Hypergeometric Series and Functions as Periods of Exponential Modules

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In recent work [D-L] it was shown that every hypergeometric series occurs as a period of an exponential module. In the present work we give second and third proofs which eliminate some of the restrictions of our previous result.

We also give some evidence supporting the suggestion [C, p. 181] of the Chudnovskys that all G-functions are generalized hypergeometric function, but we would modify their proposal so as to include the composition of hypergeometric functions in several variables with rational maps of $\mathbb{C}^n$ into $\mathbb{C}$.

We have benefitted from the advice of N. Katz and C. Sabbah.

§1. Exponential Module.

For $\nu \in \mathbb{Z}$ let $(x)_\nu = \Gamma(x+\nu)/\Gamma(x)$, an element of $\mathbb{Q}(x)$.

Let y be a hypergeometric series in n variables i.e.

$$y(a,t) = \sum_{s \in \mathbb{N}^n} \frac{\prod_{p=1+N_1}^{N} (\alpha_p)_{\ell_p(s)}}{\prod_{p=1}^{N_1} (1-\alpha_p)_{-\ell_p(s)}(-1)^{\ell_p(s)}} \, t^s$$

where $\alpha = (\alpha_1 \dots \alpha_N) \in \mathbb{C}^N$, $\alpha_p = 0$ for $1 \le p \le n$ and

$$\ell_p(s) = \sum_{j=1}^{n} A_{p,j} s_j$$

a linear form with coefficients in $\mathbb{Z}$. We insist that $\ell_j(s) = -s_j$ $(1 \le j \le n)$. We assume

(1.1) If $p \le N_1$, $\alpha_p \in \mathbb{Z}$ then $\alpha_p \in -\mathbb{N}$ and $A_{p,j} \le 0$.

(1.2) If $p > N_1$, then $\alpha_p \notin \mathbb{Z}$.

For later use we mention a weaker form of 1.2.

(1.2)′ If $\alpha_p \in \mathbb{Z}$, $p > N_1$ then $\alpha_p \in \mathbb{N}^\times, A_{p,j} \geq 0, 1 \leq j \leq n$.

Let $\Omega = \mathbb{Q}(\alpha)$, $R' = \Omega(t)[X_1, \dots, X_m, \frac{1}{X_1 X_2 \dots X_m}]$ where $m = N - n$. Let $a = (a_1, \dots, a_m) = (\alpha_{n+1}, \dots, \alpha_N)$. Let $g(t, X) \in R'$ be defined by

$$-g(X,t) = X_1 + \ldots + X_m + \sum_{j=1}^{n} t_j X^{A'^{(j)}}$$

where $A'^{(j)}$ is the j^{th} column of the matrix

$$A' = \begin{pmatrix} A_{n+1,1}, \dots, A_{n+1,n} \\ \dots \\ A_{N,1}, \dots, A_{N,n} \end{pmatrix}$$

(so $X^{A'^{(j)}} = \prod_{i=1}^{m} X^{A_{n+i,j}}$). We view g as a Laurent polynomial in X parametrized by t.

We define operators on R', $E_i = X_i \frac{\partial}{\partial X_i}$, $(1 \leq i \leq m)$, we define $g_i = E_i g$ $(1 \leq i \leq m)$ and operators $D_{a,i,t} = E_i + a_i + g_i$ $(1 \leq i \leq m)$ on R'. For $1 \leq j \leq n$ we define $\sigma_j = \frac{\partial}{\partial t_j} + \frac{\partial g}{\partial t_j}$ again operators on R'. We now define $\Omega(t)$ vector spaces

$$\mathcal{W}'_{a,t} = R' / \Sigma D_{a,i,t}\, R'$$

which we view as an $\mathcal{R} = \Omega[t, \frac{1}{t}, \sigma_1, \dots, \sigma_n]$ module since σ_j is stable on $\Sigma D_{a,i,t} R'$.

Let $\tilde{\mathcal{R}} = \Omega[t, \frac{1}{t}, \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}]$. Clearly $\tilde{\mathcal{R}} \simeq \mathcal{R}$ under the mapping $\varphi \colon \frac{\partial}{\partial t_j} \to \sigma_j$.

Let $\mathfrak{A}(a)$ be the annihilator of $y(a,t)$ in $\tilde{R}$. Let $\mathfrak{A}_1(a)$ be the annihilator in $\mathcal{R}$ of [1] (the class of 1 in $\mathcal{W}'_{a,t}$).

For $1 \leq j \leq n$ let $\delta_j = t_j \frac{\partial}{\partial t_j}$.

Let $\mathfrak{M}$ be the semigroup of $\Omega[\delta]^\times$ generated by all linear operators $\{c + d_1\delta_1 + \ldots + d_n\delta_n \mid c \notin \mathbb{Z}, (d_1, \dots, d_n) \in \mathbb{Z}^n\}$. If $\mathfrak{C}$ is an ideal of $\tilde{\mathcal{R}}$ then $(\mathfrak{C} \colon \mathfrak{M}) = \{\theta \in \mathcal{R} \mid \mathfrak{m}\theta \in \mathfrak{C}$ for some $\mathfrak{m} \in \mathfrak{M}\}$. Note that $\mathfrak{M}$ is independent of a.

Theorem A.

(i) $\mathfrak{A}(a) = ((\varphi^{-1}\mathfrak{A}_1(a)) \colon \mathfrak{M})$, *subject to 1.1, 1.2.*

(ii) *If a is generic, i.e. $a_1, \dots, a_m$ algebraically independent over $\mathbb{Q}$ then* $\varphi(\mathfrak{A}(a)) = \mathfrak{A}_1(a)$.

§2. Dual space.

Before starting the proof we recall some definitions concerning dual spaces. We follow here the exposition of [GHF] and draw upon the 1991 Princeton lectures of A. Adolphson.

Let $R'^* = \{ \sum_{u \in \mathbb{Z}^m} A_u \frac{1}{X^u} \mid A_u \in \Omega(t) \}$ an $\Omega(t)$ space with a pairing $R'^* \times R' \to \Omega(t)$ defined by $\langle \xi^*, \xi \rangle =$ constant term of $\xi^* \xi$. By means of this pairing we identify R'^* with $\mathrm{Hom}(R', \Omega(t))$ and deduce the mapping $D^*_{a,i,t} = -E_i + a_i + g_i$ adjoint to $D_{a,i,t}$ $(1 \le i \le m)$. The annihilator, $\mathcal{K}'_{a,t}$, of $\Sigma D_{a,i,t} R'$ in R'^* is via the above pairing dual to $\mathcal{W}'_{a,t}$. The action of $\partial / \partial t_j$ on $\Omega(t)$ extends to $\mathcal{K}'_{a,t}$ via $\sigma^*_i = \frac{\partial}{\partial t_j} - \frac{\partial g}{\partial t_j}$ and trivially

$$\frac{\partial}{\partial t_j} \langle \xi^*, \xi \rangle = \langle \sigma^*_j \xi^*, \xi \rangle + \langle \xi^*, \sigma_j \xi \rangle .$$

We may construct universal bases of $\mathcal{W}'_{a,t}$ and $\mathcal{K}'_{a,t}$. Let $\hat{R}$ be the subring of R' generated over $\Omega(t)$ by the set of all monomials X^u such that u is a lattice point in the cone determined by g. Let $\{\xi_i\}_{1 \le i \le q}$ be a set of monomials of $\hat{R}$ which represent a basis of $\hat{R} / \sum_{i=1}^{m} g_i \hat{R}$. Then these represent a basis of $\mathcal{W}'_{a,t}$ for generic a. More generally let $\rho_1, \dots, \rho_\mu$ be $\mathbb{Z}$-linear forms in m variables whose zero sets represent the codimension one faces of the polyhedron of g which pass through the origin and which are normalized by the conditions that each ρ_i maps $\mathbb{Z}^m$ into a subset of $\mathbb{Z}$ with unity as a greater common divisor and that each ρ_i maps the cone of this polyhedron into the positive reals. Then $\{\xi_i\}_{1 \le i \le q}$ represents a basis of $\mathcal{W}'_{a,t}$ provided [GHF, Proposition 6.4.1] no $\rho_j(a)$ lies in $\mathbb{N}^\times$ $(j = 1, 2, \dots, \mu)$.

For generic a we have a basis $\{\xi^*_{a,i,t}\}_{1 \le i \le q}$ of $\mathcal{K}'_{a,t}$ and this specializes to a basis $\{\xi_{a^{(0)},i,t}\}_{1 \le i \le 1}$ of $\mathcal{K}'_{a^{(0)},t}$ provided no $\rho_j(a^{(0)})$ lies in $\mathbb{N}^\times$. However for each $\mu \in \mathbb{Z}^m$, $\{X^u \xi^*_{a-u,i,t}\}_{1 \le i \le q}$ also represents a basis $\mathcal{K}'_{a,t}$ and this specializes to a basis of

$\mathcal{K}'_{a^{(0)},t}$ provided no $\rho_j(a^{(0)} - u) \in \mathbb{N}^{\times}$. We conclude that for any $a^{(0)}$ there exists a generic basis of $\mathcal{K}'_{a,t}$ which specializes to a basis of $\mathcal{K}'_{a^{(0)},t}$.

Let $R_0'^{*} = \{\Sigma A_u \frac{1}{X^u} \mid A_u \in \Omega\}$, then multiplication by $G(t,X) = \exp(g(X,t) - g(X,0))$ is a map of $R_0'^{*}$ into $\Omega[[t]] \otimes R'^{*}$ and the image is annihilated by σ_j^*, $1 \leq j \leq n$. Furthermore this mapping induces an Ω linear map of $\mathcal{K}'_{a,0}$ into $\mathcal{K}'_{a,t} \otimes \Omega[[t]]$ which we call $T^*_{0,t}$.

§3. Proof of Theorem A.

3.1. We now commence our proof. By an easy calculation, $\mathcal{K}'_{a,0}$ is a one dimensional Ω space with basis $\xi_0^* = \prod_{i=1}^{m} \xi_i^*$ where [GHF §13.1] $\xi_i^* = \sum_{s_i=-\infty}^{+\infty} (a_i)_{s_i}/X_i^{s_i}$, the Heaviside generalized exponential series. (We use here the hypothesis that $a_i \notin \mathbb{N}^{\times}$, but this argument could be modified so as to permit $\alpha_p \in \mathbb{N}^{\times}$, $A_{p,j} \geq 0$ if $p > N_1$). Let $\xi_t^* = T^*_{0,t}\xi_0^* = \xi_0^* G(t,X)$, a horizontal element of $\mathcal{K}'_{a,t} \otimes \Omega[[t]]$, i.e. annihilated by each σ_j^*. By a direct calculation $y(a,t) = \langle \xi_t^*, 1 \rangle$.

If $P(t, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}) \in \tilde{\mathcal{R}}$ then

$$P\left(t, \frac{\partial}{\partial t}\right) y = \langle \xi_t^*, \varphi(P) 1 \rangle$$

and then $\varphi(P)[1] = 0$ implies $P \in \mathfrak{A}(a)$, i.e.

$$\varphi(\mathfrak{A}(a)) \supset \mathfrak{A}_1(a) \ ,$$

subject to condition 1.1, 1.2.

3.2

To continue the proof we first consider the case in which a is generic. We use the notation of 3.1.

Proposition. *Let $\eta \in \mathbb{Q}[X, \frac{1}{X}]$, (i.e. independent of a). If a is generic and $0 = \langle \xi_0^*, \eta \rangle$ then $\eta = 0$.*

Proof. Let $\eta = \Sigma A_u X^u$, a finite sum. By hypothesis $0 =$ $\Sigma A_u \Gamma(a_1 + u_1) \dots \Gamma(a_m + u_m)$, a sum over the support of η. We conclude that each $A_u = 0$ since $\Gamma(x), \Gamma(x+1), \dots, \Gamma(x+\ell)$ are linearly independent over a field K if x is transcendental over K.

Corollary. *Let $\eta \in \mathbb{Q}(t)[X, \frac{1}{X}]$ (independent of a). If a is generic and $\langle \xi_t^*, \eta \rangle = 0$ then $\eta = 0$.*

Proof. We may assume $\eta \in \mathbb{Q}[t, X, \frac{1}{X}]$. For each $u \in \mathbb{N}^n$ we have

$$0 = \Big(\prod_{i=1}^{n} (\frac{\partial}{\partial t_j})^{u_j}\Big) \langle \xi_t^*, \eta \rangle = \langle \xi_t^*, \sigma_1^{u_1} \dots \sigma_n^{u_n} \eta \rangle \ .$$

Since η is analytic at $t = 0$ we may specialize at $t = 0$ and deduce $0 = \langle \xi_0^*, (\sigma^u \eta)_{t=0} \rangle$ and so by the proposition $(\sigma^u \eta)_{t=0} = 0$ for all u. But

$$\Big(\prod_{j=1}^{n} (\frac{\partial}{\partial t_j})^{u_j}\Big) (\eta G(t, X)) = G(t, X) \sigma^u \eta$$

and specializing at $t = 0$ shows that

$$\Big(\prod_{j=1}^{n} (\frac{\partial}{\partial t_j})^{u_j}\Big) (\eta G(t, X)) \mid_{t=0} = 0$$

and so by Taylor's theorem, $\eta = 0$.

Lemma. *If a is generic then $\varphi(\mathfrak{A}(a)) = \mathfrak{A}_1(a)$.*

Proof. Let $P(t, \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}) \in \mathfrak{A}(a)$ then $0 = Py = \langle \xi_t^*, \varphi(P)1 \rangle$. We may assume $P \in \mathbb{Q}[a, t, \frac{1}{t}, \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}]$. We assert that $\varphi(P)[1] = 0$. By hypothesis $\varphi(P)1 =$ $\eta \in \mathbb{Q}[a, t, \frac{1}{t}, X, \frac{1}{X}]$. By reduction modulo $\Sigma D_{a,i,t} \mathbb{Q}[a, t, \frac{1}{t}, X, \frac{1}{X}]$ we may reduce to the case in which η is independent of a, but then $0 = \langle \xi_t^*, \eta \rangle$ and by the Corollary, $\eta = 0$.

3.4

Let $a^{(0)}$ be any specialization of a. Conditions 1.1, 1.2 play no role here.

Proposition. *Let $P(a, t\frac{\partial}{\partial t}) \in \mathbb{Q}[a, t, \frac{1}{t}, \frac{\partial}{\partial t}, \dots, \frac{\partial}{\partial t_n}]$. If $Py(a,t) = 0$ then $\varphi(P(a^{(0)}, t, \frac{\partial}{\partial t}))[1] = 0$ in $\mathcal{W}'_{a^{(0)},t}$.*

Proof. We have shown $\varphi(P(a,t,\frac{\partial}{\partial t}))[1] = 0$ in $\mathcal{W}'_{a,t}$. Hence $\langle \xi^*_{a,i,t}, \varphi(P(a,t,\frac{\partial}{\partial t})1\rangle = 0$ $(1 \le i \le q)$. We may assume that the basis $\{\xi^*_{a,i,t}\}_{1\le i\le j}$ specializes to a basis $\{\xi^*_{a^{(0)},i,t}\}$ and hence the specialization, $\varphi(P(a^{(0)}, t, \frac{\partial}{\partial t}))1$, annihilates a basis of $\mathcal{K}'_{a^{(0)},t}$. This completes the proof.

Corollary. *For $1 \le j \le n$ let $\delta_j = t_j \frac{\partial}{\partial t_j}$,*

$$\theta_j = \prod_{A_{p,j}<0} \left((\alpha_p + \ell_p(\delta))_{A_{\rho,j}}\right)^{-1} \circ t_j^{-1} - \prod_{A_{p,j}>0} (\alpha_p + \ell_p(\delta))_{A_{p,j}}$$

then $\varphi(\theta_j) \in \mathfrak{A}_1(a)$ with no condition on a.

Proof. $\theta_j \in \mathfrak{A}(a)$ for a generic and specializes.

3.5

We now assume that a satisfies conditions (1.1), (1.2). Let $\mathfrak{B} = \sum_{j=1}^{n} \tilde{\mathcal{R}}\theta_j$. We have shown that

$$\varphi(\mathfrak{B}) \subset \mathfrak{A}_1(a) \subset \varphi(\mathfrak{A}(a)) .$$

To complete the proof it is enough to verify that $\mathfrak{A} = (\mathfrak{B} : \mathfrak{M})$. For this one shows that given $\theta \in \tilde{\mathcal{R}}$ there exists $\mathfrak{m} \in \mathfrak{M}$, $s \in \mathbb{N}^n$ such that $\mathfrak{m}\theta = b + t^{-s}\nu$ where $b \in \mathfrak{B}$ and $\nu \in \Omega[\delta]$. The key point here is that for $A_{\rho,j} > 0$ we know by 1.2 that $\alpha_p \notin \mathbb{Z}$. We conclude that if $\theta y = 0$ then $\nu y = 0$ which shows that $\nu = 0$ since the support of y is $\mathbb{N}^n$ and $\nu \in \Omega(\delta]$. This completes the proof of Theorem A.

§4. Periods of exponential modules.

Let V and U be $\tilde{\mathcal{R}}$ modules. We say that $u \in U$ is a *period* of $v \in V$ if $v \mapsto u$ defines an $\tilde{\mathcal{R}}$-module homomorphism of $\tilde{\mathcal{R}}v$ onto $\tilde{\mathcal{R}}u$.

Let $V' = X^a R' \exp g / \sum_{i=1}^{m} E_i R' \exp g$, let $\hat{V} = X^a \hat{R} \exp g / \Sigma E_i \hat{R} \exp g$, and let $\omega = X^a \exp g$. (For defniition of $\hat{R}$ see §2.)

We note that V' is an $\tilde{\mathcal{R}}$ module which is isomorphic to

$$R'\omega \bigwedge_{i=1}^{m} \frac{dX_i}{X_i} / d\Big(\sum_{\ell=1}^{m} R'\omega \frac{dX_1}{X_1} \wedge \ldots \wedge \widehat{\frac{dX_\ell}{X_\ell}} \wedge \ldots \wedge \frac{dX_m}{X_m}\Big)$$

and a similar result holds for $\hat{V}$, replacing R' by $\hat{R}$.

Theorem B.

Let $U_0 = \Omega[[t]]$.

1. Subject to 1.1, 1.2, y is (up to a constant factor) the only period in U_0 of $[\omega]$, the class of ω in V'.

2. Subject to 1.1, 1.2′, y is (up to a constant factor) the only period in U_0 of $[\hat{\omega}]$, the class of ω in $\hat{V}$.

Proof. That y is a period of $[\omega]$ follows from 3.1. For special values of a as suggested by 1.2′, the class $[\omega]$ in V' may be quite trivial and for that reason we use $\hat{V}$. For the dual theory we replace $D^*_{a,i,t}$ by $\hat{D}^*_{a,i,t} = \hat{\gamma}_- \circ D^*_{a,i,t}$ where $\hat{\gamma}_-$ is the natural projection of R'^* onto $\hat{R}^*$, the $\Omega(t)$ space of Laurent series $\Sigma A'_u / X^u$ where the sum is over all u such that $X^u \in \hat{R}$. We repeat the argument of 3.1 by explaining the basis of $\hat{\mathcal{K}}_{a,0}$.

If X_i^{-1} occurs in g, let $\hat{\xi}_i^* = \xi_i^*$ as defined in 3.1. If X_i^{-1} does not occur in g, let $\hat{\xi}_i^* = \sum_{s_i=0}^{\infty} (a_i)_{s_i} / X_i^{s_i}$. (If $a_i \in \mathbb{N}^\times$ then by hypothesis we will be in this second situation.) Then $\hat{\xi}_0^* = \hat{\gamma}_- \prod_{i=1}^{m} \hat{\xi}_i^*$ is a basis of $\hat{\mathcal{K}}_{a,0}$ and $y(a,t) = \langle \hat{\xi}_t^*, 1 \rangle$ where $\hat{\xi}_t^*$ is the horizontal element of $\hat{\mathcal{K}}_{a,t} \otimes \Omega[[t]]$ obtained by applying $\hat{T}^*_{0,t} = \hat{\gamma}_- \circ T^*_{0,t}$ to $\hat{\xi}_t^*$. By the same argument as in 3.1, y is a period of the class of 1 in $\hat{\mathcal{W}}_{a,t} = \hat{R}/\Sigma D_{a,i,t}\hat{R}$, i.e. of the class of ω in $\hat{V}$.

We know that $[\omega]$ is annihilated by $\mathcal{B}$. The same proofs shows that $[\hat{\omega}]$ is also annihilated by $\mathcal{B}$. Thus if $\sum_{s\in\mathbb{Z}} C(s)t^s$ is a period of either class then writing θ_j in the form

$$\theta_j = -k_j(\delta)(1+\delta_j) \circ t_j^{-1} - h_j(\delta)$$

where

$$h_j(\delta) = \prod_{A_{p,j}>0} (\alpha_p + \ell_p(\delta))_{A_{p,j}}$$
$$k_j(\delta) = \prod_{p>n, A_{p,j}<0} \left(\alpha_p + \ell_p(\delta) + A_{p,j}\right)_{-A_{p,j}}$$

then we must have the recursion relations

$$(1+s_j)k_j(s)C(s+1_j) + h_j(s)C(s) = 0 \ .$$

It follows from 1.1, 1.2′ that neither $h_j(s)$ nor $k_j(s)$ can ever be zero for $s \in \mathbb{N}^n$. The uniqueness of y as period in U_0 now follows.

§5. Application.

Let $f^{(1)}, \dots, f^{(\ell)}, h$ be Laurent polynomials in $z_1, \dots, z_n$ with possibly variable coefficients in $\mathbb{C}$. We view the periods of the differential $\omega = \frac{\prod_{j=1}^{n} z_j^{a_j}}{\prod_{i=1}^{\ell} f^{(i)b_i}} \cdot$ $\cdot \exp h \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}$ as hypergeometric functions of the variable coefficients. But are these branches of a hypergeometric series? We give a positive response subject to the following conditions.

4.1 For $1 \le j \le n$ there exists i_j such that $1 \cdot z_j$ is a term of $f^{(i_j)}$.

4.2. For $1 \le i \le \ell$ the constant term of $f^{(i)}$ is 1.

4.3 All coefficients in $f^{(1)}, \dots, f^{(\ell)}, h$ aside from those previously mentioned are generic.

We reduce to the preceding sections in two steps. We introduce new variables $X_{n+1}, \dots, X_{n+\ell}$. Then ω is obtained by a formal Laplace transform [GHF, D-L] from

$$\omega' = z_1^{a_1} \dots z_n^{a_n} X_{n+1}^{b_1} \dots X_{n+\ell}^{b_\ell} \exp g(z,x) \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n} \wedge \frac{dX_{n+1}}{X_{n+1}} \wedge \dots \wedge \frac{dX_{n+\ell}}{X_{n+\ell}}$$

where

$$g(z,X) = h(z) + \sum_{i=1}^{\ell} X_{n+i} f^{(i)}(z) .$$

We now make a change of variable. For $1 \leq i \leq n$ let $X_j = z_j X_{n+i_j}$ and eliminate the $z_1, \dots, z_n$ variables. Then ω' takes the form $X^c \exp \tilde{g}(X) \wedge_{i=1}^{n+\ell} \frac{dX_i}{X_i}$ where

$$\tilde{g} = X_1 + \dots + X_{n+\ell} + \sum_{j=1}^{\mu} t_\mu X^{A'(\mu)} .$$

Here $t_1, \dots, t_\mu$ are the generic coefficients appearing in the $f^{(i)}$ and h and the monomials are those obtained from those of h and from the non-special terms of the $f^{(i)}$.

Since we may replace z_i by z_i^{-1} condition 4.1 may be modified in an obvious way.

Example. Let ω_t be the holomorphic differential associated with a $K-3$ surface

$$\omega_t = \left[z_1 z_2 (1+z_1)(1+z_2)(1+t_1 z_1 + t_2 z_2)(1 + t_3 z_1 + t_4 z_2)\right]^{-\frac{1}{2}} dz_1 \wedge dz_2 .$$

We are indebted to William Hoyt for bringing this example to our attention. Using the inverse Laplace transform we obtain

$$\omega = (X_3 X_4 X_5 X_6)^{1/2} (z_1 z_2)^{1/2} \ \exp(g(t,X,z)) \ \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \Big(\bigwedge_{j=3}^{6} \frac{dX_j}{X_j} \Big)$$

where $-g = X_3(1+z_1) + X_4(1+z_2) + X_5(1 + t_1 z_1 + t_2 z_2) + X_6(1 + t_3 z_1 + t_4 z_2)$. Let $X_1 = z_1 X_3$, $X_2 = z_2 X_4$ and eliminate z_1, z_2 so now we have

$$-\tilde{g} = X_1 + \dots + X_6 + t_1 \frac{X_1 X_5}{X_3} + t_2 \frac{X_2 X_5}{X_4} + t_3 \frac{X_1 X_6}{X_3} + t_4 \frac{X_2 X_6}{X_4} .$$

So now we have $\omega = (X_1X_2X_5X_6)^{1/2}\exp(\tilde{g}(t,X))\bigwedge_{i=1}^{6}\frac{dX_i}{X_i}$. The matrix A' and vector a are given by

$$A'|a = \begin{pmatrix} 1 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & \frac{1}{2} \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & \frac{1}{2} \end{pmatrix} .$$

The associated hypergeometries series is

$$\sum_{s\in\mathbb{N}^4} \frac{(\frac{1}{2})_{s_1+s_3}(\frac{1}{2})_{s_2+s_4}(\frac{1}{2})_{s_1+s_2}(\frac{1}{2})_{s_3+s_4}}{(s_1+s_3)!(s_2+s_4)!s_1!s_2!s_3!s_4!} t_1^{s_1}t_2^{s_2}t_3^{s_3}t_4^{s_4}$$

§6. Hypersurfaces.

We take this opportunity to make available some old calculations (1965) involving the periods of differentials in middle dimension on the complement in $\mathbb{P}^n$ of the union of the coordinate hyperplanes and a non-singular hypersurface of degree d. This topic has been treated in several places [E_1, E_2, C-N, K].

Let $A = (A_1, \dots, A_m)$ be algebraically indpendent over $\mathbb{Q}$. Let $\Omega = \mathbb{Q}(A)$. Let $\mathcal{L}^S$ be the ring over Ω generated by all monomials, X^v, in $\Omega[X_0, X_1, \dots, X_{n+1}]$ such that

$$dv_0 = v_1 + \dots + v_{n+1}\ ,\ v_j \geq 1\ (1 \leq j \leq n+1)\ . \tag{6.1}$$

Let $X^{\omega^{(1)}}, \dots, X^{\omega^{(m)}}$ be distinct monomials of degree d in $X_1, \dots, X_{n+1}$, each of which is divisible by at least two of the variables. We define $f(A,X) = \sum_{i=1}^{n+1} X_i^d + \sum_{j=1}^{m} A_jX^{\omega^{(j)}}$ so that $f(0,X)$ defines the Fermat hypersurface.

We define endomorphisms of $\mathcal{L}^S$ as linear Ω spaces

$$D_i = E_i + X_0 f_i\ , f_i = E_i f(A,X)\ .$$

we define $\mathcal{W}_A^S = \mathcal{L}^S / \sum_{i=1}^{n+1} D_i\mathcal{L}^S$. The space $\mathcal{L}^{*^S}$ adjoint to $\mathcal{L}^S$ consists of all formal series

$$\xi^* = \left\{\Sigma B_v/X^v \mid B_v \in \Omega\ , v \text{ satisfies (6.1)}\right\}\ .$$

Let γ_-^{st} be the projections of formal Laurent series into $\mathcal{L}^{*^S}$ defined by $\gamma_-^{\mathrm{st}} 1/X^v = 1/X^v$ if v satisfies 6.1 and zero otherwise. The pairing of $\mathcal{L}^{*^S}$ with $\mathcal{L}^S$ is as defined in §2. The operator adjoint to D_i is $D_i^* = \gamma_-^{\mathrm{st}} \circ (-E_i + X_0 f_i)$ $1 \le i \le n+1$. The dual of $\mathcal{W}_A^S$ is $\mathcal{K}_A^S = \{\xi^* \in \mathcal{L}^{*^S} \mid D_i^* \xi^* = 0 \,, 1 \le i \le n+1\}$.

Let $\mathcal{A} = \{u \text{ satisfying } (6.1) \mid 1 \le u_i \le d \,, 1 \le i \le n+1\}$. Then $\{X^u\}_{u \in \mathcal{A}}$ represents a basis of $\mathcal{W}_0^S$. The dual basis is given by

$$\xi_{u,0}^* = \frac{1}{X^u \Gamma_0(\frac{u'}{d})} \sum_{s \in \mathbb{N}^{n+1}} \Gamma_0\Big(\frac{u'}{d} + \sum_{i=1}^{n+1} s_i \mathbf{1}_i\Big) \cdot \frac{1}{(-X_0)^{\Sigma s_i}} \frac{1}{\prod_{i=1}^{n+1} X_i^{d s_i}}$$

where $\Gamma_0(u') \overset{\mathrm{def}}{=} \prod_{i=1}^{n+1} \Gamma(u_i)$, $u' =$ projecton of u on the last n+1 coordinates . This symbol is used only for elements of $\mathbb{Q}^{n+1}$ none of whose coordinates lie in $-\mathbb{N}$.

We map $\mathcal{K}_0^S$ onto $\mathcal{K}_A^S$ by $T_{0,A}^* = \gamma_-^{\mathrm{st}} \circ G(A,X)$, $G(A,X) = \exp(\sum_{j=1}^{m} A_j X_0 X^{\omega^{(j)}})$. The period matrix coincides with the matrix of this transformation and so we compute $C_{u,v} = \langle T_{0,A}^* \xi_{0,u}^*, X^v \rangle$ for all $u, v \in \mathcal{A}$. It is technically simpler to replace the basis $\{\xi_{a,u}^*\}_{u \in \mathcal{A}}$ of the target space $\mathcal{K}_A^S$ by $\{\xi_{a,\tilde{u}}^*\}_{u \in \mathcal{A}}$ dual to $\{X^{\tilde{u}}\}_{u \in \mathcal{A}}$ where for each $u \in \mathcal{A}$, we choose $\tilde{u}$ satisfying 5.1 such that $(u_1, \dots, u_{n+1}) \equiv (\tilde{u}_1, \dots, \tilde{u}_{n+1})$ mod d. The precise choice will be indicated below. Thus we compute

$$C_{u,\tilde{v}} = \langle T_{0,A}^* \xi_{a,0}^*, X^{\tilde{v}} \rangle$$

and we find

$$\Gamma_0\Big(\frac{u'}{d}\Big) C_{u,\tilde{v}} = \sum_{(r,s) \in \mathbb{N}^{n+1} \times \mathbb{N}^m} \Gamma_0\Big(\frac{u'}{d} + \sum_{i=1}^{n+1} s_i \mathbf{1}_i\Big) (-1)^{\Sigma s_i} \prod_{j=1}^{m} A_j^{r_j} / r_j!$$

the sum being over all r, s such that

$$u' + d(s_1, \dots, s_{n+1}) = \tilde{v}' + \sum_{j=1} r_j \omega^{(j)} \,.$$

Consider all solutions of $u' \equiv \tilde{v}' + \Sigma r_j {\omega'}^{(j)}$ mod d. This set is a finite set independent of the choice of $\tilde{v}$. For each solution let $(\overline{r}_1, \dots, \overline{r}_m) \in [0, d-1]^j$ be a minimal representative. We may now write

$$(r_1, \ldots, r_m) = \overline{r} + d(t_1, \ldots, t_m) \quad \text{with} \quad t \in \mathbb{N}^m .$$

Thus

$$u' + d(s_1, \ldots, s_{n+-1}) = \tilde{v}' + \sum_{j=1}^{m} \overline{r}_j \omega^{(j)} + d \sum_{j=1}^{m} t_j \omega^{(j)} .$$

We now choose $\tilde{v} \equiv v'$ mod d such that for each solution $(\overline{r}_1, \ldots, \overline{r}_m)$ in our finite set, each component of $\tilde{v}' + \sum_{j=1}^{m} \overline{r}_j \omega^{(j)} - u'$ is in $\mathbb{N}$. By hypothesis these components are divisible by d. For each $\overline{r}$ we define $s^{(0)}$,

$$s^{(0)} = (s_1^{(0)}, \ldots, s_{n+1}^{(0)}) = \frac{1}{d}[\tilde{v}' + \Sigma\, \overline{r}_j \omega^{(j)} - u']$$

and so we have $s = s^{(0)} + \sum_{j=1}^{m} t_j \omega^{(j)}$. Thus

$$\Gamma_0(\frac{u'}{d}) C_{u,\tilde{v}} = \sum_{\overline{r}} \sum_{t \in \mathbb{N}^m} \varepsilon(\overline{r}, t) \prod_{j=1}^{m} \frac{A_j^{\overline{r}_j + dt_j}}{(\overline{r}_j + dt_j)!} \cdot \Gamma_0(\frac{u'}{d} + s^{(0)} + \sum_{j=1}^{m} t_j \omega^{(j)}) .$$

Here $\varepsilon(\overline{r}, t) = (-1)^{s_1 + \ldots + s_{n+1}} = (-1)^{s_1^{(0)} + \ldots + s_{n+1}^{(0)} + d(t_1 + \ldots + t_n)}$. We obtain in this way a finite sum of generalized hypergeometric functions.

As a special case one may obtain classical formulae for the roots of a polynomial equation as generalized hypergeometric functions of the coefficients [M].

The symbol $C_{u,v}$ refers to a period of

$$\omega_v = X_0^{v_0} x_1^{v_1} \ldots x_n^{v_n} \exp(X_0 f(x, 1)) \frac{dX_0}{X_0} \wedge \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n} ,$$

where $(x_1, \ldots, x_n, 1)$ refer to dehomogenized coordinates. By means of the Laplace transform relative to X_0, ω_v may be replaced by $\overline{\omega}_v = \frac{x^{v'}}{f(x,1)^{v_0}} \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n}$. The period $C_{u,v}$ involves a cycle γ_u which may be identified by an examination of the matrix at $A = 0$. Details on this method of distinguishing cycles may be found in [C-N p. 287] and the work of Tretkoff [T] on Fermat surfaces may be helpful.

§7. Modified hypergeometric series.

We consider once again the hypergeometric series $y(a,t)$ of §1. Our object is to describe the series obtained by formally differentating the series with respect to the parameters $(a_1,\dots,a_m) = (\alpha_{n+1},\dots\alpha_{n+m})$. The basic point is that by the formulae of 3.1 we have

$$\left(\frac{\partial}{\partial a_1}\right)^{\ell_1}\cdots\left(\frac{\partial}{\partial a_m}\right)^{\ell_m} y = \left\langle\left(\frac{\partial}{\partial a_1}\right)^{\ell_1}\cdots\left(\frac{\partial}{\partial a_m}\right)^{\ell_m}\xi_t^*, 1\right\rangle .$$

The formula $D_{a,i}^* = -E_i + a_i + g_i$ shows that

$$\frac{\partial}{\partial a_i}\circ D_{a,j}^* = D_{a,j}^*\circ\frac{\partial}{\partial a_j} + \delta_{i,j} .$$

Thus if $\mathcal{K}^{(r_1,\dots,r_m)}$ denotes the set of all $\xi^*\in R'^*$ such that

$$D_{a,j}^{*\,r_j}\xi^* = 0 \qquad (1\le i\le m)$$

then $\frac{\partial}{\partial a_1}$ maps $\mathcal{K}^{(r_1,\dots,r_m)}$ into $\mathcal{K}^{(1+r_1,r_2,\dots,r_m)}$. Thus the derivatives of y with respect to a may be viewed as periods of elements of spaces of the type $R'/\mathcal{D}_a^{(r)}R'$ where $\mathcal{D}_a^{(r)}$ denotes forms in $D_{a_1},\dots,D_{a,m}$ of degree r.

§8. Delsarte Sums.

We sketch a generalization of §5 which uses ideas associated with §6.

Following Delsarte [De] let us consider h, a Laurent polynomial in m variables

$$h(X) = \sum_{j=1}^{m} X^{\omega^{(j)}}$$

where $\omega^{(1)},\dots,\omega^{(m)}$ are linearly independent elements of $\mathbb{Z}^m$. The corresponding exponential module may be described briefly.

We define $\mathbb{Q}$ linear forms $L_1,\dots,L_m$ in m variables by the condition $-L_i(\omega^{(j)}) = \delta_{i,j}$ $(1\le i,j\le m)$.

Let $a = (a_1, \dots, a_m)$ be sufficiently general, $\Omega = \mathbb{Q}(a)$, $R'_0 = \Omega[X, \frac{1}{X}]$, $D_{a,i} = E_i + a_i + h_i$, $h_i = E_i h$, $E_i = X_i \frac{\partial}{\partial X_i}$. The structure of $R'_0 / \Sigma D_{a,i} R'_0$ is clarified by the remark that for $u \in \mathbb{Z}^m$

$$[X^{u+\omega^{(j)}}] = L_j(a+u)[X^u] .$$

To verify this we observe that

$$\begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = \omega \begin{pmatrix} X^{\omega^{(1)}} \\ \vdots \\ X^{\omega^{(m)}} \end{pmatrix}$$

where ω is the $m \times m$ matrix whose j^{th} column is $\omega^{(j)}$. Let $c = \omega^{-1}$. Then $X^{u+\omega^{(i)}} = \sum_{j=1}^{m} c_{ij} h_j X^u \equiv -\sum_{j=1}^{m} c_{ij}(E_j + a_j) X^u = -\sum_{j=1}^{m} c_{ij}(a_j + u_j) \cdot X^u$ modulo $\Sigma D_{a,i} R'_0$. The assertion follows from $L_i(u) = -\Sigma c_{ij} u_j$.

Let $\mathcal{A}$ be a set of representatives in $\mathbb{Z}^m$ of $\mathbb{Z}^m / \sum_{j=1}^{m} \mathbb{Z}\omega^{(j)}$. Then $\{X^u\}_{u \in \mathcal{A}}$ represents a basis $\mathcal{W}'_a = R'_0 / \Sigma D_{a,i} R'_0$. The dual basis $\{\xi^*_{u,a}\}_{u \in \mathcal{A}}$ of the dual space $\mathcal{K}'_a = \{\xi^* \in \mathcal{R}'^*_0 \,|(-E_i + a_i + h_i)\xi^* = 0, 1 \le i \le m\}$ will now be constructed

$$\xi^*_{u,a} = \frac{1}{X^u} \sum_{r \in \mathbb{Z}^m} \prod_{j=1}^{m} (-L_j(a+u))_{r_j} \cdot X^{-\sum_{j=1}^{m} r_j \omega^{(j)}} (-1)^{\sum_{j=1}^{m} r_j} .$$

Indeed if $\xi^* = \Sigma A_v / X^v$ lies in $\mathcal{K}'_a$ then

$$-(v_i + a_i) A_v = \sum_{j=1}^{m} \omega_{ij} A_{v+\omega^{(j)}}$$

and hence

$$A_{v+\omega^{(j)}} = L_j(a+v) A_v .$$

It follows that the support of $\xi^*_{u,a}$ lies on $u + \sum_{j=1}^{m} \mathbb{Z} w^{(j)}$ and $A_u = 1$. The asserted formula then follows by induction on $r_1, \dots, r_m$ and the definitions.

We now consider a deformation of the Delsarte polynomial. Let $g(t, X) = h(X) + \sum_{j=1}^{n} t_j X^{\mu^{(j)}}$ where $\{\mu^{(1)}, \dots, \mu^{(n)}\}$ is a set of distinct elements of $\mathbb{Z}^m$ disjoint from $\{\omega^{(1)}, \dots, \omega^{(m)}\}$. As in §1 we construct $\mathcal{W}'_{a,t}$ and $\mathcal{K}'_{a,t}$ using the present formula for g. We deform $\mathcal{K}'_a$ into $\mathcal{K}'_{a,t}$ by means of $T^*_{0,t} =$ multiplication by $\exp(g(t,X) - h(X))$. For $u \in \mathcal{A}$, $T^*_{0,t}\xi^*_{u,a}$ is a horizontal element of $\mathcal{K}'_{a,t}$. For $v \in \mathcal{A}$ we put $C_{u,v} = \langle T^*_{0,t}\xi^*_{u,a}, X^v \rangle$.

Trivially $C_{u,v}$ is a period of $[X^v]$, the class of X^v in $\mathcal{W}'_{a,t}$ if we identify $\mathcal{R}$ with $\tilde{\mathcal{R}}$ by means of φ as in §1. We may also view $C_{u,v}$ as a period of $X^{a+v} \exp g(x,t) \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_m}{X_m}$. We compute

$$(*) \qquad C_{u,v} = \sum_s \frac{t^s}{s_1! \ldots s_m!} \prod_{i=1}^m (-L_i(a+u))_{-L_i(v-u+\sum_{j=1}^n s_j \mu^{(j)})} ,$$

the sum being over all $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ such that

$$(**) \qquad L_i(v - u + \sum_{j=1}^n s_j \mu^{(j)}) \in \mathbb{Z} \qquad 1 \le i \le m .$$

Let $\mathcal{L}$ be the lattice in $\mathbb{Z}^n$ consisting of all $s = (s_1, \ldots, s_n)$ such that $L_i(\sum_{j=1}^n s_j \mu^{(j)}) \in \mathbb{Z}$ $(1 \le i \le m)$. For fixed u, v there exists a finite set of solutions, s, in $\mathbb{Z}^n$ modulo $\mathcal{L}$ of the equation $(**)$. Let $s^{(1)}, \ldots, s^{(q)}$ be a set of representatives of these classes. Then each solution of $(**)$ may be written uniquely in the form $s^{(j)} + w$, $j \in [1,q]$, $w \in \mathcal{L}$. Thus $C_{u,v}$ may be written as a sum $F_1 + \ldots + F_q$ where F_j is obtained by restricting the sum on the right side of $(*)$ to $s = s^{(j)} + w$ with w running over $\mathcal{L} \cap (\mathbb{N}^m - s^{(j)})$.

Thus we obtain hypergeometric functions as periods of differentials of the type discussed in §4 subject to the weaker hypothesis.

4.1′ The monomials of $h + \sum_{i=1}^{\ell} f^{(i)}(z) X_i$ include $n + \ell$ with exponents which are linearly independent over $\mathbb{Q}$ in $\mathbb{Z}^{n+\ell}$.

4.2′ The remaining monomials have algebraically independent coefficients.

§9. New Method.

In this section we improve upon Theorem A. We are indebted to C. Sabbah for proposing the method of this section.

Let $\overline{R} =$

$\Omega[t,t^{-1},X,X^{-1}] = \Omega[t_1,\dots,t_n,t_1^{-1},\dots,t_n^{-1},X_1,\dots,X_m,\ X_1^{-1},\dots X_m^{-1}]$.

We define $\overline{W}_{a,t} = \overline{R}/\Sigma D_{a,i,t}\overline{R}$ as in section 1 and we view it as an $\Omega[t,t^{-1},\sigma]$-module or an $\Omega[t,t^{-1},t\frac{\partial}{\partial t}]$-module (we identify $\Omega[t,t^{-1},\sigma]$ and $\Omega[t,t^{-1},t\frac{\partial}{\partial t}]$ via φ).

We denote by $\overline{\mathfrak{A}}_1(a)$ the annihilator of the class [1] of 1 in $\overline{W}_{a,t}$, viewed as an ideal in $\Omega[t,t^{-1},t\frac{\partial}{\partial t}]$. We will prove in this section the following theorem.

Theorem C.

1) *If the hypergeometric series $y(a,t)$ satisfies 1.1 and 1.2, we have*

$$\mathfrak{A}(a) = \overline{\mathfrak{A}}_1(a) .$$

2) *If the hypergeometric series $y(a,t)$ satisfies 1.1, 1.2′, then*

$$\mathfrak{A}(a) \subset \overline{\mathfrak{A}}_1(a) .$$

We identify the non-commutative ring $\Omega[t,t^{-1},t\frac{\partial}{\partial t}]$ with $\Omega[\tau,\tau^{-1},s]$ by an isomorphism $\gamma\colon \Omega[t,t^{-1},t\frac{\partial}{\partial t}] \to \Omega[\tau,\tau^{-1},s]$:

$$\gamma(\tau_i) = t_i^{-1}, \gamma(\tau_i^{-1}) = t_i\ , \quad \gamma(s_i) = t_i\frac{\partial}{\partial t_i}\ .$$

So the commutation relations between the τ's and the s's are

$$\tau_i s_j = s_j \tau_i \qquad i \neq j$$

$$\tau_i s_i = (s_i + 1)\tau_i\ .$$

We remark that $\Omega[s]$ and $\Omega(s)$ have a natural $\Omega[\tau,\tau^{-1},s]$-module structure, the τ_i's acting as shift operators: $\tau_i P(s) = P(s+\mathbf{1}_i)$, $\tau_i^{-1}P(s) = P(s-\mathbf{1}_i)$. If $\ell\colon \Omega^n \to \Omega$ is a linear form with integer coefficients and $\alpha \in \Omega$, we denote by $M(\ell,\alpha)(s)$ the $\Omega(s)$-module, free of rank 1, generated by the symbol $[\Gamma(\ell(s)+\alpha)]$ over $\Omega(s)$. We put on $M(\ell,\alpha)(s)$ a left $\Omega[\tau,\tau^{-1},s]$-module structure by

$$\tau_i\Big[\Gamma(\ell(s)+\alpha)\Big] = \frac{\Gamma(\ell(s+1_i)+\alpha)}{\Gamma(\ell(s)+\alpha)}\Big[\Gamma(\ell(s)+\alpha)\Big]\ .$$

We denote by $M(\ell,\alpha)$ the $\Omega[\tau,\tau^{-1},s]$-module generated by $[\Gamma(\ell(s)+\alpha)]$ in $M(\ell,\alpha)(s)$. Now let ℓ_j, $1\le j\le N$, be linear forms as above, and $\alpha_j\in\Omega$, $1\le j\le N$.

Denote by M the tensor product over $\Omega[s]$ of the modules $M(\ell_i,\alpha_i)$. We impose on M an $\Omega[\tau,\tau^{-1},s]$-module structure by

$$\tau_i(e_1\otimes\ldots\otimes e_N)=\tau_i e_1\otimes\ldots\otimes\tau_i e_N\ .$$

We denote by

$$e=\bigotimes_{1\le i\le N}\big[\Gamma(\ell_i(s)+\alpha_i)\big]\in M\ .$$

Note that in general e does not generate M as an $\Omega[\tau,\tau^{-1},s]$-module.

We will deduce the theorem from the following statement:

Proposition. *The $\Omega[\tau,\tau^{-1},s]$-module generated by e in Ω is isomorphic to the $\Omega[t,t^{-1},t\frac{\partial}{\partial t}]$-module generated by [1] in $\overline{W}_{a,t}$, by an isomorphism sending e to [1], using the isomorphism γ between $\Omega[t,t^{-1},t\frac{\partial}{\partial t}]$ and $\Omega[\tau,\tau^{-1},s]$.*

Proof. We consider

$$\pi:\Omega[t_1,\ldots,t_n,t_1^{-1},\ldots,t_n^{-1}]\to\Omega[T_1,\ldots,T_N,T_1^{-1},\ldots,T_N^{-1}]$$

given by

$$\pi(t_i)=\prod_{1\le j\le N}T_j^{A_{j,i}}\ .$$

As above, we identify $\Omega[T,T^{-1},T\frac{\partial}{\partial T}$ with $\Omega[\theta,\theta^{-1},S]$ via $\theta_i=T_i^{-1}$, $\theta_i^{-1}=T$, $S_i=T_i\frac{\partial}{\partial T_i}$. As above we consider the $\Omega(S)$-module generated by $[\Gamma(S_i+\alpha_i)]$, $\tilde M(S_i,\alpha_i)(S)$, and we denote by $\tilde M(S_i,\alpha_i)$ the sub $\Omega[\theta,\theta^{-1},S]$-module generated by $[\Gamma(S_i+\alpha_i)]$. We denote by $\tilde M$ the tensor product over $\Omega[S]$ of the $\tilde M(S_i,\alpha_i)$, $1\le i\le N$. As above we put on $\tilde M$ a natural $\Omega[\theta,\theta^{-1},S]$-module structure. We define

$$\tilde e=\bigotimes_{1\le i\le N}[\Gamma(S_i+\alpha_i)]\in\tilde M\ .$$

We remark that obviously $\tilde{e}$ generates $\tilde{M}$ as an $\Omega[\theta,\theta^{-1},S]$-module. More precisely we have:

Lemma 1. *Let $\tilde{I}$ be the left $\Omega[\theta,\theta^{-1},S]$-module generated by*

$$\theta_i - (S_i+\alpha_i)\ ,1\le i\le N\ .$$

Then $\tilde{M}$ is isomorphic to $\Omega[\theta,\theta^{-1},S]/\tilde{I}$ as an $\Omega[\theta,\theta^{-1},S]$-module, via an isomorphism sending $\tilde{e}$ to [1].

Proof. Obvious.

We will use the morphism

$$\begin{aligned}\psi\colon \Omega[S] &\to \Omega[s]\\ S_i &\to \ell_i(s)\ .\end{aligned}$$

We now remark that $M(\ell_i,\alpha_i)$ is isomorphic as an $\Omega[s]$-module to $\Omega[s]\otimes_\psi \tilde{M}(S_i,\alpha_i)$, hence

$$\begin{aligned}M = \bigotimes_{\Omega[s]} M(\ell_i,\alpha_i) &\simeq \otimes_{\Omega[s]}(\Omega[s]\otimes_\psi \tilde{M}(S_i,\alpha_i))\\ &\simeq \Omega[s]\otimes_\psi \big(\bigotimes_{\Omega[s]} \tilde{M}(S_i,\alpha_i)\big)\\ &\simeq \Omega[s]\otimes_\psi \tilde{M}\ .\end{aligned}$$

Furthermore e corresponds to $1\otimes\tilde{e}$. We may extend this isomorphism to an $\Omega[\tau,\tau^{-1},s]$-module isomorphism by putting:

$$\tau_i(f(s)\otimes m) = \tau_i f(s)\otimes \prod_{1\le j\le N}\theta_j^{A_{j,i}} m$$

for $f(s)\in\Omega[s]$, $m\in\tilde{M}$.

So we deduce the following:

Lemma 2. *The $\Omega[\tau,\tau^{-1},s]$-module generated by e in M is isomorphic to the $\Omega[\tau,\tau^{-1},s]$-module generated by [1] in*

$$W := \Omega[s]\otimes_{\psi}\Omega[\theta,\theta^{-1},S]/\tilde{I}\ .$$

Proof. If we identify $\Omega[\theta,\theta^{-1},S]$ and $\Omega[T,T^{-1},T\frac{\partial}{\partial T}]$ via $\gamma\colon \gamma(T_i)=\theta_i^{-1}$, $\gamma(T_i^{-1})=\theta_i$, $\gamma(T_i\frac{\partial}{\partial T_i})=S_i$, $\tilde{I}$ is identified with the ideal $\overline{I}$ generated by

$$T_j\frac{\partial}{\partial T_j}+\alpha_j-T_j^{-1}\ ,\qquad 1\le j\le N\ .$$

But $\Omega[T,T^{-1},T\frac{\partial}{\partial T}]/\overline{I}$ is isomorphic to the $\Omega[T,T^{-1},T\frac{\partial}{\partial T}]$-module generated by $\varphi=e^{-\sum_{1\le i\le N}T_i^{-1}}\Big(\prod_{1\le i\le N}T_i^{-\alpha_i}\Big)$ and [1] corresponds to φ. Now suppose that $\alpha_i=0$ for $1\le i\le n$, and that $\ell_i(s)=-s_i$ for $1\le i\le n$.

We will identify $\overline{R}=\Omega[t,X,t^{-1},X^{-1}]$ with $\Omega[T,T^{-1}]$ via:

$$t_i=T_i^{-1}\prod_{j>n}T_j^{A_{j,i}}\qquad 1\le i\le n$$
$$X_i=T_{i+n}^{-1}\qquad 1\le i\le m\ .$$

Hence we see that $\Omega[\theta,\theta^{-1},S]/\tilde{I}$ is isomorphic to the $\overline{R}[t\frac{\partial}{\partial t},X\frac{\partial}{\partial X}]$-module generated by $e^{g(t,X)}\prod_{1\le i\le m}X_i^{\alpha_{i+n}}$, with $-g(t,X)=\sum_{i=1}^{m}X_i+\sum_{i=1}^{n}t_i\big(\prod_{k=1}^{m}X_k^{A_{k+n,i}}\big)$. Call this last module $\mathcal{E}$. Now we just have to remark that the mapping $\psi\colon\Omega[S]\to\Omega[s]$ corresponds, via the above isomorphisms to

$$\Phi\colon\Omega\Big[t\frac{\partial}{\partial t},X\frac{\partial}{\partial X}\Big]\to\Omega[t\frac{\partial}{\partial t}]$$

defined by $\Phi(t_i\frac{\partial}{\partial t_i})=t_i\frac{\partial}{\partial t_i}$, $\Phi(X_i\frac{\partial}{\partial X_i})=0$. Hence we obtain that $W=\Omega[s]\otimes_{\psi}\Omega[\theta,\theta^{-1},S]/\tilde{I}$ is isomorphic to $\Omega[t\frac{\partial}{\partial t}]\otimes_{\Phi}\mathcal{E}$, this last module being isomorphic to $\mathcal{E}/\sum_{1\le i\le m}X_i\frac{\partial}{\partial X_i}\mathcal{E}$. Furthermore $1\otimes[1]$ corresponds to the class of $e^{g(t,X)}\prod_{1\le i\le m}X_i^{\alpha_{i+n}}$

and the isomorphism is compatible with the $\Omega[\tau,\tau^{-1},s]$-module structure on W if we put the standard left $\Omega[t,t^{-1},t\partial/\partial t]$-module structure on $\mathcal{E}/\sum_{1\le i\le m} X_i\frac{\partial}{\partial X_i}\mathcal{E}$. Now to finish the proof of the proposition we have just to observe that multiplication by $e^{g(t,X)}\prod_{1\le i\le m} X_i^{\alpha_i+n}$ gives an isomorphism of $\Omega[t,t^{-1},t\partial/\partial t]$-modules between $\overline{W}_{a,t}$ and $\mathcal{E}/\sum_{1\le i\le m} X_i\frac{\partial}{\partial X_i}\mathcal{E}$, the action of $t_i\frac{\partial}{\partial t_i}$ on $\overline{W}_{a,t}$ being given by $t_i\frac{\partial}{\partial t_i}+t_i\frac{\partial g}{\partial t_i}$.

Proof of Theorem C.

1) If 1.1 and 1.2 are satisfied then

$$C(s)=\Big(\prod_{q=1+N_1}^{N}(\alpha_q)_{\ell_q(s)}\Big)\Big(\prod_{p=1}^{N_1}(-1)^{\ell_p(s)}(1-\alpha_p)_{-\ell_p(s)}\Big)^{-1}$$

is defined for $s\in\mathbb{Z}^n$ and zero for $s\in\mathbb{Z}^n-\mathbb{N}^n$. Hence we can write

$$y(t,a)=\sum_{s\in\mathbf{Z}^n}C(s)t^s\ ,$$

and we have

$$t_iy(t,a)=\sum_{s\in\mathbf{Z}^n}C(s-1_i)t^s\ ,$$
$$t_i^{-1}y(t,a)=\sum_{s\in\mathbf{Z}^n}C(s+1_i)t^s\ ,$$
$$t_i\frac{\partial}{\partial t_i}y(t,a)=\sum_{s\in\mathbf{Z}^n}s_iC(s+1_i)t^s\ .$$

From that we deduce that if $P(t,t^{-1},t\partial/\partial t)\in\Omega[t,t^{-1},t\partial/\partial t]$, then $P(t,t^{-1},t\partial/\partial t)y(t)=0$ if and only if $P(\tau^{-1},\tau,s)C(s)=0$ in $\Omega(s)^\times$. But it is clear that

$$P(\tau^{-1},\tau,s)C(s)=0\qquad\text{in}\quad\Omega(s)^\times$$

is equivalent to

$$P(\tau^{-1},\tau,s)[e]=0\qquad\text{in}\quad M\ ,$$

and the result follows from the proposition.

2) If 1.1, 1.2′ are satisfied then $C(s)$ is defined for $s \in \mathbb{N}^n$. If $P(t, t^{-1}, t\frac{\partial}{\partial t})$ annihilates $y(t, a)$, then there exists $k \in \mathbb{N}^n$ such that $P(\tau^{-1}, \tau, s)C(s) = 0$ for $s \in k + \mathbb{N}^n$. This is enough to ensure that $P(\tau^{-1}, \tau, s)C(s) = 0$ in $\Omega(s)^{\times}$ and the result follows as in case 1).

Remark. We explain the relation between Theorem C and Theorem A. Since $\overline{R} \subset R'$ it is clear that $\overline{\mathfrak{A}} \subset \mathfrak{A}_1$ while by A1 we have $\mathfrak{A}_1 \subset \mathfrak{A}$ subject to 1.1, 1.2. Thus Theorem C1 implies that $\mathfrak{A}_1 = \mathfrak{A}$.

It follows from [GHF, eq. 1.3.3] that if a is generic then $\overline{R} \cap \Sigma D_{a,i,t} R' = \Sigma D_{a,i,t}\overline{R}$. Hence in the generic case C1 is equivalent to A1.

The implications of C2 are obscure since if 1.2 is weakened to 1.2′ the class of 1 in $\overline{\mathcal{W}}$ (and in $\mathcal{W}'$) may be trivial. It is perhaps more interesting in that case to use the class of 1 in $\hat{\mathcal{W}}$ as in Theorem B2.

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