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Hypergeometric series

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§1. Introduction

Our object is to explain in cohomological terms all hypergeometric series. This we do via the theory of "exponential modules" (see §7 for definition) previously studied [GHF]. At the same time we establish a connection with the theory of Gelfand, Kapranov, and Zelevinsky [GKZ]. By means of a formal Laplace transformation (§9) we deduce integral representations of the classical type. In an appendix we treat the full list of Horn [E.M.O.T] the Lauricella generalization of Appell's $F_1 - F_4$ as well as a number of confluent forms of Lauricella type. Many of these representations have appeared in the literature (e.g. Erdelyi, Yoshida, Humbert, Aomoto, Pastro). To our knowledge the only previous systematic list is due to Kita [K]. Kita's list goes beyond ours in that he gives the corresponding cycle of integration. On the other hand he does not treat the confluent case. Our interest lies only in the differential module and questions of convergence and confluence play no role.

In the present treatment we have imposed the condition that the multiplicative character parameters (e.g. a, b, c in the case of ${}_2F_1$) are algebraically independent. We will not at this time explain how these conditions may be eliminated but do point out that the classical representations are totally inadequate in the case of trivial parameters (e.g. $(a, b, c) = 0$ in the above example) while the exponential modules do provide differential modules associated with the specialized differential equations. In previous work [GHF] one of the authors has shown how Frobenius structure and Boyarsky principle may be associated with exponential modules.

§2. Notation

Let S be a finite set of lattice points $\mathbb{Z}^n$
$\widehat{H}_0$ be the semigroup generated by $\mathcal{S}$. (Thus $0 \in \widehat{H}_0$).
H_0 be the group generated by $\mathcal{S}$. Let H be the linear space spanned by $\mathcal{S}$.
Let $g = \sum_{u \in \mathcal{S}} \lambda_u X^u$ be the generic polynomial with support in $\mathcal{S}$. Thus

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$(\cdots, \lambda_u, \cdots)$ are algebraically independent over $\mathbb{Q}$.

Let $a = (a_1, \dots, a_n)$ be generic point of H, i.e. the transcendence degree of $\mathbb{Q}(a)$ is equal to the dimension of H. We insist that $\mathbb{Q}(a)$ be linearly disjoint from $\mathbb{Q}(\lambda)$ over $\mathbb{Q}$.

Let $\widehat{R}$ be the ring of polynomials with coefficients in $\mathbb{Q}(a, \lambda)$ and support in $\widehat{H}_0$.

Let R' be the ring of polynomials with coefficients in $\mathbb{Q}(a, \lambda)$ and support in H_0.

For $1 \leq i \leq n$ we define $E_i = X_i \frac{\partial}{\partial X_i}$, $g_i = E_i g$,

$$D_{a,i} = E_i + g_i + a_i$$

an operator on R' and on $\widehat{R}$. We define $\widehat{\mathcal{W}}_a = \widehat{R}/\sum_{i=1}^n D_{a,i}\widehat{R}$. We give $\widehat{\mathcal{W}}_a$ the structure of a differential module (see §7 for another point of view) by setting

$$\sigma_u = \frac{\partial}{\partial \lambda_u} + \frac{\partial g}{\partial \lambda_u} \qquad \forall u \in \mathcal{S}.$$

We restrict our attention to $\widehat{\mathcal{W}}_a$ but our results may be extended to $\mathcal{W}'_a = R'/\sum_{i=1}^n D_{a,i}R'$ by means of the isomorphism between $\widehat{\mathcal{W}}_a$ and $\mathcal{W}'_a$ induced by the injection $\widehat{R} \hookrightarrow R'$.

We shall have occasion to speak of

$$\mathcal{H} = \widehat{R} \cap \mathbb{Q}(\lambda)\left[a, X, \frac{1}{X_1 \cdots X_n}\right]$$

Let $\mathcal{R}$ be the non-commutative ring, $\mathcal{R} = \mathbb{Q}(a, \lambda)[\sigma]$. Following the spirit of Gelfand and his colleagues [G.K.Z], we define the Gelfand ideal, $\mathfrak{B}$ of $\mathcal{R}$ to be the left ideal

$$\mathfrak{B} = \sum_{i=1}^n \mathcal{R} Z_i + \sum_A \mathcal{R} \square_A,$$

the second sum being over the module of $\mathbb{Z}$ relations, A, among the elements of $\mathcal{S}$. Here

$$Z_i = \sum_{u \in \mathcal{S}} u_i \lambda_u \sigma_u + a_i$$

while if $\sum_{u \in \mathcal{S}} A_u u = 0$, is a relation over $\mathbb{Z}$ among the elements of $\mathcal{S}$ then for each u we write $A_u = A'_u - A''_u$, where $A'_u \cdot A''_u = 0$, and both A'_u and A''_u lie in $\mathbb{N}$ and define

$$\square_A = \prod_{u \in \mathcal{S}} \sigma_u^{A'_u} - \prod_{u \in \mathcal{S}} \sigma_u^{A''_u}.$$

Let $\mathfrak{A}$ be the left ideal of all $\theta \in \mathcal{R}$ which annihilate [1], the class of 1 in $\widehat{\mathcal{W}}_a$. We shall refer to $\mathfrak{A}$ as the exponential module ideal.

We shall show that $\mathfrak{A} = \mathfrak{B}$.

§3. Basic Propositions

The main hypothesis, a is generic element of H, will be understood throughout this section.

PROPOSITION 3.1. *Let $\xi \in \mathbb{Q}(\lambda)[X, 1/\prod_{i=1}^{n} X_i]$ and suppose that*

$$\xi \in \sum D_{a,i}\mathcal{H}$$

then $\xi = 0$. (Note that ξ is an element of $\widehat{R}$ independent of a).

PROOF. By hypothesis $\xi = \sum D_{a,i}\eta_i$, $\eta_i \in \mathcal{H}$. By definition of $\mathcal{H}$ there exists $m \in \mathbb{N}$ such that each η_i is annihilated by all derivatives of order $m-1$ relative to $a_1, \dots, a_n$. We conclude from [GHF, p. 14] that ξ may be written in the form

$$\xi = \sum_{i=1}^{m-1} \frac{(-1)^{i+1}}{i!} \sum_{j_1,\dots,j_i=1}^{n} D_{a,j_1} \cdots D_{a,j_i} \frac{\partial^i \xi}{\partial a_{j_1} \partial a_{j_2} \cdots \partial a_{j_i}}.$$

The right side is zero as element of $\widehat{R}$.

PROPOSITION 3.2. *The annihilator in $\mathcal{R}$ of 1 (as element of $\widehat{R}$) is $\sum_A \mathcal{R}\Box_A$.*

PROOF. For each element, A, of the module of relations of $\mathcal{S}$ we have $\Box_A \cdot 1 = 0$. Conversely if $P \in \mathcal{R}$, $P \cdot 1 = 0$ then we write $P = \sum B_w \sigma^w$ where each $B_w \in \mathbb{Q}(a, \lambda)$. We may assume that P is minimal in its class modulo $\sum \mathcal{R}\Box_A$ in the sense that Card $(w \mid B_w \neq 0)$ is minimal. By hypothesis $0 = \sum B_w X^{\sum_{u \in \mathcal{S}} u \cdot w_u}$. Hence if $B_w \neq 0$ then there exists $w' \neq w$ with $B_{w'} \neq 0$ such that

$$\sum u \cdot w_u = \sum u \cdot w'_u.$$

For each $u \in \mathcal{S}$ let $v_u = \min(w_u, w'_u)$ and let $A'_u = w'_u - v_u$, $A''_u = w_u - v_u$, $A_u = A'_u - A''_u$. Then $\sigma^{w'} = \sigma^w + \sigma^v \Box_A$ and hence modulo $\sum \mathcal{R}\Box_A$ we may eliminate $\sigma^{w'}$. This completes the proof.

PROPOSITION 3.3. *Let $P = \sum_v \lambda^v P_v(\sigma)$, a Laurent polynomial in λ whose coefficients, P_v, lie in $\mathbb{Q}[\sigma]$ for each multi-exponent, v. Suppose $P(\lambda, \sigma)1 \in \sum D_{a,i}\widehat{R}$. Then for each v,*

$$P_v(\sigma)1 \in \sum D_{a,i}\widehat{R}.$$

PROOF. Multiplying on the left by a suitable monomial, λ^w, we may assume that P is a polynomial in λ. We use induction on the degree of P as polynomial in

λ. Since multiplication by $X^u \in \widehat{R}$ commutes with σ_w for each $w \in \mathcal{S}$, we deduce after multiplication on the left

$$PX^u \in \sum D_{a-u,i}\widehat{R}.$$

Applying the automorphism $a \to a+u$ (which does not change P) and letting $u \in \mathcal{S}$,

$$P\sigma_u 1 = PX^u \in \sum D_{a,i}\widehat{R}.$$

On the other hand, multiplying our original relation on the left by σ_u, we have $\sigma_u P1 \in \sum D_{a,i}\widehat{R}$. Thus taking the difference

$$(\sigma_u P - P\sigma_u)1 \in \sum D_{a,i}\widehat{R}.$$

By an elementary calculation

$$\sum_v \lambda^{v-\mathbf{1}_u} v_u P_v(\sigma)1 \in \sum D_{a,i}\widehat{R}.$$

We may now use induction on the degree of P as polynomial in λ.

COROLLARY 3.4. *Let $P \in \mathbb{Q}(a)[\lambda, \frac{1}{\lambda}, \sigma]$. Let $u \in \mathcal{S}$. Suppose $\sigma_u P1 \in \sum D_{a,i}\widehat{R}$ (i.e. $\sigma_u P \in \mathfrak{A}$). Then $P1 \in \sum D_{a,i}\widehat{R}$, i.e. $P \in \mathfrak{A}$.*

PROOF. Multiplying by an element of $\mathbb{Q}[a]$ we may reduce to the case in which $P \in \mathbb{Q}[a, \lambda, \frac{1}{\lambda}, \sigma]$. Since a_i commutes with σ_v for each $v \in \mathcal{S}$, we may reduce modulo $\sum_{i=1}^n \mathbb{Q}[a, \lambda, \frac{1}{\lambda}, \sigma]Z_i$ to the case in which P lies in $\mathbb{Q}[\lambda, \frac{1}{\lambda}, \sigma]$. Thus we may write $P = \sum \lambda^v P_v(\sigma)$, $P_v \in \mathbb{Q}[\sigma]$ for each multi-exponent, v.

We use induction on the number of distinct multi-exponents, v, such that $P_v \neq 0$. By hypothesis

$$\sum \Big[\lambda^v \sigma_u P_v(\sigma) + v_u \lambda^{v-\mathbf{1}_u} P_v(\sigma)\Big]1 \in \sum D_{a,i}\widehat{R}.$$

Hence by Proposition 3.3, $\sigma_u P_v(\sigma)1 \in \sum D_{a,i}\widehat{R}$ for each multi-index v in which v_u is maximal. Since σ_u commutes with $P_v(\sigma)$, we have for each such multi-exponent, v,

$$P_v(\sigma)\sigma_u 1 \in \sum D_{a,i}\widehat{R}$$

and so

$$P_v(\sigma)X^u \in \sum D_{a,i}\widehat{R}.$$

This implies [GHF Lemma 1.0.1],

$$P_v(\sigma)1 \in \sum D_{a+u,i}\widehat{R}$$

We now apply the automorphism $a \to a - u$ to deduce

$$P_v(\sigma)1 \in \sum D_{a,i}\widehat{R}$$

for each multi-exponent v such that v_u is maximal. Let T be the set of all such v. Then $\widetilde{P} = \sum_{v \notin T} \lambda^v P_v$ is congruent to P mod $\mathfrak{A}$ and so $\sigma_u \widetilde{P}$ again lies in $\mathfrak{A}$. Since $\widetilde{P}$ is a sum $\sum_v \lambda^v P_v$ involving fewer v than P, the induction hypothesis may be applied.

§4. Equality of Gelfand ideal and exponential module ideal

THEOREM. $\mathfrak{B} = \mathfrak{A}$.

PROOF.
4.1. We easily verify that $\mathfrak{B} \subset \mathfrak{A}$.
(a) If A lies in the module of relations over $\mathbb{Z}$ among the elements of $\mathcal{S}$, then

$$\begin{aligned}\Box_A 1 &= \left(\prod_{u \in \mathcal{S}} \sigma_u^{A'_u}\right)1 - \left(\prod_{u \in \mathcal{S}} \sigma_u^{A''_u}\right)1 \\ &= \prod_{u \in \mathcal{S}} X^{u \cdot A'_u} - \prod_{u \in \mathcal{S}} X^{u \cdot A''_u} = 0.\end{aligned}$$

(So in fact $\Box_A 1 = 0$ as element of $\widehat{R}$.)
(b) $Z_i 1 = a_i + \sum_{u \in \mathcal{S}} u_i \lambda_u X^u = a_i + g_i = D_{a,i}1$.

4.2. We must now invert the inclusion. Let $P \in \mathfrak{A}$. Thus $P1 = \sum_{i=1}^n D_{a,i}\eta_i$, each $\eta_i \in \mathbb{Q}(a, \lambda)[X, \frac{1}{X}] \cap \widehat{R}$. With no loss in generality we may replace P by $h \cdot P$, where $h \in \mathbb{Q}[a, \lambda]$ is chosen such that $h\eta_i \in \mathbb{Q}[a, \lambda, X, \frac{1}{X}] \cap \widehat{R}$. Thus we may assume $P \in \mathbb{Q}[a, \lambda, \sigma]$ and that

$$P(a, \lambda, \sigma)1 = \sum_{i=1}^n D_{a,i}\eta_i$$

and each $\eta_i \in \widehat{R} \cap \mathbb{Q}[a, \lambda, X, \frac{1}{X}]$. We may reduce P modulo $\sum_{i=1}^n \mathbb{Q}[a, \lambda, \sigma] Z_i \subset \mathfrak{B}$ and assume that P is independent of a. Hence by Proposition 3.1, $P1 = 0$, and so by Proposition 3.2, $P \in \mathfrak{B}$.

§5. Hypergeometric series

Let y be a hypergeometric series in n variables, $t_1, \dots, t_n$ i.e. $y = \sum_{s \in \mathbb{N}^n} C(s) t^s$ where $N \geq N_1 \geq n$,

$$C(s) = \frac{1}{\prod_{p=1}^{N_1} (-1)^{l_p(s)}(1-\alpha_p)_{-\ell_p(s)}} \cdot \prod_{q=1+N_1}^{N} (\alpha_q)_{\ell_q(s)}$$

and $\ell_1, \dots, \ell_N$ are linear forms in $\mathbb{Z}[s_1, \dots, s_n]$. We simplify:

5.1. We insist that $\alpha_p = 0$ and $\ell_p(s) = -s_p$ for $1 \le p \le n$.

5.2. We insist that for $p \in [1, N]$, the greatest common divisor of the coefficients of ℓ_p be unity.

5.3. We insist that $(\alpha_{n+1}, \alpha_{n+2}, \dots, \alpha_N)$ be algebraically independent over $\mathbb{Q}$.

For $1 \le j \le n$ let $\delta_j = t_j \frac{\partial}{\partial t_j}$.

5.4. It follows from 5.3 that the annihilator of y in $\mathbb{Q}(\alpha)[\delta_1, \dots, \delta_n]$ is trivial. Indeed if $P(\delta)$ were such an element then $P(s)C(s) = 0$ for all $s \in \mathbb{N}^n$ and hence $P(s) = 0$ for all $s \in \mathbb{N}^n$ since by 5.3 $C(s)$ is never zero.

Let $\Omega = \mathbb{Q}(\alpha)$, $\overline{\mathcal{R}} = \Omega[t, \frac{1}{t}, \delta]$. Let $\overline{\mathfrak{A}}$ be the annihilator of y in $\overline{\mathcal{R}}$. For $1 \le p \le N$ let $l_p(s) = \sum_{j=1}^n A_{p,j} s_j$. We compute

$$\begin{aligned} \frac{C(s+\mathbf{1}_j)}{C(s)} &= \prod_{p=1}^{N} \big(\alpha_p + l_p(s)\big)_{A_{p,j}} \\ &= \prod_{A_{p,j} \ge 0} \big(\alpha_p + l_p(s)\big)_{A_{p,j}} \Big/ \prod_{A_{p,j} < 0} \big(\alpha_p + l_p(s)\big)^{-1}_{A_{p,j}}. \end{aligned}$$

We put

$$h_j(s) = \prod_{A_{p,j} \ge 0} \big(\alpha_p + l_p(s)\big)_{A_{p,j}} \tag{5.5}$$

$$k_j(s) = \prod_{A_{p,j} < 0} \Big(\big(\alpha_p + l_p(s)\big)_{A_{p,j}}\Big)^{-1} \tag{5.6}$$

and set

$$\theta_j = k_j(\delta) \circ t_j^{-1} - h_j(\delta) \in \overline{\mathcal{R}}. \tag{5.7}$$

It is clear that $\theta_j \in \overline{\mathfrak{A}}$ $(1 \le j \le n)$. Let $\overline{\mathfrak{B}} = \sum_{j=1}^n \overline{\mathcal{R}} \theta_j$.

Let $\mathcal{M}$ be the multiplicative subgroup generated by

$$\{\alpha_p + \ell_p(\delta) + u_p \mid \alpha_p \ne 0,\ u_p \in \mathbb{Z}\}.$$

We denote $(\overline{\mathfrak{B}} : \mathcal{M}) = \{\theta \in \overline{\mathcal{R}} \mid \mathfrak{m}\theta \in \overline{\mathfrak{B}} \text{ for some } \mathfrak{m} \in \mathcal{M}\}$.

5.7.1. Note that if $\alpha_p \notin \mathbb{Z}$ then $\alpha_p + l_p(\delta)$ has trivial kernel in $\Omega\langle t_1, \dots, t_n\rangle$, the space of formal Laurent series in $t_1, \dots, t_n$. It follows that the same hold for each element of $\mathcal{M}$.

PROPOSITION 5.8.

$$\overline{\mathfrak{A}} = (\overline{\mathfrak{B}} : \mathcal{M}).$$

PROOF. Certainly $\overline{\mathfrak{B}} \subset \overline{\mathfrak{A}}$. We assert that $(\overline{\mathfrak{B}} : \mathcal{M}) \subset \overline{\mathfrak{A}}$. Let $\theta \in (\overline{\mathfrak{B}} : \mathcal{M})$, hence there exists $\mathfrak{m} \in \mathcal{M}$ such that $\mathfrak{m}\theta y = 0$. Since $\theta y \in \Omega\langle t_1, \dots, t_n\rangle$, it follows from 5.7.1 that $\theta y = 0$ and hence $\theta \in \overline{\mathfrak{A}}$.

To reverse the inclusion we first show that if $\theta \in \overline{\mathcal{R}}$ then there exists $\mathfrak{m} \in \mathcal{M}$, $s \in \mathbb{N}^n$, $b \in \overline{\mathfrak{B}}$, $\nu \in \Omega[\delta]$ such that $\mathfrak{m}\theta = b + \frac{1}{t^s}\nu$. Indeed we may choose s so that $t^s\theta \in \Omega[t, \delta]$. For each $u \in \mathbb{N}^s$

$$\begin{aligned} h_1(\delta - \mathbf{1}_1 - u) \circ t^u t_1 &= t^u h_1(\delta - \mathbf{1}_1) \circ t_1 \\ &= t^u\big(-t_1\theta_1 + k_1(\delta - \mathbf{1}_1)\big) \equiv k_1(\delta - \mathbf{1}_1 - u) \circ t^u \mod \overline{\mathfrak{B}} \end{aligned}$$

and $h_1(\delta - \mathbf{1}_1 - u) \in \mathcal{M}$. It follows by induction on the degree of $t^s\theta$ as polynomial in t that $\mathfrak{m}, b, \nu$ exist as asserted.

If now $\theta \in \overline{\mathfrak{A}}$ then $\nu y = 0$ and so by 5.4 $\nu = 0$. Thus $\mathfrak{m} \in \overline{\mathfrak{B}}$ as asserted.

§6. Pullback to N-space

We use the notation of §5. Let $m + n = N$. Let $\mathfrak{G}^{(N)} = \Omega\langle T_1, \dots, T_N\rangle$ be the space of formal Laurent series in N variables with coefficients in Ω. For $1 \le j \le n$ let $\tau_j = \prod_p(-T_p)^{-A_{p,j}}$. Let $\widetilde{T} = (T_{n+1}, \dots, T_N)$. There exists a 1-1 correspondence π between $\mathbb{Z}^N$ and $\mathbb{Z}^n \times \mathbb{Z}^{\mathrm{m}}$

$$\pi(w) = \big(\pi_1(w), \pi_2(w)\big)$$

such that for $w \in \mathbb{Z}^N$

$$T^w = \pm\tau^{\pi_1(w)}\widetilde{T}^{\pi_2(w)}.$$

Let $\Delta_p = T_p\frac{\partial}{\partial T_p}$, a differential operator on $\mathfrak{G}^{(N)}$.

PROPOSITION 6.1. *$\mathfrak{G}_0^{(N)} = \Omega\langle\tau_1, \dots, \tau_n\rangle$ is a subspace of $\mathfrak{G}^{(N)}$.*

$\mathfrak{G}^{(N)}$ may be identified with $\mathfrak{G}_0^{(N)}\langle T_{n+1}, \dots, T_N\rangle = \{\sum_{v\in\mathbb{Z}^m} \xi_v\widetilde{T}^v \mid \xi_v \in \mathfrak{G}_0^{(N)}\}$ and in particular if $\sum_{v\in\mathbb{Z}^n} \xi_v\widetilde{T}^v = 0$ as element of $\mathfrak{G}^{(N)}$ then each $\xi_v = 0$.

PROOF. If $\xi = \sum k_w T^w \in \mathfrak{G}^{(N)}$ then $\xi = \sum_{v\in\mathbb{Z}^m} \xi_v\widetilde{T}^v$ where $\xi_v = \sum_{u\in\mathbb{Z}^n} \pm k_{\pi^{-1}(u,v)}\tau^u$.

Conversely if $\xi_v = \sum_{u\in\mathbb{Z}^n} H_{u,v}\tau^u$, then $\sum \xi_v\widetilde{T}^v = \sum_{u,v} H_{u,v}T^{\pi^{-1}(u,v)} = \sum_{w\in\mathbb{Z}^N} \pm H_{\pi(w)}T^w$. Again if $\sum \xi_v\widetilde{T}^v = 0$ as element of $\mathfrak{G}^{(N)}$ then $H_{\pi(w)} = 0$ for all $w \in \mathbb{Z}^N$ and hence $H_{u,v} = 0$ for all u, v. This completes the proof.

Let χ be the mapping of $\mathfrak{G}^{(n)} = \Omega\langle t_1, \dots, t_n\rangle$ into $\mathfrak{G}^{(N)}$ defined by $\chi(t^u) = \tau^u$ for all $u \in \mathbb{Z}^n$.

Let A be the $N \times m$ matrix defined by

$$\begin{pmatrix} \ell_1(s) \\ \vdots \\ \ell_N(s) \end{pmatrix} = A \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}.$$

We write the matrix A in the form

(6.1.1) $$A = \begin{pmatrix} -I_n \\ A' \end{pmatrix}$$

where A' is a $m \times n$ matrix. Let

(6.1.2) $$B = (A', I_m)$$

an $m \times N$ matrix. The rows of B span the space of N-tuples, w, such that $wA = 0$.

For $1 \leq i \leq m$ let $\rho_i = \sum_{p=1}^{N} B_{ip}\Delta_p$, an element of $\widetilde{\mathcal{R}} = \Omega[T, \frac{1}{T}, \Delta]$, (where $\Delta = (\Delta_1, \dots, \Delta_N)$).

Let $\mathfrak{C}$ be the set of all $\eta \in \Omega[T, \frac{1}{T}, \Delta]$ which annihilate $\mathfrak{G}_0^{(N)}$, the image under χ of $\mathfrak{G}^{(n)}$.

PROPOSITION 6.2.

$$\mathfrak{C} = \sum_{C=1}^{m} \widetilde{\mathcal{R}}\rho_i.$$

PROOF.

(6.2.1) $$-\begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_N \end{pmatrix} \circ \chi = \chi \circ A \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix}.$$

Hence $-\rho_i \circ \chi = -\sum_{p=1}^{N} B_{i,p}\Delta_p \circ \chi = \chi \circ (B_{11}, \dots, B_{1,N})A \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix} = 0$. Thus ρ_i annihilates $\mathfrak{G}_0^{(N)}$.

Conversely letting $\mathfrak{C}_0 = \sum_{i=1}^{m} \widetilde{\mathcal{R}}\rho_i$ we know that

(6.2.2) $$\widetilde{\mathcal{R}} = \Omega[T, \frac{1}{T}, \Delta_1, \dots, \Delta_n] + \mathfrak{C}_0.$$

If $\eta \in \mathfrak{C}$ then we may assume that (after reduction modulo $\mathfrak{C}_0$)

(6.2.3) $$\eta = \sum_{v \in \mathbb{Z}^m} \widetilde{T}^v P_v(\tau, \Delta_1, \dots, \Delta_n),$$

a finite sum over v and where each $P_v \in \Omega[\tau, \frac{1}{\tau}, \Delta_1, \dots, \Delta_n]$. For $j \leq n$, $\Delta_j \circ \chi = \chi \circ \delta_j$ and hence Δ_j is stable on $\mathfrak{G}_0^{(N)}$. The same then holds for $P_v(\tau, \Delta_1, \dots, \Delta_n)$. Thus if $\xi \in \mathfrak{G}_0^{(N)}$ we have

$$0 = \eta\xi = \sum_{v \in \mathbb{Z}^m} \widetilde{T}^v(P_v(\tau, \Delta_1, \dots, \Delta_n)\xi)$$

and so by Proposition 6.1, P_v annihilates ξ.

Thus we may assume that $\eta \in \Omega[\tau, \frac{1}{\tau}, \Delta_1, \dots, \Delta_n]$. Let us write $\eta = \sum_{u \in \mathbb{N}^n} Q_u(\tau)\Delta_1^{u_1} \cdots \Delta_n^{u_n}$, a finite sum where each $\mathbb{Q}_u$ lies in $\Omega[\tau, \frac{1}{\tau}]$. For $\xi \in \mathfrak{G}^{(n)}$ we have

$$0 = \eta\chi(\xi) = \chi\Big(\sum_{u \in \mathbb{N}^n} Q_u(t)\delta^u \xi\Big).$$

Since χ is injective, we conclude that $\sum_{u \in \mathbb{N}^n} Q_u(t)\delta^u$ is an elenent of $\overline{\mathcal{R}}$ which annihilates $\mathfrak{G}^{(n)}$. We conclude that $Q_u(t) = 0$ for all $u \in \mathbb{N}^n$. Thus η is identically zero. This completes the proof.

COROLLARY 6.3. *For each $\eta \in \overline{\mathcal{R}}$ there exists $\eta^* \in \widetilde{\mathcal{R}}$ such that*

$$\chi \circ \eta = \eta^* \circ \chi.$$

PROOF. As a ring $\overline{\mathcal{R}}$ is generated by the operation of multiplication by $t_j^{\pm 1}$ and the operation δ_j. We may choose $t_j^* = \tau_j$, $\delta_j^* = \Delta_j$ $(1 \leq j \leq n)$.

REMARK. η^* is unique mod $\mathfrak{C}$.

Let $\overline{\mathfrak{A}}$ be the annihilator of y in $\overline{R}$. Let

$$\mathfrak{A}^* = \mathfrak{C} + \sum_{\eta \in \overline{\mathfrak{A}}} \widetilde{\mathcal{R}}\eta^*$$

an ideal of $\widetilde{\mathcal{R}}$.

COROLLARY 6.4. *$\mathfrak{A}^*$ is the annihilator of $\chi(y)$ in $\widetilde{\mathcal{R}}$.*

PROOF. Trivially $\mathfrak{A}^*$ lies in the annihilator. To show the converse let η lie in the annihilator. Reducing modulo $\mathfrak{C}$ we may assume that η is of the form (6.2.3). Following the proof of Proposition 6.2, (with $\xi = \chi(y)$) we conclude that we may assume $\eta \in \Omega[\tau, \frac{1}{\tau}, \Delta_1, \dots, \Delta_n]$ and writing $\eta = \sum_{u \in \mathbb{N}^n} \mathbb{Q}_u(\tau)\Delta_1^{u_1} \cdots \Delta_n^{u_n}$ we deduce

$$0 = \chi\Big(\sum_{u \in \mathbb{N}^n} \mathbb{Q}_u(t)\delta^u y\Big).$$

Thus $\sum_{u\in\mathbb{N}^n} \mathbb{Q}_u(t)\delta^u \in \overline{\mathfrak{A}}$ and η lies in $\mathfrak{A}^*$.

PROPOSITION 6.5. *Let* $\omega_j = \prod_{A_{p,j}>0}(-T_p)^{-A_{p,j}}$. *Then*

$$\omega_j\theta_j^* = \prod_{A_{p,j}<0} ((\alpha_p - \Delta_p)_{A_{p,j}})^{-1}(-T_p)^{A_{p,j}} - \prod_{A_{p,j}>0} (-T_p)^{-A_{p,j}}(\alpha_p - \Delta_p)_{A_{p,j}}.$$

PROOF.

$$\theta_j = \prod_{A_{p,j}<0} ((\alpha_p + \ell_p(\delta))_{A_{p,j}})^{-1}\, t_j^{-1} - \prod_{A_{p,j}>0} (\alpha_p + \ell_p(\delta))_{A_{p,j}}$$

while $(\ell_p(\delta))^* = -\Delta_p$. Hence

$$\theta_j^* = \prod_{A_{p,j}<0} ((\alpha_p - \Delta_p)_{A_{p,j}})^{-1}\, \tau_j^{-1} - \prod_{A_{p,j}>0} (\alpha_p - \Delta_p)_{A_{p,j}}.$$

The assertion now follows by multiplication by ω_j.

Let $\widetilde{\mathcal{M}}$ be the multiplicative semigroup in $\widetilde{\mathcal{R}}$ generated by $\{\alpha_p - \Delta_p + u_p \mid \alpha_p \neq 0,\ u_p \in \mathbb{Z}\}$.

By the same arguments as in 5.7.1, each element of $\widetilde{\mathcal{M}}$ has trivial kernel in $\Omega\langle T_1, \dots, T_N\rangle$.

LEMMA 6.6. *Let* $\widetilde{\mathfrak{A}} = \sum_{j=1}^n \widetilde{\mathcal{R}}\theta_j^* + \sum_{i=1}^m \widetilde{\mathcal{R}}\rho_i$. *The annihilator of* $\chi(y)$ *in* $\widetilde{\mathcal{R}}$ *is* $(\widetilde{\mathfrak{A}} : \widetilde{\mathcal{M}})$.

PROOF. Certainly $\widetilde{\mathfrak{A}} \subset$ annihilator of $\chi(y)$. Let $\mathfrak{m} \in \widetilde{\mathcal{M}}$, $\theta \in \widetilde{\mathcal{R}}$ and suppose $\mathfrak{m}\theta \in \widetilde{\mathfrak{A}}$. Thus $\mathfrak{m}\theta\chi(y) = 0$ while $\theta\chi(y) \in \Omega\langle T_1, \dots, T_N\rangle$ and by the above argument $\theta\chi(y) = 0$, i.e. $(\widetilde{\mathfrak{A}} : \widetilde{\mathcal{M}})$ lies in the annihilator of $\chi(y)$.

Now let η^* lie in the annihilator. Reducing modulo $\mathfrak{C}$, η^* has the form (6.2.3) and proceeding as in the proof of Proposition 6.2 we may further reduce to the case in which $\eta^* \circ \chi = \chi \circ \eta$ where η lies in the annihilator of y. Thus by Proposition 5.8 there exists $\mathfrak{m} \in \mathcal{M}$ such that $\mathfrak{m}\eta \in \sum \overline{\mathcal{R}}\theta_j$ and so

$$\mathfrak{m}^*\eta^* \in \sum \widetilde{\mathcal{R}}\theta_j^* + \mathfrak{C}.$$

Recall that since $\mathfrak{m} \in \mathcal{M}$, $\mathfrak{m}$ is a product of elements $(\alpha_p + \ell_p(\delta) + u_p)$ where $\alpha_p \neq 0$, $u_p \in \mathbb{Z}$. Thus we may take $\mathfrak{m}^*$ to be a product of elements $(\alpha_p - \Delta_p + u_p)$, i.e. $\mathfrak{m}^* \in \widetilde{\mathcal{M}}$. Thus $\eta^* \subset (\widetilde{\mathfrak{A}} : \widetilde{\mathcal{M}})$ which completes the proof.

LEMMA 6.7. *Let* $\mathcal{M}_0$ *be the semigroup generated by* $\{\Delta_p + u_p \mid \alpha_p \neq 0,\ u_p \in \mathbb{Z}\}$.

Let

$$(6.7.1) \qquad a = B\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} \alpha_{n+1} \\ \vdots \\ \alpha_N \end{pmatrix}$$

$$(6.7.2) \qquad L_i = \rho_i + a_i = \sum_{p=1}^{N} B_{i,p}\Delta_p + a_i \quad (1 \le i \le m)$$

$$(6.7.3) \qquad \Theta_j = \prod_{A_{p,j}<0} \left(\frac{\partial}{\partial T_p}\right)^{-A_{p,j}} - \prod_{a_{p,j}>0} \left(\frac{\partial}{\partial T_p}\right)^{A_{p,j}}.$$

Let

$$\mathfrak{A}' = \sum_{j=1}^{n} \widetilde{\mathcal{R}}\Theta_j + \sum_{i=1}^{m} \widetilde{\mathcal{R}}L_i.$$

The annihilator of $z = T^{-\alpha}\chi(y)$ *in* $\widetilde{\mathcal{R}}$ *is* $(\mathfrak{A}' : \mathcal{M}_0)$.

PROOF. The annihilator of $z = T^{-\alpha}\circ$ annihilator of $\chi(y) \circ T^{\alpha}$

$$= (T^{-\alpha} \circ \widetilde{\mathfrak{A}} \circ T^{\alpha} : T^{-\alpha}\widetilde{\mathcal{M}} \circ T^{\alpha}).$$

Thus it is enough to check that

$$T^{-\alpha} \circ \rho_i \circ T^{\alpha} = \rho_i + a_i = L_i$$

$$T^{-\alpha} \circ \omega_j\theta_j^* \circ T^{\alpha} = \Theta_j$$

$$T^{-\alpha} \circ \widetilde{\mathcal{M}} \circ T^{\alpha} = \mathcal{M}_0.$$

This completes the proof.

We now return to the space $\widehat{\mathcal{W}}_a$ discussed in §2 but here we put

$$(6.7.4) \qquad g(X_1, \dots, X_m) = T_1X^{A'^{(1)}} + \dots + T_nX^{A'^{(n)}} + T_{n+1}X_1 + \dots + T_NX_m$$

where $A'^{(j)}$ is the j^{th} column of A'.

Let $\mathcal{R} = \Omega[T, \frac{1}{T}, \sigma_{T_1}, \dots, \sigma_{T_N}]$ viewed as operators on $\widehat{\mathcal{W}}_a$, with $\sigma_{T_i} = \frac{\partial}{\partial T_i} + \frac{\partial g}{\partial T_i}$.

THEOREM 6.8. *The mapping* $\varphi : \widetilde{\mathcal{R}} \to \mathcal{R}$ *defined by* $\varphi(\frac{\partial}{\partial T_i}) = \sigma_{T_i}$ *is an isomorphism mapping the annihilator of* z *onto* $\mathfrak{A}$, *the annihilator of* [1].

PROOF.

$$\varphi(L_i) = a_i + \sum_{p=1}^{N} B_{ip}T_p\sigma_{T_p}$$

and $B_{i,p}$ is the exponent of X_i in the monomial multipled by T_p in g ($B_{i,p} = 0$ if $p > n$, $p \neq n+i$, $B_{i,p} = 1$ if $p = n+i$, $B_{i,p} = A'_{i,p}$ if $p \leq n$). Thus $\varphi(L_i) = Z_i$ in the terminology of §2. The set S of §2 now consists of the columns of B.

The relations among the columns of $B = (A', I_m)$ are generated by the columns of A. Corresponding to $A^{(j)}$, the j^{th} column of A, we have (cf. §2)

$$\square_{A^{(j)}} = \prod_{A_{p,j}<0} \sigma_{T_p}^{-A_{p,j}} - \prod_{A_{p,j}>0} \sigma_{T_p}^{A_{p,j}}$$

and this is precisely $\varphi(\Theta_j)$.

This shows that φ maps $\mathfrak{A}'$ onto $\mathfrak{B}$ in the notation of §2.

If $\theta z = 0$, $\theta \in \widetilde{\mathcal{R}}$ then there exists $\mathfrak{m} \in \mathcal{M}_0$ such that $\mathfrak{m}\theta \in \mathfrak{A}'$. Thus $\varphi(\mathfrak{m})\varphi(\theta)[1] = 0$. We assert that $\varphi(\theta) \in \mathfrak{A}$. We use induction on the degree of $\mathfrak{m}$ as polynomial in Δ. We may assume $\mathfrak{m} = (\Delta_p + u_p)\mathfrak{m}_1$ with $\mathfrak{m}_1 \in \mathcal{M}_0$, $u_p \in \mathbb{Z}$. Thus $(T_p\sigma_p + u_p)\varphi(\mathfrak{m}_1)\varphi(\theta)[1] = 0$ and so

$$(T_p\sigma_p)T_p^{u_p}\varphi(\mathfrak{m}_1)\varphi(\theta)[1] = 0.$$

By Corollary 3.4, $\varphi(\mathfrak{m}_1)\varphi(\theta)[1] = 0$. Hence by induction $\varphi(\theta)[1] = 0$. This shows that φ maps $(\mathfrak{A}' : M_0)$ into $\mathfrak{A}$. By Theorem 4 we conclude

$$\varphi(\mathfrak{A}') = \mathfrak{A} = \varphi((\mathfrak{A}' : \mathcal{M}_0)).$$

This completes the proof.

Since φ is an isomorphism we deduce two consequences.

COROLLARY 6.9. *The annihilator of* $z, (\mathfrak{A}' : \mathcal{M}_0) = \mathfrak{A}'$.
The annihilator of $\chi(y), \mathfrak{A}^* = (\widetilde{\mathfrak{A}} : \widetilde{\mathcal{M}}) = \widetilde{\mathfrak{A}}$.

REMARK. Hypergeometric series are solutions of the hypergeometric D-modules defined in [L.S.2]. In fact the inverse mapping by χ of such hypergeometric D-modules are up to a twisting exactly the D-modules of [G.K.Z.] which seem to be defined only in the case of regular singularities. In §8 we shall remove the necessity of both the transformation χ and twisting and identify hypergeometric D-modules with the D-modules of [GHF].

§7. Exponential modules

Let $\widehat{H}_0, H_0, \mathcal{S}, g$ be as in §2 but we now drop the requirement that the λ_u be algebraically independent over $\mathbb{Q}$. We may still view $\widehat{R}$ as a $\mathbb{Q}(a, \lambda)$ space. Let $\lambda_1, \dots, \lambda_q$ be a transcendence basis for $\mathbb{Q}(\lambda)$ over $\mathbb{Q}$.

LEMMA 7.1. *The mapping* $\xi \mapsto X^a\xi \exp g(x)$ *of* $\widehat{R}$ *into* $\widehat{R}X^a \exp g(x)$ *induces an isomorphism,*

$$\widehat{\mathcal{W}}_a \cong \widehat{R} X^a \exp g(x) \Big/ \sum_{i=1}^{n} E_i\big(\widehat{R} X^a \exp g(X)\big)$$

as differential modules over $\mathbb{Q}(a, \lambda)$, *the operator* $\sigma_j = \frac{\partial}{\partial \lambda_j} + \frac{\partial g}{\partial \lambda_j}$ *on* $\widehat{\mathcal{W}}_a$ *corresponding to* $\frac{\partial}{\partial \lambda_j}$ *on the right side.*

PROOF. Obvious.

We now remove the twist.

COROLLARY 7.2. *With* g *as in* 6.7.4, $\widetilde{\mathcal{R}} = \Omega[T, \frac{1}{T}, \Delta]$ *operates on*

$$T^\alpha \widehat{R} X^a \exp g \Big/ \sum_{i=1}^{m} E_i(\widehat{R} T^\alpha X^a \exp g)$$

and the annihilator in $\widetilde{\mathcal{R}}$ *of the class of* $T^\alpha X^a \exp g$ *is precisely the same as the annihilator* $\widetilde{\mathfrak{A}}$ *of* $\chi(y)$.

PROOF. By Theorem 6.8 and the lemma, the annihilator in $\widetilde{\mathcal{R}}$ of $[X^a \exp g]$ coincides with that of z. Twisting by T^α, we deduce that the annihilator of $[T^\alpha x^a \cdot \exp g]$ coincides with that of $\chi(y)$.

§8. Normalization (Removal of pullback)

Let $\widetilde{g} \in \mathbb{Q}[t, z]$, $t = (t_1, \dots, t_n)$, $z = (z_1, \dots, z_m)$, $m = N - n$ be defined by

$$-\widetilde{g}(t, z) = z_1 + \cdots + z_m + \sum_{j=1}^{n} t_j z^{A'^{(j)}},$$

the columns $A'^{(j)}$ $(1 \le j \le n)$ being precisely as in §5. So here $\mathcal{S}$ is precisely as in the proof of Theorem 6.8. Let $\widetilde{R}$ be the ring in $\mathbb{Q}(\alpha, t)[z, \frac{1}{z}]$ generated by the monomials z^u, $u \in \mathcal{S}$. Let $\widetilde{\mathcal{W}}_a$ be the $\mathbb{Q}(\alpha, t)$ space defined precisely as in §2 with a, α related as in (6.7.1). Then $\widetilde{\mathcal{W}}_a$ is a differential module with operators $\widetilde{\sigma}_j = \frac{\partial}{\partial t_j} + \frac{\partial g}{\partial t_j}$. To avoid confusion we shall use $\widetilde{E}_i$ to denote $z_i \frac{\partial}{\partial z_i}$ as an operator on $\widetilde{R}$.

We recall y in §5 and its annihilator $\overline{\mathfrak{A}}$ in $\overline{\mathcal{R}} = \Omega[t, \frac{1}{t}, \delta]$.

Let φ be the isomorphism of $\overline{\mathcal{R}}$ with $\mathcal{R}_1 = \Omega[t, \frac{1}{t}, \widetilde{\sigma}]$ defined by $\varphi(\delta_j) = t_j \widetilde{\sigma}_j$ $(1 \le j \le n)$.

THEOREM. *The image of* $\overline{\mathfrak{A}}$ *under* φ *is precisely the annihilator in* $\mathcal{R}_1$ *of* $[1]$, *the class of* 1 *in* $\widetilde{\mathcal{W}}_a$.

PROOF. Using the natural isomorphism of §7 it is enough to show that $\overline{\mathfrak{A}} = \overline{\mathfrak{A}}_1$, the annihilator in $\overline{\mathcal{R}}$ of the class of $z^a \exp\widetilde{g}$ in $\widetilde{V}_a \stackrel{def}{=} z^a \widetilde{R} \exp\widetilde{g} / \sum_{i=1}^m \widetilde{E}_i(z^a \widetilde{R} \exp\widetilde{g})$.

Let $\widehat{R}$ be the ring over $\Omega(T)$ generated by the monomials of g in (6.7.4) and let

$$\widehat{V}_a = T^\alpha X^a \widehat{R} \exp g \Big/ \sum_{i=1}^m E_i(T^\alpha X^a \widehat{R} \exp g)$$

where $E_i = X_i \frac{\partial}{\partial X_i}$. This is an $\Omega(T)$ space on which $\widetilde{\mathcal{R}} = \Omega[T, \frac{1}{T}, \Delta_1, \dots, \Delta_N]$ operates. (The class of 1 in $\widetilde{\mathcal{W}}_a$ corresponds to the class of $T^\alpha X^a \exp g$ and so involves a twist by T^α.)

We use the map χ of

$$z^a \exp\widetilde{g}\widetilde{\mathcal{R}} \text{ into } T^\alpha X^a \exp g\widehat{R}$$

defined by

$$\chi(z_i) = -X_i T_{i+n} \text{ for } 1 \le i \le m$$

$$\chi(t_j) = \tau_j = \prod_{p=1}^N (-T_p)^{-A_{p,j}} \text{ for } 1 \le j \le n.$$

We have the relations

$$E_i \circ \chi = \chi \circ \widetilde{E}_i \text{ for } 1 \le i \le m,$$

$$\Delta_p \circ \chi = \chi \circ \widetilde{E}_{p-n} - \chi \circ \ell_p(\delta) \text{ for } 1 \le p \le N,\ \widetilde{E}_{p-n} = 0 \text{ if } p < n.$$

Thus classes of $\widetilde{V}_a$ are mapped into classes of $\widehat{V}_a$ and Δ_p is stable on the image of $\widetilde{V}_a$ in $\widehat{V}_a$. In particular χ induces a mapping of $\widetilde{V}_a$ into an $\Omega(\tau)$ linear submodule of $\widehat{V}_a$.

We sketch the remainder of the proof. By the method of proof of Proposition 6.1, χ is injective. Our relations show that $\sum_{p=1}^N B_{i,p}\Delta_p$ lies in the annihilator of $\chi(\widetilde{V}_a)$. By the proof of Proposition 6.2, the annihilator of the image is again $\mathfrak{C}_0$. By the proof of Corollary 6.5 the annihilator of $[T^\alpha X^a \exp g]$ in $\widetilde{\mathcal{R}}$ is precisely $\mathfrak{C} + \sum_{\eta \in \mathfrak{A}_1} \widetilde{\mathcal{R}}\eta^*$, the sum being over all $\eta \in \overline{\mathfrak{A}}_1$, the annihilator in $\overline{R}$ of $[z^a \exp\widetilde{g}]$. But we have shown (Corollary 7.2) that the annihilator in $\widetilde{\mathcal{R}}$ of $[T^\alpha X^a \exp g]$ is $\widetilde{\mathfrak{A}}$ which by Corollary 6.9 coincides with $\mathfrak{A}^* = \mathfrak{C} + \sum_{\eta \in \overline{\mathfrak{A}}} \widetilde{\mathcal{R}}\eta^*$. This shows that $\mathfrak{C} + \sum_{\eta \in \overline{\mathfrak{A}}} \widetilde{\mathcal{R}}\eta^* = \mathfrak{C} + \sum_{\eta \in \overline{\mathfrak{A}}_1} \widetilde{\mathcal{R}}\eta^*$. It follows from the injectivity of χ that $\overline{\mathfrak{A}} = \overline{\mathfrak{A}}_1$.

§9. Laplace isomorphism

Let $R' = K[X_1, \dots, X_m, \frac{1}{X_1 \cdots X_m}]$ where K is a field containing $\mathbb{Q}(a)$. We assume that

$$g = X_m f^{(m)} + \cdots + X_\ell f^{(\ell)} + f^{(\ell-1)} \tag{9.0.1}$$

where $f^{(j)}(m \geq j \geq \ell - 1)$ lies in $K[X_1, \dots, X_{\ell-1}, \frac{1}{X_1 X_2 \cdots X_{\ell-1}}]$ (and so is independent of $X_l, \dots, X_m$). We assume that $a_m, a_{m-1}, \dots, a_\ell$ are each outside $-\mathbb{Z}$ (If $a_p \in \mathbb{Z}$, $p \geq l$, then we must insist that $a_p \geq 1$ and must remove from R' all monomials X^u with $u_p < 0$). We make no hypothesis concerning $a_1, \dots, a_{\ell-1}$.

For $m \geq p \geq \ell - 1$ we put $g^{(p)} = f^{(\ell-1)} + \sum_{j=\ell}^{p} X_j f^{(j)}$ (and so in particular, $g^{(\ell-1)} = f^{(\ell-1)}$).

We define $\theta_p = \frac{X_1^{a_1} \cdots X_p^{a_p}}{f^{(m)a_m} \cdots f^{(p+1)a_{p+1}}} \exp g^{(p)}$.

We define R_p to be the K span of all expressions $X_1^{u_1} \cdots X_p^{u_p} / f^{(m)^{u_m}} \cdots f^{(p+1)^{u_{p+1}}}$ for all $(u_1, \dots, u_m) \in \mathbb{Z}^m$. (Thus $R_m = R'$.)

We define T_p to be the mapping $\theta_p R_p \to \theta_{p-1} R_{p-1}$ defined by linearity and the condition

$$T_p \theta_p \frac{X_1^{u_1} \cdots X_p^{u_p}}{f^{(m)u_m} \cdots f^{(p+1)u_{p+1}}} \to \theta_{p-1} \frac{X_1^{u_1} \cdots X_{p-1}^{u_{p-1}}}{f^{(m)u_m} \cdots f^{(p+1)u_{p+1}} f^{(p)u_p}} (a_p)_{u_p} / (-1)^{u_p}.$$

Thus X_p does not appear in the image.

LEMMA 9.1 (cf. [GHF, §10.2]). *For $p \geq \ell$, T_p maps $\theta_p R_p$ onto $\theta_{p-1} R_{p-1}$. For $1 \leq i \leq p$ the diagram*

$$\begin{array}{ccc} \theta_p R_p & \xrightarrow{T_p} & \theta_{p-1} R_{p-1} \\ {\scriptstyle E_i}\downarrow & & \downarrow{\scriptstyle E_i} \\ \theta_p R_p & \xrightarrow[T_p]{} & \theta_{p-1} R_{p-1} \end{array}$$

commutes. In particular $E_p(\theta_p R_p) \subset \ker(T_p, \theta_p R_p)$.

THEOREM 9.2 (cf. [GHF Lemma 11.1.1]). *For $p \geq \ell$, T_p induces an isomorphism*

$$\theta_p R_p \Big/ \sum_{i=1}^{p} E_i(\theta_p R_p) \simeq \theta_{p-1} R_{p-1} \Big/ \sum_{i=1}^{p-1} E_i(\theta_{p-1} R_{p-1}).$$

PROOF. Let $\eta \in R_p$ and suppose $T_p(\eta \theta_p) = \sum_{i=1}^{p-1} E_i(\eta_i \theta_{p-1})$, where $\eta_i \in R_{p-1}$ $(1 \leq i \leq p-1)$. We assert that

$$\eta \theta_p \in E_p(\theta_p R_p) + \sum_{j=1}^{p-1} E_j(\theta_p R_p). \tag{9.2.1}$$

By surjectivity there exist $\xi_i, \dots, \xi_{p-1} \in R_p$ such that

$$T_p \theta_p \xi_i = \eta_i \theta_{p-1} \quad 1 \le i \le p-1$$

and hence

$$T_p \sum_{i=1}^{p-1} E_i(\theta_p \xi_i) = \sum_{i=1}^{p-1} E_i(\eta_i \theta_{p-1}),$$

which shows that

$$T_p(\eta\theta_p - \sum_{i=1}^{p-1} E_i(\theta_p \xi_i)) = 0.$$

Thus we may assume $\eta \in R_p$,

$$T_p(\eta\theta_p) = 0. \tag{9.2.3}$$

We assert that

$$\eta\theta_p \in E_p(R_p\theta_p). \tag{9.2.4}$$

We write

$$\eta = \sum C_u \frac{X_1^{u_1} \cdots X_p^{u_p}}{f^{(m)u_m} \cdots f^{(p+1)u_{p+1}}}. \tag{9.2.5}$$

Let $S = \{u_p \mid C_u \neq 0$ for some choice of $u_1, \dots, u_{p-1}\}$. Let $M = \sup_{u_p \in S} u_p$, $M' = \inf_{u_p \in S} u_p$.

We use induction on $M - M'$. By (9.2.3)

$$0 = \sum_{u \in S} C_u \frac{X_1^{u_1} \cdots X_{p-1}^{u_{p-1}}}{f^{(m)u_m} \cdots f^{(p)u_p}} \epsilon(u_p)$$

where $\epsilon(u_p) = (a_p)_{u_p}(-1)^{u_p}$.

Thus

$$\begin{aligned} &\epsilon(M) \sum_{u_p = M} \frac{C_u X_1^{u_1} \cdots X_{p-1}^{u_{p-1}}}{f^{(m)u_m} \cdots f^{(p+1)u_{p+1}}} \\ &= -f^{(p)M} \sum_{u_p \le M-1} C_u \epsilon(u_p) \frac{X_1^{u_1} \cdots X_{p-1}^{u_{p-1}}}{f^{(m)u_m} \cdots f^{(p)u_p}}. \end{aligned} \tag{9.2.6}$$

Let

$$\xi_1 = \sum_{u_p \le M-1} C_u \frac{X_1^{u_1} \cdots X_p^{u_p}}{f^{(m)u_m} \cdots f^{(p+1)u_{p+1}}}$$

$$\xi_0 = \sum_{u_p = M} C_u \frac{X_1^{u_1} \cdots X_{p-1}^{u_{p-1}}}{f^{(m)u_m} \cdots f^{(p+1)u_{p+1}}}.$$

Thus

$$\eta = \xi_1 + X_p^M \xi_0.$$

If $M = M'$ then $\xi_1 = 0$ and hence the right side of (9.2.6) is zero and so the same holds for ξ_0, i.e. $\eta = 0$. Thus we may assume $M \neq M'$.

By (9.2.6) we have

$$\xi_0 = f^{(p)} \xi_2$$

where

$$\xi_2 = -f^{(p)(M-1)} \sum_{u_p \leq M-1} C_u \cdot \frac{\epsilon(u_p)}{\epsilon(M)} \frac{X_1^{u_1} \cdots X_{p-1}^{u_{p-1}}}{f^{(m)u_m} \cdots f^{(p)u_p}} \in R_p.$$

Thus

$$\eta = \xi_1 + X_p f^{(p)} X_p^{M-1} \xi_2.$$

Now ξ_2 is independent of X_p and hence

$$E_p(\theta_p X_p^{M-1} \xi_2) = (a_p + M - 1)\theta_p X_p^{M-1} \xi_2 + X_p f^{(p)} X_p^{M-1} \xi_2 \theta_p.$$

Thus

$$\theta_p \eta = \theta_p \xi_1 + E_p(\theta_p X_p^{M-1} \xi_2) - (a_p + M - 1)\theta_p X_p^{M-1} \xi_2.$$

Thus we may replace η by $\xi_1 - (a_p + M - 1) X_p^{M-1} \xi_2$ and the support S' of this element has the property that $\sup_{u \in S'} u_p - \inf_{u_p \in S'} u_p \leq M - M' - 1$. The assertion now follows from the induction hypothesis.

Corollary 9.3.

$$T = T_\ell \circ T_{\ell+1} \circ \cdots \circ T_m$$

induces an isomorphism of $\theta_m R' / \sum_{i=1}^m E_i(\theta_m R')$ *with* $\theta_{\ell-1} R_{\ell-1} / \sum_{i=1}^{\ell-1} E_i(\theta_{\ell-1} R_{\ell-1})$.

We recall (§2), $\widehat{R}$ is defined to be the subring of R' generated by all monomials occuring in g. We define $\widehat{R}_{\ell-1}$ a subring of $R_{\ell-1}$ by the condition

$$T(\widehat{R}\theta_m) = \widehat{R}_{\ell-1}\theta_{\ell-1}.$$

Corollary 9.4. *If*

$$\widehat{R} \cap \sum_{i=1}^m D_{a,i} R' = \sum_{i=1}^m D_{a,i} \widehat{R} \tag{9.4.1}$$

then T induces an isomorphism of

$$\widehat{R}\theta_m \Big/ \sum_{i=1}^{m} E_i(\theta_m \widehat{R}) \quad \textit{with} \quad \theta_{\ell-1}\widehat{R}_{\ell-1} \Big/ \sum_{i=1}^{\ell-1} E_i(\theta_{\ell-1}\widehat{R}_{\ell-1}).$$

PROOF. It is enough to show that if $\eta \in \widehat{R}$ and

$$T(\eta\theta_m) \in \sum_{i=1}^{\ell-1} E_i(\theta_{\ell-1}R_\ell)$$

then $\eta\theta_m \in \sum_{i=1}^{m} E_i(\theta_m\widehat{R}_m)$. By the preceding corollary we know that $\eta\theta_m \in \sum_{i=1}^{m} E_i(\theta_m R_m) = \sum_{i=1}^{m} E_i(\theta_m R')$ and so the assertion follows from (9.4.1).

REMARK 9.4.2. Hypothesis 9.4.1 is known to be valid if a is generic and more generally if no lattice point lies on any face of codimension one passing through the origin of the polyhedron of g.

§10. Integral representation

Integral representation of hypergeometric functions associated with

$$-g = X_1 + \cdots + X_m + \sum_{j=1}^{m} t_j X^{A'(j)}$$

are obtained by application of Corollary 9.3. For many purposes (e.g. p-adic cohomology as in [GHF]) it is enough to consider $\theta_m R' / \sum_{i=1}^{m} E_i(\theta_m R')$, but for comparison with the classical literature (Erdelyi, Humbert) and the recent work of Yoshida and Aomoto and Kita we regroup the terms of g as indicated by (9.0.1). There is no unique way to arrive at this regrouping, particularly since we may introduce multiplicative changes in variable,

$$\begin{pmatrix} \log z_1 \\ \vdots \\ \log z_m \end{pmatrix} = C \begin{pmatrix} \log X_1 \\ \vdots \\ \log X_m \end{pmatrix} \tag{10.1}$$

where $C \in G\ell(m, \mathbb{Z})$. Of course such a change in variables must be accompanied by a corresponding change in the vector $a = (a_1, \ldots, a_m)$.

We illustrate this by working out representations for the Horn list, the Lauricella functions, and ${}_kF_{k-1}$ and a few of the confluent forms of the Lauricella functions.

In the non-confluent case g is homogeneous of degree 1 and hence $g = X_1 h(z)$, $z = (z_2, z_3, \ldots, z_m)$. It follows that in all such cases there is an integral representation of the form $(z_2^{a_2} \cdots z_m^{a_m} / h(z)^{a_1+a_2+\cdots+a_m}) \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_m}{z_m}$.

Appendix

Notes on the Appendix.

We give some details for all the elements of Horn's list, for the four Lauricella series, for some of their confluent variations and for ${}_{\kappa}F_{\kappa-1}$.

The information is in the following form following the name of the function.

1. The general term of the series representation
2. $(n, N, m) =$ (number of variables, number of factors in the general term, the number of X variables)
3. The signs of $t_1, t_2, \dots, t_n$
4. The matrix A' of equation 6.1.1 with a listing of the corresponding $\alpha_{n+1}, \dots, \alpha_{n+m}$
5. $-g$ with the correct signs for $t_1, \dots, t_n$
6. The change in variables used to achieve a more standard integral representation
7. The transformed representation of g
8. X^a in terms of new variables
9. The integral representation.

We give no indication of cycles.

M. Kita has treated all the complete series of Horn as well as the four series of Lauricella and has given appropriate cycles. His differentials usually but not always agree with ours. Kita has also given a general integral formula for all hypergeometric series of non-confluent type.

F_1

1. $$\frac{(\alpha)_{s_1+s_2}(\beta)_{s_1}(\beta')_{s_2}}{(\gamma)_{s_1+s_2}s_1!s_2!}$$

2. $(n, N, m) = (2, 6, 4)$

3. $+t_1, +t_2$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1-\gamma \\ 1 & 1 & \alpha \\ 1 & 0 & \beta \\ 0 & 1 & \beta' \end{array}\right)$$

5. $$-g = X_1 + X_2 + X_3 + X_4 + t_1\frac{X_2X_3}{X_1} + t_2\frac{X_2X_4}{X_1}$$

6. $(X_1, X_2, X_3, X_4) = (X_1, -zX_1, X_3, X_4)$

7. $-g = X_1(1-z) + X_3(1-zt_1) + X_4(1-zt_2)$

8. $X^a = X_1^{1+\alpha-\gamma} z^\alpha X_3^\beta X_4^{\beta'}$

9. $\dfrac{z^\alpha}{(1-z)^{1+\alpha-\gamma}} \dfrac{1}{(1-t_1z)^\beta} \dfrac{1}{(1-t_2z)^{\beta'}} \dfrac{dz}{z}$

6′. $(X_1, X_2, X_3, X_4) = (X_1, X_2, -X_1z_1, -X_1z_2)$

7′. $-g = X_1(1-z_1-z_2) + X_2(1-t_1z_1-t_2z_2)$

8′. $X^a = X_1^{1-\gamma+\beta+\beta'} X_2^\alpha z_1^\beta z_2^{\beta'}$

9′. $\dfrac{z_1^\beta z_2^{\beta'}}{(1-z_1-z_2)^{1-\gamma+\beta+\beta'}} \dfrac{1}{(1-t_1z_2-t_2z_2)^\alpha} \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

10. [E.M.O.T. p. 230, 231]

F_2

1. $\dfrac{(\alpha)_{s_1+s_2}(\beta)_{s_1}(\beta')_{s_2}}{(\gamma)_{s_1}(\gamma')_{s_2} s_1! s_2!}$

2. $(n, N, m) = (2, 7, 4)$

3. $+t_1, +t_2$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\gamma \\ 0 & -1 & 1-\gamma' \\ 1 & 1 & \alpha \\ 1 & 0 & \beta \\ 0 & 1 & \beta' \end{array}\right)$$

5. $-g = X_1 + X_2 + X_3 + X_4 + X_5 + t_1 \dfrac{X_3X_4}{X_1} + t_2 \dfrac{X_3X_5}{X_2}$

6. $(X_1, X_2, X_3, X_4, X_5) = (X_1, X_2, X_3, -z_1X_1, -z_2X_2)$

7. $-g = X_1(1-z_1) + X_2(1-z_2) + X_3(1-t_1z_1-t_2z_2)$

8. $X^a = X_1^{1+\beta-\gamma} X_2^{1+\beta'-\gamma'} X_3^\alpha z_1^\beta z_2^{\beta'}$

9. $$\frac{z_1^{\beta} z_2^{\beta'}}{(1-z_1)^{1+\beta-\gamma}(1-z_2)^{1+\beta'-\gamma'}(1-t_1z_1-t_2z_2)^{\alpha}}\,\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}$$

10. [E.M.O.T. p. 230]

F_3

1. $$\frac{(\alpha)_{s_1}(\alpha')_{s_2}(\beta)_{s_1}(\beta')_{s_2}}{(\gamma)_{s_1+s_2}s_1!s_2!}$$

2. $(n, N, m) = (2, 7, 5)$

3. $+t_1, +t_2$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1-\gamma \\ 1 & 0 & \alpha \\ 0 & 1 & \alpha' \\ 1 & 0 & \beta \\ 0 & 1 & \beta' \end{array}\right)$$

5. $$-g = X_1 + X_2 + X_3 + X_4 + X_5 + t_1\frac{X_2X_4}{X_1} + t_2\frac{X_3X_5}{X_1}$$

6. $(X_1, X_2, X_3, X_4, X_5) = (X_1, X_2, X_3, -X_1z_1, -X_1z_2)$

7. $-g = X_1(1 - z_1, -z_2) + X_2(1 - t_1z_1) + X_3(1 - t_2z_2)$

8. $X^a = X_1^{1+\beta+\beta'-\gamma}X_2^{\alpha}X_3^{\alpha'}z_1^{\beta}z_2^{\beta'}$

9. $$\frac{z_1^{\beta} z_2^{\beta'}}{(1-z_1-z_2)^{1+\beta+\beta'-\gamma}(1-t_1z_1)^{\alpha}(1-t_2z_2)^{\alpha'}}\,\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}$$

10. [E.M.O.T. p. 230]

F_4

1. $$\frac{(\alpha)_{s_1+s_2}(\beta)_{s_1+s_2}}{(\gamma)_{s_1}(\gamma')_{s_2}s_1!s_2!}$$

2. $(n, N, m) = (2, 6, 4)$

3. $+t_1, +t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \begin{pmatrix} -1 & 0 & \vline & 1-\gamma \\ 0 & -1 & \vline & 1-\gamma' \\ 1 & 1 & \vline & \alpha \\ 1 & 1 & \vline & \beta \end{pmatrix}$

5. $-g = X_1 + X_2 + X_3 + X_4 + t_1 \dfrac{X_3 X_4}{X_1} + t_2 \dfrac{X_3 X_4}{X_2}$

6. $(X_1, X_2, X_3, X_4) = (-X_3 z_1, -X_3 z_2, X_3, X_4)$

7. $-g = X_3(1 - z_1 - z_2) + X_4 \left(1 - t_1 \dfrac{1}{z_1} - t_2 \dfrac{1}{z_2}\right)$

8. $X^a = z_1^{1-\gamma} z_2^{1-\gamma'} X_3^{2+\alpha-\gamma_1-\gamma_2} X_4^{\beta}$

9. $\dfrac{z_1^{1-\gamma_1} z_2^{1-\gamma_2}}{(1 - z_1 - z_2)^{2+\alpha-\gamma-\gamma'}} \dfrac{1}{(1 - t_1 \frac{1}{z_1} - t_2 \frac{1}{z_2})^{\beta}} \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

10. [Yo. p. 329], [Pa, p. 120]

G_1

1. $\dfrac{(\alpha)_{s_1+s_2} (\beta)_{s_2-s_1} (\beta')_{s_1-s_2}}{s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. $-t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \begin{pmatrix} 1 & 1 & \vline & \alpha \\ -1 & 1 & \vline & \beta \\ 1 & -1 & \vline & \beta' \end{pmatrix}$

5. $-g = X_1 + X_2 + X_3 - t_1 \dfrac{X_1 X_3}{X_2} - t_2 \dfrac{X_1 X_2}{X_3}$

6. $(X_1, X_2, X_3) = (X_1, X_3 z, X_3)$

7. $-g = X_3(1 + z) + X_1 \left(1 - t_1 \dfrac{1}{z} - t_2 z\right)$

8. $X^a = X_1^{\alpha} z^{\beta} X_3^{\beta+\beta'}$

9. $\dfrac{z^{\beta}}{(1-t_1\frac{1}{z}-t_2 z)^{\alpha}}\dfrac{1}{(1+z)^{\beta+\beta'}}\dfrac{dz}{z}$

10. [Er 1950, p. 156, $z=\dfrac{u}{1-u}$]

G_2

1. $\dfrac{(\alpha)_{s_1}(\alpha')_{s_2}(\beta)_{s_2-s_1}(\beta')_{s_1-s_2}}{s_1!s_2!}$

2. $(n,N,m)=(2,6,4)$

3. $-t_1,-t_2$

4. $(A'\mid\boldsymbol{\alpha})=\left(\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & \alpha' \\ -1 & 1 & \beta \\ 1 & -1 & \beta' \end{array}\right)$

5. $-g=X_1+X_2+X_3+X_4-t_1\dfrac{X_1X_4}{X_3}-t_2\dfrac{X_2X_3}{X_4}$

6. $(X_1,X_2,X_3,X_4)=(X_1,X_2,zX_4,X_4)$

7. $-g=X_1\left(1-t_1\dfrac{1}{z}\right)+X_2(1-t_2z)+X_4(1+z)$

8. $X^a=X_1^{\alpha}X_2^{\alpha'}z^{\beta}X_4^{\beta+\beta'}$

9. $\dfrac{z^{\beta}}{(1-t_1\frac{1}{z})^{\alpha}(1-t_2z)^{\alpha'}}\dfrac{1}{(1+z)^{\beta+\beta'}}\dfrac{dz}{z}$

10. [Er 1950, p. 148]

G_3

1. $\dfrac{(\alpha)_{2s_2-s_1}(\alpha')_{2s_1-s_2}}{s_1!s_2!}$

2. $(n,N,m)=(2,4,2)$

3. $-t_1,-t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{matrix} -1 & 2 \\ 2 & -1 \end{matrix} \;\middle|\; \begin{matrix} \alpha \\ \alpha' \end{matrix}\right)$

5. $-g = X_1 + X_2 - t_1 \dfrac{X_2^2}{X_1} - t_2 \dfrac{X_1^2}{X_2}$

6. $(X_1, X_2) = (X_1, X_1 z)$

7. $-g = X_1 \left(1 + z - t_1 z^2 - t_2 \dfrac{1}{z}\right)$

8. $X^a = X_1^{\alpha+\alpha'} z^{\alpha'}$

9. $\dfrac{z^{\alpha'}}{(1 + z - t_1 z^2 - t_2 \frac{1}{z})^{\alpha+\alpha'}} \dfrac{dz}{z}$

10. [Er 1950, p. 158]

H_1

1. $\dfrac{(\alpha)_{s_1 - s_2} (\beta)_{s_1+s_2} (\gamma)_{s_2}}{(\delta)_{s_1} s_1! s_2!}$

2. $(n, N, m) = (2, 6, 4)$

3. $t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{matrix} -1 & 0 \\ 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1-\delta \\ \alpha \\ \beta \\ \gamma \end{matrix}\right)$

5. $-g = X_1 + X_2 + X_3 + X_4 + t_1 \dfrac{X_2 X_3}{X_1} - t_2 \dfrac{X_3 X_4}{X_2}$

6. $(X_1, X_2, X_3, X_4) = (X_1, X_1 z_1, X_3, X_1 z_1 z_2)$

7. $-g = X_1(1 + z_1 + z_1 z_2) + X_3(1 + t_1 z_1 - t_2 z_2)$

8. $X^a = X_1^{1+\alpha+\gamma-\delta} z_1^{\alpha+\gamma} z_2^{\gamma} X_3^{\beta}$

9. $\dfrac{z_1^{\alpha+\gamma} z_2^{\gamma}}{(1+t_1 z_1 - t_2 z_2)^{\beta}(1+z_1+z_1 z_2)^{1-\delta+\alpha+\gamma}} \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

10. [Yo, p. 330]

H_2

1. $\dfrac{(\alpha)_{s_1-s_2}(\beta)_{s_1}(\gamma)_{s_2}(\delta)_{s_2}}{(\epsilon)_{s_1} s_1! s_2!}$

2. $(n, N, m) = (2, 7, 5)$

3. $t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\epsilon \\ 1 & -1 & \alpha \\ 1 & 0 & \beta \\ 0 & 1 & \gamma \\ 0 & 1 & \delta \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + X_4 + X_5 + t_1 \dfrac{X_2 X_3}{X_1} - t_2 \dfrac{X_4 X_5}{X_2}$

6. $(X_1, X_2, X_3, X_4, X_5) = (X_1, X_1 z_1, X_3, X_4, X_1 z_1 z_2)$

7. $-g = X_1(1 + z_1 + z_1 z_2) + X_3(1 + t_1 z_1) + X_4(1 - t_2 z_2)$

8. $X^a = X_1^{1-\epsilon+\alpha+\delta} z_1^{\alpha+\delta} z_2^{\delta} X_3^{\beta} X_4^{\gamma}$

9. $\dfrac{z_1^{\alpha+\delta} z_2^{\delta}}{(1+z_1+z_1 z_2)^{1+\delta+\alpha-\epsilon}(1+t_1 z_1)^{\beta}(1-t_2 z_2)^{\gamma}} \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

10. [Yo, p. 330]

H_3

1. $\dfrac{(\alpha)_{2s_1+s_2}(\beta)_{s_2}}{(\gamma)_{s_1+s_2} s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. t_1, t_2

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1-\gamma \\ 2 & 1 & \alpha \\ 0 & 1 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + t_1 \dfrac{X_2^2}{X_1} + t_2 \dfrac{X_2 X_3}{X_1}$

6. $(X_1, X_2, X_3) = (X_1, -X_1 z, X_3)$

7. $-g = X_1(1 - z + t_1 z^2) + X_3(1 - t_2 z)$

8. $X^a = X_1^{1+\alpha-\gamma} z^\alpha X_3^\beta$

9. $\dfrac{z^\alpha}{(1 - z + t_1 z^2)^{1+\alpha-\gamma}(1 - t_2 z)^\beta} \dfrac{dz}{z}$

H_4

1. $\dfrac{(\alpha)_{2s_1+s_2}(\beta)_{s_2}}{(\gamma)_{s_1}(\delta)_{s_2} s_1! s_2!}$

2. $(n, N, m) = (2, 6, 4)$

3. t_1, t_2

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\gamma \\ 0 & -1 & 1-\delta \\ 2 & 1 & \alpha \\ 0 & 1 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + X_4 + t_1 \dfrac{X_3^2}{X_1} + t_2 \dfrac{X_3 X_4}{X_2}$

6. $(X_1, X_2, X_3, X_4) = (X_1, X_2, -X_1 z_1, -X_2 z_2)$

7. $-g = X_1(1 - z_1 + t_1 z_1^2 + t_2 z_1 z_2) + X_2(1 - z_2)$

8. $X^a = X_1^{1+\alpha-\gamma} X_2^{1+\beta-\delta} z_1^\alpha z_2^\beta$

9. $\dfrac{z_1^\alpha z_2^\beta}{(1 - z_1 + t_1 z_1^2 + t_2 z_1 z_2)^{1+\alpha-\gamma}(1 - z_2)^{1+\beta-\delta}} \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

H_5

1. $\dfrac{(\alpha)_{2s_1+s_2}(\beta)_{s_2-s_1}}{(\gamma)_{s_2}s_1!s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. $-t_1, t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} 0 & -1 & 1-\gamma \\ 2 & 1 & \alpha \\ -1 & 1 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 - t_1\dfrac{X_2^2}{X_3} + t_2\dfrac{X_2X_3}{X_1}$

6. $(X_1, X_2, X_3) = (X_2/z_1z_2, X_2, X_2/z_1)$

7. $-g = X_2\left(1 - t_1z_1 + t_2z_2 + \dfrac{1}{z_1} + \dfrac{1}{z_1z_2}\right)$

8. $X^a = z_1^{\gamma-1-\beta}X_2^{1+\alpha+\beta-\gamma}z_2^{\gamma-1}$

9. $\dfrac{z_1^{\gamma-\beta-1}z_2^{\gamma-1}}{(1 - t_1z_1 + t_2z_2 + \frac{1}{z_1} + \frac{1}{z_1z_2})^{1+\alpha+\beta-\gamma}}\dfrac{dz_1}{z_1}\wedge\dfrac{dz_2}{z_2}$

H_6

1. $\dfrac{(\alpha)_{2s_1-s_2}(\beta)_{s_2-s_1}(\gamma)_{s_2}}{s_1!s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. $-t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} 2 & -1 & \alpha \\ -1 & 1 & \beta \\ 0 & 1 & \gamma \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 - t_1\dfrac{X_1^2}{X_2} - t_2\dfrac{X_2X_3}{X_1}$

6. $(X_1, X_2, X_3) = (X_1, X_1/z, X_3)$

7. $-g = X_1\left(1 + \frac{1}{z} - t_1 z\right) + X_3\left(1 - t_2\frac{1}{z}\right)$

8. $X^a = X_1^{\alpha+\beta} z^{-\beta} X_3^{\gamma}$

9. $\frac{z^{-\beta}}{(1 + \frac{1}{z} - t_1 z)^{\alpha+\beta}} \frac{1}{(1 - t_2\frac{1}{z})^{\gamma}} \frac{dz}{z}$

H_7

1. $\frac{(\alpha)_{2s_1 - s_2}(\beta)_{s_2}(\gamma)_{s_2}}{(\delta)_{s_1} s_1! s_2!}$

2. $(n, N, m) = (2, 6, 4)$

3. $t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\delta \\ 2 & -1 & \alpha \\ 0 & 1 & \beta \\ 0 & 1 & \gamma \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + X_4 + t_1\frac{X_2^2}{X_1} - t_2\frac{X_3 X_4}{X_2}$

6. $(X_1, X_2, X_3, X_4) = (X_1, X_1 z_1, X_3, X_1 z_2)$

7. $-g = X_1(1 + z_1 + t_1 z_1^2 + z_2) + X_3\left(1 - t_2\frac{z_2}{z_1}\right)$

8. $X^a = X_1^{1+\alpha+\gamma-\delta} z_1^{\alpha} z_2^{\gamma} X_3^{\beta}$

9. $\frac{z_1^{\alpha} z_2^{\gamma}}{(1 + z_1 + t_1 z_1^2 + z_2)^{1+\alpha+\gamma-\delta}} \frac{1}{(1 - t_2\frac{z_2}{z_1})^{\beta}} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$

ϕ_1

1. $\frac{(\alpha)_{s_1+s_2}(\beta)_{s_1}}{(\gamma)_{s_1+s_2} s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. t_1, t_2

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1-\gamma \\ 1 & 1 & \alpha \\ 1 & 0 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + t_1 \dfrac{X_2 X_3}{X_1} + t_2 \dfrac{X_2}{X_1}$

6. $(X_1, X_2, X_3) = (X_1, -X_1 z, X_3)$

7. $-g = -t_2 z + X_1(1-z) + X_3(1 - t_1 z)$

8. $X^a = X_1^{1+\alpha-\gamma} z^\alpha X_3^\beta$

9. $\dfrac{z^\alpha \exp(t_2 z)}{(1-z)^{1+\alpha-\gamma}(1-t_1 z)^\beta} \dfrac{dz}{z}$

10. [Hu, p. 79]

ϕ_2

1. $\dfrac{(\beta)_{s_1} (\beta')_{s_2}}{(\gamma)_{s_1+s_2} s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. t_1, t_2

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1-\gamma \\ 1 & 0 & \beta \\ 0 & 1 & \beta' \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + t_1 \dfrac{X_2}{X_1} + t_2 \dfrac{X_3}{X_1}$

6. $(X_1, X_2, X_3) = (X_1, -X_1 z_1, -X_1 z_2)$

7. $-g = X_1(1 - z_1 - z_2) - t_1 z_1 - t_2 z_2$

8. $X^a = X_1^{1+\beta+\beta'-\gamma} z_1^\beta z_2^{\beta'}$

9. $\dfrac{z_1^\beta z_2^{\beta'}}{(1 - z_1 - z_2)^{1+\beta+\beta'-\gamma}} \exp(t_1 z_1 + t_2 z_2) \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

6′. $(X_1, X_2, X_3) = (-z, X_2, X_3)$

7′. $-g = -z + X_2\left(1 - t_1\frac{1}{z}\right) + X_3\left(1 - t_2\frac{1}{z}\right)$

8′. $X^a = z^{1-\gamma} X_2^{\beta} X_3^{\beta'}$

9′. $\dfrac{z^{1-\gamma}}{(1-t_1\frac{1}{z})^{\beta}(1-t_2\frac{1}{z})^{\beta'}} \exp z \dfrac{dz}{z}$

10. [Hu, p. 74], [Er 1939, p. 227]

ϕ_3

1. $\dfrac{(\beta)_{s_1}}{(\gamma)_{s_1+s_2} s_1! s_2!}$

2. $(n, N, m) = (2, 4, 2)$

3. t_1, t_2

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1-\gamma \\ 1 & 0 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 + t_1\dfrac{X_2}{X_1} + t_2\dfrac{1}{X_1}$

6. $(X_1, X_2) = (-z, X_2)$

7. $-g = X_2\left(1 - t_1\frac{1}{z}\right) - \left(z + t_2\frac{1}{z}\right)$

8. $X^a = z^{1-\gamma} X_2^{\beta}$

9. $\dfrac{z^{1-\gamma}\exp(z + t_2\frac{1}{z})}{(1-t_1\frac{1}{z})^{\beta}} \dfrac{dz}{z}$

ψ_1

1. $\dfrac{(\alpha)_{s_1+s_2}(\beta)_{s_1}}{(\gamma)_{s_1}(\gamma')_{s_2} s_1! s_2!}$

2. $(n, N, m) = (2, 6, 4)$

3. t_1, t_2

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\gamma \\ 0 & -1 & 1-\gamma' \\ 1 & 1 & \alpha \\ 1 & 0 & \beta \end{array}\right)$$

5. $-g = X_1 + X_2 + X_3 + X_4 + t_1 \dfrac{X_3 X_4}{X_1} + t_2 \dfrac{X_3}{X_2}$

6. $(X_1, X_2, X_3, X_4) = \left(X_1, -\dfrac{1}{z_2}, X_3, -X_1 z_1\right)$

7. $-g = -\dfrac{1}{z_2} + X_1(1-z_1) + X_3(1 - t_1 z_1 - t_2 z_2)$

8. $X^a = X_1^{1+\beta-\gamma} z_1^\beta z_2^{\gamma'-1} X_3^\alpha$

9. $\dfrac{z_1^\beta z_2^{\gamma'-1} \exp(\frac{1}{z_2})}{(1-z_1)^{1+\beta-\gamma}(1-t_1 z_1 - t_2 z_2)^\alpha} \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

ψ_2

1. $\dfrac{(\alpha)_{s_1+s_2}}{(\gamma)_{s_1}(\gamma')_{s_2} s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. t_1, t_2

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\gamma \\ 0 & -1 & 1-\gamma' \\ 1 & 1 & \alpha \end{array}\right)$$

5. $-g = X_1 + X_2 + X_3 + t_1 \dfrac{X_3}{X_1} + t_2 \dfrac{X_3}{X_2}$

6. $(X_1, X_2, X_3) = (-z_1, -z_2, X_3)$

7. $-g = -(z_1 + z_2) + X_3 \left(1 - t_1 \dfrac{1}{z_1} - t_2 \dfrac{1}{z_2}\right)$

8. $X^a = z_1^{1-\gamma} z_2^{1-\gamma'} X_3^\alpha$

9. $$\frac{z_1^{1-\gamma} z_2^{1-\gamma'}}{(1 - t_1 \frac{1}{z_1} - t_2 \frac{1}{z_2})^\alpha} \exp(z_1 + z_2) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

Ξ_1

1. $$\frac{(\alpha)_{s_1} (\alpha')_{s_2} (\beta)_{s_1}}{(\gamma)_{s_1+s_2} s_1! s_2!}$$

2. $(n, N, m) = (2, 6, 4)$

3. t_1, t_2

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1-\gamma \\ 1 & 0 & \alpha \\ 0 & 1 & \alpha' \\ 1 & 0 & \beta \end{array} \right)$$

5. $-g = X_1 + X_2 + X_3 + X_4 + t_1 \dfrac{X_2 X_4}{X_1} + t_2 \dfrac{X_3}{X_1}$

6. $(X_1, X_2, X_3, X_4) = (X_1, -X_1 z_1, -X_1 z_2, X_4)$

7. $-g = X_1(1 - z_1 - z_2) + X_4(1 - t_1 z_1) - t_2 z_2$

8. $X^a = X_1^{1-\gamma+\alpha+\alpha'} z_1^\alpha z_2^{\alpha'} X_4^\beta$

9. $$\frac{z_1^\alpha z_2^{\alpha'}}{(1 - z_1 - z_2)^{1+\alpha+\alpha'-\gamma} (1 - t_1 z_1)^\beta} \exp(t_2 z_2) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

10. [Hu, p. 79]

Ξ_2

1. $$\frac{(\alpha)_{s_1} (\beta)_{s_1}}{(\gamma)_{s_1+s_2} s_1! s_2!}$$

2. $(n, N, m) = (2, 5, 3)$

3. t_1, t_2

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1-\gamma \\ 1 & 0 & \alpha \\ 1 & 0 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + t_1 \dfrac{X_2 X_3}{X_1} + t_2 \dfrac{1}{X_1}$

6. $(X_1, X_2, X_3) = (z_2, X_2, -z_1 z_2)$

7. $-g = X_2(1 - z_1 t_1) + z_2 - z_1 z_2 + t_2 \dfrac{1}{z_2}$

8. $X^a = z_1^{\beta} z_2^{1-\gamma+\beta} X_2^{\alpha}$

9. $\dfrac{z_1^{\beta} z_2^{1-\gamma+\beta}}{(1 - z_1 t_1)^{\alpha}} \exp\left(z_1 z_2 - z_2 - t_2 \dfrac{1}{z_2} \right) \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

Γ_1

1. $\dfrac{(\alpha)_{s_1} (\beta)_{s_2 - s_1} (\beta')_{s_1 - s_2}}{s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. $-t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} 1 & 0 & \alpha \\ -1 & 1 & \beta \\ 1 & -1 & \beta' \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 - t_1 \dfrac{X_1 X_3}{X_2} - t_2 \dfrac{X_2}{X_3}$

6. $(X_1, X_2, X_3) = (X_1, X_3 z, X_3)$

7. $-g = X_1 \left(1 + t_1 \dfrac{1}{z}\right) + X_3(1 - z) + t_2 z$

8. $X^a = X_1^{\alpha} z^{\beta} X_3^{\beta+\beta'}$

9. $\dfrac{z^{\beta}}{(1 + t_1 \frac{1}{z})^{\alpha} (1 - z)^{\beta+\beta'}} \exp(-t_2 z) \dfrac{dz}{z}$

10. [Er 1940, p. 351]

Γ_2

1. $\dfrac{(\beta)_{s_2-s_1}(\beta')_{s_1-s_2}}{s_1!s_2!}$

2. $(n, N, m) = (2, 4, 2)$

3. $-t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 1 & \beta \\ 1 & -1 & \beta' \end{array}\right)$

5. $-g = X_1 + X_2 - t_1\dfrac{X_2}{X_1} - t_2\dfrac{X_1}{X_2}$

6. $(X_1, X_2) = (X_1, -X_1 z)$

7. $-g = X_1(1-z) + t_1 z + t_2\dfrac{1}{z}$

8. $X^a = X_1^{\beta+\beta'} z^{\beta'}$

9. $\dfrac{z^{\beta'}}{(1-z)^{\beta+\beta'}} \exp\left(-t_1 z - t_2\dfrac{1}{z}\right)\dfrac{dz}{z}$

$\mathbf{H}_1$

1. $\dfrac{(\alpha)_{s_1-s_2}(\beta)_{s_1+s_2}}{(\delta)_{s_1}s_1!s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. $t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\delta \\ 1 & -1 & \alpha \\ 1 & 1 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + t_1\dfrac{X_2X_3}{X_1} - t_2\dfrac{X_3}{X_2}$

6. $(X_1, X_2, X_3) = \left(-\frac{1}{z_1 z_2}, \frac{1}{z_2}, X_3\right)$

7. $-g = \frac{1}{z_2} - \frac{1}{z_1 z_2} + X_3(1 - t_1 z_1 - t_2 z_2)$

8. $X^a = z_1^{\delta-1} z_2^{\delta-\alpha-1} X_3^{\beta}$

9. $\frac{z_1^{\delta-1} z_2^{\delta-\alpha-1}}{(1 - t_1 z_1 - t_2 z_2)^{\beta}} \exp\left(\frac{1}{z_1 z_2} - \frac{1}{z_2}\right) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$

$\mathbf{H}_2$

1. $\frac{(\alpha)_{s_1-s_2}(\beta)_{s_1}(\gamma)_{s_2}}{(\delta)_{s_1} s_1! s_2!}$

2. $(n, N, m) = (2, 6, 4)$

3. $t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\delta \\ 1 & -1 & \alpha \\ 1 & 0 & \beta \\ 0 & 1 & \gamma \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + X_4 + t_1 \frac{X_2 X_3}{X_1} - t_2 \frac{X_4}{X_2}$

6. $(X_1, X_2, X_3, X_4) = \left(-\frac{1}{z_1 z_2}, \frac{1}{z_2}, X_3, X_4\right)$

7. $-g = -\frac{1}{z_1 z_2} + \frac{1}{z_2} + X_3(1 - t_1 z_1) + X_4(1 - t_2 z_2)$

8. $X^a = z_1^{\delta-1} z_2^{-\alpha-1+\delta} X_3^{\beta} X_4^{\gamma}$

9. $\frac{z_1^{\delta-1} z_2^{\delta-\alpha-1}}{(1 - t_1 z_1)^{\beta}(1 - t_2 z_2)^{\gamma}} \exp\left(\frac{1}{z_1 z_2} - \frac{1}{z_2}\right) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$

$\mathbf{H}_3$

1. $\frac{(\alpha)_{s_1-s_2}(\beta)_{s_1}}{(\delta)_{s_1} s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. $t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\delta \\ 1 & -1 & \alpha \\ 1 & 0 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + t_1 \dfrac{X_2 X_3}{X_1} - t_2 \dfrac{1}{X_2}$

6. $(X_1, X_2, X_3) = \left(-\dfrac{1}{z_1 z_2}, \dfrac{1}{z_2}, X_3\right)$

7. $-g = X_3(1 - t_1 z_1) + \dfrac{1}{z_2} - \dfrac{1}{z_1 z_2} - t_2 z_2$

8. $X^a = z_1^{\delta-1} z_2^{\delta-1-\alpha} X_3^{\beta}$

9. $\dfrac{z_1^{\delta-1} z_2^{\delta-1-\alpha}}{(1 - t_1 z_1)^{\beta}} \exp\left(\dfrac{1}{z_1 z_2} - \dfrac{1}{z_2} + t_2 z_2\right) \dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$

$\mathbf{H}_4$

1. $\dfrac{(\alpha)_{s_1 - s_2} (\gamma)_{s_2}}{(\delta)_{s_1} s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. $t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\delta \\ 1 & -1 & \alpha \\ 0 & 1 & \gamma \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + t_1 \dfrac{X_2}{X_1} - t_2 \dfrac{X_3}{X_2}$

6. $(X_1, X_2, X_3) = (X_1, -X_1 z_1, -X_1 z_1 z_2)$

7. $-g = X_1(1 - z_1 - z_1 z_2) - (t_1 z_1 + t_2 z_2)$

8. $X^a = X_1^{1+\alpha+\gamma-\delta} z_1^{\alpha+\gamma} z_2^{\gamma}$

9. $$\frac{z_1^{\alpha+\gamma} z_2^{\gamma}}{(1 - z_1 - z_1 z_2)^{1+\alpha+\gamma-\delta}} \exp(t_1 z_1 + t_2 z_2) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

$\mathbf{H}_5$

1. $$\frac{(\alpha)_{s_1 - s_2}}{(\delta)_{s_1} s_1! s_2!}$$

2. $(n, N, m) = (2, 4, 2)$

3. $t_1, -t_2$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1 - \delta \\ 1 & -1 & \alpha \end{array}\right)$$

5. $$-g = X_1 + X_2 + t_1 \frac{X_2}{X_1} - t_2 \frac{1}{X_2}$$

6. $$(X_1, X_2) = \left(-\frac{1}{z_1 z_2}, \frac{1}{z_2}\right)$$

7. $$-g = -\frac{1}{z_1 z_2} + \frac{1}{z_2} - t_1 z_1 - t_2 z_2$$

8. $X^a = z_1^{\delta-1} z_2^{\delta-\alpha-1}$

9. $$z_1^{\delta-1} z_2^{\delta-\alpha-1} \exp\left(\frac{1}{z_1 z_2} - \frac{1}{z_2} + t_1 z_1 + t_2 z_2\right) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

$\mathbf{H}_6$

1. $$\frac{(\alpha)_{2s_1 + s_2}}{(\gamma)_{s_1 + s_2} s_1! s_2!}$$

2. $(n, N, m) = (2, 4, 2)$

3. t_1, t_2

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & -1 & 1 - \gamma \\ 2 & 1 & \alpha \end{array}\right)$$

5. $$-g = X_1 + X_2 + t_1 \frac{X_2^2}{X_1} + t_2 \frac{X_2}{X_1}$$

6. $(X_1, X_2) = \left(-\frac{X_2}{z}, X_2\right)$

7. $-g = X_2\left(1 - \frac{1}{z} - zt_1\right) - zt_2$

8. $X^a = z^{\gamma-1}X_2^{1+\alpha-\gamma}$

9. $\frac{z^{\gamma-1}}{(1 - \frac{1}{z} - zt_1)^{1+\alpha-\gamma}} \exp(t_2 z)\frac{dz}{z}$

$\mathbf{H}_7$

1. $\frac{(\alpha)_{2s_1+s_2}}{(\gamma)_{s_1}(\delta)_{s_2}s_1!s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. t_1, t_2

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\gamma \\ 0 & -1 & 1-\delta \\ 2 & 1 & \alpha \end{array}\right)$

5. $-g = X_1 + X_2 + X_3 + t_1\frac{X_3^2}{X_1} + t_2\frac{X_3}{X_2}$

6. $(X_1, X_2, X_3) = \left(-\frac{X_3}{z_1}, -\frac{1}{z_2}, X_3\right)$

7. $-g = -\frac{1}{z_2} + X_3\left(1 - \frac{1}{z_1} - t_1 z_1 - t_2 z_2\right)$

8. $X^a = z_1^{\gamma-1}z_2^{\delta-1}X_3^{1+\alpha-\gamma}$

9. $\frac{z_1^{\gamma-1}z_2^{\delta-1}}{(1 - \frac{1}{z_1} - t_1 z_1 - t_2 z_2)^{1+\alpha-\gamma}} \exp\left(\frac{1}{z_2}\right)\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$

$\mathbf{H}_8$

1. $\frac{(\alpha)_{2s_1-s_2}(\beta)_{s_2-s_1}}{s_1!s_2!}$

2. $(n, N, m) = (2, 4, 2)$

3. $-t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} 2 & -1 & \alpha \\ -1 & 1 & \beta \end{array}\right)$

5. $-g = X_1 + X_2 - t_1 \dfrac{X_1^2}{X_2} - t_2 \dfrac{X_2}{X_1}$

6. $(X_1, X_2) = (X_1, X_1 z)$

7. $-g = X_1 \left(1 + z - t_1 \dfrac{1}{z}\right) - t_2 z$

8. $X^a = X_1^{\alpha+\beta} z^\beta$

9. $\dfrac{z^\beta}{(1 + z - t_1 \frac{1}{z})^{\alpha+\beta}} \exp(t_2 z) \dfrac{dz}{z}$

$\mathbf{H}_9$

1. $\dfrac{(\alpha)_{2s_1 - s_2} (\beta)_{s_2}}{(\delta)_{s_1} s_1! s_2!}$

2. $(n, N, m) = (2, 5, 3)$

3. $t_1, -t_2$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\delta \\ 2 & -1 & \alpha \\ 0 & 1 & \beta \end{array}\right)$

5. $g = X_1 + X_2 + X_3 + t_1 \dfrac{X_2^2}{X_1} - t_2 \dfrac{X_3}{X_2}$

6. $(X_1, X_2, X_3) = (X_2 z_1, X_2, X_2 z_2)$

7. $-g = X_2 \left(1 + z_1 + z_2 + t_1 \dfrac{1}{z_1}\right) - t_2 z_2$

8. $X^a = z_1^{1-\delta} z_2^\beta X_2^{\alpha+\beta+1-\delta}$

9. $$\frac{z_1^{1-\delta} z_2^{\beta}}{(1+z_1+z_2+t_1\frac{1}{z_1})^{\alpha+\beta+1-\delta}} \exp(t_2 z_2) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

$\mathbf{H}_{10}$

1. $$\frac{(\alpha)_{2s_1-s_2}}{(\delta)_{s_1} s_1! s_2!}$$

2. $(n, N, m) = (2, 4, 2)$

3. $t_1, -t_2$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\delta \\ 2 & -1 & \alpha \end{array}\right)$$

5. $$-g = X_1 + X_2 + t_1 \frac{X_2^2}{X_1} - t_2 \frac{1}{X_2}$$

6. $(X_1, X_2) = (-z_1, -z_2)$

7. $$-g = z_1 - z_2 - t_1 \frac{z_2^2}{z_1} + t_2 \frac{1}{z_2}$$

8. $X^a = z_1^{1-\delta} z_2^{\alpha}$

9. $$z_1^{1-\delta} z_2^{\alpha} \exp\left(z_1 + z_2 + t_1 \frac{z_2^2}{z_1} - t_2 \frac{1}{z_2}\right) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

$\mathbf{H}_{11}$

1. $$\frac{(\alpha)_{s_1-s_2} (\beta)_{s_2} (\gamma)_{s_2}}{(\delta)_{s_1} s_1! s_2!}$$

2. $(n, N, m) = (2, 6, 4)$

3. $t_1, -t_2$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{cc|c} -1 & 0 & 1-\delta \\ 1 & -1 & \alpha \\ 0 & 1 & \beta \\ 0 & 1 & \gamma \end{array}\right)$$

5. $-g = X_1 + X_2 + X_3 + X_4 + t_1 \frac{X_2}{X_1} - t_2 \frac{X_3 X_4}{X_2}$

6. $(X_1, X_2, X_3, X_4) = (X_1, -X_1 z_1, -X_1 z_1 z_2, X_4)$

7. $-g = X_1(1 - z_1 - z_1 z_2) + X_4(1 - t_2 z_2) - t_1 z_1$

8. $X^a = X_1^{1+\alpha+\beta-\delta} z_1^{\alpha+\beta} z_2^{\beta} X_4^{\gamma}$

9. $\frac{z_1^{\alpha+\beta} z_2^{\beta}}{(1 - z_1 - z_1 z_2)^{1+\alpha+\beta-\delta}} \frac{1}{(1 - t_2 z_2)^{\gamma}} \exp(t_1 z_1) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$

$\mathbf{F}_A$

1. $\frac{(\alpha)_{s_1+\cdots+s_n} (\beta_1)_{s_1} \cdots (\beta_n)_{s_n}}{(\gamma_1)_{s_1} \cdots (\gamma_n)_{s_n} s_1! \cdots s_n!}$

2. $(n, N, m) = (n, 3n+1, 2n+1)$

3. $+t_1, \dots, +t_n$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{ccccc|c} -1 & 0 & 0 & \cdots & 0 & 1-\gamma_1 \\ 0 & -1 & 0 & \cdots & 0 & 1-\gamma_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & -1 & 1-\gamma_n \\ 1 & 0 & 0 & \cdots & 0 & \beta_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & \beta_n \\ 1 & 1 & 1 & \cdots & 1 & \alpha \end{array}\right)$$

5. $-g = X_1 + X_2 + \cdots X_{2n+1} + t_1 X_{2n+1} \frac{X_{n+1}}{X_1} + t_2 X_{2n+1} \frac{X_{n+2}}{X_2} + \cdots$
$+ t_n X_{2n+1} \frac{X_{2n}}{X_n}$

6. $(X_1, \dots, X_{2n+1}) = (X_1, \dots, X_n, -X_1 z_1, -X_2 z_2, \dots, -X_n z_n, X_{2n+1})$

7. $-g = \sum_{i=1}^{n} X_i(1 - z_i) + X_{2n+1}(1 - t_1 z_1 - t_2 z_2 - \cdots - t_n z_n)$

8. $X^a = X_1^{1+\beta_1-\gamma_1} X_2^{1+\beta_2-\gamma_2} \cdots X_n^{1+\beta_n-\gamma_n} z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} X_{2n+1}^{\alpha}$

9. $$\frac{z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n}}{(1-z_1)^{1+\beta_1-\gamma_1} \cdots (1-z_n)^{1+\beta_n-\gamma_n}} \frac{1}{(1-t_1z_1-t_2z_2\cdots-t_nz_n)^{\alpha}} \frac{dz_1}{z_1} \wedge \cdots \cdots \wedge \frac{dz_n}{z_n}$$

$\mathbf{F}_B$

1. $$\frac{(\alpha_1)_{s_1}\cdots(\alpha_n)_{s_n}(\beta_1)_{s_1}\cdots(\beta_n)_{s_n}}{(\gamma)_{s_1+\cdots+s_n} s_1!\cdots s_n!}$$

2. $(n, N, m) = (n, 3n+1, 2n+1)$

3. $t_1, t_2, \ldots, t_n$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & 0 & \cdots & 0 & \alpha_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \alpha_n \\ 1 & 0 & 0 & \cdots & 0 & \beta_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \beta_n \\ -1 & -1 & -1 & \cdots & -1 & 1-\gamma \end{array}\right)$$

5. $$-g = X_1 + X_2 + \cdots + X_{2n+1} + t_1 \frac{X_1X_{n+1}}{X_{2n+1}} + \cdots + t_n \frac{X_nX_{2n}}{X_{2n+1}}$$

6. $$(X_1, \ldots, X_{2n+1}) = (X_1, \ldots, X_n, -X_{2n+1}z_1, \ldots, -X_{2n+1}z_n, X_{2n+1})$$

7. $$-g = X_{2n+1}(1 - z_1 - z_2 - \cdots - z_n) + \sum_{i=1}^{n} X_i(1 - t_iz_i)$$

8. $$X^a = X_1^{\alpha_1} \cdots X_n^{\alpha_n} z_1^{\beta_1} \cdots z_n^{\beta_n} X_{2n+1}^{1+\beta_1+\cdots+\beta_n-\gamma}$$

9. $$\frac{z_1^{\beta_1}\cdots z_n^{\beta_n}}{\prod_{i=1}^{n}(1-t_iz_i)^{\alpha_i}} \frac{1}{(1-z_1-z_2-\cdots-z_n)^{1+\beta_1+\cdots+\beta_n-\gamma}} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

$\mathbf{F}_C$

1. $$\frac{(\alpha)_{s_1+s_2+\cdots+s_n}(\beta)_{s_1+\cdots+s_n}}{(\gamma_1)_{s_1}\cdots(\gamma_n)_{s_n} s_1!\cdots s_n!}$$

2. $(n, N, m) = (n, 2n+2, n+2)$

3. $t_1, \dots, t_n$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{ccccc|c} -1 & 0 & 0 & \cdots & 0 & 1-\gamma_1 \\ 0 & 0 & -1 & \cdots & 0 & 1-\gamma_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & -1 & 1-\gamma_n \\ 1 & 1 & 1 & \cdots & 1 & \alpha \\ 1 & 1 & 1 & \cdots & 1 & \beta \end{array}\right)$$

5. $$-g = X_1 + \cdots + X_{n+2} + t_1 \frac{X_{n+1}X_{n+2}}{X_1} + \cdots + t_n \frac{X_{n+1}X_{n+2}}{X_n}$$

6. $$(X_1, \dots, X_{n+2}) = \left(-\frac{X_{n+1}}{z_1}, \dots, -\frac{X_{n+1}}{z_n}, X_{n+1}, X_{n+2}\right)$$

7. $$-g = X_{n+1}\left(1 - \tfrac{1}{z_1} - \cdots - \tfrac{1}{z_n}\right) + X_{n+2}(1 - t_1 z_1 - \cdots - t_n z_n)$$

8. $$X^a = z_1^{\gamma_1 - 1} \cdots z_n^{\gamma_n - 1} X_{n+1}^{n+\alpha-\gamma_1-\cdots-\gamma_n} X_{n+2}^{\beta}$$

9. $$\frac{z_1^{\gamma_1-1} \cdots z_n^{\gamma_n-1}}{(1 - \frac{1}{z_1} - \cdots - \frac{1}{z_n})^{n+\alpha-\gamma_1-\cdots-\gamma_n}} \frac{1}{(1 - t_1 z_1 - \cdots - t_n z_n)^{\beta}} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

$\mathbf{F}_D$

1. $$\frac{(\alpha)_{s_1+\cdots+s_n}(\beta_1)_{s_1}\cdots(\beta_n)_{s_n}}{(\gamma_1)_{s_1+\cdots+s_n} s_1! \cdots s_n!}$$

2. $(n, N, m) = (n, 2n+2, n+2)$

3. $t_1, t_2, \dots, t_n$

4. $$(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & \beta_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \beta_n \\ 1 & 1 & 1 & \cdots & 1 & \alpha \\ -1 & -1 & -1 & \cdots & -1 & 1-\gamma \end{array}\right)$$

5. $-g = X_1 + X_2 + \cdots + X_{n+2} + t_1 X_1 \dfrac{X_{n+1}}{X_{n+2}} + \cdots + t_n X_n \dfrac{X_{n+1}}{X_{n+2}}$

6. $(X_1, \ldots, X_{n+2}) = (X_1, \ldots, X_n, -X_{n+2}z, X_{n+2})$

7. $-g = X_1(1 - t_1 z) + \cdots + X_n(1 - t_n z) + X_{n+2}(1 - z)$

8. $X^a = X_1^{\beta_1} \cdots X_n^{\beta_n} z^\alpha X_{n+2}^{\alpha+1-\gamma}$

9. $\dfrac{z^\alpha}{\prod_{i=1}^{n}(1 - t_i z)^{\beta_i}} \dfrac{1}{(1-z)^{1+\alpha-\gamma}}$

6′. $(X_1, \ldots, X_{n+2}) = (-X_{n+2}z_1, \ldots, -X_{n+2}z_n, X_{n+1}, X_{n+2})$

7′. $-g = X_{n+2}(1 - z_1 - \cdots - z_n) + X_{n+1}(1 - t_1 z_1 - t_2 z_2 - \cdots - t_n z_n)$

8′. $X^a = z_1^{\beta_1} \cdots z_n^{\beta_n} X_{n+1}^\alpha X_{n+2}^{1+\beta_1+\cdots+\beta_n-\gamma}$

9′. $\dfrac{z_1^{\beta_1} \cdots z_n^{\beta_n}}{(1 - t_1 z_1 - \cdots - t_n z_n)^\alpha (1 - z_1 - \cdots - z_n)^{1+\beta_1+\cdots+\beta_n-\gamma}} \dfrac{dz_1}{z_1} \wedge \cdots \wedge \dfrac{dz_n}{z_n}$

$\phi_2^{(n)}$

1. $\dfrac{(\beta_1)_{s_1} \cdots (\beta_n)_{s_n}}{(\gamma)_{s_1+\cdots+s_n} s_1! \cdots s_n!}$

2. $(n, N, m) = (n, 2n+1, n+1)$

3. $t_1, \ldots, t_n$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & \beta_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \beta_n \\ -1 & -1 & -1 & \cdots & -1 & 1-\gamma \end{array}\right)$

5. $-g = X_1 + \cdots + X_{n+1} + t_1 \dfrac{X_1}{X_{n+1}} + \cdots + t_n \dfrac{X_n}{X_{n+1}}$

6. $(X_1, \ldots, X_{n+1}) = (-X_{n+1}z_1, \ldots, -X_{n+1}z_n, X_{n+1})$

7. $-g = X_{n+1}(1 - z_1 - \cdots - z_n) - t_1 z_1 - \cdots - t_n z_n$

8. $X^a = z_1^{\beta_1} \cdots z_n^{\beta_n} X_{n+1}^{1+\beta_1+\cdots+\beta_n-\gamma}$

9. $\dfrac{z_1^{\beta_1} \cdots z_n^{\beta_n}}{(1-z_1-\cdots-z_n)^{1+\beta_1+\cdots+\beta_n-\gamma}} \exp(t_1z_1+\cdots+t_nz_n)\dfrac{dz_1}{z_2} \wedge \cdots \wedge \dfrac{dz_n}{z_n}$

6′. $(X_1, \dots, X_{n+1}) = (X_1, \dots, X_n, -z)$

7′. $-g = X_1\left(1 - t_1\dfrac{1}{z}\right) + \cdots + X_n\left(1 - t_n\dfrac{1}{z}\right) - z$

8′. $X^a = X_1^{\beta_1} \cdots X_n^{\beta_n} z^{1-\gamma}$

9′. $\dfrac{z^{1-\gamma}}{\prod_{j=1}^{n}(1-t_j/z)^{\beta_j}} \exp z \dfrac{dz}{z}$

$\psi_2^{(n)}$

1. $\dfrac{(\beta)_{s_1+\cdots+s_n}}{(\gamma_1)_{s_1}\cdots(\gamma_n)_{s_n} s_1!\cdots s_n!}$

2. $(n, N, m) = (n, 2n+1, n+1)$

3. $t_1, \dots, t_n$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{ccccc|c} -1 & 0 & 0 & \cdots & 0 & 1-\gamma_1 \\ 0 & -1 & 0 & \cdots & 0 & 1-\gamma_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & -1 & 1-\gamma_n \\ 1 & 1 & 1 & \cdots & 1 & \beta \end{array}\right)$

5. $-g = X_1 + \cdots + X_{n+1} + t_1\dfrac{X_{n+1}}{X_1} + \cdots + t_n\dfrac{X_{n+1}}{X_n}$

6. $(X_1, \dots, X_{n+1}) = \left(-\dfrac{X_{n+1}}{z_1}, \dots, -\dfrac{X_{n+1}}{z_n}, X_{n+1}\right)$

7. $-g = X_{n+1}\left(1 - \dfrac{1}{z_1} - \cdots - \dfrac{1}{z_n}\right) - (t_1z_1 + \cdots + t_nz_n)$

8. $X^a = z_1^{\gamma_1-1} \cdots z_n^{\gamma_n-1} X_{n+1}^{\beta+n-\gamma_1-\cdots-\gamma_n}$

9. $$\frac{z_1^{\gamma_1-1}\cdots z_n^{\gamma_n-1}}{(1-\frac{1}{z_1}-\cdots-\frac{1}{z_n})^{\beta+n-\gamma_1-\cdots-\gamma_n}}\exp(t_1z_1+\cdots+t_nz_n)\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_n}{z_n}$$

ϕ_D

1. $$\frac{(\alpha)_{s_1+\cdots+s_n}(\beta_1)_{s_1}\cdots(\beta_{n-1})_{s_{n-1}}}{(\gamma)_{s_1+\cdots+s_n}\cdot s_1!\cdots s_n!}$$

2. $(n,N,m)=(n,2n+1,n+1)$

3. $t_1,\ldots,t_n$

4. $$(A'\mid\boldsymbol{\alpha})=\left(\begin{array}{cccccc|c}1&0&0&\cdots&0&0&\beta_1\\0&1&0&\cdots&0&0&\beta_2\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdot\\0&0&0&\cdots&1&0&\beta_{n-1}\\1&1&1&\cdots&1&1&\alpha\\-1&-1&-1&\cdots&-1&-1&1-\gamma\end{array}\right)$$

5. $$-g=X_1+\cdots+X_{n+1}+t_1X_1\frac{X_n}{X_{n+1}}+\cdots+t_{n-1}X_{n-1}\frac{X_n}{X_{n+1}}+t_n\frac{X_n}{X_{n+1}}$$

6. $(X_1,\ldots,X_{n+1})=(X_1,\ldots,X_{n-1},-X_{n+1}z,X_{n+1})$

7. $-g=X_1(1-t_1z)+\cdots+X_{n-1}(1-t_{n-1}z)+X_{n+1}(1-z)-t_nz$

8. $X^a=X_1^{\beta_1}\cdots X_{n-1}^{\beta_{n-1}}z^{\alpha}X_{n+1}^{1+\alpha-\gamma}$

9. $$\frac{z^{\alpha}}{\prod_{i=1}^{n-1}(1-t_iz)^{\beta_i}}\frac{1}{(1-z)^{1+\alpha-\gamma}}\exp(t_nz)\frac{dz}{z}$$

${}_kF_{k-1}$

1. $$\frac{(\alpha_1)_s\cdots(\alpha_k)_s}{(\gamma_1)_s\cdots(\gamma_{k-1})_s s!}$$

2. $(n,N,m)=(1,2k,2k-1)$

3. $(-1)^kt$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{c|c} 1 & \alpha_1 \\ 1 & \alpha_2 \\ \cdots & \cdots \\ 1 & \alpha_k \\ -1 & 1-\gamma_1 \\ -1 & 1-\gamma_2 \\ \cdots & \cdots \\ -1 & 1-\gamma_{k-1} \end{array} \right)$

5. $-g = X_1 + \cdots + X_{2k-1} + (-1)^k t \dfrac{X_1 \cdots X_k}{X_{k+1} \cdots X_{k+k-1}}$

6. $(X_1, \ldots, X_{2k-1}) = (-X_{k+1}z_1, -X_{k+2}z_2, \ldots, -X_{k+(k-1)}z_{k-1}, X_k, X_{k+1}, \ldots, X_{2k-1})$

7. $-g = \sum_{i=1}^{k-1} X_{k+i}(1-z_i) + X_k(1 - tz_1 \cdots z_{k-1})$

8. $X^a = z_1^{\alpha_1} \cdots z_{k-1}^{\alpha_{k-1}} X_k^{\alpha_k} X_{k-1}^{1+\alpha_1-\gamma_1} \cdots X_{k+(k-1)}^{1+\alpha_{k-1}-\gamma_{k-1}}$

9. $\dfrac{z_1^{\alpha_1} \cdots z_{k-1}^{\alpha_{k-a}}}{\prod_{i=1}^{k-1} (1-z_i)^{1+\alpha_i-\gamma_i}} \dfrac{1}{(1-tz_1 \cdots z_{k-1})^{\alpha_k}} \dfrac{dz_1}{z_1} \wedge \cdots \wedge \dfrac{dz_{k-1}}{z_{k-1}}$

10. [GO, p. 281]

${}_kF_{k+l}$, $l \geq 0$.

1. $\dfrac{(\alpha_1)_s \cdots (\alpha_k)_s}{(\gamma_1)_s \cdots (\gamma_{k+l})_s s!}$

2. $(n, N, m) = (1, 2k+l+1, 2k+l)$

3. $(-1)^{k+l+1}t$

4. $(A' \mid \boldsymbol{\alpha}) = \left(\begin{array}{c|c} 1 & \alpha_1 \\ 1 & \alpha_2 \\ \vdots & \vdots \\ 1 & \alpha_k \\ -1 & 1-\gamma_1 \\ \vdots & \vdots \\ -1 & 1-\gamma_{k+l} \end{array}\right)$

5. $-g = X_1 + \cdots + X_{2k+l} + (-1)^{k+l+1} t \dfrac{X_1 \cdots X_k}{X_{k+l+1} \cdots X_{2k+l}} \cdot \dfrac{1}{X_{k+1} \cdots X_{k+l}}$

6. $(X_1, \ldots, X_{2k+l}) = (-X_{k+l+1} z_1, \ldots, -X_{2k+l} z_k, -z_{k+1}, \ldots, -z_{k+l}, X_{k+l+1}, \ldots, X_{2k+l})$
(i.e. $X_i = -X_{k+l+1} z_i$ if $1 \leq i \leq k$, $X_i = -z_i$ if $k+1 \leq i \leq k+l$, $X_i = X_i$ if $k+l \leq i$)

7. $-g = \sum_{i=1}^{k} X_{k+l+i}(1 - z_i) - \sum_{i=1}^{l} z_{k+i} - t \dfrac{z_1 \cdots z_k}{z_{k+1} \cdots z_{k+l}}$

8. $X^a = z_1^{\alpha_1} \cdots z_k^{\alpha_k} z_{k+1}^{1-\gamma_1} \cdots z_{k+l}^{1-\gamma_l} X_{k+l+1}^{1-\gamma_{l+1}+\alpha_1} \cdots X_{k+l+k}^{1-\gamma_{l+k}+\alpha_k}$

9. $\dfrac{z_1^{\alpha_1} \cdots z_k^{\alpha_k} z_{k+1}^{1-\gamma_1} \cdots z_{k+l}^{1-\gamma_k}}{\prod_{i=1}^{k} (1 - z_i)^{1-\gamma_{l+i}+\alpha_i}} \exp\Big(\sum_{i=1}^{l} z_{k+i} + t \dfrac{z_1 \cdots z_k}{z_{k+1} \cdots z_{k+l}}\Big)$

10. [Er 1937]

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