mathscr{R}_{\mathscr{C}}}$ belongs to the full group of $T$.

*Remark* 7.2. The concept of an intermitted transformation $T_E$ makes sense for any equivalence relation $E$ for which intersection of any orbit of $T$ with any $E$-class is either empty or infinite. In particular, intermitted transformations can be considered for any conservative $T \in [G \curvearrowright X]$ in a full group of a locally compact group action. For instance, with a cocompact cross section $\mathscr{C}$ we can associate an equivalence relation of lying in same cell of the Voronoi tessellation (see Appendix B.2). Such an equivalence relation does have the aforementioned transversal property, and hence intermitted transformation is well-defined.

Note also the following connection with the more familiar construction of the induced transformation. Let $T \in \mathrm{Aut}(X, \mu)$, let $A \subseteq X$ be a set of positive measure, and define $\mathscr{A}$ to be the equivalence relation with two classes: $A$ and $X \setminus A$. Induced transformations $T_A$ and $T_{X \setminus A}$ commute and satisfy $T_A \circ T_{X \setminus A} = T_{\mathscr{A}}$.

The next lemma forms the core of this section. It shows that the operation of taking an intermitted transformation does not increase the norm. As we discuss later in Remark 7.5, the analog of this statement is false even for $\mathbb{R}^2$-flows, which perhaps justifies the technical nature of the argument.

**Lemma 7.3.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a conservative automorphism and let $\mathscr{C}$ be a cross section. Let also $Y$ be the set of points where $T$ and $T_{\mathscr{R}_{\mathscr{C}}}$ differ: $Y = \{x \in X : Tx \neq T_{\mathscr{R}_{\mathscr{C}}} x\}$. One has $\int_Y |\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)| \, d\mu \leq \int_Y |\rho_T(x)| \, d\mu$.*

*Proof.* By the definition of $Y$, for any $x \in Y$ the arc from $x$ to $Tx$ jumps over at least one point of $\mathscr{C}$. We may therefore represent $|\rho_T(x)|$ as the sum of the distance from $x$ to the first point of $\mathscr{C}$ along the arc plus the rest of the arc. More formally, for $x \in X$ let $\pi_{\mathscr{C}}(x)$ be the unique $c \in \mathscr{C}$ such that $x \in c + [0, \mathrm{gap}_{\mathscr{C}}(c))$. Define $\alpha : Y \to \mathbb{R}^{\geq 0}$ by

$$\alpha(x) = \begin{cases} |\rho(x, \sigma_{\mathscr{C}}(\pi_{\mathscr{C}}(x)))|, & \text{if } \rho(x, Tx) > 0, \\ |\rho(x, \pi_{\mathscr{C}}(x))| & \text{if } \rho(x, Tx) < 0. \end{cases}$$

Note that $\alpha(x) \leq |\rho_T(x)|$, and set $\beta(x) = |\rho_T(x)| - \alpha(x)$, so that $\int_Y |\rho_T(x)| \, d\mu = \int_Y \alpha(x) \, d\mu + \int_Y \beta(x) \, d\mu$. For instance, in the context of Figure 4, $\alpha(x_4) = \rho(x_4, c_2)$ and $\beta(x_4) = \rho(c_2, x_5)$. Let us partition $Y = Y' \sqcup Y''$, where

$$Y' = \{x \in Y : \rho(x, Tx) \text{ and } \rho(x, T_{\mathscr{R}_{\mathscr{C}}} x) \text{ have the same sign or } T_{\mathscr{R}_{\mathscr{C}}} x = x\},$$

and $Y'' = Y \setminus Y'$ consists of those $x \in Y$ for which the signs of $\rho(x, Tx)$ and $\rho(x, T_{\mathscr{R}_{\mathscr{C}}} x)$ are different. For example, referring to the same figure, $x_0 \in Y''$, while $x_2 \in Y'$.

To prove the lemma it is enough to show two inequalities:

$$\int_{Y'} |\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)| \, d\mu \leq \int_Y \alpha(x) \, d\mu, \tag{3}$$

$$\int_{Y''} |\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)| \, d\mu \leq \int_Y \beta(x) \, d\mu. \tag{4}$$

The first one is straightforward, since equality of signs of $\rho(x, Tx)$ and $\rho(x, T_{\mathscr{R}_{\mathscr{C}}} x)$ implies that $T_{\mathscr{R}_{\mathscr{C}}} x$ is closer than $x$ to the point $c \in \mathscr{C}$ over which goes the arc from $x$ to $Tx$. For example, the point $x_2$ in Figure 4 satisfies $|\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x_2)| = \rho(x_2, x_4) \leq \rho(x_2, c_2) = \alpha(x_2)$. Thus $|\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)| \leq \alpha(x)$ for all $x \in Y'$ and so

$$\int_{Y'} |\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)| \, d\mu \leq \int_{Y'} \alpha(x) \, d\mu \leq \int_Y \alpha(x) \, d\mu,$$

which gives (3). The other inequality will take us a bit more work.

For $x \in Y''$, let $N(x) \geq 1$ be the smallest integer such that the sign of $\rho(x, T^{N(x)+1} x)$ is opposite to that of $\rho_T(x)$. In less formal words, $N(x)$ is the smallest integer such that the arc from $T^{N(x)} x$ to $T^{N(x)+1} x$ jumps over $x$. In particular, points $T^k x$, $1 \leq k \leq N(x)$, are all on the same side relative to $x$, while $T^{N(x)+1} x$ is on the other side of it. We consider the map $\eta : Y'' \to X$ given by $\eta(x) = T^{N(x)} x$. Properties of this map will be crucial for establishing the inequality (4), so let us provide some explanations first.

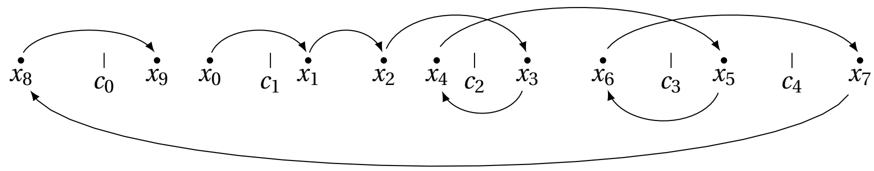


FIGURE 4. Dynamics of a conservative orbit.

Consider once again Figure 4, which shows a partial orbit of a point $x_0$ for $x_i = T^i x_0$ up to $i \leq 9$ and several points $c_i \in \mathscr{C}$. First, as we have already noted before $x_0 \in Y$, since $\neg x_0 \mathscr{R}_{\mathscr{C}} x_1$; moreover, $x_0 \in Y''$, since $x_9 = T_{\mathscr{R}_{\mathscr{C}}} x_0$ is to the left of $x_0$, while $x_1$ is to the right of it, so $\rho(x_0, x_1)$ and $\rho(x_0, x_9)$ have the opposite signs. Also, $N(x_0) = 7$, because $x_8$ is the first point in the orbit to left of $x_0$, thus $\eta(x_0) = x_7$. In particular, generally $T^{N(x)+1} x \neq T_{\mathscr{R}_{\mathscr{C}}} x$, but $T^{N(x)+1} x = T_{\mathscr{R}_{\mathscr{C}}} x$ is the case for $x \in Y''$ whenever $T^{N(x)+1} x$ and $x$ are $\mathscr{R}_{\mathscr{C}}$-equivalent.

The next point in the orbit $x_1 \not\in Y$, whereas $x_2 \in Y$ but $x_2 \not\in Y''$, because $T_{\mathscr{R}_{\mathscr{C}}} x_2 = x_4$ and both $\rho(x_2, x_3)$ and $\rho(x_2, x_4)$ are positive. The point $x_3$ belongs to $Y''$ and has $N(x_3) = 1$ with $\eta(x_3) = x_4$. Points $x_4, x_5, x_6 \in Y$, but whether any of them are elements of $Y''$ is not clear from Figure 4, as the orbit segment is too short to clarify the values of $T_{\mathscr{R}_{\mathscr{C}}} x_i$, $i = 4, 5, 6$. However, if $x_4, x_5, x_6$ happen to lie in $Y''$, then $N(x_5) = 1$ with $\eta(x_5) = x_6$, and $N(x_4) = 3$, $N(x_6) = 1$, $\eta(x_4) = \eta(x_6) = x_7 = \eta(x_0)$. In particular, the function $x \mapsto \eta(x)$ is not necessarily one-to-one, but we are going to argue that it is always finite-to-one.

*Claim* 1. *If $x, y \in Y''$ are distinct points such that $\eta(x) = \eta(y)$, then $\neg x \mathscr{R}_{\mathscr{C}} y$.*

*Proof of the claim.* Suppose $x, y \in Y''$ satisfy $\eta(x) = \eta(y)$. The definition of $\eta$ implies that $x$ and $y$ must belong to the same orbit of $T$, and we may assume without loss of generality that $y = T^{k_0} x$ for some $k_0 \geq 1$. If the orbit of $x$ and $y$ is aperiodic, it implies that that $N(x) > k_0$ and $N(y) + k_0 = N(x)$, $N(y) \geq 1$. However, even if the orbit is periodic, either $N(y) + k_0 = N(x)$ for the smallest positive integer $k_0$ such that $y = T^{k_0} x$ or $N(x) + k_0' = N(y)$ for the smallest positive integer $k_0'$ such that $x = T^{k_0'} y$. Interchanging the roles of $x$ and $y$ if necessary, we may therefore assume that $N(y) + k_0 = N(x)$ holds for some $k_0 \geq 1$, $T^{k_0} x = y$, regardless of the type of orbit we consider.

Suppose $x$ and $y$ are $\mathscr{R}_{\mathscr{C}}$-equivalent. Let $k \geq 1$ be the smallest natural number for which $x$ and $T^k x$ are $\mathscr{R}_{\mathscr{C}}$-equivalent. By the assumption $x \mathscr{R}_{\mathscr{C}} y$ and the choice of $k_0$ we have $k \leq k_0 < N(x)$. By the definition of $N(x)$, all points $T^i x$, $1 \leq i \leq N(x)$, are on the same side of $x$. In particular, this applies to $Tx$ and $T^k x$, which shows that $\rho(x, Tx)$ and $\rho(x, T_{\mathscr{R}_{\mathscr{C}}} x)$ have the same sign, thus $x \not\in Y''$. $\square_{\text{claim}}$

The above claim implies that the function $x \mapsto \eta(x)$ is finite-to-one for the arc from $\eta(x)$ to $T\eta(x)$ intersects only finitely many $\mathscr{R}_{\mathscr{C}}$-equivalence classes, and the preimage of $\eta(x)$ picks at most one point from each such class. Note also that $\eta(x) \in Y$ for all $x \in Y''$, but $\eta(x)$ may not be an element of $Y''$. Among the $\mathscr{R}_{\mathscr{C}}$-equivalence classes that the arc from $\eta(x)$ to $T\eta(x)$ goes over two are special—the intervals that contain $T\eta(x)$ and $\eta(x)$, respectively. Our goal will be to bound the sum of $|\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)|$ over the points $x$ with the same $\eta(x)$ value by $\beta(\eta(x))$ (see Claim 3 below). For a typical point $x$ we can bound $|\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)|$ simply by the length of the interval of its $\mathscr{R}_{\mathscr{C}}$-class. For example, Figure 4 does not specify $T_{\mathscr{R}_{\mathscr{C}}} x_4$, but we can be sure that $|\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x_4)| \leq \rho(c_1, c_2)$. In view of Claim 1, such an estimate comes close to showing that the sum of $|\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)|$ over $x$ with the same image $\eta(x)$ is bounded by $|\rho(\eta(x), T\eta(x))|$. It merely comes close, due to the two special $\mathscr{R}_{\mathscr{C}}$-classes mentioned above, where our estimate needs to be improved. The next claim shows that one of these special cases is of no concern as $x$ is never $\mathscr{R}_{\mathscr{C}}$-equivalent to $\eta(x)$.

*Claim* 2. *For all $x \in Y''$ we have $\neg x \mathscr{R}_{\mathscr{C}} \eta(x)$.*

*Proof of the claim.* Suppose towards the contradiction that $x \mathscr{R}_{\mathscr{C}} \eta(x)$, and let $k \geq 1$ be the smallest integer for which $x \mathscr{R}_{\mathscr{C}} T^k(x)$; in particular, $T_{\mathscr{R}_{\mathscr{C}}} x = T^k x$. Note that $k \leq N(x)$ by the assumption, and by the definition of $N(x)$, $\rho(T^k x, x)$ has the same sign as $\rho_T(x)$, whence $x \not\in Y''$. $\square_{\text{claim}}$

Pick some $y \in Y$ for which the preimage $\eta^{-1}(y)$ is non-empty, and let $z_1, \ldots, z_n \in Y''$ be all the elements in this preimage. For instance, in the situation depicted in Figure 4, we may have $n = 3$ and $z_1 = x_0$, $z_2 = x_4$, $z_3 = x_6$, and $y = x_7$. The following claim unlocks the path towards the inequality (4).

*Claim* 3. *In the above notation, $\sum_{i=1}^{n} |\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(z_i)| \leq \beta(y)$.*

*Proof of the claim.* Recall that the arc from $y$ to $Ty$ crosses at least one point in $\mathscr{C}$. If $c \in \mathscr{C}$ is the first such point, then $\beta(y)$ is defined to be $|\rho(c, Ty)|$. For instance, in the notation of Figure 4, $\beta(x_7) = |\rho(c_4, x_8)|$. Each point $z_i$ is located under the arc from $y$ to $Ty$, and by Claim 2, no point $z_i$ belongs to the interval from $c$ to $y$. In the language of our concrete example, no point $z_i$ can be between $c_4$ and $x_7$. As discussed before, $|\rho_{T_{\mathscr{R}_\mathscr{C}}}(x)|$ is always bounded by the length of the gap to which $x$ belongs. This is sufficient to prove the claim if no $z_i$ is equivalent to $Ty$, as in this case the whole $\mathscr{R}_\mathscr{C}$-equivalence class of every $z_i$ is fully contained under the interval between $c$ and $Ty$, and distinct $z_i$ represent distinct $\mathscr{R}_\mathscr{C}$-classes by Claim 1. This is the situation depicted in Figure 4, and our argument boils down to the inequalities

$$|\rho_{T_{\mathscr{R}_\mathscr{C}}}(x_0)|+|\rho_{T_{\mathscr{R}_\mathscr{C}}}(x_4)|+|\rho_{T_{\mathscr{R}_\mathscr{C}}}(x_5)| \le |\rho(c_0,c_1)|+|\rho(c_1,c_2)|+|\rho(c_2,c_3)| \le |\rho(c_0,c_4)| \le \beta(x_7).$$

Suppose there is some $z_i$ such that $z_i \mathscr{R}_\mathscr{C} Ty$. By Claim 1 such $z_i$ must be unique, and we assume without loss of generality that $z_1 \mathscr{R}_\mathscr{C} Ty$. For example, this situation would occur if in Figure 4 $Tx_7$ were equal to $x_9$. Let $c'$ be the first element of $\mathscr{C}$ over which goes the arc from $z_1$ to $Tz_1$ (it would be the point $c_1$ in Figure 4). It is enough to show that $|\rho_{T_{\mathscr{R}_\mathscr{C}}}(z_1)| \le |\rho(T_{\mathscr{R}_\mathscr{C}} z_1, c')|$, as we can use the previous estimate for all other $|\rho_{T_{\mathscr{R}_\mathscr{C}}}(z_i)|$, $i \ge 2$. Note that $T_{\mathscr{R}_\mathscr{C}} z_1 = Ty$, and $z_1 \in Y''$ by assumption, which implies that the signs of $\rho(z_1, T_{\mathscr{R}_\mathscr{C}} z_1)$ and $\rho(z_1, c')$ are different. The latter is equivalent to saying that $z_1$ is between $T_{\mathscr{R}_\mathscr{C}} z_1$ and $c'$, i.e., $|\rho(T_{\mathscr{R}_\mathscr{C}} z_1, c')| = |\rho_{T_{\mathscr{R}_\mathscr{C}}}(z_1)| + |\rho(z_1, c')|$, and the claim follows. $\square_{\text{claim}}$

We are now ready to finish the proof of this lemma. We have already shown that $\eta$ is finite-to-one, so let $Y''_n \subseteq Y''$, $n \ge 1$, be such that $x \mapsto \eta(x)$ is $n$-to-one on $Y''_n$. Let $R_n = \eta(Y''_n)$, and recall that $R_n \subseteq Y$. Sets $R_n$ are pairwise disjoint. Let $\phi_{k,n} : R_n \to Y''_n$, $1 \le k \le n$, be Borel bijections that pick the $k$th point in the preimage: $Y''_n = \bigsqcup_{i=1}^n \phi_{k,n}(R_n)$. Note that maps $\phi_{k,n} : R_n \to \phi_{k,n}(R_n)$ are measure-preserving, since they belong to the pseudo full group of $T$, and $\sum_{k=1}^n |\rho_{T_{\mathscr{R}_\mathscr{C}}}(\phi_{k,n}(x))| \le \beta(x)$ for all $x \in R_n$ by Claim 3. One now has

$$\begin{aligned}
\int_{Y''_n} |\rho_{T_{\mathscr{R}_\mathscr{C}}}(x)|\, d\mu &= \sum_{k=1}^n \int_{\phi_{k,n}(R_n)} |\rho_{T_{\mathscr{R}_\mathscr{C}}}(x)|\, d\mu \\
\because \phi_{n,k} \text{ are measure-preserving} &= \int_{R_n} \sum_{k=1}^n |\rho_{T_{\mathscr{R}_\mathscr{C}}}(\phi_{k,n}^{-1}(x))|\, d\mu \\
\because \text{Claim 3} &\le \int_{R_n} \beta(x)\, d\mu.
\end{aligned}$$

Summing these inequalities over $n$ we get

$$\int_{Y''} |\rho_{T_{\mathscr{R}_\mathscr{C}}}(x)|\, d\mu = \sum_{n=1}^\infty \int_{Y''_n} |\rho_{T_{\mathscr{R}_\mathscr{C}}}(x)|\, d\mu \le \sum_{n=1}^\infty \int_{R_n} \beta(x)\, d\mu \le \int_Y \beta(x)\, d\mu,$$

where the last inequality is based on the fact that sets $R_n$ are pairwise disjoint. This finished the proof of the inequality (4) as well as the lemma. $\square$

Several important facts follow easily from Lemma 7.3. For one, it implies that for any cross section $\mathscr{C}$ the intermitted transformation $T_{\mathscr{R}_\mathscr{C}}$ belongs to $[\mathbb{R} \curvearrowright X]_1$. In fact, we have the following inequality on the norms.

**Corollary 7.4.** *For any intermitted transformation $T_{\mathscr{R}_\mathscr{C}}$ one has $\|T_{\mathscr{R}_\mathscr{C}}\|_1 \le \|T\|_1$.*

*Proof.* By the definition of the set $Y$ in Lemma 7.3, $\rho_{T_{\mathscr{R}_\mathscr{C}}}(x) = \rho_T(x)$ for all $x \notin Y$, hence

$$\begin{aligned}
\int_X |\rho_{T_{\mathscr{R}_\mathscr{C}}}(x)|\, d\mu &= \int_{X\setminus Y} |\rho_{T_{\mathscr{R}_\mathscr{C}}}(x)|\, d\mu + \int_Y |\rho_{T_{\mathscr{R}_\mathscr{C}}}(x)|\, d\mu \\
\because \text{Lemma 7.3} &\le \int_{X\setminus Y} |\rho_T(x)|\, d\mu + \int_Y |\rho_T(x)|\, d\mu = \int_X |\rho_T(x)|\, d\mu,
\end{aligned}$$

which shows $\|T_{\mathscr{R}_\mathscr{C}}\|_1 \le \|T\|_1$. $\square$

*Remark* 7.5. As we discussed in Remark 7.2, the concept of an intermitted transformation applies wider than the case of one-dimensional flows. We mention, however, that the analog of Lemma 7.3 and Corollary 7.4 does not hold even for free measure-preserving $\mathbb{R}^2$-flows. Consider an annulus depicted in Figure 5a and let $T$ be the rotation by an angle $\alpha$ around the center of this annulus. Let the equivalence relation $E$ consist of two classes, each composing half of the ring. For a point $x$ such that $\neg xETx$, $T_E x$ will be close to the other side of the class. It is easy to arrange the parameters (the angle $\alpha$ and the radii of the annulus) so that $\|\rho_{T_E}(x)\| > \|\rho_T(x)\|$ for all $x$ such that $Tx \ne T_E x$.

Every free measure-preserving flow $\mathbb{R}^2 \curvearrowright X$ admits a tiling of its orbits by rectangles. The transformation $T \in [\mathbb{R}^2 \curvearrowright X]_1$ can be defined similarly to Figure 5a on each rectangle of the tiling by splitting each tile into two

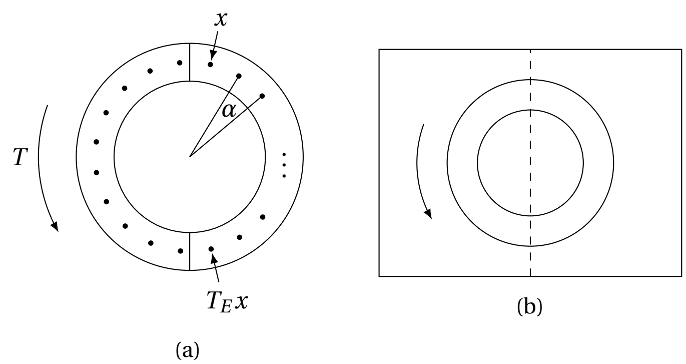


FIGURE 5. Construction of a conservative transformation $T$ with $\|T_E\|_1 > \|T\|_1$.

equivalence classes as in 5b. The resulting transformation $T$ will have bounded orbits and satisfy $\|T_E\|_1 > \|T\|_1$ relative to the equivalence relation $E$ whose classes are the half tiles.

When the gaps in a cross section $\mathscr{C}$ are large, $x$ and $Tx$ will often be $\mathscr{R}_{\mathscr{C}}$-equivalent, and it therefore natural to expect that $T_{\mathscr{R}_{\mathscr{C}}}$ will be close to $T$. This intuition is indeed valid, and the following approximation result is the most important consequence of Lemma 7.3.

**Lemma 7.6.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a conservative transformation. For any $\epsilon > 0$ there exists $M$ such for any cross section $\mathscr{C}$ with $\mathrm{gap}_{\mathscr{C}}(c) \geq M$ for all $c \in \mathscr{C}$ one has $\|T \circ T^{-1}_{\mathscr{R}_{\mathscr{C}}}\|_1 < \epsilon$.*

*Proof.* Let $A_K = \{x \in X : |\rho_T(x)| \geq K\}$, $K \in \mathbb{R}^{\geq 0}$, be the set of points whose cocycle is at least $K$ in the absolute value. Since $T \in [\mathbb{R} \curvearrowright X]_1$, we may pick $K \geq 1$ is so large that $\int_{A_K} |\rho_T(x)|\, d\mu < \epsilon/4$. Pick any real $M$ such that $2K^2/M < \epsilon/4$. We claim that it satisfies the conclusion of the lemma. To verify this we pick a cross section $\mathscr{C}$ with all gaps having size at least $M$. Set as before $Y = \{x \in X : Tx \neq T_{\mathscr{R}_{\mathscr{C}}} x\}$. Since $\left\| T \circ T^{-1}_{\mathscr{R}_{\mathscr{C}}} \right\|_1 = \int_Y D(Tx, T_{\mathscr{R}_{\mathscr{C}}} x)\, d\mu$, our task is to estimate this integral. This can be done in a rather crude way. We can simply use the triangle inequality $D(Tx, T_{\mathscr{R}_{\mathscr{C}}} x) \leq |\rho_T(x)| + |\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)|$, and deduce

$$\int_Y D(Tx, T_{\mathscr{R}_{\mathscr{C}}} x)\, d\mu \leq \int_Y |\rho_T(x)|\, d\mu + \int_Y |\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)|\, d\mu \leq 2 \int_Y |\rho_T(x)|\, d\mu,$$

where the last inequality is based on Lemma 7.3. It remains to show that $\int_Y |\rho_T(x)|\, d\mu < \epsilon/2$.

Let $\widetilde{X} = \{c + [K, \mathrm{gap}_{\mathscr{C}}(c) - K] : c \in \mathscr{C}\}$ be the region that leaves out intervals of length $K$ on both sides of each point in $\mathscr{C}$. Note that for any $x \in \widetilde{X} \setminus A_K$ one has $x \mathscr{R}_{\mathscr{C}} Tx$ and thus $T_{\mathscr{R}_{\mathscr{C}}} x = Tx$ for such points. Therefore, $Y \subseteq A_K \sqcup B_K$, where $B_K = X \setminus (\widetilde{X} \cup A_K)$, and thus

$$\int\limits_Y |\rho_T(x)|\, d\mu \leq \int\limits_{A_K} |\rho_T(x)|\, d\mu + \int\limits_{B_K} |\rho_T(x)|\, d\mu < \epsilon/4 + K \cdot 2K/M < \epsilon/2. \qquad \square$$

**Lemma 7.7.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a conservative transformation. For any $\epsilon > 0$ there exists a periodic transformation $P \in [T]$ such that $\|T \circ P^{-1}\|_1 < \epsilon$.*

*Proof.* By Lemma 7.6, we can find a cocompact cross section $\mathscr{C}$ such that $\|T \circ T^{-1}_{\mathscr{R}_{\mathscr{C}}}\| < \epsilon/2$. Let $\tilde{M}$ be an upper bound for gaps in $\mathscr{C}$. Recall that the cocycle $|\rho_{T_{\mathscr{R}_{\mathscr{C}}}}(x)|$ is uniformly bounded by $\tilde{M}$, and, in fact, the same is true for any element in the full group of $T_{\mathscr{R}_{\mathscr{C}}}$. In particular, we may use Rokhlin's Lemma to find a periodic $P \in [T_{\mathscr{R}_{\mathscr{C}}}]$ such that $\|T_{\mathscr{R}_{\mathscr{C}}} \circ P^{-1}\| < \epsilon/2\tilde{M}$, and conclude that $\|T_{\mathscr{R}_{\mathscr{C}}} \circ P^{-1}\|_1 < \epsilon/2$. We therefore have

$$\|T \circ P^{-1}\|_1 \leq \|T \circ T^{-1}_{\mathscr{R}_{\mathscr{C}}}\|_1 + \|T_{\mathscr{R}_{\mathscr{C}}} \circ P^{-1}\|_1 < \epsilon. \qquad \square$$

**Corollary 7.8.** *If $T \in [\mathbb{R} \curvearrowright X]_1$ is conservative then $T$ belongs to the derived full group $D([\mathbb{R} \curvearrowright X]_1)$, in particular its index satisfies $\mathscr{I}(T) = 0$.*

*Proof.* Follows directly from Lemma 7.7 and Corollary C.11. $\square$

## 8. Dissipative and Monotone Transformations

The previous section studied conservative transformations, whereas this one concentrates on dissipative ones. Our goal will be to show that any dissipative $T \in [\mathbb{R} \curvearrowright X]_1$ of index $\mathscr{I}(T) = 0$ belongs to the derived subgroup $D([\mathbb{R} \curvearrowright X]_1)$. We begin however by describing some general aspects of dynamics of dissipative automorphisms.

Recall that according to Proposition B.22, any transformation $T \in [\mathbb{R} \curvearrowright X]$ induces a $T$-invariant partition of the phase space $X = D \sqcup C$ such that $T|_C$ is conservative and $T|_D$ is dissipative. Formally speaking, a transformation is said to be dissipative if the partition trivializes to $D = X$. For the purpose of this section it is however convenient to widen this notion just a bit by allowing $T$ to have fixed points.

**Definition 8.1.** A transformation $T \in [\mathbb{R} \curvearrowright X]$ is said to be **dissipative** if $D = \operatorname{supp} T$, where $D$ is the dissipative element of the Hopf's decomposition for $T$.

8.1. **Orbit limits and Monotone transformations.** We begin by showing that dynamics of dissipative transformations in $\mathrm{L}^1$ full groups of $\mathbb{R}$-flows is similar to those in $\mathrm{L}^1$ full groups of $\mathbb{Z}$ actions. We do so by establishing an analog of Belinskaja's result [Bel68, Thm. 3.2]. Recall that a sequence of reals is said to have an almost constant sign if all but finitely many elements of the sequence have the same sign.

**Proposition 8.2.** *Let $S$ be a measure-preserving transformation of the real line which commensurates the set $\mathbb{R}^-$, suppose that $S$ is dissipative. Then for almost all $x \in \mathbb{R}$, the sequence of reals $(S^k(x) - x)_{k\in\mathbb{N}}$ has almost constant sign.*

*Proof.* Let $Q$ be the set of reals $x$ such that $(S^k(x) - x)_{k\in\mathbb{N}}$ does not have almost constant sign, and suppose by contradiction that $Q$ has positive measure. Since $S$ is dissipative, we can then find a Borel fundamental domain $A \subseteq \mathbb{R}$ for $S$ which intersects $Q$ with positive measure, and we thus obtain a set $Q' := Q \cap A$ all whose translates are disjoint, such that for all $x \in Q'$, $(S^k(x) - x)_{k\in\mathbb{N}}$ does not have almost constant sign.

Since $S$ is dissipative, for almost all $x \in Q'$, the sequence of absolute values $(|S^k(x)|)_{k\in\mathbb{N}}$ tends to $+\infty$ (see Proposition B.20). In particular, there are infinitely many points $y$ in the $S$-orbit of $x$ such that $y < 0$ but $S(y) > 0$. Since the map $Q' \times \mathbb{Z} \to \mathbb{R}$ which maps $(x, k)$ to $S^k(x)$ is measure-preserving, this yields that the set of $y < 0$ such that $S(y) > 0$ has infinite measure, contradicting the fact that $S$ commensurates the set $\mathbb{R}^-$. □

**Corollary 8.3.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipative transformation. For almost all $x \in X$, the sequence $(\rho(x, T^k(x)))_{k\in\mathbb{N}}$, $x \in \operatorname{supp} T$ has an almost constant sign.*

*Proof.* Let $T \in [\mathbb{R} \curvearrowright X]_1$. For all $x \in X$, denote by $T_x$ the measure-preserving transformation of $\mathbb{R}$ induced by $T$ on the $\mathbb{R}$-orbit of $x$. By the proof of Proposition 5.8, the integral

$$\int_X \lambda(\mathbb{R}^{\geq 0} \triangle (T_x(\mathbb{R}^{\geq 0})))\, d\mu(x)$$

is finite, in particular for almost every $x \in X$, the transformation $T_x$ commensurates the set $\mathbb{R}^{\geq 0}$. The conclusion now follows directly from the previous proposition. □

For any dissipative transformation in an $\mathrm{L}^1$ full group of a free locally compact Polish group action and for almost every $x \in X$, $\rho(x, T^n x) \to \infty$ as $n \to \infty$, in the sense that $\rho(x, T^n x)$ eventually escapes any compact subset of the acting group. In the context of flows, Corollary 8.3 strengthens this statement and implies that $\rho(x, T^n x)$ must converge to either $+\infty$ or $-\infty$.

**Corollary 8.4.** *If $T \in [\mathbb{R} \curvearrowright X]_1$ is dissipative, then for almost every point $x \in \operatorname{supp} T$ either $\lim_{n\to\infty} \rho(x, T^n x) = +\infty$ or $\lim_{n\to\infty} \rho(x, T^n x) = -\infty$.* □

In view of this corollary, there is a canonical $T$-invariant decomposition of $\operatorname{supp} T$ into "positive" and "negative" orbits.

**Definition 8.5.** Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipative automorphism. Its support is partitioned into $\vec{X} \sqcup \overleftarrow{X}$, where

$$\vec{X} = \{x \in \operatorname{supp} T : \lim_{n\to\infty} \rho(x, T^n x) = +\infty\},$$
$$\overleftarrow{X} = \{x \in \operatorname{supp} T : \lim_{n\to\infty} \rho(x, T^n x) = -\infty\}.$$

The set $\vec{X}$ is said to be **positive evasive** and $\overleftarrow{X}$ is **negative evasive**.

According to Corollary 8.3, for almost every $x \in \operatorname{supp} T$ *eventually* either all $T^n x$ are to the right of $x$ or all are to the left of it. There are points $x$ for which the adverb "eventually" can, in fact, be dropped.

**Corollary 8.6.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipative transformation and let*

$$\vec{A} = \{x \in \vec{X} : \rho(x, T^n x) > 0 \text{ for all } n \geq 1\},$$
$$\overleftarrow{A} = \{x \in \overleftarrow{X} : \rho(x, T^n x) < 0 \text{ for all } n \geq 1\}.$$

*The set $A = \vec{A} \sqcup \overleftarrow{A}$ is a complete section for $T|_{\operatorname{supp} T}$.*

*Proof.* We need to show that almost every orbit of $T$ intersects $A$. Let $x \in \operatorname{supp} T$ and suppose for definiteness that $x \in \vec{X}$. Since $\lim_{n\to\infty} \rho(x, T^n x) = +\infty$, there is $n_0 = \max\{n \in \mathbb{N} : \rho(x, T^n x) \leq 0\}$, and therefore $T^{n_0} x \in \vec{A}$. □

**Definition 8.7.** A dissipative transformation $T \in [\mathbb{R} \curvearrowright X]_1$ is **monotone** if $\rho(x, Tx) > 0$ for almost all $x \in \vec{X}$, and $\rho(x, Tx) < 0$ for almost all $x \in \overleftarrow{X}$.

**Corollary 8.8.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipative transformation. There is a complete section $A \subseteq \operatorname{supp} T$ and a periodic transformation $P \in [\mathbb{R} \curvearrowright X]_1 \cap [T]$ such that $T = P \circ T_A$ and $T_A$ is monotone.*

*Proof.* Take $A$ to be as in Corollary 8.6 and note that $P = T \circ T_A^{-1}$ is periodic and satisfies the conclusions of the corollary. □

As we discussed at the beginning of the section, our goal is to show that the index of the kernel map coincides with the derived subgroup of $[\mathbb{R} \curvearrowright X]_1$. Note that if $T = P \circ T_A$ is as above, then $\mathscr{I}(T) = \mathscr{I}(T_A)$, and coupled with the results of Section 7 it will suffice to show that all monotone transformations of index zero belong to $D([\mathbb{R} \curvearrowright X]_1)$. This will be the focus of the rest of this section and will take some effort to achieve, but the main strategy is to show that such automorphisms can be approximated by periodic maps, which is the content of Theorem 8.15 below.

8.2. **Arrival and Departure sets.** Throughout the rest of this section we fix a cross section $\mathscr{C} \subset X$ and a monotone transformation $T \in [\mathbb{R} \curvearrowright X]_1$. The **arrival set** $A_{\mathscr{C}}$ is the set of the first visitors to $E_{\mathscr{C}}$ classes: $A_{\mathscr{C}} = \{x \in \operatorname{supp} T : \neg x E_{\mathscr{C}} T^{-1} x\}$. Analogously, the **departure set** $D_{\mathscr{C}}$ is defined to be $D_{\mathscr{C}} = \{x \in \operatorname{supp} T : \neg x E_{\mathscr{C}} T x\}$. We also let $\vec{A}_{\mathscr{C}}$ denote $A_{\mathscr{C}} \cap \vec{X}$ and $\overleftarrow{A}_{\mathscr{C}} = A_{\mathscr{C}} \cap \overleftarrow{X}$; likewise for $\vec{D}_{\mathscr{C}}$ and $\overleftarrow{D}_{\mathscr{C}}$. Note that $T(D_{\mathscr{C}}) = A_{\mathscr{C}}$, and thus $T^{-1}(A_{\mathscr{C}}) = D_{\mathscr{C}}$. There is, however, another useful map from $A_{\mathscr{C}}$ onto $D_{\mathscr{C}}$.

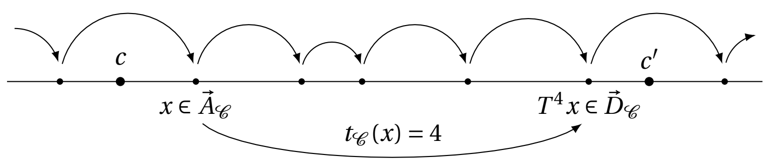


FIGURE 6. Arrival and Departure sets.

We define the **transfer value** $t_{\mathscr{C}} : A_{\mathscr{C}} \to \mathbb{N}$ by the condition $t_{\mathscr{C}}(x) = \min\{n \geq 0 : T^n x \in D_{\mathscr{C}}\}$ and the **transfer function** $\tau_{\mathscr{C}} : A_{\mathscr{C}} \to D_{\mathscr{C}}$ is defined to be $\tau_{\mathscr{C}}(x) = T^{t_{\mathscr{C}}(x)} x$. Note that $\tau_{\mathscr{C}}$ is measure-preserving. The transfer value introduces a partition of the arrival set $A_{\mathscr{C}} = \bigsqcup_{n\in\mathbb{N}} A^n_{\mathscr{C}}$, where $A^n_{\mathscr{C}} = t^{-1}_{\mathscr{C}}(n)$; by applying the transfer function, it also produces a partition for the departure set: $D_{\mathscr{C}} = \bigsqcup_{n\in\mathbb{N}} D^n_{\mathscr{C}}$, where $D^n_{\mathscr{C}} = \tau_{\mathscr{C}}(A^n_{\mathscr{C}})$.

In plain words, $t_{\mathscr{C}}(x) + 1$ is the number of points in $[x]_{E_T} \cap [x]_{E_{\mathscr{C}}}$. Therefore if $\lambda^{\mathscr{C}}_c(A^n_{\mathscr{C}}) \geq \lambda^{\mathscr{C}}_c(A^m_{\mathscr{C}})$ for some $n \geq m$ then also $\lambda^{\mathscr{C}}_c([A^n_{\mathscr{C}}]_{E_T}) \geq \lambda^{\mathscr{C}}_c([A^m_{\mathscr{C}}]_{E_T})$. In Subsections 8.3 and 8.4 we modify the transformation $T$ on the arrival and departure sets and we want to do this in a way that affects as many orbits as possible as measured by $\lambda^{\mathscr{C}}_c$. This amounts to using sets $A^n_{\mathscr{C}}$ (and $D^n_{\mathscr{C}}$) with as high values of $n$ as possible. The next lemma will be helpful in conducting such a selection in a measurable way across all of $c \in \mathscr{C}$.

**Lemma 8.9.** *Let $A \subseteq X$ be a measurable set with a measurable partition $A = \bigsqcup_n A_n$ and let $\xi : \mathscr{C} \to \mathbb{R}^{\geq 0}$ be a measurable function such that $\xi(c) \leq \lambda^{\mathscr{C}}_c(A)$ for all $c \in \mathscr{C}$. There are measurable $\nu : \mathscr{C} \to \mathbb{N}$ and $r : \mathscr{C} \to \mathbb{R}^{\geq 0}$ such that for any $c \in \mathscr{C}$ for which $\xi(c) > 0$ one has*

$$\lambda^{\mathscr{C}}_c\Big(\Big(\bigsqcup_{n>\nu(c)} A_n\Big) \cup \big(A_{\nu(c)} \cap (c + [0, r(c)])\big)\Big) = \xi(c).$$

*Proof.* For $c \in \mathscr{C}$ such that $\xi(c) > 0$ set $\nu(c) = \min\{n \in \mathbb{N} : \lambda^{\mathscr{C}}_c\big(\bigsqcup_{k>n} A_k\big) < \xi(c)\}$. Note that one necessarily has $\lambda^{\mathscr{C}}_c\big(A_{\nu(c)}\big) \geq \xi(c) - \lambda^{\mathscr{C}}_c\big(\bigsqcup_{n>\nu(c)} A_n\big)$. Set

$$r(c) = \min\Big\{a \geq 0 : \lambda^{\mathscr{C}}_c\big(A_{\nu(c)} \cap (c + [0, a])\big) = \xi(c) - \lambda^{\mathscr{C}}_c\Big(\bigsqcup_{n>\nu(c)} A_n\Big)\Big\}.$$

These functions $v$ and $r$ satisfy the conclusions of the lemma. □

**Definition 8.10.** Consider the partition of the positive arrival set $\vec{A}_{\mathscr{C}} = \bigsqcup_n \vec{A}^n_{\mathscr{C}}$ and let $\xi : \mathscr{C} \to \mathbb{R}^{\geq 0}$, $r : \mathscr{C} \to \mathbb{R}^{\geq 0}$, and $v : \mathscr{C} \to \mathbb{N}$ be as in Lemma 8.9. The set $\vec{A}^{\bullet}_{\mathscr{C}}$ defined by the condition

$$\vec{A}^{\bullet}_{\mathscr{C}}(c) = \bigsqcup_{n > \vec{v}(c)} \vec{A}^n_{\mathscr{C}}(c) \cup \big(A^{\vec{v}(c)}_{\mathscr{C}} \cap (c + [0, \vec{r}(c)])\big) \quad \text{for all } c \in \mathscr{C}$$

is said to be the **positive $\xi$-copious arrival set**. The **positive $\xi$-copious departure set** is given by $\vec{D}^{\bullet}_{\mathscr{C}} = \tau_{\mathscr{C}}(\vec{A}^{\bullet}_{\mathscr{C}})$. The definitions of the **negative $\xi$-copious arrival** and **departure sets** uses the partition $\overleftarrow{A}_{\mathscr{C}} = \bigsqcup_n \overleftarrow{A}^n_{\mathscr{C}}$ of the negative arrival set and is analogous.

Copious sets maximize measure $\lambda^{\mathscr{C}}_c$ of their saturation under the action of $T$. In other words, among all subsets $A' \subseteq \vec{A}_{\mathscr{C}}$ for which $\lambda^{\mathscr{C}}_c(A') = \xi(c)$, the measure $\lambda^{\mathscr{C}}_c([A']_{E_T})$ is maximal when $A'(c) = \vec{A}^{\bullet}_{\mathscr{C}}(c)$. In particular, if $\lambda^{\mathscr{C}}_c(\vec{A}^{\bullet}_{\mathscr{C}})$ is close to $\lambda^{\mathscr{C}}_c(\vec{A}_{\mathscr{C}})$, then we expect $\lambda^{\mathscr{C}}_c([\vec{A}^{\bullet}_{\mathscr{C}}]_{E_T})$ to be close to $\lambda^{\mathscr{C}}_c([\vec{A}_{\mathscr{C}}]_{E_T})$. The following lemma quantifies this intuition.

**Lemma 8.11.** *Let $\xi : \mathscr{C} \to \mathbb{R}^{\geq 0}$ be such that $\xi(c) \leq \lambda^{\mathscr{C}}_c(\vec{A}_{\mathscr{C}})$ for all $c \in \mathscr{C}$, and let $\vec{A}^{\bullet}_{\mathscr{C}}$ be the $\xi$-copious arrival set constructed in Lemma 8.9. If there exists $1/2 > \delta > 0$ such that $\xi(c) \geq (1-\delta)\lambda^{\mathscr{C}}_c(\vec{A}_{\mathscr{C}})$ for all $c \in \mathscr{C}$, then*

$$\lambda^{\mathscr{C}}_c([\vec{A}_{\mathscr{C}}(c) \setminus \vec{A}^{\bullet}_{\mathscr{C}}(c)]_{E_T}) \leq \frac{\delta}{1-\delta}\lambda^{\mathscr{C}}_c(\vec{X}) \quad \textit{for all } c \in \mathscr{C},$$

*and therefore also $\mu([\vec{A}_{\mathscr{C}} \setminus \vec{A}^{\bullet}_{\mathscr{C}}]_{E_T}) \leq \frac{\delta}{1-\delta}\mu(\vec{X})$.*

*An analogous statement is valid for the negative arrival set $\overleftarrow{A}_{\mathscr{C}}$.*

*Proof.* Let $v$ be as in Lemma 8.9 and note that $\bigsqcup_{k > v(c)} \vec{A}^k_{\mathscr{C}}(c) \subseteq \vec{A}^{\bullet}_{\mathscr{C}}(c) \subseteq \bigsqcup_{k \geq v(c)} \vec{A}^k_{\mathscr{C}}(c)$ whenever $c \in \mathscr{C}$ satisfies $\xi(c) > 0$. Recall that for $x \in \vec{A}^n_{\mathscr{C}}$ we have $x E_{\mathscr{C}} T^k$ for all $0 \leq k \leq n$ and sets $T^k(\vec{A}^n_{\mathscr{C}})$ are pairwise disjoint. In particular,

$$\lambda^{\mathscr{C}}_c(\vec{X}) \geq \lambda^{\mathscr{C}}_c\big(\big[\bigsqcup_{k \geq v(c)} \vec{A}^k_{\mathscr{C}}(c)\big]_{E_T}\big) \geq (v(c)+1)\lambda^{\mathscr{C}}_c\big(\bigsqcup_{k \geq v(c)} \vec{A}^k_{\mathscr{C}}(c)\big) \geq (v(c)+1)\lambda^{\mathscr{C}}_c(\vec{A}^{\bullet}_{\mathscr{C}}) = (v(c)+1)\xi(c). \tag{5}$$

Note also that $\xi(c) \geq (1-\delta)\lambda^{\mathscr{C}}_c(\vec{A}_{\mathscr{C}})$ implies

$$\lambda^{\mathscr{C}}_c(\vec{A}_{\mathscr{C}} \setminus \vec{A}^{\bullet}_{\mathscr{C}}) \leq \xi(c)\delta/(1-\delta). \tag{6}$$

For any $c \in \mathscr{C}$ we have

$$\begin{aligned}
\lambda^{\mathscr{C}}_c([\vec{A}_{\mathscr{C}}(c) \setminus \vec{A}^{\bullet}_{\mathscr{C}}(c)]_{E_T}) &\leq \lambda^{\mathscr{C}}_c(\{T^k x : x \in \vec{A}_{\mathscr{C}}(c) \setminus \vec{A}^{\bullet}_{\mathscr{C}}(c), 0 \leq k \leq \vec{v}(c)\}) \\
&\leq (\vec{v}(c)+1)\lambda^{\mathscr{C}}_c(\vec{A}_{\mathscr{C}} \setminus \vec{A}^{\bullet}_{\mathscr{C}}) \\
\because (6) &\leq (\vec{v}(c)+1)\vec{\xi}(c)\delta/(1-\delta) \\
\because (5) &\leq \lambda^{\mathscr{C}}_c(\vec{X})\delta/(1-\delta).
\end{aligned}$$

The inequality for the measure $\mu$ follows by disintegrating $\mu$ into $\int_{\mathscr{C}} \lambda^{\mathscr{C}}_c(\cdot)$.

The argument for the negative arrival set is completely analogous. □

8.3. **Coherent modifications.** We remind the reader that our goal is to show that any dissipative $T \in [\mathbb{R} \curvearrowright X]_1$ of index $\mathscr{I}(T) = 0$ can be approximated by periodic transformations. One approach to "loop" the orbits of $T$ is by mapping $\vec{D}_{\mathscr{C}}(c)$ to $\overleftarrow{A}_{\mathscr{C}}(c)$ and $\overleftarrow{D}_{\mathscr{C}}(c)$ to $\vec{A}_{\mathscr{C}}(c)$ (cf. Figure 11). For such a modification to work, measures $\lambda^{\mathscr{C}}_c(\vec{D}_{\mathscr{C}}(c))$ and $\lambda^{\mathscr{C}}_c(\overleftarrow{A}_{\mathscr{C}}(c))$ have to be equal. Recall that $\mathscr{I}(T) = 0$ implies that for almost every $c \in \mathscr{C}$ the measure of points $x$ such that $x \leq c < Tx$ equals the measure of those $y$ for which $Ty < c \leq y$. If one could guarantee that $T(\vec{D}_{\mathscr{C}}(c)) = \vec{A}_{\mathscr{C}}(\sigma_{\mathscr{C}}(c))$, then the aforementioned modification would indeed work. In the case of $\mathbb{Z}$ actions, discreteness of the acting group allows one to find a cross section $\mathscr{C}$ for which this condition does hold. Whereas for the flows we have to deal with the possibility that $T(\vec{D}_{\mathscr{C}}(c))$ can be "scattered" (see Figure 9) along the orbit and be unbounded, which is the key reason for the increased complexity compared to the argument for $\mathbb{Z}$ actions.

Since we can't hope to "loop" all the orbits of $T$, we will do the next best thing, and apply the modification of Figure 11 on "most" orbits as measured by $\lambda^{\mathscr{C}}_c$. Copious sets discussed in Subsection 8.2 have large saturations under $T$, but, generally speaking, fail to satisfy $T(\vec{D}^{\bullet}_{\mathscr{C}}(c)) = \vec{A}^{\bullet}_{\mathscr{C}}(\sigma_{\mathscr{C}}(c))$ for the same reason as do the sets $\vec{D}_{\mathscr{C}}(c)$. Our plan is to use the "$\epsilon$ of room" provided by the difference $\vec{D}_{\mathscr{C}}(c) \setminus \vec{D}^{\bullet}_{\mathscr{C}}(c)$ in order to modify $T$ into some $T'$ with the same arrival and departure sets as $T$, but for which also $T'(\vec{D}^{\bullet}_{\mathscr{C}}(c)) = \vec{A}^{\bullet}_{\mathscr{C}}(\sigma_{\mathscr{C}}(c))$ holds. In this subsection we describe two abstract modifications of dissipative transformations, and the approximation strategy outlined above will later be implemented in Subsection 8.4.

Since we are about to consider arrival and departure sets of different transformations, we use the notation $\vec{A}_{\mathscr{C}}[U]$ to denote the positive arrival set constructed for a transformation $U$; likewise for negative arrival and departure sets, etc.

**Lemma 8.12.** *Let $\phi$ and $\phi'$ be measure-preserving transformations on $X$ subject to the following conditions:*

*(1)* $\operatorname{supp}(\phi) \subseteq D_{\mathscr{C}}$, $\operatorname{supp}(\phi') \subseteq A_{\mathscr{C}}$*;*
*(2)* $\phi(\vec{D}_{\mathscr{C}}) = \vec{D}_{\mathscr{C}}$, $\phi(\overleftarrow{D}_{\mathscr{C}}) = \overleftarrow{D}_{\mathscr{C}}$, *and* $\phi'(\vec{A}_{\mathscr{C}}) = \vec{A}_{\mathscr{C}}$, $\phi'(\overleftarrow{A}_{\mathscr{C}}) = \overleftarrow{A}_{\mathscr{C}}$*;*
*(3)* $xE_{\mathscr{C}}\phi(x)$ *and* $xE_{\mathscr{C}}\phi'(x)$ *for all* $x \in \operatorname{supp} T$.

*The transformation $Ux = \phi' T\phi(x)$ is monotone, $Ux = Tx$ for all $x \notin D_{\mathscr{C}}$, and the sets $D_{\mathscr{C}}$, $A_{\mathscr{C}}$ remain the same:*

$$\vec{X}[U] = \vec{X} \qquad \overleftarrow{X}[U] = \overleftarrow{X},$$
$$\vec{D}_{\mathscr{C}}[U] = \vec{D}_{\mathscr{C}} \qquad \overleftarrow{D}_{\mathscr{C}}[U] = \overleftarrow{D}_{\mathscr{C}},$$
$$\vec{A}_{\mathscr{C}}[U] = \vec{A}_{\mathscr{C}} \qquad \overleftarrow{A}_{\mathscr{C}}[U] = \overleftarrow{A}_{\mathscr{C}}.$$

*Moreover, the integral of lengths of "departing arcs" remains unchanged:*

$$\int_{D_{\mathscr{C}}} |c_U(x)|\, d\mu = \int_{D_{\mathscr{C}}} |c_T(x)|\, d\mu,$$

*and the following estimate on $\int_X D(Tx, Ux)\, d\mu$ is available:*

$$\int_X D(Tx, Ux)\, d\mu \le 2\int_{D_{\mathscr{C}}} |c_T(x)|\, d\mu.$$

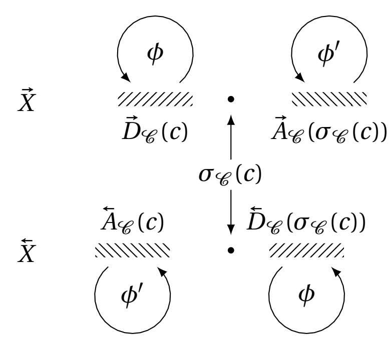


FIGURE 7. The transformation $U = \phi' T\phi$ defined in Lemma 8.12.

*Proof.* Figure 7 illustrates the definition of the transformation $U$. Equality of the arrival and departure sets is straightforward to verify. Note that $\phi(\vec{D}_{\mathscr{C}}(c)) = \vec{D}_{\mathscr{C}}(c)$ for all $c \in \mathscr{C}$, and therefore $\int_{\vec{D}_{\mathscr{C}}} c_\phi(x)\, d\mu = 0$. In fact, the following four integrals vanish:

$$\int_{\vec{D}_{\mathscr{C}}} c_\phi(x)\, d\mu = \int_{\overleftarrow{D}_{\mathscr{C}}} c_\phi(x)\, d\mu = \int_{\vec{A}_{\mathscr{C}}} c_{\phi'}(x)\, d\mu = \int_{\overleftarrow{A}_{\mathscr{C}}} c_{\phi'}(x)\, d\mu = 0. \tag{7}$$

Observe that $c_U$ is positive on $\vec{D}_{\mathscr{C}}$ and negative on $\overleftarrow{D}_{\mathscr{C}}$, thus

$$\begin{aligned}
\int_{D_{\mathscr{C}}} |c_U(x)|\, d\mu &= \int_{\vec{D}_{\mathscr{C}}} c_{\phi' T\phi}(x)\, d\mu - \int_{\overleftarrow{D}_{\mathscr{C}}} c_{\phi' T\phi}(x)\, d\mu \\
\because \text{cocycle identity} &= \int_{\vec{D}_{\mathscr{C}}} c_\phi(x)\, d\mu + \int_{\vec{D}_{\mathscr{C}}} c_T(\phi(x))\, d\mu + \int_{\vec{D}_{\mathscr{C}}} c_{\phi'}(T\phi(x))\, d\mu \\
&\quad - \int_{\overleftarrow{D}_{\mathscr{C}}} c_\phi(x)\, d\mu - \int_{\overleftarrow{D}_{\mathscr{C}}} c_T(\phi(x))\, d\mu - \int_{\overleftarrow{D}_{\mathscr{C}}} c_{\phi'}(T\phi(x))\, d\mu \\
&= \int_{\vec{D}_{\mathscr{C}}} c_\phi(x)\, d\mu + \int_{\vec{D}_{\mathscr{C}}} c_T(x)\, d\mu + \int_{\vec{A}_{\mathscr{C}}} c_{\phi'}(x)\, d\mu
\end{aligned}$$

$$- \int_{\overleftarrow{D}_{\mathscr{C}}} c_\phi(x)\, d\mu - \int_{\overleftarrow{D}_{\mathscr{C}}} c_T(x)\, d\mu - \int_{\overleftarrow{A}_{\mathscr{C}}} c_{\phi'}(x)\, d\mu$$

$$\because \text{ Eq. (7)} = \int_{\vec{D}_{\mathscr{C}}} c_T(x)\, d\mu - \int_{\overleftarrow{D}_{\mathscr{C}}} c_T(x)\, d\mu = \int_{D_{\mathscr{C}}} |c_T(x)|\, d\mu.$$

Finally, note that for any $x \in D_{\mathscr{C}}$, the arc from $x$ to $Tx$ intersects the arc from $T^{-1}\phi' T\phi(x)$ to $\phi' T\phi(x)$ (both arcs go over the same point of $\mathscr{C}$), and therefore

$$D(Tx, Ux) \leq |c_T(x)| + |c_T(T^{-1}\phi' T\phi(x))|.$$

Integration over $D_{\mathscr{C}}$ yields $\int_X D(Tx, Ux)\, d\mu = \int_{D_{\mathscr{C}}} D(Tx, Ux)\, d\mu \leq 2\int_{D_{\mathscr{C}}} |c_T(x)|\, d\mu$. □

**Lemma 8.13.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a monotone transformation, let $F \subseteq D_{\mathscr{C}}$ be such that $\lambda_c^{\mathscr{C}}(\vec{F}) = \lambda_c^{\mathscr{C}}(\overleftarrow{F})$ for all $c \in \mathscr{C}$ and the function $\mathscr{C} \ni c \mapsto \lambda_c^{\mathscr{C}}(F)$ is $E$-invariant (i.e., $\lambda_c^{\mathscr{C}}(F) = \lambda_{c'}(F)$ whenever $c$ and $c'$ belong to the same orbit of the flow). Let $Z \subseteq A_{\mathscr{C}}$ be the arrival subset that corresponds to $F$, i.e., $Z = T(F)$. Let $\psi : \vec{F} \to \overleftarrow{Z}$ and $\psi' : \overleftarrow{F} \to \vec{Z}$ be any measure-preserving transformations such that $\psi(x) E_{\mathscr{C}} x$ and $\psi'(x) E_{\mathscr{C}} x$ for all $x$ in the corresponding domains. Define $V : X \to X$ by the following formula:*

$$Vx = \begin{cases} \psi(x) & \text{if } x \in \vec{F}, \\ \psi'(x) & \text{if } x \in \overleftarrow{F}, \\ Tx & \text{otherwise.} \end{cases}$$

*The transformation $V$ is a measure-preserving automorphism from the full group $[\mathbb{R} \curvearrowright X]$ and $Vx = Tx$ for all $x \notin F$. The integral of distances $D(Tx, Vx)$ can be estimated as follows:*

$$\int_X D(Tx, Vx)\, d\mu \leq 2\int_{D_{\mathscr{C}}} |c_T(x)|\, d\mu.$$

The following figure illustrates the notions of Lemma 8.13.

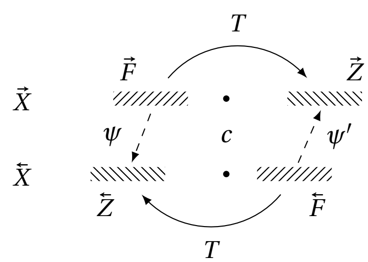


FIGURE 8. The transformation $V$ defined in Lemma 8.13.

*Proof.* It is straightforward to verify that $V$ is a measure-preserving transformation. For the integral inequality note that for any $x \in \vec{F}$ one has $D(Tx, Vx) \leq |c_T(x)| + |c_T(T^{-1}x)|$, and therefore

$$\int_{\vec{F}} D(Tx, Vx)\, d\mu \leq \int_{\vec{F}} |c_T(x)|\, d\mu + \int_{\overleftarrow{F}} |c_T(x)|\, d\mu = \int_F |c_T(x)|\, d\mu \leq \int_{D_{\mathscr{C}}} |c_T(x)|\, d\mu.$$

A similar inequality holds for $\int_{\overleftarrow{F}} d(Tx, Vx)\, d\mu$, and the lemma follows. □

8.4. **Periodic approximations.** We now have all the ingredients necessary to prove that monotone transformations can be approximated by periodic automorphisms. Our arguments follow the approach outlined at the beginning of Subsection 8.3.

In the following lemma we assume that the Lebesgue measure of those $x \in \vec{X}$ that jump over any given $c \in \mathscr{C}$ is bounded from above by some $\beta$, and that most of such jumps — of measure at least $\gamma$ — are between adjacent $E_{\mathscr{C}}$-classes. We are going to construct a periodic approximation $P$ of the transformation $T$ with an explicit bound on $\int_X D(Tx, Px)\, d\mu$, which can be made small for a sufficiently sparse cross section $\mathscr{C}$. When the flow is ergodic, this lemma alone suffices to conclude that $T \in D([\mathbb{R} \curvearrowright X]_1)$. Theorem 8.15 builds upon Lemma 8.14 and treats the general case.

**Lemma 8.14.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a monotone transformation, let $K > 0$ be a positive real, and let $J = \{x \in \operatorname{supp} T : |c_T(x)| \geq K\}$. Let $\mathscr{C}$ be a cross section such that $\operatorname{gap}_{\mathscr{C}}(c) > K$ for all $c \in \mathscr{C}$. Let $0 < \gamma < \beta$ be reals such that for all $c \in \mathscr{C}$:*

$$\begin{aligned}
&\lambda_c^{\mathscr{C}}(\{x \in \vec{X} : x < \sigma_{\mathscr{C}}(c) \leq Tx, TxE_{\mathscr{C}}\sigma_{\mathscr{C}}(c)\}) > \gamma,\\
&\lambda_c^{\mathscr{C}}(\{x \in \overleftarrow{X} : Tx < c \leq x, TxE_{\mathscr{C}}\sigma_{\mathscr{C}}^{-1}(c)\}) > \gamma,\\
&\lambda(\{x \in \vec{X} : x < \sigma_{\mathscr{C}}(c) \leq Tx\}) < \beta.\\
&\lambda(\{x \in \overleftarrow{X} : Tx < c \leq x\}) < \beta.
\end{aligned}$$

*There exists a periodic transformation $P \in [\mathbb{R} \curvearrowright X]_1$ such that $\operatorname{supp} P \subseteq \operatorname{supp} T$ and*

$$\int_X D(Tx, Px)\, d\mu \leq 5\int_{D_{\mathscr{C}}} |c_T(x)|\, d\mu + \int_J |c_T(x)|\, d\mu + \frac{K(\beta - \gamma)}{\gamma}\mu(\operatorname{supp} T).$$

*Proof.* Let $D_{\mathscr{C}}$ and $A_{\mathscr{C}}$ be the departure and the arrival sets of $T$. Figure 9 depicts the arrival $\vec{A}_{\mathscr{C}}(c)$ and the departure $\vec{D}_{\mathscr{C}}(c)$ sets for an element $c$ of the cross section $\mathscr{C}$. Note that preimages $T^{-1}(\vec{A}_{\mathscr{C}}(c))$ may come from different (possibly, infinitely many) $E_{\mathscr{C}}$-equivalence classes; likewise, images $T(\vec{D}_{\mathscr{C}}(c))$ of the departure set may visit several $E_{\mathscr{C}}$-equivalence classes.

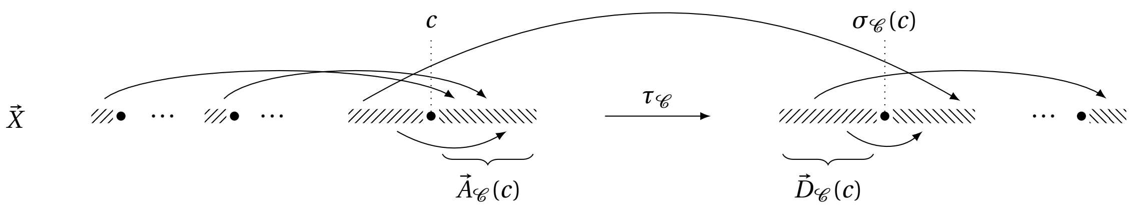


FIGURE 9. The arrival $\vec{A}_{\mathscr{C}}(c)$ and the departure $\vec{D}_{\mathscr{C}}(c)$ sets for some $c \in \mathscr{C}$.

Set $\xi(c) = \gamma$ to be the constant function; in view of the assumptions on $\gamma$, we may apply Lemma 8.9 to get positive and negative $\xi$-copious arrival sets $\vec{A}^{\bullet}_{\mathscr{C}} \subseteq A_{\mathscr{C}}$, $\overleftarrow{A}^{\bullet}_{\mathscr{C}} \subseteq A_{\mathscr{C}}$, as well as the corresponding departure sets $\vec{D}^{\bullet}_{\mathscr{C}} = \tau_{\mathscr{C}}(\vec{A}^{\bullet}_{\mathscr{C}})$ and $\overleftarrow{D}^{\bullet}_{\mathscr{C}} = \tau_{\mathscr{C}}(\overleftarrow{A}^{\bullet}_{\mathscr{C}})$. Set $A^{\bullet}_{\mathscr{C}} = \vec{A}^{\bullet}_{\mathscr{C}} \sqcup \overleftarrow{A}^{\bullet}_{\mathscr{C}}$ and $D^{\bullet}_{\mathscr{C}} = \vec{D}^{\bullet}_{\mathscr{C}} \sqcup \overleftarrow{D}^{\bullet}_{\mathscr{C}}$. We have $\lambda(A^{\bullet}_{\mathscr{C}}(c)) = 2\gamma = \lambda(D^{\bullet}_{\mathscr{C}}(c))$ for all $c \in \mathscr{C}$. Let

$$\begin{aligned}
A^{\circ}_{\mathscr{C}} &= \left\{x \in \vec{A}_{\mathscr{C}} : T^{-1}xE_{\mathscr{C}}\sigma_{\mathscr{C}}^{-1}(\pi_{\mathscr{C}}(x))\right\} \cup \left\{x \in \overleftarrow{A}_{\mathscr{C}} : T^{-1}xE_{\mathscr{C}}\sigma_{\mathscr{C}}(\pi_{\mathscr{C}}(x))\right\},\\
D^{\circ}_{\mathscr{C}} &= \left\{x \in \vec{D}_{\mathscr{C}} : TxE_{\mathscr{C}}\sigma_{\mathscr{C}}(\pi_{\mathscr{C}}(x))\right\} \cup \left\{x \in \overleftarrow{D}_{\mathscr{C}} : TxE_{\mathscr{C}}\sigma_{\mathscr{C}}^{-1}(\pi_{\mathscr{C}}(x))\right\},
\end{aligned}$$

be the set of arcs that jump from/to *the next* $E_{\mathscr{C}}$-equivalence class. By the assumptions of the lemma, we have $\lambda(\vec{D}^{\circ}_{\mathscr{C}}(c)) \geq \gamma$ and $\lambda(\vec{A}^{\circ}_{\mathscr{C}}(c)) \geq \gamma$ for all $c \in \mathscr{C}$. Let $\phi$ be any measure-preserving transformation such that:

- $\phi$ is supported on $D_{\mathscr{C}}$;
- $\phi(\vec{D}_{\mathscr{C}}) = \vec{D}_{\mathscr{C}}$ and $\phi(\overleftarrow{D}_{\mathscr{C}}) = \overleftarrow{D}_{\mathscr{C}}$;
- $\phi(x)E_{\mathscr{C}}x$ for all $x \in X$;

and moreover

$$\phi(D^{\bullet}_{\mathscr{C}}) \subseteq D^{\circ}_{\mathscr{C}}. \tag{8}$$

Select a transformation $\phi'$ such that

- $\phi'$ is supported on $A_{\mathscr{C}}$;
- $\phi'(\vec{A}_{\mathscr{C}}) = \vec{A}_{\mathscr{C}}$ and $\phi'(\overleftarrow{A}_{\mathscr{C}}) = \overleftarrow{A}_{\mathscr{C}}$;
- $\phi'(x)E_{\mathscr{C}}x$ for all $x \in X$;

and moreover

$$\phi'(T \circ \phi(D^{\bullet}_{\mathscr{C}})) = A^{\bullet}_{\mathscr{C}}. \tag{9}$$

Figure 10 illustrates these maps. Note that while in general $\tau_{\mathscr{C}}\big(\vec{A}^{\circ}(c)\big) \neq \vec{D}^{\circ}(c)$, one has $\tau_{\mathscr{C}}\big(\vec{A}^{\bullet}(c)\big) = \vec{D}^{\bullet}(c)$ for all $c \in \mathscr{C}$ by the definition of the $\xi$-copious departure set.

Let $U$ be the transformation obtained by applying Lemma 8.12 to $T$, $\phi$ and $\phi'$. The automorphism $U$ satisfies $U(\vec{D}^{\bullet}_{\mathscr{C}}(c)) = \vec{A}^{\bullet}_{\mathscr{C}}(\sigma_{\mathscr{C}}(c))$ and $U(\overleftarrow{D}^{\bullet}_{\mathscr{C}}(c)) = \overleftarrow{A}^{\bullet}_{\mathscr{C}}(\sigma_{\mathscr{C}}^{-1}(c))$ for all $c \in \mathscr{C}$. Choose a measure-preserving transformation

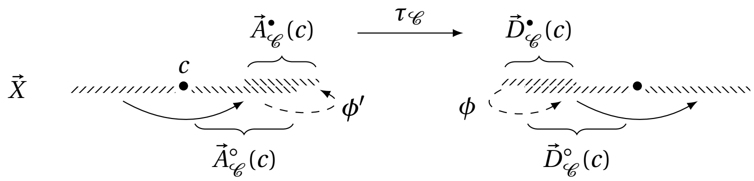


FIGURE 10. Automorphism $\phi$ maps $D^\bullet_{\mathscr{C}}(c)$ into $D^\circ_{\mathscr{C}}(c)$ and $(\phi')^{-1}$ sends $A^\bullet_{\mathscr{C}}(c)$ into $A^\circ_{\mathscr{C}}(c)$.

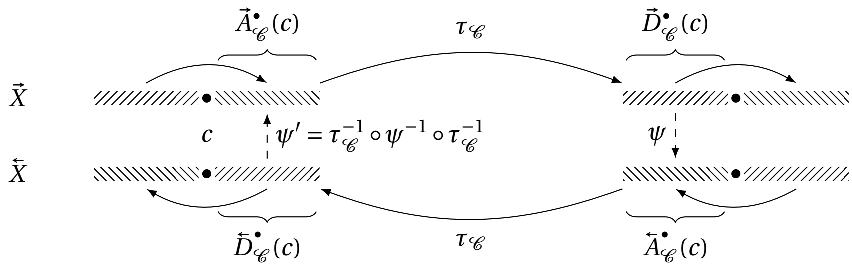


FIGURE 11. Construction of the automorphism $V$ from $U$, $\psi$, and $\psi'$.

$\psi : \vec{D}^\bullet_{\mathscr{C}} \to \overleftarrow{A}^\bullet_{\mathscr{C}}$ such that $xE_{\mathscr{C}}\psi(x)$ for all $x$ in the domain of $\psi$. Set $\psi' = \tau_{\mathscr{C}}^{-1} \circ \psi^{-1} \circ \tau_{\mathscr{C}}^{-1} : \overleftarrow{D}^\bullet_{\mathscr{C}} \to \vec{A}^\bullet_{\mathscr{C}}$. Let $V$ be the transformation that is produced by Lemma 8.13 applied to $U$, $\psi$, and $\psi'$ (see Figure 11).
Finally, set $P : X \to X$ to be

$$Px = \begin{cases} Vx & \text{if } x \in [D^\bullet_{\mathscr{C}}]_{E_V}, \\ x & \text{otherwise.} \end{cases}$$

We claim that $P$ satisfies the conclusions of the lemma. It is periodic since $\psi' \circ \tau_{\mathscr{C}} \circ \psi \circ \tau_{\mathscr{C}}$ is the identity map and $\operatorname{supp} P \subseteq \operatorname{supp} T$ by construction. It remains to estimate $\int_X D(Tx, Px)\, d\mu$.

$$\begin{aligned} \int_X D(Tx,Px)\,d\mu &\le \int_X D(Tx,Ux)\,d\mu + \int_X D(Ux,Vx)\,d\mu + \int_X D(Vx,Px)\,d\mu \\ &\le [\text{Estimates of Lemma 8.12 and Lemma 8.13}] \\ &\le 4\int_{D_{\mathscr{C}}} |c_T(x)|\,d\mu + \int_X D(Vx,Px)\,d\mu. \end{aligned}$$

We concentrate on estimating $\int_X D(Vx,Px)\,d\mu$. Recall that $Tx = Ux = Vx$ for all $x \notin D_{\mathscr{C}}$, hence $c_T(x) = c_V(x)$ for $x \notin D_{\mathscr{C}}$. Set $\Psi = (\vec{X} \cup \overleftarrow{X}) \setminus [D^\bullet_{\mathscr{C}}]_{E_V}$ and note that $Vx = Ux$ for $x \in \Psi$, and therefore using the conclusion of Lemma 8.12 we have

$$\int_{D_{\mathscr{C}}\cap\Psi} |c_V(x)|\,d\mu = \int_{D_{\mathscr{C}}\cap\Psi} |c_U(x)|\,d\mu \le \int_{D_{\mathscr{C}}} |c_U(x)|\,d\mu = \int_{D_{\mathscr{C}}} |c_T(x)|\,d\mu. \tag{10}$$

The integral $\int_X D(Vx,Px)\,d\mu$ can now be estimated as follows.

$$\begin{aligned} \int_X D(Vx,Px)\,d\mu &= \int_\Psi |c_V(x)|\,d\mu \\ &\le \int_{\Psi\setminus D_{\mathscr{C}}} |c_V(x)|\,d\mu + \int_{D_{\mathscr{C}}\cap\Psi} |c_V(x)|\,d\mu \\ \because Tx = Vx \text{ for } x \notin D_{\mathscr{C}} \text{ and Eq. (10)} &\le \int_{\Psi\setminus D_{\mathscr{C}}} |c_T(x)|\,d\mu + \int_{D_{\mathscr{C}}} |c_T(x)|\,d\mu. \end{aligned}$$

Finally, we consider the integral $\int_{\Psi\setminus D_{\mathscr{C}}} |c_T(x)|\,d\mu$ and partition its domain $\Psi \setminus D_{\mathscr{C}}$ as $(J \cap (\Psi \setminus D_{\mathscr{C}})) \sqcup ((\Psi \setminus D_{\mathscr{C}}) \setminus J)$, which yields

$$\int_{\Psi\setminus D_{\mathscr{C}}} |c_T(x)|\,d\mu \le \int_J |c_T(x)|\,d\mu + K\mu(\Psi) \le \int_J |c_T(x)|\,d\mu + \frac{K(\beta-\gamma)}{\gamma}\mu(\operatorname{supp} T),$$

where the last inequality follows from Lemma 8.11 with $\delta = 1-\gamma/\beta$. Combining all the inequalities together, we get

$$\int_X D(Tx,Px)\,d\mu \le 5\int_{D_{\mathscr{C}}} |c_T(x)|\,d\mu + \int_J |c_T(x)|\,d\mu + \frac{K(\beta-\gamma)}{\gamma}\mu(\operatorname{supp} T). \qquad \square$$

Lemma 8.14 allows us to approximate with a periodic transformation a monotone $T$ for which the Lebesgue measure of points jumping over any given $c \in X$ is roughly constant across orbits. To deal with the general case, we simply need to split the phase space $X$ into countably many segments invariant under the flow, and apply Lemma 8.14 on each of them separately. Small care needs to be taken to ensure that values $(\beta-\gamma)/\gamma$, which appear in the formulation of Lemma 8.14, remain uniformly small across the partition of $X$. Details are presented in the following theorem.

**Theorem 8.15.** *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a monotone transformation that belongs to the kernel of the index map, hence $\lambda(\{x \in \operatorname{supp} T : x < c \le Tx\}) = \lambda(\{y \in \operatorname{supp} T : Ty < c \le y\})$ for almost all $c \in X$. For any $\epsilon > 0$ there exists a periodic transformation $P \in [\mathbb{R} \curvearrowright X]_1$ such that $\operatorname{supp} P \subseteq \operatorname{supp} T$ and $\int_X D(Tx,Px)\,d\mu < \epsilon$.*

*Proof.* Let $K_\epsilon \ge 1$ be such that for the set $J_\epsilon = \{x \in \operatorname{supp} T : |c_T(x)| \ge K_\epsilon\}$ one has $\int_{J_\epsilon} |c_T(x)|\,d\mu < \epsilon/18$. Pick a cross section $\mathscr{C}$ with gaps so large that $2K_\epsilon^2/\operatorname{gap}(c) < \epsilon/15$ for all $c \in \mathscr{C}$, which ensures

$$K_\epsilon \cdot \mu(D_{\mathscr{C}} \setminus J_\epsilon) \le \epsilon/15. \tag{11}$$

Note also that Eq. (11) holds for any cross section $\mathscr{C}' \subseteq \mathscr{C}$, since $D_{\mathscr{C}'} \subseteq D_{\mathscr{C}}$ and $\operatorname{gap}_{\mathscr{C}'}(c) \ge \operatorname{gap}_{\mathscr{C}}(c)$ for all $c \in \mathscr{C}'$.

For any positive real $\alpha > 0$ there exists a positive $\delta(\alpha) = \delta > 0$ so small that $\delta < \alpha$ and $2\delta/(\alpha-\delta) < \epsilon/3K_\epsilon$. We may therefore pick countably many positive reals $\alpha_n > 0$, $n \ge 1$, such that $\mathbb{R}^{>0} = \bigcup_n (\alpha_n - \delta_n/2, \alpha_n + \delta_n/2)$ and

$$\Big(\frac{2\delta_n}{\alpha_n - \delta_n}\Big) < \frac{\epsilon}{3K_\epsilon} \quad \forall n \ge 1. \tag{12}$$

Define intervals $I_n = (\alpha_n - \delta_n/2, \alpha_n + \delta_n/2)$, $n \ge 1$.

Let $\zeta : \mathscr{C} \to R^{\ge 0}$ be the map that measures the set of forward arcs over its argument:

$$\zeta(c) = \lambda\big(\{x \in \vec{X} : x < c \le Tx\}\big).$$

Set $\mathscr{C}_1 = \zeta^{-1}(I_1)$ and construct inductively $\mathscr{C}_n = \zeta^{-1}(I_n) \setminus \big[\bigcup_{k<n} \mathscr{C}_k\big]_E$. Sets $\mathscr{C}_n$ are pairwise disjoint, and moreover, $\neg c_1 E c_2$ for all $c_1 \in \mathscr{C}_{n_1}$, $c_2 \in \mathscr{C}_{n_2}$, $n_1 \ne n_2$. Let $\chi_n : \mathscr{C}_n \to \mathbb{N}$, $n \ge 1$, be the function defined by

$$\chi_n(c) = \min\Big\{m \in \mathbb{N} : \lambda\big(\{x \in \vec{X} : x < c \le Tx, D(x,c) \le m, D(Tx,c) \le m\} > \zeta(c) - \delta_n/2\big)\Big\}.$$

Set $\mathscr{C}'_{n,1} = \chi_n^{-1}(1)$ and define inductively $\mathscr{C}'_{n,m} = \chi^{-1}(m) \setminus \big[\bigcup_{k<m} \mathscr{C}'_{n,k}\big]_E$. Let $X_{n,m}$ denote the saturated set $[\mathscr{C}'_{n,m}]_E$. Finally, for all $m,n \ge 1$, let $\mathscr{C}_{n,m} \subseteq \mathscr{C}'_{n,m}$ be such that $\operatorname{gap}_{\mathscr{C}_{n,m}}(c) > m$ for all $c \in \mathscr{C}_{n,m}$. Sets $\mathscr{C}_{n,m}$ and $X_{n,m}$ satisfy the following conditions:

(1) $\mathscr{C}_{n,m}$ is a cross section for the restriction of the flow onto $X_{n,m}$;
(2) sets $X_{n,m}$, $m,n \ge 1$, are pairwise disjoint.
(3) $\zeta(c) \in I_n$ and $\lambda_c^{\mathscr{C}}\big(\{x \in \vec{X} : x < \sigma_{\mathscr{C}_{n,m}}(c) \le Tx\}\big) > \alpha_n - \delta_n$ for all $c \in \mathscr{C}_{n,m}$.

Let $T_{n,m}$ denote the restriction of $T$ onto $X_{n,m}$. Apply Lemma 8.14 to the transformation $T_{n,m}$, cross section $\mathscr{C}_{n,m}$, which has gaps at least $K_\epsilon$, and $\beta = \alpha_n + \delta_n$, $\gamma = \alpha_n - \delta_n$. Let $P_{n,m}$ be the resulting periodic transformation on $X_{n,m}$. Set $P = \bigsqcup_{n,m} P_{n,m}$. We claim that $P$ satisfies conclusions of the theorem. Set $\mathscr{C}' = \bigsqcup_{n,m} \mathscr{C}_{n,m}$ and note that $\mathscr{C}' \subseteq \mathscr{C}$, whence $D_{\mathscr{C}'} \subseteq D_{\mathscr{C}}$.

$$\begin{aligned}
\int_X D(Tx,Px)\,d\mu &= \sum_{n,m} \int_{X_{n,m}} D(T_{n,m}x, P_{n,m}x)\,d\mu \\
\because \text{Lemma 8.14} &\le 5\sum_{n,m}\int_{D_{\mathscr{C}_{n,m}}} |c_T(x)|\,d\mu + \sum_{n,m}\int_{J_\epsilon \cap X_{n,m}} |c_T(x)|\,d\mu + \sum_{n,m} K_\epsilon\Big(\frac{2\delta_n}{\alpha_n-\delta_n}\Big)\mu(X_{n,m}) \\
\because \text{Eq. (12)} &\le 5\int_{D_{\mathscr{C}}} |c_T(x)|\,d\mu + \int_{J_\epsilon} |c_T(x)|\,d\mu + (\epsilon/3)\mu(X) \\
&\le 5\int_{D_{\mathscr{C}} \setminus J_\epsilon} |c_T(x)|\,d\mu + 6\int_{J_\epsilon} |c_T(x)|\,d\mu + \epsilon/3 \\
\because \text{choice of } K_\epsilon &< 5K_\epsilon \mu(D_{\mathscr{C}} \setminus J_\epsilon) + \epsilon/3 + \epsilon/3 \\
\because \text{Eq. (11)} &\le \epsilon,
\end{aligned}$$

and the theorem follows. $\square$

**Corollary 8.16.** *Let $\mathbb{R} \curvearrowright X$ be a measure-preserving flow and $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipative transformation. If $\mathscr{I}(T) = 0$, then $T \in D([\mathbb{R} \curvearrowright X]_1)$.*

*Proof.* By Corollary 8.8, there is a monotone transformation $U$ and a periodic transformation $P$ such that $T = U \circ P$. Since $P \in D([\mathbb{R} \curvearrowright X]_1)$ by Corollary C.11, it remains to show that $U$ belongs to the derived subgroup. The latter follows from Theorem 8.15, since $\mathscr{I}(U) = \mathscr{I}(T) - \mathscr{I}(P) = 0$. □

## 9. CONCLUSIONS

Our objective in this last section is to draw several conclusions regarding the structure of the $L^1$ full groups of measure-preserving flows. The analysis conducted in Sections 7 and 8 leads to the most technically challenging result of our work, which is the following theorem.

**Theorem 9.1.** *Let $\mathscr{F} : \mathbb{R} \curvearrowright X$ be a free measure-preserving flow on a standard probability space. The kernel of the index map coincides with the closed derived subgroup $D([\mathscr{F}]_1)$.*

*Proof.* Inclusion $D([\mathscr{F}]_1) \subseteq \ker \mathscr{I}$ is automatic since the image of $\mathscr{I}$ is abelian. For the other direction pick a transformation $T \in \ker \mathscr{I}$ and consider its Hopf's decomposition $X = C \sqcup D$ provided by Proposition B.22. We have $T = T_C \circ T_D$, where $T_C \in [\mathscr{F}]_1$ is conservative and $T_D \in [\mathscr{F}]_1$ is dissipative. According to Corollary 7.8, $\mathscr{I}(T_C) = 0$ and $T_C \in D([\mathscr{F}]_1)$, whence $\mathscr{I}(T_D) = \mathscr{I}(T) - \mathscr{I}(T_C) = 0$. Therefore, the dissipative part $T_D$ satisfies the assumptions of Corollary 8.16, which yields $T_D \in D([\mathscr{F}]_1)$, and hence $T \in D([\mathscr{F}]_1)$ as desired. □

Empowered with the result above and Corollary 4.18, we can estimate the topological ranks of $L^1$ full groups of flows. We recall the following well-known inequalities.

**Proposition 9.2.** *Let $\phi : G \to H$ be a surjective continuous homomorphism of Polish groups. The topological rank* $\mathrm{rk}(G)$ *satisfies*

$$\mathrm{rk}(H) \leq \mathrm{rk}(G) \leq \mathrm{rk}(H) + \mathrm{rk}(\ker \phi).$$

**Proposition 9.3.** *Let $\mathscr{F} : \mathbb{R} \curvearrowright X$ be a free measure-preserving flow on a standard probability space $(X, \mu)$. The topological rank* $\mathrm{rk}([\mathscr{F}]_1)$ *is finite if and only if the flow has finitely many ergodic components. Moreover, if $\mathscr{F}$ has exactly $n$ ergodic components then*

$$n + 1 \leq \mathrm{rk}([\mathscr{F}]_1) \leq n + 3.$$

*Proof.* Let $\mathscr{E}$ be the space of probability invariant ergodic measures of the flow, and let $p$ be the probability measure on $\mathscr{E}$ such that $\mu = \int_{\mathscr{E}} \nu \, dp(\nu)$ (see Appendix B.3). Proposition 5.6 shows that the index map $\mathscr{I} : [\mathscr{F}]_1 \to L^1(\mathscr{E}, p)$ is continuous and surjective. An application of Proposition 9.2 yields

$$\mathrm{rk}(L^1(\mathscr{E}, p)) \leq \mathrm{rk}([\mathscr{F}]_1) \leq \mathrm{rk}(L^1(\mathscr{E}, p)) + \mathrm{rk}(\ker \mathscr{I}) = \mathrm{rk}(L^1(\mathscr{E}, p)) + 2. \tag{13}$$

where the last equality is based on Theorem 9.1 and Corollary 4.18. Since $L^1(\mathscr{E}, \nu)$ is a Banach space, its topological rank is finite if and only if its dimension is finite, which is equivalent to $(\mathscr{E}, p)$ being purely atomic with finitely many atoms. We have shown that $\mathrm{rk}([\mathscr{F}]_1)$ is finite if and only if the flow has only finitely many ergodic components. The moreover part of the proposition follows from the inequality (13) and the observation that $\mathrm{rk}(L^1(\mathscr{E}, p)) = \dim(L^1(\mathscr{E}, p)) + 1$. □

As already mentioned in the introduction, we conjecture that the topological rank completely remembers the number of ergodic components.

**Conjecture 9.4.** *Let $\mathscr{F}$ be a measure-preserving flow. If it has exactly $n$ ergodic components, then* $\mathrm{rk}([\mathscr{F}]_1) = n + 1$.

$L^1$ full groups of $\mathbb{Z}$ actions are known to contain almost all information about the action. Given a measure-preserving action of a boundedly generated Polish group, we can always twist it by a continuous automorphism of the group and still get the same $L^1$ full group. In the case of $\mathbb{Z}$-actions, this takes a particularly simple form since the only automorphism of $\mathbb{Z}$ is given by $n \mapsto -n$. It follows from the results of R. M. Belinskaja [Bel68] and the reconstruction theorems that this is actually the only way two get the same $L^1$ full group for $\mathbb{Z}$-actions: if $a_i : \mathbb{Z} \curvearrowright X_i$, $i = 1, 2$, are two free ergodic measure-preserving actions of $\mathbb{Z}$ with isomorphic $L^1$ full groups, then $a_1$ is isomorphic either to $a_2$ or to the inverse of $a_2$. We conclude by asking whether the analogous theorem holds for flows, noting that the only continuous automorphisms of $\mathbb{R}$ are multiplications by nonzero scalars.

**Question 9.5.** *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be free ergodic measure-preserving flows with isomorphic $L^1$ full groups. Is it true that there is $\alpha \in \mathbb{R}^*$ such that $\mathscr{F}_1$ and $\mathscr{F}_2 \circ m_\alpha$ are isomorphic, where $m_\alpha$ denotes the multiplication by $\alpha$?*

## APPENDIX A. NORMED GROUPS

**Definition A.1.** A **norm** on a group $G$ is a map $\|\cdot\| : G \to \mathbb{R}^{\geq 0}$ such that for all $g, h \in G$

(1) $\|g\| = 0$ if and only if $g = e$;
(2) $\|g\| = \|g^{-1}\|$;
(3) $\|gh\| \leq \|g\| + \|h\|$.

If $G$ is moreover a topological group, a norm $\|\cdot\|$ on $G$ is called **compatible** when it induces the same neighborhoods of the identity as the topology.

There is a correspondence between (compatible) left-invariant metrics on a group and (compatible) norms on it. Indeed, given a left-invariant metric $d$ on $G$ the function $\|g\| = d(e, g)$ is a norm. Conversely, from a norm $\|\cdot\|$ one can recover the left-invariant metric $d$ via $d(g, h) = \|g^{-1}h\|$.

Analogously, there is a correspondence between norms and right invariant metrics given by $d(g, h) = \|hg^{-1}\|$. Note, however, that there are metrics that are neither left- nor right-invariant, which nonetheless induce a group norm via the same formula $\|g\| = d(g, e)$. Consider for example a Polish group $G$ with a compatible left-invariant metric $d'$ on it. If $G$ is not a CLI group, the metric $d'$ is not complete, but the metric

$$d(g, f) = (d'(f, g) + d'(f^{-1}, g^{-1}))/2$$

is complete. Since $d(g, e) = d'(g, e)$, we see that $d$ induces the same norm $\|\cdot\|$ as does the left-invariant metric $d'$.

The language of group norms thus contains the same information as the formalism of left-invariant (or right-invariant) metrics, but it has the stylistic advantage of removing the need of making a choice between the invariant side, when such a choice is immaterial.

There is a canonical way to push a norm onto a factor group.

**Proposition A.2** (see [Fre04, Thm. 2.2.10])**.** *Let $(G, \|\cdot\|)$ be a Polish normed group, and let $H \trianglelefteq G$ be a closed normal subgroup of $G$. The function $\|gH\|^{G/H} = \inf\{\|gh\| : h \in H\}$ is a norm on $G/H$ which is compatible with the quotient topology. In particular $(G/H, \|\cdot\|^{G/H})$ is a Polish normed group.*

**Definition A.3.** A compatible norm $\|\cdot\|$ on a locally compact Polish group $G$ is **proper** if all balls $\{g \in G : \|g\| \leq r\}$ are compact.

R. A. Struble [Str74] showed that all locally compact Polish groups admit a proper norm.

## APPENDIX B. ACTIONS OF LOCALLY COMPACT SECOND COUNTABLE GROUPS

In this section of the appendix we collect some well-known facts related to actions of locally compact second countable groups. This exposition is provided for reader's convenience and completeness of the paper. We recall that by a result of G. W. Mackey [Mac62], any Boolean measure-preserving action of a locally compact second countable group can be realized as a spatial Borel action, so we may switch to pointwise formulations, whenever convenient for the exposition.

B.1. **Disintegration of measure.** Let $\mathscr{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space $(X, \mu)$, and let $\pi : X \to Y$ be a measurable reduction to the identity relation on some standard Lebesgue space $(Y, \nu)$, $\pi(x) = \pi(y)$ if and only if $x \mathscr{R} y$. Suppose that $\nu$ is a $\sigma$-finite measure on $Y$ that is equivalent to the push forward $\pi_* \mu$. A **disintegration** of $\mu$ relative to $(\pi, \nu)$ is a collection of measures $(\mu_y)_{y \in Y}$ on $X$ such that for all Borel sets $A \subseteq X$

(1) $\mu_y(X \setminus \pi^{-1}(y)) = 0$ for $\nu$-almost all $y \in Y$;
(2) the map $Y \ni y \mapsto \mu_y(A) \in \mathbb{R}$ is measurable;
(3) $\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$.

A theorem of D. Maharam [Mah50] asserts that $\mu$ can be disintegrated over any $(\pi, \nu)$ as above. In fact, existence of a disintegration can be proved in a setup considerably more general (see, for example, D. H. Fremlin [Fre06, Thm. 452I]), but in the framework of standard Lebesgue spaces, disintegration is essentially unique. While the context of our paper is purely ergodic theoretical, we note that the disintegration result holds in the descriptive set theoretical setting as well, as discussed in [Mah84] and [GM89]. Without striving for generality, we formulate here a specific version, which suits our needs.

**Theorem B.1** (Disintegration of Measure)**.** *Let $(X,\mu)$ be a standard Lebesgue space, $(Y,\nu)$ be a $\sigma$-finite standard Lebesgue space, and let $\pi : X \to Y$ be a measurable function. If $\pi_*\mu$ is equivalent to $\nu$, then there exists a disintegration $(\mu_y)_{y\in Y}$ of $\mu$ over $(\pi,\nu)$. Moreover, such a disintegration is essentially unique in the sense that if $(\mu'_y)_{y\in Y}$ is another disintegration, then $\mu_y = \mu'_y$ for $\nu$-almost all $y \in Y$.*

*Remark* B.2. It is more common to formulate the disintegration theorem with the assumption that $\pi_*\mu = \nu$, when one can additionally ensure that $\mu_y(X) = \mu(X)$ for $\nu$-almost all $y$. Weakening the equality $\pi_*\mu = \nu$ to mere equivalence is a simple consequence, for if $(\mu_y)_{y\in Y}$ is a disintegration of $\mu$ over $(\pi,\pi_*\mu)$, then $\big(\frac{d\pi_*\mu}{d\nu}(y)\cdot\mu_y\big)_{y\in Y}$ is a disintegration of $\mu$ over $(\pi,\nu)$.

Let $X_a \subseteq X$ be the set of atoms of the disintegration, i.e., $X_a = \{x \in X : \mu_y(x) > 0 \text{ for some } y \in Y\}$, and let $F$ be the equivalence relation on $X_a$, where two atoms within the same fiber are equivalent whenever they have the same measure: $x_1 F x_2$ if and only if $\mu_{\pi(x_1)}(x_1) = \mu_{\pi(x_2)}(x_2)$ and $\pi(x_1) = \pi(x_2)$. The equivalence relation $F$ is measurable and has finite classes $\mu$-almost surely. Let $X_n$, $n \geq 1$, be the union of $F$-equivalence classes of size exactly $n$, thus $X_a = \bigsqcup_{n\geq 1} X_n$. Set also $X_0 = X \setminus X_a$ to be the atomless part of the disintegration and let $\mathscr{R}_n$ denote the restriction of $\mathscr{R}$ onto $X_n$.

Consider the group $[\mathscr{R}] \leq \mathrm{Aut}(X,\mu)$ of measure-preserving automorphisms for which $x\mathscr{R}Tx$ holds $\mu$-almost surely. Every $T \in [\mathscr{R}]$ preserves $\nu$-almost all measures $\mu_y$, since $(T_*\mu_y)_{y\in Y}$ is a disintegration of $T_*\mu = \mu$, which has to coincide with $(\mu_y)_{y\in Y}$ by uniqueness of the disintegration. In particular, the partition $X = \bigsqcup_{n\in\mathbb{N}} X_n$ is invariant under the full group $[\mathscr{R}]$, and for any $T \in [\mathscr{R}]$ the restriction $T\restriction_{X_n} \in [\mathscr{R}_n]$ for every $n \in \mathbb{N}$. Conversely, for a sequence $T_n \in [\mathscr{R}_n]$, $n \in \mathbb{N}$, one has $T = \bigsqcup_n T_n \in [\mathscr{R}]$. We therefore have an isomorphism of (abstract) groups $[\mathscr{R}] \cong \prod_{n\in\mathbb{N}}[\mathscr{R}_n]$.

The groups $[\mathscr{R}_n]$ can be described quite explicitly. First, let us consider the case $n \geq 1$, thus $X_n \subseteq X_a$. All equivalence classes of the restriction of $F$ onto $X_n$ have size $n$. Let $Y_n \subseteq X_n$ be a measurable transversal, i.e., a measurable set intersecting every $F$-class in a single point, and let $\nu_n = \mu\restriction_{Y_n}$. Every $T \in [\mathscr{R}_n]$ produces a permutation of $\mu$-almost every $F$-class, so we can view it as an element of $\mathrm{L}^0(Y_n,\nu_n,\mathfrak{S}_n)$, where $\mathfrak{S}_n$ is the group of permutations of an $n$-element set. This identification works in both directions and produces an isomorphism $[\mathscr{R}_n] \cong \mathrm{L}^0(Y_n,\nu_n,\mathfrak{S}_n)$. Note also that all $\nu_n$ are atomless if so is $\mu$. We allow for $\mu(X_n) = 0$, in which case $\mathrm{L}^0(Y_n,\nu_n,\mathfrak{S}_n)$ is the trivial group.

Let us now go back to the equivalence relation $\mathscr{R}_0 = \mathscr{R} \cap X_0 \times X_0$, and recall that measures $\mu_y\restriction_{X_0}$ are atomless. Let $Y_0 = \{y : \mu_y(X_0) > 0\}$ be the encoding of fibers with non-trivial atomless components and put $\nu_0 = \nu\restriction_{Y_0}$. In particular, for every $y \in Y_0$ the space $(X_0,\mu_y)$ is isomorphic to the interval $[0,\mu_y(X_0)]$ endowed with the Lebesgue measure. In fact, one can select such isomorphisms in a measurable way across all $y \in Y_0$. More precisely, there is a measurable isomorphism $\psi : X_0 \to \{(y,r) \in Y_0 \times \mathbb{R} : 0 \leq r \leq \mu_y(X_0)\}$ such that for all $y \in Y_0$

- $\psi(\pi^{-1}(y) \cap X_0) = \{y\} \times [0,\mu_y(X_0)]$;
- $\psi_*\mu_y\restriction_{X_0}$ coincides with the Lebesgue measure on $\{y\} \times [0,\mu_y(X_0)]$.

The reader may find further details in [GM89, Thm. 2.3], where the same construction is discussed in a more refined setting of Borel disintegrations.

Using the isomorphism $\psi$, we can identify each $\pi^{-1}(y) \cap X_0$, $y \in Y_0$, with $[0,\mu_y(X_0)]$. Since every $T \in [\mathscr{R}_0]$ preserves $\nu$-almost every measure $\mu_y$, we may rescale these intervals and view any $T \in [\mathscr{R}_0]$ as an element of $\mathrm{L}^0(Y_0,\nu_0,\mathrm{Aut}([0,1],\lambda))$. Conversely, every $f \in \mathrm{L}^0(Y_0,\nu_0,\mathrm{Aut}([0,1],\lambda))$ gives rise to $T_f \in [\mathscr{R}_0]$ via the notationally convoluted but natural

$$T_f(x) = \psi^{-1}\Big(\pi(x), \big(f(\pi(x))\cdot \mathrm{proj}_2(\psi(x))/\mu_{\pi(x)}(X_0)\big)\mu_{\pi(x)}(X_0)\Big),$$

which, in plain words, simply applies $f(\pi(x))$ upon the corresponding fiber identified with $[0,1]$ using $\psi$. This map is an isomorphism between $[\mathscr{R}_0]$ and $\mathrm{L}^0(Y_0,\nu_0,\mathrm{Aut}([0,1],\lambda))$.

Let us say that $\mathscr{R}$ has **atomless classes** if $\mu_y$ is atomless $\nu$-almost surely or, equivalently, $\mu(X_a) = 0$ in the notation above. We may summarize the discussion so far into the following proposition.

**Proposition B.3.** *Let $\mathscr{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space $(X,\mu)$. There are (possibly empty) standard Lebesgue spaces $(Y_k,\nu_k)$, $k \in \mathbb{N}$, such that the full group $[\mathscr{R}] \leq \mathrm{Aut}(X,\mu)$ is (abstractly) isomorphic to*

$$\mathrm{L}^0(Y_0,\nu_0,\mathrm{Aut}([0,1],\lambda)) \times \prod_{n\geq 1}\mathrm{L}^0(Y_n,\nu_n,\mathfrak{S}_n),$$

*where $\mathfrak{S}_n$ is the group of permutations of a $n$-element set. If $\mu$ is atomless, then so are the spaces $(Y_n,\nu_n)$, $n \geq 1$. If $\mathscr{R}$ has atomless classes, then all $(Y_n,\nu_n)$, $n \geq 1$, are negligible and $[\mathscr{R}]$ is isomorphic to $\mathrm{L}^0(Y_0,\nu_0,\mathrm{Aut}([0,1],\lambda))$.*

We can further refine the product in Proposition B.3 by decomposing the spaces $(Y_n, \nu_n)$ into individual atoms and the atomless remainders. More specifically, let $(Z, \nu_Z)$ be a standard Lebesgue space and $G$ be a Polish group. Given a measurable partition $Z = Z_0 \sqcup Z_1$, every function $f \in \mathrm{L}^0(Z, \nu_Z, G)$ can be associated with a pair $(f_0, f_1) \in \mathrm{L}^0(Z_0, \nu_{Z,0}, G) \times \mathrm{L}^0(Z_1, \nu_{Z,1}, G)$, $\nu_{Z,i} = \nu_Z \restriction_{Z_i}$ and $f_i = f \restriction_{Z_i}$, which is an isomorphism of the topological groups. The same consideration applies to finite or countably infinite partitions.

**Proposition B.4.** *Let $(Z, \nu_Z)$ be a standard Lebesgue space and $G$ be a Polish group. For any finite or countably infinite measurable partition $Z = \bigsqcup_{n \in I} Z_n$, there is an isomorphism of topological groups $\mathrm{L}^0(Z, \nu_Z, G)$ and $\prod_{n \in I} \mathrm{L}^0(Z_n, \nu_{Z,n}, G)$, where $\nu_{Z,n}$ is the restriction of $\nu_Z$ onto $Z_n$.*

Applying Proposition B.4 to the partition of $(Z, \nu_Z)$ into an atomless part $Z_0$ and individual atoms $Z_k = \{z_k\}$ (if any), and noting that for a singleton $Z_k$ the group $\mathrm{L}^0(Z_k, \nu_{Z,k}, G)$ is naturally isomorphic to $G$, we get the following corollary.

**Corollary B.5.** *Let $(Z, \nu_Z)$ be a standard Lebesgue space and $G$ be a Polish group. Let $Z_a \subseteq Z$ be the set of atoms of $Z$ and $Z_0 = Z \setminus Z_a$ be the atomless part. The group $\mathrm{L}^0(Z, \nu_Z, G)$ is isomorphic to $\mathrm{L}^0(Z_0, \nu_Z \restriction_{Z_0}, G) \times G^{|Z_a|}$.*

Combining the discussion above with Proposition B.3, we obtain a very concrete representation for $[\mathscr{R}]$. In the formulation below $G^0$ is understood to be the trivial group.

**Proposition B.6.** *Let $\mathscr{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space $(X, \mu)$. There are cardinals $\kappa_n \leq \aleph_0$ and $\epsilon_n \in \{0, 1\}$ such that*

$$[\mathscr{R}] \cong \mathrm{L}^0([0,1], \lambda, \mathrm{Aut}([0,1], \lambda))^{\epsilon_0} \times \mathrm{Aut}([0,1], \lambda)^{\kappa_0} \times \Big(\prod_{n \geq 1} \mathrm{L}^0([0,1], \lambda, \mathfrak{S}_n)^{\epsilon_n} \times \mathfrak{S}_n^{\kappa_n}\Big).$$

*If $\mu$ is atomless, then $\kappa_n = 0$ for all $n \geq 1$; if $\mathscr{R}$ has atomless classes, then $\epsilon_n = 0$ for all $n \geq 1$.*

So far we viewed $[\mathscr{R}]$ as an abstract group. This is because neither of the two natural topologies on $\mathrm{Aut}(X, \mu)$ play well with the full group construction—$[\mathscr{R}]$ is generally not closed in the weak topology, and not separable in the uniform topology whenever $\mu(X_0) > 0$. Nonetheless, the isomorphism given in Proposition B.3 shows that there is a natural Polish topology on $[\mathscr{R}]$, which arises when we view groups $\mathrm{L}^0(Y_0, \nu_0, \mathrm{Aut}([0,1], \lambda))$ and $\mathrm{L}^0(Y_n, \nu_n, \mathfrak{S}_n)$ as Polish groups in the topology of convergence in measure. It is with respect to this topology we formulate Proposition B.7.

**Proposition B.7.** *Let $\mathscr{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space $(X, \mu)$. The set of periodic elements is dense in $[\mathscr{R}]$.*

*Proof.* Rokhlin's Lemma implies that any $T \in [\mathscr{R}]$ can be approximated in the uniform topology by periodic elements from $[T] \subseteq [\mathscr{R}]$. Since the uniform topology is stronger than the Polish topology on $[\mathscr{R}]$, the proposition follows. □

B.2. **Tessellations.** An important feature of locally compact group actions is the fact that they all admit measurable cross sections. This was proved by J. Feldman, P. Hahn, and C. Moore in [FHM78], whereas a Borel version of the result was obtained by A. S. Kechris in [Kec92].

**Definition B.8.** Let $G \curvearrowright X$ be a Borel action of a locally compact second countable group. A **cross section** is a Borel set $\mathscr{C} \subseteq X$ which is both

- a **complete section** for $\mathscr{R}_G$: it intersects every orbit of the action and
- **lacunary**: for some neighborhood of the identity $1_G \in U \subseteq G$ one has $U \cdot c \cap U \cdot c' = \varnothing$ for all distinct $c, c' \in \mathscr{C}$.

A cross section $\mathscr{C}$ is $K$-**cocompact**, where $K \subseteq G$ is a compact set, if $K \cdot \mathscr{C} = X$; a cross section is **cocompact** if it is $K$-cocompact for some compact $K \subseteq G$.

Any action $G \curvearrowright X$ admits a $K$-cocompact cross section, whenever $K \subseteq G$ is a compact neighborhood of the identity (see [Slu17, Thm. 2.4]). We also remind the following well-known lemma on the possibility to partition a cross section into pieces with a prescribed lacunarity parameter.

**Lemma B.9.** *Let $G \curvearrowright X$ be a Borel action of a locally compact Polish group and $\mathscr{C}$ be a cross section for the action. For any compact neighborhood of the identity $V \subseteq G$ there exists a finite Borel partition $\mathscr{C} = \bigsqcup_i \mathscr{C}_i$ such that each $\mathscr{C}_i$ is $V$-lacunary.*

*Proof.* Set $K = (V \cup V^{-1})^2$ and let $U \subseteq G$ be a compact neighborhood of the identity small enough for $\mathscr{C}$ to be $U$-lacunary. Define a binary relation $\mathscr{G}$ on $\mathscr{C}$ by declaring $(c, c') \in \mathscr{G}$ whenever $c \in K \cdot c'$ and $c \neq c'$. Note that $\mathscr{G}$ is symmetric since so is $K$. We view $\mathscr{G}$ as a Borel graph on $\mathscr{C}$ and claim that it is locally finite. More specifically, if $\lambda$ is a right Haar measure, then the degree of each $c \in \mathscr{C}$ is at most $\lfloor \frac{\lambda(U \cdot K)}{\lambda(U)} \rfloor - 1$.

Indeed, let $c_0, \ldots, c_N \in \mathscr{C}$ be distinct elements such that $c_i \in K \cdot c_0$ for all $i \leq N$; in particular $(c_i, c_0) \in \mathscr{G}$ for $i \geq 1$. Let $k_i \in K$ be such that $k_i \cdot c_0 = c_i$. Lacunarity of $\mathscr{C}$ asserts that sets $U \cdot c_i = U k_i \cdot c_0$ are supposed to be pairwise disjoint, which necessitates $U k_i$ to be pairwise disjoint for $0 \leq i \leq N$. Clearly $U k_i \subseteq UK$ as $k_i \in K$. Using the right-invariant of $\lambda$ we have $\lambda(UK) \geq \lambda\big(\bigsqcup_{i \leq N} U k_i\big) = (N+1)\lambda(U)$, and thus $N + 1 \leq \frac{\lambda(UK)}{\lambda(U)}$ as claimed.

We may now use [KST99, Prop. 4.6] to deduce existence of a finite partition $\mathscr{C} = \bigsqcup_i \mathscr{C}_i$ such that no two points in $\mathscr{C}_i$ are adjacent. In other words, if $c, c' \in \mathscr{C}_i$ are distinct, then $c \notin K \cdot c'$, and therefore $V \cdot c \cap V \cdot c' = \varnothing$, which shows that $\mathscr{C}_i$ are $V$-lacunary. □

Every cross section $\mathscr{C}$ gives rise to a smooth subrelation of $\mathscr{R}_G$ by associating to $x \in X$ "the closest point" of $\mathscr{C}$ in the same orbit. Such a subrelation is known as the Voronoi tessellation. For the purposes of Section 4, we need a slightly more abstract concept of a tessellation which may not correspond to Voronoi domains. While far from being the most general, the following treatment is sufficient for our needs.

**Definition B.10.** Let $G \curvearrowright X$ be an action of a locally compact second countable group on a standard Borel space and let $\mathscr{C} \subseteq X$ be a cross section. A **tessellation** over $\mathscr{C}$ is a Borel set $\mathscr{W} \subseteq \mathscr{C} \times X$ such that

1. all fibers $\mathscr{W}_c = \{x \in X : (c, x) \in \mathscr{W}\}$ are pairwise disjoint for $c \in \mathscr{C}$;
2. for all $c \in \mathscr{C}$ elements of $\mathscr{W}_c$ are $\mathscr{R}_G$-equivalent to $c$, i.e., $\{c\} \times \mathscr{W}_c \subseteq \mathscr{R}_G$;
3. fibers cover the phase space, $X = \bigsqcup_{c \in \mathscr{C}} \mathscr{W}_c$.

A tessellation $\mathscr{W}$ over $\mathscr{C}$ is $N$-**lacunary** for an open $N \subseteq G$ if $\{(c, N \cdot c) : c \in \mathscr{C}\} \subseteq \mathscr{W}$. It is $K$-**cocompact**, $K \subseteq G$, if $\mathscr{W} \subseteq \{(c, K \cdot c) : c \in \mathscr{C}\}$.

Any tessellation $\mathscr{W}$ can be viewed as a (flipped) graph of a function, since for any $x \in X$ there is a unique $c \in \mathscr{C}$ such that $(c, x) \in \mathscr{W}$. We denote such $c$ by $\pi_{\mathscr{W}}(x)$, which produces a Borel map $\pi_{\mathscr{W}} : X \to \mathscr{C}$. There is a natural equivalence relation $\mathscr{R}_{\mathscr{W}}$ associated with the tessellation. Namely, $x_1$ and $x_2$ are $\mathscr{R}_{\mathscr{W}}$-equivalent whenever they belong to the same fiber, i.e., $\pi_{\mathscr{W}}(x_1) = \pi_{\mathscr{W}}(x_2)$. In view of the item (2), $\mathscr{R}_{\mathscr{W}} \subseteq \mathscr{R}_G$ and moreover, every $\mathscr{R}_G$-class consists of countably many $\mathscr{R}_{\mathscr{W}}$-classes.

Voronoi tessellations provide a specific way of constructing tessellations over a given cross section. Suppose that the group $G$ is endowed with a compatible *proper* norm $\|\cdot\|$. Let $D : \mathscr{R}_G \to \mathbb{R}^{\geq 0}$ be the associated metric on the orbits of the action (as in Section 2.2) and let $\preceq_{\mathscr{C}}$ be a Borel linear order on $\mathscr{C}$. The **Voronoi tessellation** over the cross section $\mathscr{C}$ relative to a proper norm $\|\cdot\|$ is the set $\mathscr{V}_{\mathscr{C}} \subseteq \mathscr{C} \times X$ defined by

$$\begin{aligned}\mathscr{V}_{\mathscr{C}} = \big\{(c, x) \in \mathscr{C} \times X \ : c \mathscr{R}_G x \text{ and for all } c' \in \mathscr{C} \text{ such that } c' \mathscr{R}_G x \text{ either } D(c, x) < D(c', x) \text{ or}\\ (D(c, x) = D(c', x) \text{ and } c \preceq_{\mathscr{C}} c')\big\}.\end{aligned}$$

Properness of the norm ensures that for each $x \in X$ there are only finitely many candidates $c$ which minimize $D(c, x)$, and hence each $x \in X$ is associated with a unique $c \in \mathscr{C}$.

For the sake of Section 4, we need a definition of the Voronoi tessellation for norms that may not be proper. The set $\mathscr{V}_{\mathscr{C}}$ specified as above may in this case fail to satisfy item (3) of the definition of a tessellation, as for some $x \in X$ there may be infinitely many $c \in \mathscr{C}$ that minimize $D(c, x)$, none of which are $\preceq_{\mathscr{C}}$-minimal. We therefore need a different way to resolve the points on the "boundary" between the regions, which can be done, for example, by delegating this task to a proper norm.

**Definition B.11.** Let $\|\cdot\|$ be a compatible norm on $G$ and let $\mathscr{C}$ be a cross section. Pick a compatible proper norm $\|\cdot\|'$ on $G$ and a Borel linear order $\preceq_{\mathscr{C}}$ on $\mathscr{C}$. Let $D$ and $D'$ be the metrics on orbits of the action associated with the norms $\|\cdot\|$ and $\|\cdot\|'$ respectively. The **Voronoi tessellation** over the cross section $\mathscr{C}$ relative to the norm $\|\cdot\|$ is the set $\mathscr{V}_{\mathscr{C}} \subseteq \mathscr{C} \times X$ defined by

$$\begin{aligned}\mathscr{V}_{\mathscr{C}} = \big\{(c, x) \in \mathscr{C} \times X \ : c \mathscr{R}_G x \text{ and for all } c' \in \mathscr{C} \text{ such that } c' \mathscr{R}_G x \text{ either } D(c, x) < D(c', x) \text{ or}\\ (D(c, x) = D(c', x) \text{ and } D'(c, x) < D'(c', x)) \text{ or}\\ (D(c, x) = D(c', x) \text{ and } D'(c, x) = D'(c', x) \text{ and } c \preceq_{\mathscr{C}} c')\big\}.\end{aligned}$$

The definition of the Voronoi tessellation does depend on the choice of the norm $\|\cdot\|'$ and the linear order $\preceq_{\mathscr{C}}$ on the cross section, but its key properties are the same regardless of these choices. We therefore often do not specify explicitly which $\|\cdot\|'$ and $\preceq_{\mathscr{C}}$ are picked. Note also that if the cross section is cocompact, then every region of the Voronoi tessellation is bounded, i.e., $\sup_{x\in X} D(x,\pi_{\mathscr{V}_{\mathscr{C}}}(x)) < +\infty$.

Our goal is to show that equivalence relations $\mathscr{R}_{\mathscr{W}}$ are atomless in the sense of Subsection B.1 as long as each orbit of the action is uncountable. To this end we first need the following lemma.

**Lemma B.12.** *Let $G$ be a locally compact second countable group acting on a standard Lebesgue space $(X,\mu)$ by measure-preserving automorphisms. Suppose that almost every orbit of the action is uncountable. If $\mathscr{A} \subseteq X$ is a measurable set such that the intersection of $\mathscr{A}$ with almost every orbit of $G$ is countable, then $\mu(\mathscr{A}) = 0$.*

*Proof.* Pick a proper norm $\|\cdot\|$ on $G$, let $\mathscr{C}$ be a cross section for the action, let $B_{2r} \subseteq G$ be an open ball around the identity of sufficiently small radius $2r > 0$ such that $B_{2r}\cdot c \cap B_{2r}\cdot c' = \varnothing$ whenever $c, c' \in \mathscr{C}$ are distinct, and let $\mathscr{V}_{\mathscr{C}}$ be the Voronoi tessellation over $\mathscr{C}$ relative to $\|\cdot\|$. Note that $B_{2r}\cdot c$ is fully contained in the $\mathscr{R}_{\mathscr{V}_{\mathscr{C}}}$-class of $c$ and set $X = B_r\cdot\mathscr{C}$. Let also $(g_n)_{n\in\mathbb{N}}$ be a countable dense subset of $G$.

We claim that it is enough to consider the case when $\mathscr{A}$ intersects each $\mathscr{R}_{\mathscr{V}_{\mathscr{C}}}$-class in at most one point. Indeed, the restriction of $\mathscr{R}_{\mathscr{V}_{\mathscr{C}}}$ onto $\mathscr{A}$ is a smooth countable equivalence relation, so one can write $\mathscr{A} = \bigsqcup_{n\in\mathbb{N}} \mathscr{A}'_n$, where each $\mathscr{A}'_n$ intersects each $\mathscr{R}_{\mathscr{V}_{\mathscr{C}}}$-class in at most one point. To simplify notations, we assume that $\mathscr{A}$ already possesses this property.

Let $\gamma : X \to \mathbb{N}$ be defined by $\gamma(x) = \min\{n\in\mathbb{N} : x\mathscr{R}_{\mathscr{V}_{\mathscr{C}}} g_n x \text{ and } g_n x \in X\}$. Let $\mathscr{A}_n = \mathscr{A}\cap\gamma^{-1}(n)$ and note that sets $\mathscr{A}_n$ partition $\mathscr{A}$. It is therefore enough to show that $\mu(\mathscr{A}_n) = 0$ for any $n\in\mathbb{N}$. Pick $n_0\in\mathbb{N}$. The action is measure-preserving and therefore $\mu(\mathscr{A}_{n_0}) = \mu(g_{n_0}\mathscr{A}_{n_0})$. Set $\mathscr{B}_0 = g_{n_0}\mathscr{A}_{n_0}$ and note that for any $x\in\mathscr{B}_0$ and $g\in B_r\subseteq G$ one has $gx\mathscr{R}_{\mathscr{V}_{\mathscr{C}}} x$. If the action were free, we could easily conclude that $\mu(\mathscr{B}_0) = 0$, since sets $g\mathscr{B}_0$, $g\in B_r$, would be pairwise disjoint. In general, we need to exhibit a little more care and construct a countable family of pairwise disjoint sets $\mathscr{B}_n$ as follows.

For $x\in\mathscr{B}_0$ let $\tau_n(x) = \min\{m\in\mathbb{N} : x\mathscr{R}_{\mathscr{V}_{\mathscr{C}}} g_m x \text{ and } g_m x \notin \bigcup_{k\le n}\mathscr{B}_n\}$. The value $\tau_n(x)$ is well-defined because the stabilizer of $x$ is closed and must be nowhere dense in $B_r$ due to the orbit $G\cdot x$ being uncountable. Put $\mathscr{B}_{n+1} = \{g_{\tau_n(x)}x : x\in\mathscr{B}_0\}$ and note that $\mu(\mathscr{B}_n) = \mu(\mathscr{B}_0)$. We get a pairwise disjoint infinite family of sets $\mathscr{B}_n$ all having the same measure. Since $\mu$ is finite, we conclude that $\mu(\mathscr{B}_0) = 0$ and the lemma follows. □

**Corollary B.13.** *Let $G$ be a locally compact second countable group acting on a standard Lebesgue space $(X,\mu)$ by measure-preserving automorphisms, let $\mathscr{C}$ be a cross section for the action and let $\mathscr{W}\subseteq\mathscr{C}\times X$ be a tessellation. If $\mu$-almost every orbit of $G$ is uncountable, then $\mathscr{R}_{\mathscr{W}}$ is atomless.*

*Proof.* Consider the disintegration $(\mu_c)_{c\in\mathscr{C}}$ of $\mathscr{R}_W$ relative to $(\pi_{\mathscr{W}},\nu)$, where $\pi_{\mathscr{W}} : X\to\mathscr{C}$ and $\nu = (\pi_{\mathscr{W}})_*\mu$. Let $X_a\subseteq X$ be the set of atoms of the disintegration. Since $\nu$-almost every $\mu_c$ is finite, fibers $\pi_{\mathscr{W}}^{-1}(c)$ have countably many atoms. Since every tessellation has only countably many tiles within each orbit, we conclude that $X_a$ has countable intersection with almost every orbit of the action. Lemma B.12 applies and shows that $\mu(X_a) = 0$, hence $\mathscr{R}_{\mathscr{W}}$ is atomless as required. □

Consider the full group $[\mathscr{R}_{\mathscr{W}}]$ which by Proposition B.3 and Corollary B.13 is isomorphic to $\mathrm{L}^0(Y,\nu,\mathrm{Aut}([0,1],\lambda))$ for some standard Lebesgue space $(Y,\nu)$. This full group can naturally be viewed as a subgroup of $[\mathscr{R}_G]$ and the topology induced on $[\mathscr{R}_{\mathscr{W}}]$ from the full group $[\mathscr{R}_G]$ coincides with the topology of converges in measure on $\mathrm{L}^0(Y,\nu,\mathrm{Aut}([0,1],\lambda))$ (see Section 3 of [CLM16]). We therefore have the following corollary.

**Corollary B.14.** *Let $G$ be a locally compact second countable group acting on a standard Lebesgue space $(X,\mu)$ by measure-preserving automorphisms, let $\mathscr{C}$ be a cross section for the action and let $\mathscr{W}\subseteq\mathscr{C}\times X$ be a tessellation and $\pi_{\mathscr{W}} : X\to\mathscr{C}$ be the corresponding reduction. If $\mu$-almost every orbit of $G$ is uncountable, then the subgroup $[\mathscr{R}_{\mathscr{W}}]\le[\mathscr{R}_G]$ is isomorphic as a topological group to $\mathrm{L}^0(\mathscr{C},(\pi_{\mathscr{W}})_*\mu,\mathrm{Aut}([0,1],\lambda))$. If moreover all orbits of the action have measure zero, then $(\pi_{\mathscr{W}})_*\mu$ is non-atomic and $[\mathscr{R}_{\mathscr{W}}]$ is isomorphic to $\mathrm{L}^0([0,1],\lambda,\mathrm{Aut}([0,1],\lambda))$.*

B.3. **Ergodic decomposition.** Let $G\curvearrowright X$ be a free measure-preserving action of a locally compact group on a standard probability space $(X,\mu)$. The space $\mathscr{E} = \mathrm{EINV}(G\curvearrowright X)$ of invariant ergodic probability measures of this action possesses a structure of a standard Borel space. The Ergodic Decomposition Theorem of V. S. Varadarajan [Var63, Thm. 4.2] asserts that there is an essentially unique Borel $\mathscr{R}_G$-invariant surjection $X\ni x\mapsto\nu_x\in\mathscr{E}$ and a probability measure $p$ on $\mathscr{E}$ such that $\mu = \int_{\mathscr{E}}\nu\, dp(\nu)$ in the sense that for all Borel $A\subseteq X$ one has $\mu(A) = \int_{\mathscr{E}}\nu(A)\, dp(\nu)$.

There is a one-to-one correspondence between measurable $\mathscr{R}_G$-invariant functions $h : X\to\mathbb{R}$ and measurable functions $\tilde{h} : \mathscr{E}\to\mathbb{R}$ given by $\tilde{h}(\nu_x) = h(x)$. For measures $\mu$ and $p$ as above, this correspondence gives an isometric isomorphism between $\mathrm{L}^1(\mathscr{E},\mathbb{R})$ and the subspace of $\mathrm{L}^1(X,\mathbb{R})$ that consists of $\mathscr{R}_G$-invariant functions.

B.4. **Orbital transformations.** Let $G \curvearrowright X$ be a free measure-preserving action of a locally compact second countable group on a standard probability space. Fix a right invariant Haar measure $\lambda$ on $G$. Any orbit $[x]_{\mathscr{R}_G}$ can be identified with the group itself via the map $G \ni g \mapsto gx \in [x]_{\mathscr{R}_G}$ and $\lambda$ can be pushed via this identification onto orbits resulting in a collection $(\lambda_x)_{x\in X}$ of measures on $X$ defined by $\lambda_x(A) = \lambda(\{g \in G : gx \in A\})$. Right invariance of the measure ensures that $\lambda_x$ depends only on the orbit $[x]_{\mathscr{R}_G}$ and is independent of the choice of the base point, $\lambda_x = \lambda_y$ whenever $x\mathscr{R}_G y$.

Freeness of the action also allows us to identify the equivalence relation $\mathscr{R}_G$ with $X \times G$ via $\Phi : X \times G \to \mathscr{R}_G$, $\Phi(x, g) = (x, gx)$. The push forward $\Phi_*(\mu \times \lambda)$ of the product measure is denoted by $M$ and can equivalently be defined by

$$M(A) = \int_X \lambda_x(A_x)\, d\mu(x),$$

where $A \subseteq \mathscr{R}_G$ and $A_x = \{y \in X : (x, y) \in A\}$.

In general, the flip transformation $\sigma : \mathscr{R}_G \to \mathscr{R}_G$, $\sigma(x, y) = (y, x)$, is not $M$-invariant. Set $\Psi : X \times G \to X \times G$ to be $\Psi = \Phi^{-1} \circ \sigma \circ \Phi$, which amounts to $\Psi(x, g) = (gx, g^{-1})$. Following the computation as in [CLM18, Prop. A.11], one can easily check that $\Psi_*(\mu\times\lambda) = \mu\times\widehat{\lambda}$, where $\widehat{\lambda}$ is the associated *left* invariant measure, $\widehat{\lambda}(A) = \lambda(A^{-1})$. If we define the measure $\widehat{M}$ on $\mathscr{R}_G$ to be

$$\widehat{M}(A) = \Phi_*(\mu \times \widehat{\lambda}) = \int_X \widehat{\lambda}_x(A_x)\, d\mu(x),$$

then $\sigma_* M = \widehat{M}$. In particular, $\sigma$ is $M$-invariant if and only if $\lambda = \widehat{\lambda}$, i.e., $G$ is unimodular.

Let $\Delta : G \to \mathbb{R}^*_+$ be the *left Haar modulus* function given for $g \in G$ by $\lambda(gA) = \Delta(g)\lambda(A)$. Recall that $\Delta : G \to \mathbb{R}^*_+$ is a continuous homomorphism (see, for instance, [Nac65, Prop. 7]), measures $\lambda$ and $\widehat{\lambda}$ belong to the same measure class and $\frac{d\widehat{\lambda}}{d\lambda}(g) = \Delta(g^{-1})$ for all $g \in G$ (see [Nac65, p. 79]).

A function $f : \mathscr{R}_G \to \mathbb{R}$ is $M$-integrable if and only if $X\times G \ni (x, g) \mapsto f(x, gx)$ is $(\mu\times\lambda)$-integrable, which together with the expression for the Radon-Nikodym derivative $\frac{d(\mu\times\widehat{\lambda})}{d(\mu\times\lambda)} = \frac{d\widehat{\lambda}}{d\lambda}$ and Fubini's Theorem yields the following identity:

$$\int_X \int_G f(x, g\cdot x)\, d\lambda(g) d\mu(x) = \int_X \int_G \Delta(g) f(g\cdot x, x)\, d\lambda(g) d\mu(x). \tag{14}$$

When the group $G$ is unimodular, this expression attains a very symmetric form and is known as the *mass transport principle*.

Any automorphism $T \in [G \curvearrowright X]$ induces for each $x \in X$ a transformation of the $\sigma$-finite measure space $(X, \lambda_x)$. In general, $T$ does not preserve $\lambda_x$, however, it is always non-singular, and the Radon-Nikodym derivative $\frac{dT_*\lambda_x}{d\lambda_x}$ can be described explicitly. Note that the full group $[G \curvearrowright X]$ admits two natural actions on $\mathscr{R}_G$: the *left* action $l$ is given by $l_T(x, y) = (Tx, y)$, and the *right* action $r$ is defined as $r_T(x, y) = (x, Ty)$. A straightforward verification (see [CLM18, Lem. A.9]) shows that $l$ is always $M$-invariant. Since $r_T \circ \sigma = \sigma \circ l_T$, we therefore have

$$(r_T)_*\widehat{M} = (r_T \circ \sigma)_* M = (\sigma \circ l_T)_* M = \sigma_* M = \widehat{M}.$$

Let $\Theta = \Phi^{-1} \circ r_T \circ \Phi$, i.e., $\Theta(x, g) = (x, \rho_{Tg}(x))$. The equality $(r_T)_*\widehat{M} = \widehat{M}$ is equivalent to $\Theta_*(\mu \times \widehat{\lambda}) = \mu \times \widehat{\lambda}$. The latter implies that each Borel $B \subseteq G$ and all measurable $A \subseteq X$ we have

$$\begin{aligned}
\int_A \widehat{\lambda}(B)\, d\mu(x) &= (\mu \times \widehat{\lambda})(A \times B) = \Theta_*(\mu \times \widehat{\lambda})(A \times B)\\
&= (\mu \times \widehat{\lambda})\big(\{(x, g) \in X \times G : (x, \rho_{Tg}(x)) \in A \times B\}\big)\\
\text{Fubini's Theorem} &= \int_A \widehat{\lambda}(\{g \in G : \rho_{Tg}(x) \in B\})\, d\mu(x)\\
&= \int_A \widehat{\lambda}(\{g \in G : gx \in T^{-1}Bx\})\, d\mu(x),
\end{aligned}$$

which is possible only if $\widehat{\lambda}(\{g \in G : gx \in T^{-1}Bx\}) = \widehat{\lambda}(B)$ for $\mu$-almost all $x$. Passing to the measures on the orbits, this translates for each $B$ into $\widehat{\lambda}_x(T^{-1}Bx) = \widehat{\lambda}_x(Bx)$. If $(B_n)_{n\in\mathbb{N}}$ is a countable algebra of Borel sets in $G$ that generates the whole Borel $\sigma$-algebra on $G$, then for each $x \in X$, $(B_n x)_{n\in\mathbb{N}}$ is an algebra of Borel subsets of the orbit $[x]_{\mathscr{R}_G}$, which generates the Borel $\sigma$-algebra on it. We have established that for $\mu$-almost all $x \in X$ the two measures, $\widehat{\lambda}_x$ and $T_*\widehat{\lambda}_x$, coincide on each $B_n x$, $n \in \mathbb{N}$, thus $\mu$-almost surely $\widehat{\lambda}_x = T_*\widehat{\lambda}_x$.

Equality $\frac{d\widehat{\lambda}}{d\lambda}(g) = \Delta(g^{-1})$ translates into $\frac{d\widehat{\lambda}_x}{d\lambda_x}(y) = \Delta(\rho(x,y)^{-1}) = \Delta(\rho(y,x))$ and the Radon-Nikodym derivative $\frac{dT_*\lambda_x}{d\lambda_x}$ can now be computed as follows.

$$\frac{dT_*\lambda_x}{d\lambda_x}(y) = \frac{dT_*\lambda_x}{dT_*\widehat{\lambda}_x}(y)\cdot\frac{dT_*\widehat{\lambda}_x}{d\widehat{\lambda}_x}(y)\cdot\frac{d\widehat{\lambda}_x}{d\lambda_x}(y)$$

$$T \text{ preservers } \widehat{\lambda}_x = \frac{dT_*\lambda_x}{dT_*\widehat{\lambda}_x}(y)\cdot\frac{d\widehat{\lambda}_x}{d\lambda_x}(y) = \frac{d\lambda_x}{d\widehat{\lambda}_x}(T^{-1}y)\cdot\frac{d\widehat{\lambda}_x}{d\lambda_x}(y) = \Big(\frac{d\widehat{\lambda}_x}{d\lambda_x}(T^{-1}y)\Big)^{-1}\cdot\frac{d\widehat{\lambda}_x}{d\lambda_x}(y)$$
$$= \Delta(\rho(x,T^{-1}y)^{-1})^{-1}\Delta(\rho(x,y)^{-1}) = \Delta\big(\rho(x,T^{-1}y)\cdot\rho(y,x)\big) = \Delta(c_{T^{-1}}(y)).$$

We summarize the content of this Subsection into a proposition.

**Proposition B.15.** *Let $G$ be a second countable locally compact group acting freely $G \curvearrowright X$ on a standard probability space $(X,\mu)$. Let $\lambda$ be a right Haar measure on $G$, $\Delta : G \to \mathbb{R}^*_+$ be the corresponding Haar modulus, and let $(\lambda_x)_{x\in X}$ be the family of measures obtained by pushing $\lambda$ onto orbits via the action map. Each $T \in [G \curvearrowright X]$ induces a non-singular transformation of $(X,\lambda_x)$ for almost every $x \in X$, and moreover one has $\lambda_x(T^{-1}A) = \int_A \Delta(c_{T^{-1}}(y))\,d\lambda_x(y)$ for all Borel sets $A \subseteq X$. If $G$ is unimodular, then $T_*\lambda_x = \lambda_x$ for almost all $x \in X$.*

For future reference, we isolate a simple lemma, which is an immediate consequence of Fubini's Theorem.

**Lemma B.16.** *Let $G$ be a second countable locally compact group acting freely on a standard probability space $(X,\mu)$. Let $\lambda$, $\widehat{\lambda}$, $(\lambda_x)_{x\in X}$, and $(\widehat{\lambda})_{x\in X}$ be as above. For any Borel set $A \subseteq X$ the following are equivalent:*

*(1) $\mu(A) = 0$;*
*(2) $\lambda_x(A) = 0$ for $\mu$-almost all $x \in X$;*
*(3) $\widehat{\lambda}_x(A) = 0$ for $\mu$-almost all $x \in X$.*

*Proof.* (1) $\iff$ (2) Using Fubini's Theorem on $(X\times G, \mu\times\lambda)$ to rearrange the order of quantifiers, one has:

$$\mu(A) = 0 \iff \forall g \in G\ \forall^\mu x \in X\ gx \notin A \iff \forall^\mu x \in X\ \forall^\lambda g \in G\ gx \notin A \iff \forall^\mu x \in X\ \lambda_x(A) = 0.$$

(2) $\iff$ (3) is evident, since $\lambda$ and $\widehat{\lambda}$ are equivalent measures, hence so are $\lambda_x$ and $\widehat{\lambda}_x$ for all $x \in X$. □

B.5. **Hopf decomposition.** An important tool in the theory of non-singular transformations on $\sigma$-finite measure spaces is the Hopf decomposition, which partitions the phase space into the so-called *dissipative* and *recurrent* parts reflecting different dynamics of the transformation. As outlined in Subsection B.4, any element of the full group of a free action of a locally compact second countable group induces a non-singular transformation on almost every orbit, where the latter is endowed with the push forward of the right Haar measure. We show in Proposition B.22 that the Hopf decomposition can be done in a measurable way simultaneously across all orbits. Let us begin by reminding the relevant definitions; the reader may consult [Kre85, Sec. 1.3] for further details.

**Definition B.17.** Let $S$ be a non-singular transformation of a $\sigma$-finite measure space $(\Omega,\lambda)$. A measurable set $A \subseteq \Omega$ is said to be:

- **wandering** if $A \cap S^k(A) = \varnothing$ for all $k \geq 1$;
- **recurrent** if $A \subseteq \bigcup_{k\geq 1} S^k(A)$;
- **infinitely recurrent** if $A \subseteq \bigcap_{n\geq 1}\bigcup_{k\geq n} S^k(A)$.

The inclusions above are understood to hold up to a null set. The transformation $S$ is:

- **dissipative** if the phase space $\Omega$ is a countable union of wandering sets;
- **conservative** if there are no wandering sets of positive measure;
- **recurrent** if every set of positive measure is recurrent;
- **infinitely recurrent** if every set of positive measure is infinitely recurrent.

It turns out that the properties of being conservative, recurrent, and infinitely recurrent are all mutually equivalent.

**Proposition B.18.** *Let $S$ be a non-singular transformation of a $\sigma$-finite measure space $(\Omega,\lambda)$. The following are equivalent:*

*(1) $S$ is conservative;*
*(2) $S$ is recurrent;*
*(3) $S$ is infinitely recurrent.*

Among the properties introduced in Definition B.17, only recurrence and dissipativity are therefore different. In fact, any non-singular transformation admits a canonical decomposition, known as the Hopf decomposition, into these two types of action.

**Proposition B.19** (Hopf decomposition)**.** *Let $S$ be a non-singular transformation of a $\sigma$-finite measure space $(\Omega,\lambda)$. There exists an $S$-invariant partition $\Omega = D\sqcup C$ such that $S\restriction_D$ is dissipative and $S\restriction_C$ is recurrent (equivalently, conservative). Moreover, if $\Omega = D'\sqcup C'$ is another partition with this property then $\lambda(D\triangle D')=0$ and $\lambda(C\triangle C')=0$.*

We also note the following consequence of dissipativity in case the measure is preserved.

**Proposition B.20.** *Let $S$ be a measure-preserving transformation of a $\sigma$-finite measure space $(\Omega,\lambda)$ and let $\Omega = D\sqcup C$ be its Hopf decomposition. For every set $A\subseteq\Omega$ of finite measure, almost every point in $D$ eventually escapes $A$: $\forall^* x\in D\ \exists N\ \forall n\geq N\ T^n x\notin A$.*

*Proof.* We may as well assume $D=\Omega$. Let $A\subseteq\Omega$ have finite measure. Let $Q$ be a wandering set whose translates cover $\Omega$. Consider the map $\Phi: Q\times\mathbb{Z}\to\Omega$ which maps $(x,n)$ to $T^n(x)$, and observe that $\Phi$ is measure-preserving if we endow $Q\times\mathbb{Z}$ with the product of the measure induced by $\lambda$ on $Q$ and the counting measure on $\mathbb{Z}$.

So if there is a positive measure set of $x\in Q$ such that $S^n(x)\in A$ for infinitely many $n\in\mathbb{N}$, by Fubini's theorem we would have that $A$ has infinite measure, a contradiction. The same conclusion is true if we replace $Q$ by any of its $S$-translates, and since these translates cover $\Omega$ the proof is finished. ☐

Fix an element $T\in[G\curvearrowright X]$ of the full group of a free measure-preserving action of a locally compact second countable group. Let us pick a cocompact cross section $\mathscr{C}$ and let $\mathcal{V}_{\mathscr{C}}$ be the associated Voronoi tessellation (see Appendix B.2). Set $\pi_{\mathscr{C}}: X\to\mathscr{C}$ to be the projection map given by the condition $(\pi_{\mathscr{C}}(x),x)\in\mathcal{V}_{\mathscr{C}}$ for all $x\in X$. Define the dissipative and conservative sets as follows:

$$D=\left\{x\in X: \exists n\in\mathbb{N}\ \forall k\in\mathbb{Z} \text{ such that } |k|\geq n \text{ one has } \pi_{\mathscr{C}}(x)\neq\pi_{\mathscr{C}}(T^k x)\right\},$$

$$C=\left\{x\in X: \forall n\in\mathbb{N}\ \exists k_1,k_2\in\mathbb{Z} \text{ such that } k_1\leq -n, n\leq k_2 \text{ and } \pi_{\mathscr{C}}(T^{k_1}x)=\pi_{\mathscr{C}}(x)=\pi_{\mathscr{C}}(T^{k_2}x)\right\}.$$

In plain words, the dissipative set $D$ consists of those points $x$ whose orbit has a finite intersection with the Voronoi region of $x$. The conservative set $C$, on the other hand, collects all the points whose orbit is bi-recurrent in the region. We argue in Proposition B.22 that sets $D$ and $C$ induce the Hopf decomposition for $T\restriction_{[x]_{\mathscr{R}_T}}$ for almost every $x\in X$; in particular, $D\sqcup C$ is a partition of $X$, which is independent of the choice of the cross section $\mathscr{C}$.

**Lemma B.21.** *Sets $D$ and $C$ partition the phase space: $X = D\sqcup C$.*

*Proof.* Define sets $N_+$ and $N_-$ according to

$$N_+=\{x\in X\setminus(D\sqcup C): \forall k\geq 1\ \pi_{\mathscr{C}}(T^k x)\neq\pi_{\mathscr{C}}(x)\},$$

$$N_-=\{x\in X\setminus(D\sqcup C): \forall k\geq 1\ \pi_{\mathscr{C}}(T^{-k} x)\neq\pi_{\mathscr{C}}(x)\},$$

and note that $X\setminus(D\sqcup C)\subseteq\bigcup_{k\in\mathbb{Z}}T^k(N_+\cup N_-)$. To show that $X=D\sqcup C$ it is enough to verify that $\mu(N_+)=0=\mu(N_-)$.

This is done by noting that these sets admit pairwise disjoint copies using piecewise translations by powers of $T$. In view of the fact that $T$ is measure-preserving, this implies that $N_+$ and $N_-$ are null. To be more precise, set $N_-^0=N_-$ and define inductively $N_-^n=\{T^{k(x)}x: x\in N_-^{n-1}\}$, where $k(x)\geq 1$ is the smallest natural number such that $\pi_{\mathscr{C}}(T^{k(x)}x)=\pi_{\mathscr{C}}(x)$. Note that $k(x)$ is well-defined, for otherwise $x$ would belong to $D$. Sets $N_-^n$, $n\in\mathbb{N}$, are pairwise disjoint, and have the same measure since $T$ is measure-preserving. We conclude that $\mu(N_-)=0$. The argument for $\mu(N_+)=0$ is similar. ☐

**Proposition B.22** (Hopf decomposition)**.** *Let $G\curvearrowright X$ be a free action of a locally compact second countable group on a standard probability space $(X,\mu)$. Let $\lambda$ be a right Haar measure on $G$ and $(\lambda_x)_{x\in X}$ be the push-forward of $\lambda$ onto the orbits as described in Appendix B.4. For any element $T\in[G\curvearrowright X]$ the measurable $T$-invariant partition $X=D\sqcup C$ defined above satisfies that for $\mu$-almost all $x\in X$ the partition $[x]_{\mathscr{R}_G}=([x]_{\mathscr{R}_G}\cap D)\sqcup([x]_{\mathscr{R}_G}\cap C)$ is the Hopf decomposition for $T\restriction_{[x]_{\mathscr{R}_G}}$ on $([x]_{\mathscr{R}_G},\lambda_x)$. Moreover, there is only one partition $X=D\sqcup C$ satisfying this property up to null sets.*

*Proof.* According to Proposition B.15, we may assume that for all $x\in X$ the map $T\restriction_{[x]_{\mathscr{R}_G}}: [x]_{\mathscr{R}_G}\to[x]_{\mathscr{R}_G}$ is a non-singular transformation with respect to $\lambda_x$ and satisfies $\lambda_x(TA)=\int_A\Delta(c_T(y))\,d\lambda_x(y)$ for all Borel $A\subseteq X$.

Let $[x]_{\mathscr{R}_G}=D_x\sqcup C_x$, $x\in X$, denote the Hopf's decomposition for $T\restriction_{[x]_{\mathscr{R}_G}}$. For any $c\in\mathscr{C}$, the set

$$\widetilde{W}_c=\left\{x\in(\mathcal{V}_{\mathscr{C}})_c: T^k x\notin(\mathcal{V}_{\mathscr{C}})_c \text{ for all } k\geq 1\right\}$$

is a wandering set and therefore $\widetilde{W}_c \subseteq D_x$ up to a null set. If $x \in D$ is such that $x \in (\mathcal{V}_{\mathscr{C}})_c$, $c \in \mathscr{C}$, then $[x]_{\mathscr{R}_G} \cap (\mathcal{V}_{\mathscr{C}})_c$ is finite, and therefore $[x]_{\mathscr{R}_G} \cap (\mathcal{V}_{\mathscr{C}})_c \subseteq \bigcup_{k\in\mathbb{Z}} T^k \widetilde{W}_c$, whence also

$$[x]_{\mathscr{R}_G} \cap D \subseteq \bigcup_{c\in\mathscr{C}\cap[x]_{\mathscr{R}_G}} \bigcup_{k\in\mathbb{Z}} T^k \widetilde{W}_c \subseteq D_x.$$

*Claim.* We have $\lambda_x([x]_{\mathscr{R}_G} \cap C \cap D_x) = 0$ for each $x \in X$.

*Proof of the claim.* Otherwise we can find $c \in \mathscr{C} \cap [x]_{\mathscr{R}_G}$ and a wandering set $W \subseteq [x]_{\mathscr{R}_G} \cap (\mathcal{V}_{\mathscr{C}})_c \cap C$ of positive measure, $\lambda_x(W) > 0$. Construct a sequence of sets $W_n$ by setting $W_0 = W$ and

$$W_n = \{T^{k_n(y)} y : y \in W_0 \text{ and } k_n(y) \text{ is minimal such that } \pi_{\mathscr{C}}(T^{k_n(y)}) = \pi_{\mathscr{C}}(y) \text{ and } T^{k_n(y)} y \not\in \bigcup_{k<n} W_k\},$$

where the value of $k_n(y)$ is well-defined for each $y \in W_0$ and $n \in \mathbb{N}$, since all points in $C$ return to their Voronoi domain infinitely often. Define a transformation $S_n : W_0 \to W_n$ as $S_n(y) = T^{k_n(y)} y$, and note that for all $n \in \mathbb{N}$ one has $c_{S_n}(y) \in \rho((\mathcal{V}_{\mathscr{C}})_c, (\mathcal{V}_{\mathscr{C}})_c)$. The region $\rho((\mathcal{V}_{\mathscr{C}})_c, (\mathcal{V}_{\mathscr{C}})_c)$ is precompact, since $\mathscr{C}$ is cocompact, and therefore using continuity of the Haar modulus $\Delta : G \to \mathbb{R}^*_+$ one can pick $\epsilon > 0$ such that $\Delta(c_{S_n}(y)) > 0$ for all $y \in W_0$ and all $n \in \mathbb{N}$.

Since $S_n$ is composed of powers of $T$, Proposition B.15 ensures that

$$\lambda_x(S_n W_0) = \int_{W_0} \Delta(c_{S_n}(y))\, d\lambda_x(y),$$

whence $\lambda_x(S_n W_0) \geq \epsilon\lambda_x(W_0)$ for each $n \in \mathbb{N}$. We now arrive at a contradiction, as $W_n$, $n \in \mathbb{N}$, form a pairwise disjoint infinite family of subsets of $(\mathcal{V}_{\mathscr{C}})_c$ whose measure is uniformly bounded away from zero by $\epsilon\lambda_x(W_0)$, which is impossible, since $\lambda_x((\mathcal{V}_{\mathscr{C}})_c) < \infty$ by cocompactness of $\mathscr{C}$. This finishes the proof of the claim. $\square_{\text{claim}}$

We have established by now that $D \cap [x]_{\mathscr{R}_G} \subseteq D_x$ and, up to a null set, $C \cap [x]_{\mathscr{R}_G} \subseteq C_x$ by the claim above. Finally, $\mu(X \setminus (D \sqcup C)) = 0$ implies via Lemma B.16 $\lambda_x((D \cap [x]_{\mathscr{R}_G}) \sqcup (C \cap [x]_{\mathscr{R}_G})) = 0$ for $\mu$-almost all $x \in X$, and therefore $\lambda_x((D \cap [x]_{\mathscr{R}_G}) \triangle D_x) = 0 = \lambda_x((C \cap [x]_{\mathscr{R}_G}) \triangle C_x)$ $\mu$-almost surely. Sets $D$ and $C$ thus satisfy the conclusion of the proposition.

For the uniqueness part of the proposition, suppose $D, C$ and $D', C'$ are two partitions of $X$ such that

$$\lambda_x(D \triangle D_x) = 0 = \lambda_x(D' \triangle D_x) \text{ and } \lambda_x(C \triangle C_x) = 0 = \lambda_x(C' \triangle C_x)$$

for $\mu$-almost all $x \in X$. One therefore also has $\forall^\mu x \in X\ \lambda_x(D \triangle D') = 0 = \lambda_x(C \triangle C')$, and hence $\mu(D \triangle D') = 0$ by Lemma B.16. $\square$

## Appendix C. Polish finitely full groups and their symmetric group

In this section, we introduce and study the class of Polish finitely full groups. It encompasses $\mathrm{L}^1$ full groups of measure-preserving actions of Polish normed groups and allows us to put some the proofs from [LM18] in a unified context (e.g. it would also apply to similarly defined $\mathrm{L}^p$ full groups). Specifically, from a Polish finitely full group we will define a closed subgroup analogous to Nekrashevych's alternating topological full group [Nek17]. We will show that this subgroup is equal to the topological derived group under a mild condition we call induction friendliness, and that it is topologically simple as soon as the original group was ergodic.

**Definition C.1.** A group of measure-preserving transformations $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ is **finitely full** if for any partition $X = A_1 \sqcup \cdots \sqcup A_n$ and $g_1, \ldots, g_n \in \mathbb{G}$ such that sets $g_1 A_1, \ldots, g_n A_n$ also partition $X$, the element $T \in \mathrm{Aut}(X,\mu)$ obtained as the reunion over $i \in \{1, \ldots, n\}$ of the restrictions $g_i \restriction_{A_i}$ belongs to $\mathbb{G}$.

For any subgroup $G \leq \mathrm{Aut}(X,\mu)$, there is the smallest finitely full group containing $G$; such a subgroup is denoted by $[\![G]\!]$. Note that if $H \leq \mathrm{Aut}(X,\mu)$ is a finite group, then the finitely full group it generates coincides with the full group: $[H] = [\![H]\!]$. This, in particular, applies to the group generated by a periodic transformation with bounded periods.

Consider a finitely full group $\mathbb{G}$, and suppose additionally that it is equipped with a Polish group topology compatible with the standard Borel structure induced by that of $\mathrm{Aut}(X,\mu)$. Standard automatic continuity results (see e.g. [BK96, Sec. 1.6]) guarantee that such a topology—as long as it exists—is necessarily unique, and it must refine the weak topology. We refer to such groups $\mathbb{G}$ as **Polish finitely full groups**.

We will also need a definition of aperiodicity which makes sense for arbitrary subgroups of $\mathrm{Aut}(X,\mu)$. Such a definition was already worked out by H. Dye [Dye59, Sec. 2] when he introduced type II subgroups. Here is an equivalent version which suffices for our purposes.

**Definition C.2.** A subgroup $G \leq \mathrm{Aut}(X,\mu)$ is called *aperiodic* when it admits a a countable weakly dense subgroup whose action on $(X,\mu)$ has only infinite orbits.

It can be checked that when $G$ is aperiodic, *every* countable weakly dense subgroup $\Gamma \leq G$ will have all its orbits infinite.

C.1. **Derived subgroup and symmetric subgroup.** Our next goal is to identify when the closed derived subgroup of a Polish finitely full group is topologically generated by involutions. We start by an important observation on the topology of Polish finitely full groups.

**Proposition C.3.** *Suppose $\mathbb{G}$ is a Polish finitely full group, and let $U \in \mathbb{G}$ be a periodic transformation with bounded periods. The topology induced by $\mathbb{G}$ on the full group of $U$ is equal to the uniform topology.*

*Proof.* By automatic continuity [BK96, Thm. 1.2.6], the topology induced by $\mathbb{G}$ on the full group of $U$ is refined by the uniform topology. On the other hand, the topology of $\mathbb{G}$ refines the weak topology, and the weak topology induces the uniform topology on the full group of $U$ since $U$ is periodic. Thus the topology induced by $\mathbb{G}$ refines the uniform topology, hence they are equal. □

We will make use of the following well-known lemma, which can be proved using the standard exhaustion argument.

**Lemma C.4.** *Let $G \leq \mathrm{Aut}(X,\mu)$ be aperiodic. For any measurable set $A \subseteq X$ there is an involution $U \in [G]$ whose support is equal to $A$. Moreover, if $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ is dense in the weak topology, then the involution $U$ can be chosen such that there exists a partition $A = \bigsqcup_n (A_n \sqcup B_n)$ satisfying, $g_n(A_n) = B_n$, $U\restriction_{A_n} = g_n\restriction_{A_n}$, and $U\restriction_{B_n} = g_n^{-1}\restriction_{B_n}$ (some of the sets $A_n$, $B_n$ may be empty).* □

A straightforward useful consequence is that aperiodic finitely full groups admit many involutions in the sense of [Fre04, p. 384]:

**Corollary C.5.** *Let $\mathbb{G}$ be a finitely full aperiodic group. Then for every $A \subseteq X$, there is a nontrivial involution $g \in \mathbb{G}$ whose support is contained in $A$.* □

The two first items of the following definition constitute analogues of Nekrashevych's symmetric and alternating topological full groups, but in our setup they actually coincide as we will see shortly.

**Definition C.6.** Given a Polish finitely full group $\mathbb{G}$, we let

- $\mathfrak{S}(\mathbb{G})$ be the closed subgroup of $\mathbb{G}$ generated by involutions, which we call the **symmetric subgroup of** $\mathbb{G}$.
- $\mathfrak{A}(\mathbb{G})$ be the closed subgroup of $\mathbb{G}$ generated by 3-cycles,
- $D(\mathbb{G})$ be the closed subgroup generated by commutators (also known as the *topological derived subgroup*).

All these groups are closed normal subgroups of $\mathbb{G}$, and $\mathfrak{A}(\mathbb{G}) \leq \mathfrak{S}(\mathbb{G}) \cap D(\mathbb{G})$ because every 3-cycle is a commutator of two involutions.

**Proposition C.7.** $\mathfrak{A}(\mathbb{G}) = \mathfrak{S}(\mathbb{G})$ *for any aperiodic finitely full group* $\mathbb{G}$.

*Proof.* We need to show that every involution is a limit of products of 3-cycles. Let $U \in \mathbb{G}$ be an involution, and let $D$ denote its fundamental domain; thus $\mathrm{supp}\, U = D \sqcup U(D)$. By Lemma C.4 one can find an involution $V \in [\mathbb{G}]$ whose support is equal to $D$. Since $\mathbb{G}$ is finitely full, we may write $D$ as an increasing union $D = \bigcup_n D_n$, $D_n \subseteq D_{n+1}$, where each $D_n$ is $V$-invariant, and for every $n \in \mathbb{N}$ the transformation $V_n$ induced by $V$ on $D_n$ belongs to the group $\mathbb{G}$ itself. Let $U_n$ denote the restriction of $U$ onto $D_n \sqcup U(D_n)$ and note that $U_n \to U$ in the uniform topology, and hence also in the topology of $\mathbb{G}$ by Proposition C.3. Our plan is to use the following permutation identity

$$(12)(34) = (12)(23)(24)(23) = (123)(423), \tag{15}$$

where $U_n$ corresponds to $(12)(34)$, $V_n$ to $(13)$, and $U_nV_nU_n$ corresponds to $(24)$. To this end let $C_n$ be a fundamental domain for $V_n$, put $W_n = U\restriction_{C_n \sqcup U(C_n)}$ (which corresponds to the involution $(12)$), and, at last, set $S_n = W_nV_nW_n$ (corresponding to $(23) = (12)(13)(12)$). Figure 12 illustrates the relations between these sets and transformations.

Equation (15) translates into $U_n = \big(W_n(W_nV_nW_n)\big)\big((U_nV_nU_n)(W_nV_nW_n)\big)$, so $U_n$ is a product of two 3-cycles, hence it belongs to $\mathfrak{A}(\mathbb{G})$. Since by construction $U_n \to U$, we conclude that $U \in \mathfrak{A}(\mathbb{G})$. □

It is unclear whether $\mathfrak{A}(\mathbb{G}) = D(\mathbb{G})$ holds for all finitely full groups, but here is a convenient sufficient condition.

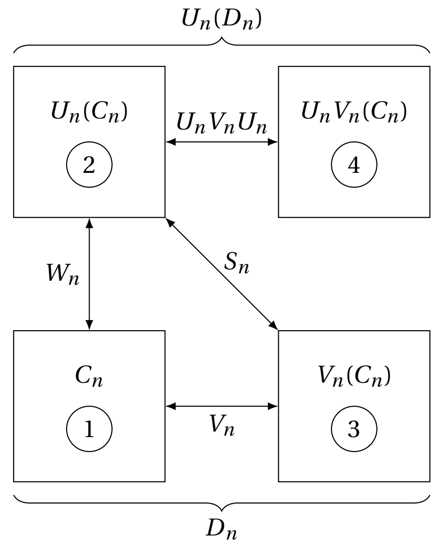


FIGURE 12. Involution $U_n$ is a products of 3-cycles via (12)(34) = (123)(234).

**Definition C.8.** A Polish finitely full group $\mathbb{G}$ is called **induction friendly** if it is stable under taking induced transformations and moreover, if $T \in \mathbb{G}$ and $A_n$ is an increasing sequence of $T$-invariant sets such that $\bigcup_n A_n = A$, then $T_{A_n} \to T_A$.

We insist on the fact that in the above definition, the sets $A_n$ are taken to be $T$-invariant, so the fact that $T_{A_n}$ belongs to $\mathbb{G}$ is a consequence of the fact that $\mathbb{G}$ is finitely full.

Observe that $\mathrm{L}^1$ full groups of measure-preserving actions of Polish normed groups are finitely full and also induction friendly. Indeed finite fulness follows from a straightforward computation, while induction friendliness is a direct consequence of the Lebesgue dominated convergence theorem. It is unclear that there exists a Polish finitely full group which is not induction friendly. Let us first observe an easy consequence of induction friendliness.

**Lemma C.9.** *Let $\mathbb{G}$ be an induction friendly Polish finitely full group. Then every periodic element belongs to $\mathfrak{S}(\mathbb{G})$.*

*Proof.* Suppose $T$ is periodic. For $n \in \mathbb{N}$ let $A_n$ be the set of $x \in X$ whose $T$-orbit has cardinality at most $n$. Then each $A_n$ is $T$-invariant and $\bigcup_n A_n = X$. Moreover, $T_{A_n}$ is periodic, so it can be written as the product of two involutions of its full group, and since $\mathbb{G}$ is finitely full and $T_{A_n}$ has bounded periods these two involutions belong to $\mathbb{G}$. The conclusion follows since by induction friendliness $T_{A_n} \to T$. □

**Proposition C.10.** *If $\mathbb{G}$ is an aperiodic induction friendly Polish finitely full group, then $\mathfrak{S}(\mathbb{G}) = D(\mathbb{G})$.*

*Proof.* Since $\mathfrak{A}(\mathbb{G}) \leq D(\mathbb{G})$ always holds, Proposition C.7 implies $\mathfrak{S}(\mathbb{G}) \leq D(\mathbb{G})$; so we concentrate on proving the reverse inclusion. For any $T \in \mathbb{G}$, if we let $\tilde{T}$ be the transformation induced by $T$ on its set of aperiodic points, then $T\tilde{T}^{-1}$ is periodic and hence $T\mathfrak{S}(\mathbb{G}) = \tilde{T}\mathfrak{S}(\mathbb{G})$ by the previous lemma. Moreover by construction $\tilde{T}$ is aperiodic on its support. In order to show that for all elements $T, U \in \mathbb{G}$ their commutator $[T, U] \in \mathfrak{S}(\mathbb{G})$, it therefore suffices to establish this containment for automorphisms that are aperiodic on their supports.

Given an automorphism $T \in \mathbb{G}$, let us say that a set $A \subseteq X$ is a *$T$-complete cross section* if it intersects every orbit in the support of $T$, i.e., if $\operatorname{supp} T \subseteq \bigcup_{i\in\mathbb{Z}} T^i(A)$. If $A$ is a $T$-complete cross section, then the transformation $T \circ T_A^{-1} \in \mathbb{G}$ is periodic, and thus is an element of $\mathfrak{S}(\mathbb{G})$ by the discussion above; in particular, $T_A\mathfrak{S}(\mathbb{G}) = T\mathfrak{S}(\mathbb{G})$.

Let now $T, U \in \mathbb{G}$ be aperiodic on their supports; we will show that $[T, U] \in \mathfrak{S}(\mathbb{G})$. Let $A$ be a $T$-complete cross section. Consider the induced transformation $U_{X\setminus A}$ and note that it commutes with $T_A$, since their supports are disjoint. We would be done if $X \setminus A$ were a $U$-complete cross section. Indeed, in this case $T\mathfrak{S}(\mathbb{G}) = T_A\mathfrak{S}(\mathbb{G})$, $U\mathfrak{S}(\mathbb{G}) = U_{X\setminus A}\mathfrak{S}(\mathbb{G})$, and $[T_A, U_{X\setminus A}]$ is trivial, hence $[T, U] \in \mathfrak{S}(\mathbb{G})$.

Motivated by this observation, we argue as follows. Pick a vanishing nested sequence $(A_n)_{n\in\mathbb{N}}$ of $T$-complete cross sections: $A_n \supseteq A_{n+1}$, $\bigcup_{i\in\mathbb{Z}} T^i(A_n) \supseteq \operatorname{supp} T$ for all $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$. Such a sequence of cross sections exists since $T$ is assumed to be aperiodic on its support. Define inductively sets $B'_n$, $n \in \mathbb{N}$, by setting $B'_0 = X \setminus A_0$, and

$$B'_n = (X \setminus A_n) \setminus \bigcup_{k<n} \bigcup_{i\in\mathbb{Z}} U^i(B'_k).$$

By construction, saturations under $U$ of the sets $B'_n$ are pairwise disjoint, and the saturation of their union is the whole space, $\bigcup_{i\in\mathbb{Z}} U^i\big(\bigcup_{n\in\mathbb{N}} B'_n\big) = X$, because cross sections $A_n$ vanish.

Let $B_n = \bigsqcup_{k<n} B'_k$, $B = \bigsqcup_{k\in\mathbb{N}} B'_k$, and note that $U_{B_n}, U_B \in \mathbb{G}$, and $U_{B_k} \to U_B$ by the induction friendliness of $\mathbb{G}$. Observe that for each $n$ transformations $T_{A_n}$ and $U_{B_n}$ have disjoint supports and, therefore, commute. Since all sets $A_n$ are $T$-complete cross sections, one has $[T, U_{B_n}] \in \mathfrak{S}(\mathbb{G})$. Taking the limit as $n \to \infty$, this yields $[T, U_B] \in \mathfrak{S}(\mathbb{G})$. Finally, since $B$ is a $U$-complete cross section, we conclude that $[T, U] \in \mathfrak{S}(\mathbb{G})$ as claimed. □

**Corollary C.11.** *Let $G$ be a Polish normed group, and let $G \curvearrowright X$ be an aperiodic Borel measure-preserving action on a standard probability space $(X,\mu)$. The three subgroups of $[G \curvearrowright X]_1$ introduced in Definition C.6 coincide:*

$$D([G \curvearrowright X]_1) = \mathfrak{A}([G \curvearrowright X]_1) = \mathfrak{S}([G \curvearrowright X]_1).$$

*Proof.* The corollary follows immediately from Propositions C.7 and C.10, since $[G \curvearrowright X]_1$ is both finitely full and induction friendly. □

C.2. **Topological simplicity of the symmetric group.** We now move on to showing that symmetric subgroups of ergodic Polish finitely full groups are always topologically simple. We follow closely [LM18, Sec. 3.4].

**Lemma C.12.** *Let $\mathbb{G}$ be an aperiodic Polish finitely full group, let $U, V \in \mathbb{G}$ be two involutions whose supports are disjoint and have the same $\mathbb{G}$-conditional measure. Then $U$ and $V$ are approximately conjugate in $\mathfrak{S}(\mathbb{G})$.*

*Proof.* Let $A$ (resp. $B$) be a fundamental domain of the restriction of $U$ (resp. $V$) to its support. Then $\mu_{\mathbb{G}}(A) = \mu_{\mathbb{G}}(B)$ so there is an involution $T \in [\mathbb{G}]$. such that $T(A) = B$.

Since $\mathbb{G}$ is finitely full, we find an increasing sequence of subsets $(A_n)$ of $A$ such that the involution $T_n$ induced by $T$ on $A_n \cup U(A_n)$ belongs to $\mathbb{G}$, and $\bigcup_n A_n = A$. Let $B_n = T(A_n) = T_n(A_n)$. We now define for every $n$ an involution $T'_n \in \mathbb{G}$ which almost conjugates $U$ to $V$: for $x \in X$ we let

$$T'_n(x) = \begin{cases} T(x) & \text{if } x \in A_n \sqcup B_n \\ VTU(x) & \text{if } x \in U(A_n) \\ UTV(x) & \text{if } x \in V(B_n) \\ x & \text{otherwise.} \end{cases}$$

For all $n \in \mathbb{N}$ and all $x \in X$, an easy calculation yields that:

- if $x \in (A \cup U(A)) \setminus (A_n \cup U(A_n))$, then $T'_n U T'_n(x) = U(x)$,
- else if $x \in (B_n \cup V(B_n)$, then $T'_n U T'_n(x) = V(x)$ and
- else $T'_n U T'_n(x) = x$.

We thus have $d_u(T'_n U T'_n, V) \to 0$. So by Proposition C.3 applied to the full group of the involution $UV$ (which contains both $U$ and $V$), we have $T'_n U T'_n \to V$. □

**Lemma C.13.** *Let $\mathbb{G}$ be a Polish finitely full group, let $U \in \mathbb{G}$ be an involution, and let $A$ be a $U$-invariant subset contained in* $\operatorname{supp} U$. *Let $V \in \mathbb{G}$ be an involution such that $V(A)$ is disjoint from* $\operatorname{supp} U$. *Then for all $\mathbb{G}$-invariant function $f \le 2\mu_{\mathbb{G}}(A)$, there is an involution $W \in \mathbb{G}$ such that $UWUW$ is an involution whose support has $\mathbb{G}$-conditional measure $f$.*

*Proof.* Let $B \subseteq A$ be a fundamental domain for the restriction of $U$ to $A$, then $\mu_{\mathbb{G}}(B) = \frac{\mu_{\mathbb{G}}(A)}{2}$. By Maharam's lemma, there is $C \subseteq B$ such that $\mu_{\mathbb{G}}(C) = \frac{f}{4}$. Let $D = C \sqcup U(C)$, then $D$ is $U$-invariant and satisfies $\mu_{\mathbb{G}}(D) = \frac{f}{2}$. Consider the involution $W \in \mathbb{G}$ defined by

$$W(x) = \begin{cases} V(x) & \text{if } x \in D \sqcup V(D) \\ x & \text{otherwise.} \end{cases}$$

Then a straightforward computation shows $UWUW$ is an involution which coincides with $U$ on $D$, with $VUV$ on $V(D)$, and is trivial elsewhere. So the support of $UWUW$ is equal to $D \sqcup V(D)$, and hence has $\mathbb{G}$-conditional measure $f$. □

**Proposition C.14.** *Let $\mathbb{G}$ be an aperiodic Polish finitely full group, let $T \in \mathbb{G}$, and denote by $A$ the $\mathbb{G}$-saturation of* $\operatorname{supp} T$. *Then the closed subgroup of $\mathbb{G}$ generated by the $\mathfrak{S}(\mathbb{G})$-conjugates of $T$ contains $\mathfrak{S}(\mathbb{G})_A$.*

*Proof.* Denote by $G$ the closed subgroup of $\mathbb{G}$ generated by the $\mathfrak{S}(\mathbb{G})$-conjugates of $T$.

We can find $B \subseteq \operatorname{supp} T$ whose $T$-translates cover $\operatorname{supp} T$, satisfying $B \cap T(B) = \emptyset$. The fact that the $T$-translates of $B$ cover $\operatorname{supp} T$ implies that the $\mathbb{G}$-translates of $B$ cover $A$, so $\mu_{\mathbb{G}}(B)(x) > 0$ for all $x \in A$. By Maharam's lemma, we then find $C \subseteq B$ whose $\mathbb{G}$-conditional measure is everywhere less than $1/4$, and strictly positive on $A$. Let $D = C \sqcup T(C)$ and $V \in [\mathbb{G}]$ such that $V(C \sqcup T(C))$ is disjoint from $C \sqcup T(C)$.

Let $W \in [\mathbb{G}]$ be an involution such that $\operatorname{supp} W = C$. Using the fact that $\mathbb{G}$ is finitely full, that $T \in \mathbb{G}$ and that $V, W \in [\mathbb{G}]$, we obtain an increasing sequence of $W$-invariant subsets $C_n$ of $C$ such that $\bigcup_n C_n = C$ and for all $n \in \mathbb{N}$

(i) $W_{C_n} \in \mathbb{G}$,
(ii) $V_{(C_n \sqcup T(C_n) \sqcup V(C_n \sqcup T(C_n))} \in \mathbb{G}$.

Then for every $n$, the element $W_{C_n} T W_{C_n} T^{-1}$ belongs to $G$, and it is an involution whose support is equal to $C_n \sqcup T(C_n)$ and has conditional measure at most $2\mu_{\mathbb{G}}(C) \leq 1/2$. Let us define for brefivty

$$\tilde{U}_n = W_{C_n} T W_{C_n} T^{-1} \in G \text{ and } \tilde{V}_n = V_{(C_n \sqcup T(C_n) \sqcup V(C_n \sqcup T(C_n))} \in \mathbb{G}.$$

For every $n \in \mathbb{N}$, let $A_n$ be the $\mathbb{G}$-saturation of $C_n$, then $A = \bigcup_n A_n$ and the union is increasing. Every involution supported on $A$ is thus the uniform limit of the involutions it induces on $A_n$, so by C.3 it suffices to show that $\mathbb{G}$ contains all the involutions which are supported on some $A_n$.

To this end, we fix $n \in \mathbb{N}$ and an involution $U$ supported on $A_n$. Let $D$ be a fundamental domain for the restriction of $U$ to its support. Using Maharam's lemma repeatedly, we then partition $D$ into a countable family $(D_k)$ such that for every $k \in \mathbb{N}$,

$$\mu_{\mathbb{G}}(D_k) \leq \frac{\mu_{\mathbb{G}}(\operatorname{supp} \tilde{U}_n)}{2}. \tag{16}$$

If we let $E_k = D_k \sqcup U(D_k)$, the sequence $(E_k)$ forms a partition of $\operatorname{supp} U$ into $U$-invariant sets. We thus have $U = \lim_{k \to +\infty} \prod_{i=0}^k U_{E_k}$ in the uniform topology and in the topology of $\mathbb{G}$ as well by Proposition C.3). Moreover, inequation (16) implies that the support of $U_{E_k}$ has $\mathbb{G}$-conditional measure at most $\mu_{\mathbb{G}}(\operatorname{supp} \tilde{U}_n)$. The set $\tilde{V}_n(\operatorname{supp} \tilde{U}_n)$ is disjoint from $\operatorname{supp} \tilde{U}_n$ by construction. Lemma C.13 provides an involution in $G$ whose support has the same conditional measure as that of $U_{E_k}$. Lemma C.12 then yields that each $U_{E_k}$ belongs to $G$. We conclude that $U \in G$ as wanted. □

**Theorem C.15.** *Let $\mathbb{G} \leq \operatorname{Aut}(X, \mu)$ be an aperiodic Polish finitely full group, let $N \leq \mathfrak{S}(\mathbb{G})$ be a closed normal subgroup of $\mathfrak{S}(\mathbb{G})$. Then there is a unique $\mathbb{G}$-invariant set $A$ such that $N = \mathfrak{S}(\mathbb{G})_A$.*

*Proof.* Observe that whenever $A_1$ and $A_2$ are $\mathbb{G}$-invariant, if $U \in \mathbb{G}$ is an involution supported in $A_1 \cup A_2$, then $U$ decomposes as the product of one involution supported in $A_1$, and one supported in $A_2$. It follows that the closed group generated by $\mathfrak{S}(\mathfrak{G})_{A_1} \cup \mathfrak{S}(\mathfrak{G})_{A_2}$ is equal to $\mathfrak{S}(\mathfrak{G})_{A_1 \cup A_2}$. Also, by Proposition C.3, whenever $(A_n)$ is an increasing union of $\mathbb{G}$-invariant sets, we have

$$\overline{\bigcup_n \mathfrak{S}(\mathbb{G})_{A_n}} = \mathfrak{S}(\mathbb{G})_{\bigcup_n A_n}.$$

The set $\{A \in \operatorname{MAlg}(X, \mu) : A \text{ is } \mathbb{G}\text{-invariant and } \mathfrak{S}(\mathbb{G})_A \leq N\}$ is thus directed and closed under countable union: it thus admits a unique maximum which is the set $A$ we seek. We then have $\mathfrak{S}(\mathbb{G}) \leq N$, and the reverse inclusion is a direct consequence of the previous proposition.

Finally, uniqueness is proved as follows: assume by contradiction $\mathfrak{S}(\mathbb{G})_{A_1} = \mathfrak{S}(\mathbb{G})_{A_2}$ but $A_1 \neq A_2$. By symmetry we may as well assume $\mu(A_1 \setminus A_2) > 0$. Corollary C.5 then provides an involution $V \in \mathbb{G}$ whose support is nontrivial and contained in $A_1 \setminus A_2$, thus $V \in \mathfrak{S}(\mathbb{G})_{A_1}$ but $V \not\in \mathfrak{S}(\mathbb{G})_{A_2}$, a contradiction. □

**Corollary C.16.** *If $\mathbb{G} \leq \operatorname{Aut}(X, \mu)$ is an aperiodic Polish finitely full group, then $\mathfrak{S}(\mathbb{G})$ is topologically simple if and only if $\mathbb{G}$ is ergodic.*

*Proof.* If $\mathbb{G}$ is ergodic, then by the above result $\mathfrak{S}(\mathbb{G})$ is clearly topologically simple. Conversely, suppose that $\mathbb{G}$ is not ergodic and let $A \subseteq X$ be a $\mathbb{G}$-invariant set with $\mu(A) \not\in \{0, 1\}$. Then $\mathfrak{S}(\mathbb{G})_A$ is a normal sugroup of $\mathbb{G}$ which is neither trivial nor equal to $\mathfrak{S}(\mathbb{G})$ as a consequence of Corollary C.5 applied to $A$ and its complement. □

Specifying to $\mathrm{L}^1$ full groups, we obtain the following result.

**Corollary C.17.** *Let $G$ be a Polish normed group, and let $G \curvearrowright X$ be an aperiodic Borel measure-preserving action on a standard probability space $(X, \mu)$. Then then the topological derived group of the $\mathrm{L}^1$ full group of the action is topologically simple if and only if the action is ergodic.*

*Proof.* The corollary follows immediately from Corollary C.11 and C.10, since $[G \curvearrowright X]_1$ is necessarily finitely full and induction friendly by Proposition 2.16. □

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