

# Motivic Integration and the Grothendieck Group of Pseudo-Finite Fields

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## Abstract

Motivic integration is a powerful technique to prove that certain quantities associated to algebraic varieties are birational invariants or are independent of a chosen resolution of singularities. We survey our recent work on an extension of the theory of motivic integration, called arithmetic motivic integration. We developed this theory to understand how  $p$ -adic integrals of a very general type depend on  $p$ . Quantifier elimination plays a key role.

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## 1. Introduction

Motivic integration was first introduced by Kontsevich [20] and further developed by Batyrev [3][4], and Denef-Loeser [8][9][12]. It is a powerful technique to prove that certain quantities associated to algebraic varieties are birational invariants or are independent of a chosen resolution of singularities. For example, Kontsevich used it to prove that the Hodge numbers of birationally equivalent projective Calabi-Yau manifolds are equal. Batyrev [3] obtained his string-theoretic Hodge numbers for canonical Gorenstein singularities by motivic integration. These are the right quantities to establish several mirror-symmetry identities for Calabi-Yau varieties. For more applications and references we refer to the survey papers [11] and [21]. Since then, several other applications to singularity theory were discovered, see e.g. Mustață [24].

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In the present paper, we survey our recent work [10] on an extension of the theory of motivic integration, called arithmetic motivic integration. We developed this theory to understand how  $p$ -adic integrals of a very general type depend on  $p$ . This is used in recent work of Hales [18] on orbital integrals related to the Langlands program. Arithmetic motivic integration is tightly linked to the theory of quantifier elimination, a subject belonging to mathematical logic. The roots of this subject go back to Tarski's theorem on projections of semi-algebraic sets and to the work of Ax-Kochen-Ersov and Macintyre on quantifier elimination for Henselian valued fields (cf. section 4). We will illustrate arithmetic motivic integration starting with the following concrete application. Let  $X$  be an algebraic variety given by equations with integer coefficients. Denote by  $N_{p,n}$  the cardinality of the image of the projection  $X(\mathbf{Z}_p) \rightarrow X(\mathbf{Z}/p^{n+1})$ , where  $\mathbf{Z}_p$  denotes the  $p$ -adic integers. A conjecture of Serre and Oesterlé states that  $P_p(T) := \sum_n N_{p,n} T^n$  is rational. This was proved in 1983 by Denef [7] using quantifier elimination, expressing  $P_p(T)$  as a  $p$ -adic integral over a domain defined by a formula involving quantifiers. This gave no information yet on how  $P_p(T)$  depends on  $p$ . But recently, using arithmetic motivic integration, we proved:

**Theorem 1.1** *There exists a canonically defined rational power series  $P(T)$  over the ring  $K_0^{mot}(\text{Var}_{\mathbf{Q}}) \otimes \mathbf{Q}$ , such that, for  $p \gg 0$ ,  $P_p(T)$  is obtained from  $P(T)$  by applying to each coefficient of  $P(T)$  the operator  $N_p$ .*

Here  $K_0(\text{Var}_{\mathbf{Q}})$  denotes the Grothendieck ring of algebraic varieties over  $\mathbf{Q}$ , and  $K_0^{mot}(\text{Var}_{\mathbf{Q}})$  is the quotient of this ring obtained by identifying two varieties if they have the same class in the Grothendieck group of Chow motives (this is explained in the next section). Moreover the operator  $N_p$  is induced by associating to a variety over  $\mathbf{Q}$  its number of rational points over the field with  $p$  elements, for  $p \gg 0$ .

As explained in section 8 below, this theorem is a special case of a much more general theorem on  $p$ -adic integrals. There we will also see how to canonically associate a “virtual motive” to quite general  $p$ -adic integrals. A first step in the proof of the above theorem is the construction of a canonical morphism from the Grothendieck ring  $K_0(\text{PFF}_{\mathbf{Q}})$  of the theory of pseudo-finite fields of characteristic zero, to  $K_0^{mot}(\text{Var}_{\mathbf{Q}}) \otimes \mathbf{Q}$ . Pseudo-finite fields play a key role in the work of Ax [1] that leads to quantifier elimination for finite fields [19][14][5]. The existence of this map is interesting in itself, because any generalized Euler characteristic, such as the topological Euler characteristic or the Hodge-Deligne polynomial, can be evaluated on any element of  $K_0^{mot}(\text{Var}_{\mathbf{Q}}) \otimes \mathbf{Q}$ , and hence also on any logical formula in the language of fields (possibly involving quantifiers). All this will be explained in section 2. In section 3 we state Theorem 3.1, which is a stronger version of Theorem 1.1 that determines  $P(T)$ . A proof of Theorem 3.1 is outlined in section 7, after giving a survey on arithmetic motivic integration in section 6.

## 2. The Grothendieck group of pseudo-finite fields

Let  $k$  be a field of characteristic zero. We denote by  $K_0(\text{Var}_k)$  the Grothendieck ring of algebraic varieties over  $k$ . This is the group generated by symbols  $[V]$  with

$V$  an algebraic variety over  $k$ , subject to the relations  $[V_1] = [V_2]$  if  $V_1$  is isomorphic to  $V_2$ , and  $[V \setminus W] = [V] - [W]$  if  $W$  is a Zariski closed subvariety of  $V$ . The ring multiplication on  $K_0(\text{Var}_k)$  is induced by the cartesian product of varieties. Let  $\mathbf{L}$  be the class of the affine line over  $k$  in  $K_0(\text{Var}_k)$ . When  $V$  is an algebraic variety over  $\mathbf{Q}$ , and  $p$  a prime number, we denote by  $N_p(V)$  the number of rational points over the field  $\mathbf{F}_p$  with  $p$  elements on a model  $\tilde{V}$  of  $V$  over  $\mathbf{Z}$ . This depends on the choice of a model  $\tilde{V}$ , but two different models will yield the same value of  $N_p(V)$ , when  $p$  is large enough. This will not cause any abuse later on. For us, an algebraic variety over  $k$  does not need to be irreducible; we mean by it a reduced separated scheme of finite over  $k$ .

To any projective nonsingular variety over  $k$  one associates its Chow motive over  $k$  (see [27]). This is a purely algebro-geometric construction, which is made in such a way that any two projective nonsingular varieties,  $V_1$  and  $V_2$ , with isomorphic associated Chow motives, have the same cohomology for each of the known cohomology theories (with coefficients in a field of characteristic zero). In particular, when  $k$  is  $\mathbf{Q}$ ,  $N_p(V_1) = N_p(V_2)$ , for  $p \gg 0$ . For example two elliptic curves define the same Chow motive iff there is a surjective morphism from one to the other. We denote by  $K_0^{mot}(\text{Var}_k)$  the quotient of the ring  $K_0(\text{Var}_k)$  obtained by identifying any two nonsingular projective varieties over  $k$  with equal associated Chow motives. From work of Gillet and Soulé [15], and Guillén and Navarro Aznar [17], it directly follows that there is a unique ring monomorphism from  $K_0^{mot}(\text{Var}_k)$  to the Grothendieck ring of the category of Chow motives over  $k$ , that maps the class of a projective nonsingular variety to the class of its associated Chow motive. What is important for the applications, is that any generalized Euler characteristic, which can be defined in terms of cohomology (with coefficients in a field of characteristic zero), factors through  $K_0^{mot}(\text{Var}_k)$ . With a generalized Euler characteristic we mean any ring morphism from  $K_0(\text{Var}_k)$ , for example the topological Euler characteristic and the Hodge-Deligne polynomial when  $k = \mathbf{C}$ . For  $[V]$  in  $K_0^{mot}(\text{Var}_k)$ , with  $k = \mathbf{Q}$ , we put  $N_p([V]) = N_p(V)$ ; here again this depends on choices, but two different choices yield the same value for  $N_p([V])$ , when  $p$  is large enough.

With a ring formula  $\varphi$  over  $k$  we mean a logical formula build from polynomial equations over  $k$ , by taking Boolean combinations and using existential and universal quantifiers. For example,  $(\exists x)(x^2 + x + y = 0 \text{ and } 4y \neq 1)$  is a ring formula over  $\mathbf{Q}$ . The mean purpose of the present section is to associate in a canonical way to each such formula  $\varphi$  an element  $\chi_c([\varphi])$  of  $K_0^{mot}(\text{Var}_k) \otimes \mathbf{Q}$ . One of the required properties of this association is the following, when  $k = \mathbf{Q}$ : If the formulas  $\varphi_1$  and  $\varphi_2$  are equivalent when interpreted in  $\mathbf{F}_p$ , for all large enough primes  $p$ , then  $\chi_c([\varphi_1]) = \chi_c([\varphi_2])$ . The natural generalization of this requirement, to arbitrary fields  $k$  of characteristic zero, is the following: If the formulas  $\varphi_1$  and  $\varphi_2$  are equivalent when interpreted in  $K$ , for all pseudo-finite fields  $K$  containing  $k$ , then  $\chi_c([\varphi_1]) = \chi_c([\varphi_2])$ . We recall that a pseudo-finite field is an infinite perfect field that has exactly one field extension of any given finite degree, and over which each absolutely irreducible variety has a rational point. For example, infinite ultraproducts of finite fields are pseudo-finite. J. Ax [1] proved that two ring formulas over  $\mathbf{Q}$  are equivalent when interpreted in  $\mathbf{F}_p$ , for all large enough primes  $p$ , if and only if they are equivalent

when interpreted in  $K$ , for all pseudo-finite fields  $K$  containing  $\mathbf{Q}$ . This shows that the two above mentioned requirements are equivalent when  $k = \mathbf{Q}$ . In fact, we will require much more, namely that the association  $\varphi \mapsto \chi_c([\varphi])$  factors through the Grothendieck ring  $K_0(\text{PFF}_k)$  of the theory of pseudo-finite fields containing  $k$ . This ring is the group generated by symbols  $[\varphi]$ , where  $\varphi$  is any ring formula over  $k$ , subject to the relations  $[\varphi_1 \text{ or } \varphi_2] = [\varphi_1] + [\varphi_2] - [\varphi_1 \text{ and } \varphi_2]$ , whenever  $\varphi_1$  and  $\varphi_2$  have the same free variables, and the relations  $[\varphi_1] = [\varphi_2]$ , whenever there exists a ring formula  $\psi$  over  $k$  that, when interpreted in any pseudo-finite field  $K$  containing  $k$ , yields the graph of a bijection between the tuples of elements of  $K$  satisfying  $\varphi_1$  and those satisfying  $\varphi_2$ . The ring multiplication on  $K_0(\text{PFF}_k)$  is induced by the conjunction of formulas in disjoint sets of variables. We can now state the following variant of a theorem of Denef and Loeser [10].

**Theorem 2.1.** *There exists a unique ring morphism*

$$\chi_c : K_0(\text{PFF}_k) \longrightarrow K_0^{\text{mot}}(\text{Var}_k) \otimes \mathbf{Q}$$

satisfying the following two properties:

- (i) For any formula  $\varphi$  which is a conjunction of polynomial equations over  $k$ , the element  $\chi_c([\varphi])$  equals the class in  $K_0^{\text{mot}}(\text{Var}_k) \otimes \mathbf{Q}$  of the variety defined by  $\varphi$ .
- (ii) Let  $X$  be a normal affine irreducible variety over  $k$ ,  $Y$  an unramified Galois cover<sup>1</sup> of  $X$ , and  $C$  a cyclic subgroup of the Galois group  $G$  of  $Y$  over  $X$ . For such data we denote by  $\varphi_{Y,X,C}$  a ring formula, whose interpretation in any field  $K$  containing  $k$ , is the set of  $K$ -rational points on  $X$  that lift to a geometric point on  $Y$  with decomposition group  $C$  (i.e. the set of points on  $X$  that lift to a  $K$ -rational point of  $Y/C$ , but not to any  $K$ -rational point of  $Y/C'$  with  $C'$  a proper subgroup of  $C$ ). Then

$$\chi_c([\varphi_{Y,X,C}]) = \frac{|C|}{|N_G(C)|} \chi_c([\varphi_{Y,Y/C,C}]),$$

where  $N_G(C)$  is the normalizer of  $C$  in  $G$ .

Moreover, when  $k = \mathbf{Q}$ , we have for all large enough primes  $p$  that  $N_p(\chi_c([\varphi]))$  equals the number of tuples in  $\mathbf{F}_p$  that satisfy the interpretation of  $\varphi$  in  $\mathbf{F}_p$ .

The proof of the uniqueness goes as follows: From quantifier elimination for pseudo-finite fields (in terms of Galois stratifications, cf. the work of Fried and Sacerdote [14][13, §26]), it follows that every ring formula over  $k$  is equivalent (in all pseudo-finite fields containing  $k$ ) to a Boolean combination of formulas of the form  $\varphi_{Y,X,C}$ . Thus by (ii) we only have to determine  $\chi_c([\varphi_{Y,Y/C,C}])$ , with  $C$  a cyclic group. But this follows directly from the following recursion formula:

$$|C| [Y/C] = \sum_{A \text{ subgroup of } C} |A| \chi_c([\varphi_{Y,Y/A,A}]).$$

This recursion formula is a direct consequence of (i), (ii), and the fact that the formulas  $\varphi_{Y,Y/C,A}$  yield a partition of  $Y/C$ . The proof of the existence of the morphism  $\chi_c$  is based on the following. In [2], del Baño Rollin and Navarro Aznar

<sup>1</sup>Meaning that  $Y$  is an integral étale scheme over  $X$  with  $Y/G \cong X$ , where  $G$  is the group of all endomorphisms of  $Y$  over  $X$ .

associate to any representation over  $\mathbf{Q}$  of a finite group  $G$  acting freely on an affine variety  $Y$  over  $k$ , an element in the Grothendieck group of Chow motives over  $k$ . By linearity, we can hence associate to any  $\mathbf{Q}$ -central function  $\alpha$  on  $G$  (i.e. a  $\mathbf{Q}$ -linear combination of characters of representations of  $G$  over  $\mathbf{Q}$ ), an element  $\chi_c(Y, \alpha)$  of that Grothendieck group tensored with  $\mathbf{Q}$ . Using Emil Artin's Theorem, that any  $\mathbf{Q}$ -central function  $\alpha$  on  $G$  is a  $\mathbf{Q}$ -linear combination of characters induced by trivial representations of cyclic subgroups, one shows that  $\chi_c(Y, \alpha) \in K_0^{mot}(\text{Var}_k) \otimes \mathbf{Q}$ . For  $X := Y/G$  and  $C$  any cyclic subgroup of  $G$ , we define  $\chi_c([\varphi_{Y,X,C}]) := \chi_c(Y, \theta)$ , where  $\theta$  sends  $g \in G$  to 1 if the subgroup generated by  $g$  is conjugate to  $C$ , and else to 0. Note that  $\theta$  equals  $|C|/|N_G(C)|$  times the function on  $G$  induced by the characteristic function on  $C$  of the set of generators of  $C$ . This implies our requirement (ii), because of Proposition 3.1.2.(2) of [10]. The map  $(Y, \alpha) \mapsto \chi_c(Y, \alpha)$  satisfies the nice compatibility relations stated in Proposition 3.1.2 of loc. cit. This compatibility (together with the above mentioned quantifier elimination) is used, exactly as in loc. cit., to prove that the above definition of  $\chi_c([\varphi_{Y,X,C}])$  extends by additivity to a well-defined map  $\chi_c : K_0(\text{PFF}_k) \rightarrow K_0^{mot}(\text{Var}_k) \otimes \mathbf{Q}$ . In loc. cit., Chow motives with coefficients in the algebraic closure of  $\mathbf{Q}$  are used, but we can work as well with coefficients in  $\mathbf{Q}$ , since here we only have to consider representations of  $G$  over  $\mathbf{Q}$ .

### 3. Arc spaces and the motivic Poincaré series

Let  $X$  be an algebraic variety defined over a field  $k$  of characteristic zero. For any natural number  $n$ , the  $n$ -th jet space  $\mathcal{L}_n(X)$  of  $X$  is the unique algebraic variety over  $k$  whose  $K$ -rational points correspond in a bijective and functorial way to the rational points on  $X$  over  $K[t]/t^{n+1}$ , for any field  $K$  containing  $k$ . The arc space  $\mathcal{L}(X)$  of  $X$  is the reduced  $k$ -scheme obtained by taking the projective limit of the varieties  $\mathcal{L}_n(X)$  in the category of  $k$ -schemes.

We will now give the definition of the motivic Poincaré series  $P(T)$  of  $X$ . This series is called the arithmetic Poincaré series in [10], and is very different from the geometric Poincaré series studied in [8]. For notational convenience we only give the definition here when  $X$  is a subvariety of some affine space  $\mathbf{A}_k^m$ . For the general case we refer to section 5 below or to our paper [10]. By Greenberg's Theorem [16], for each  $n$  there exists a ring formula  $\varphi_n$  over  $k$  such that, for all fields  $K$  containing  $k$ , the  $K$ -rational points of  $\mathcal{L}_n(X)$ , that can be lifted to a  $K$ -rational point of  $\mathcal{L}(X)$ , correspond to the tuples satisfying the interpretation of  $\varphi_n$  in  $K$ . (The correspondence is induced by mapping a polynomial over  $K$  to the tuple consisting of its coefficients.) Clearly, when two formulas satisfy this requirement, then they are equivalent when interpreted in any field containing  $k$ , and hence define the same class in  $K_0(\text{PFF}_k)$ . Now we are ready to give the definition of  $P(T)$ :

$$P(T) := \sum_n \chi_c([\varphi_n]) T^n$$

**Theorem 3.1.** *The motivic Poincaré series  $P(T)$  is a rational power series over the ring  $K_0^{mot}(\text{Var}_k)[\mathbf{L}^{-1}] \otimes \mathbf{Q}$ , with denominator a product of factors of the*

form  $1 - \mathbf{L}^a T^b$ , with  $a, b \in \mathbf{Z}$ ,  $b > 0$ . Moreover if  $k = \mathbf{Q}$ , the Serre Poincaré series  $P_p(T)$ , for  $p \gg 0$ , is obtained from  $P(T)$  by applying the operator  $N_p$  to each coefficient of the numerator and denominator of  $P(T)$ .

In particular we see that the degrees of the numerator and the denominator of  $P_p(T)$  remain bounded for  $p$  going to infinity. This fact was first proved by Macintyre [23] and Pas [26].

## 4. Quantifier elimination for valuation rings

Let  $R$  be a ring and assume it is an integral domain. We will define the notion of a DVR-formula over  $R$ . Such a formula can be interpreted in any discrete valuation ring  $A \supset R$  with a distinguished uniformizer  $\pi$ . It can contain variables that run over the discrete valuation ring, variables that run over the value group  $\mathbf{Z}$ , and variables that run over the residue field. A DVR-formula over  $R$  is build from quantifiers with respect to variables that run over the discrete valuation ring, or over the value group, or over the residue field, Boolean combinations, and expressions of the following form:  $g_1(x) = 0$ ,  $\text{ord}(g_1(x)) \leq \text{ord}(g_2(x)) + L(a)$ ,  $\text{ord}(g_1(x)) \equiv L(a) \pmod{d}$ , where  $g_1(x)$  and  $g_2(x)$  are polynomials over  $R$  in several variables  $x$  running over the discrete valuation ring, where  $L(a)$  is a polynomial of degree  $\leq 1$  over  $\mathbf{Z}$  in several variables  $a$  running over the value group, and  $d$  is any positive integer (not a variable). Moreover we also allow expressions of the form  $\varphi(\overline{\text{ac}}(h_1(x)), \dots, \overline{\text{ac}}(h_r(x)))$ , where  $\varphi$  is a ring formula over  $R$ , to be interpreted in the residue field,  $h_1(x), \dots, h_r(x)$  are polynomials over  $R$  in several variables  $x$  running over the discrete valuation ring, and  $\overline{\text{ac}}(v)$ , for any element  $v$  of the discrete valuation ring, is the residue of the angular component  $\text{ac}(v) := v\pi^{-\text{ord}v}$ . For the discrete valuation rings  $\mathbf{Z}_p$  and  $K[[t]]$ , we take as distinguished uniformizer  $\pi$  the elements  $p$  and  $t$ .

**Theorem 4.1 (Quantifier Elimination of Pas [26])** *Suppose that  $R$  has characteristic zero. For any DVR-formula  $\theta$  over  $R$  there exists a DVR-formula  $\psi$  over  $R$ , which contains no quantifiers running over the valuation ring and no quantifiers running over the value group, such that*

- (1)  $\theta \longleftrightarrow \psi$  holds in  $K[[t]]$ , for all fields  $K$  containing  $R$ .
- (2)  $\theta \longleftrightarrow \psi$  holds in  $\mathbf{Z}_p$ , for all primes  $p \gg 0$ , when  $R = \mathbf{Z}$ .

The Theorem of Pas is one of several quantifier elimination results for Henselian valuation rings, and goes back to the work of Ax-Kochen-Ersov and Cohen on the model theory of valued fields, which was further developed by Macintyre, Delon [6], and others, see e.g. Macintyre's survey [22].

Combining the Theorem of Pas with the work of Ax mentioned in section 2, one obtains

**Theorem 4.2 (Ax-Kochen-Ersov Principle, version of Pas)** *Let  $\sigma$  be a DVR-formula over  $\mathbf{Z}$  with no free variables. Then the following are equivalent:*

- (i) *The interpretation of  $\sigma$  in  $\mathbf{Z}_p$  is true for all primes  $p \gg 0$ .*
- (ii) *The interpretation of  $\sigma$  in  $K[[t]]$  is true for all pseudo-finite fields  $K$  of characteristic zero.*

## 5. Definable subassignments and truncations

Let  $h : \mathcal{C} \rightarrow \text{Sets}$  be a functor from a category  $\mathcal{C}$  to the category of sets. We shall call the data for each object  $C$  of  $\mathcal{C}$  of a subset  $h'(C)$  of  $h(C)$  a *subassignment* of  $h$ . The point in this definition is that  $h'$  is not assumed to be a subfunctor of  $h$ . For  $h'$  and  $h''$  two subassignments of  $h$ , we shall denote by  $h' \cap h''$  and  $h' \cup h''$ , the subassignments  $C \mapsto h'(C) \cap h''(C)$  and  $C \mapsto h'(C) \cup h''(C)$ , respectively.

Let  $k$  be a field of characteristic zero. We denote by  $\text{Field}_k$  the category of fields which contain  $k$ . For  $X$  a variety over  $k$ , we consider the functor  $h_X : K \mapsto X(K)$  from  $\text{Field}_k$  to the category of sets. Here  $X(K)$  denotes the set of  $K$ -rational points on  $X$ . When  $X$  is a subvariety of some affine space, then a subassignment  $h$  of  $h_X$  is called *definable* if there exists a ring formula  $\varphi$  over  $k$  such that, for any field  $K$  containing  $k$ , the set of tuples that satisfy the interpretation of  $\varphi$  in  $K$ , equals  $h(K)$ . Moreover we define the *class*  $[h]$  of  $h$  in  $K_0(\text{PFF}_k)$  as  $[\varphi]$ . More generally, for any algebraic variety  $X$  over  $k$ , a subassignment  $h$  of  $h_X$  is called *definable* if there exists a finite cover  $(X_i)_{i \in I}$  of  $X$  by affine open subvarieties and definable subassignments  $h_i$  of  $h_{X_i}$ , for  $i \in I$ , such that  $h = \cup_{i \in I} h_i$ . The *class*  $[h]$  of  $h$  in  $K_0(\text{PFF}_k)$  is defined by linearity, reducing to the affine case.

For any algebraic variety  $X$  over  $k$  we denote by  $h_{\mathcal{L}(X)}$  the functor  $K \mapsto X(K[[t]])$  from  $\text{Field}_k$  to the category of sets. Here  $X(K[[t]])$  denotes the set of  $K[[t]]$ -rational points on  $X$ . When  $X$  is a subvariety of some affine space, then a subassignment  $h$  of  $h_{\mathcal{L}(X)}$  is called *definable* if there exists a DVR-formula  $\varphi$  over  $k$  such that, for any field  $K$  containing  $k$ , the set of tuples that satisfy the interpretation of  $\varphi$  in  $K[[t]]$ , equals  $h(K)$ . More generally, for any algebraic variety  $X$  over  $k$ , a subassignment  $h$  of  $h_{\mathcal{L}(X)}$  is called *definable* if there exists a finite cover  $(X_i)_{i \in I}$  of  $X$  by affine open subvarieties and definable subassignments  $h_i$  of  $h_{\mathcal{L}(X_i)}$ , for  $i \in I$ , such that  $h = \cup_{i \in I} h_i$ . A family of definable subassignments  $h_n$ ,  $n \in \mathbf{Z}$ , of  $h_{\mathcal{L}(X)}$  is called a *definable family of definable subassignments* if on each affine open of a suitable finite affine covering of  $X$ , the family  $h_n$  is given by a DVR-formula containing  $n$  as a free variable running over the value group.

Let  $X$  be a variety over  $k$ . Let  $h$  be a definable subassignment of  $h_{\mathcal{L}(X)}$ , and  $n$  a natural number. The *truncation of  $h$  at level  $n$* , denoted by  $\pi_n(h)$ , is the subassignment of  $h_{\mathcal{L}_n(X)}$  that associates to any field  $K$  containing  $k$  the image of  $h(K)$  under the natural projection map from  $X(K[[t]])$  to  $\mathcal{L}_n(X)(K)$ . Using the Quantifier Elimination Theorem of Pas, we proved that  $\pi_n(h)$  is a definable subassignment of  $h_{\mathcal{L}_n(X)}$ , so that we can consider its class  $[\pi_n(h)]$  in  $K_0(\text{PFF}_k)$ . Using the notion of truncations, we can now give an alternative (but equivalent) definition of the motivic Poincaré series  $P(T)$ , which works for any algebraic variety  $X$  over  $k$ , namely  $P(T) := \sum_n \chi_c([\pi_n(h_{\mathcal{L}(X)})])T^n$ .

A definable subassignment  $h$  of  $h_{\mathcal{L}(X)}$  is called *weakly stable at level  $n$*  if for any field  $K$  containing  $k$  the set  $h(K)$  is a union of fibers of the natural projection map from  $X(K[[t]])$  to  $\mathcal{L}_n(X)(K)$ . If  $X$  is nonsingular, with all its irreducible components of dimension  $d$ , and  $h$  is a definable subassignment of  $h_{\mathcal{L}(X)}$ , which is weakly stable at level  $n$ , then it is easy to verify that

$$[\pi_n(h)]\mathbf{L}^{-nd} = [\pi_m(h)]\mathbf{L}^{-md}$$

for all  $m \geq n$ . Indeed this follows from the fact that the natural map from  $\mathcal{L}_m(X)$  to  $\mathcal{L}_n(X)$  is a locally trivial fibration for the Zariski topology with fiber  $\mathbf{A}_k^{(m-n)d}$ , when  $X$  is nonsingular.

## 6. Arithmetic motivic integration

Here we will outline an extension of the theory of motivic integration, called arithmetic motivic integration. If the base field  $k$  is algebraically closed, then it coincides with the usual motivic integration.

We denote by  $\widehat{K}_0^{mot}(\text{Var}_k)[\mathbf{L}^{-1}]$  the completion of  $K_0^{mot}(\text{Var}_k)[\mathbf{L}^{-1}]$  with respect to the filtration of  $K_0^{mot}(\text{Var}_k)[\mathbf{L}^{-1}]$  whose  $m$ -th member is the subgroup generated by the elements  $[V]\mathbf{L}^{-i}$  with  $i - \dim V \geq m$ . Thus a sequence  $[V_i]\mathbf{L}^{-i}$  converges to zero in  $\widehat{K}_0^{mot}(\text{Var}_k)[\mathbf{L}^{-1}]$ , for  $i \mapsto +\infty$ , if  $i - \dim V_i \mapsto +\infty$ .

**Definition-Theorem 6.1.** *Let  $X$  be an algebraic variety of dimension  $d$  over a field  $k$  of characteristic zero, and let  $h$  be a definable subassignment of  $h_{\mathcal{L}(X)}$ . Then the limit*

$$\nu(h) := \lim_{n \rightarrow \infty} \chi_c([\pi_n(h)])\mathbf{L}^{-(n+1)d}$$

*exists in  $\widehat{K}_0^{mot}(\text{Var}_k)[\mathbf{L}^{-1}] \otimes \mathbf{Q}$  and is called the arithmetic motivic volume of  $h$ .*

We refer to [10, §6] for the proof of the above theorem. If  $X$  is nonsingular and  $h$  is weakly stable at some level, then the theorem follows directly from what we said at the end of the previous section. When  $X$  is nonsingular affine, but  $h$  general, the theorem is proved by approximating  $h$  by definable subassignments  $h_i$  of  $h_{\mathcal{L}(X)}$ ,  $i \in \mathbf{N}$ , which are weakly stable at level  $n(i)$ . For  $h_i$  we take the subassignment obtained from  $h$  by adding, in the DVR-formula  $\varphi$  defining  $h$ , the condition  $\text{ord}g(x) \leq i$ , for each polynomial  $g(x)$  over the valuation ring, that appears in  $\varphi$ . (Here we assume that  $\varphi$  contains no quantifiers over the valuation ring.) It remains to show then that  $\chi_c([\pi_n(\text{ord}g(x) > i)])\mathbf{L}^{-(n+1)d}$  goes to zero when both  $i$  and  $n \gg i$  go to infinity, but this is easy.

**Theorem 6.2.** *Let  $X$  be an algebraic variety of dimension  $d$  over a field  $k$  of characteristic zero, and let  $h, h_1$  and  $h_2$  be definable subassignments of  $h_{\mathcal{L}(X)}$ .*

- (1) *If  $h_1(K) = h_2(K)$  for any pseudo-finite field  $K \supset k$ , then  $\nu(h_1) = \nu(h_2)$ .*
- (2)  *$\nu(h_1 \cup h_2) = \nu(h_1) + \nu(h_2) - \nu(h_1 \cap h_2)$*
- (3) *If  $S$  is a subvariety of  $X$  of dimension  $< d$ , and if  $h \subset h_{\mathcal{L}(S)}$ , then  $\nu(h) = 0$ .*
- (4) *Let  $h_n$ ,  $n \in \mathbf{N}$ , be a definable family of definable subassignments of  $h_{\mathcal{L}(X)}$ . If  $h_n \cap h_m = \emptyset$ , for all  $n \neq m$ , then  $\sum_n \nu(h_n)$  is convergent and equals  $\nu(\bigcup_n h_n)$ .*
- (5) *Change of variables formula. Let  $p : Y \rightarrow X$  be a proper birational morphism of nonsingular irreducible varieties over  $k$ . Assume for any field  $K$  containing  $k$  that the jacobian determinant of  $p$  at any point of  $p^{-1}(h(K))$  in  $Y(K[[t]])$  has  $t$ -order equal to  $e$ . Then  $\nu_X(h) = \mathbf{L}^{-e} \nu_Y(p^{-1}(h))$ . Here  $\nu_X, \nu_Y$  denote the arithmetic motivic volumes relative to  $X, Y$ , and  $p^{-1}(h)$  is the subassignment of  $h_{\mathcal{L}(Y)}$  given by  $K \mapsto p^{-1}(h(K)) \cap Y(K[[t]])$ .*

Assertion (1) is a direct consequence of the definitions. Assertions (2) and (4) are proved by approximating the subassignments by weakly stable ones. Moreover



for (4) we also need the fact that  $h_n = \emptyset$  for all but a finite number of  $n$ 's, when all the  $h_n$ , and their union, are weakly stable (at some level depending on  $n$ ). Assertion (5) follows from the fact that for  $n \gg e$  the map  $\mathcal{L}_n(Y) \rightarrow \mathcal{L}_n(X)$  induced by  $p$  is a piecewise trivial fibration with fiber  $\mathbf{A}_k^e$  over the image in  $\mathcal{L}_n(X)$  of the points of  $\mathcal{L}(Y)$  where the jacobian determinant of  $p$  has  $t$ -order  $e$ . See [10] for the details.

## 7. About the proof of Theorem 3.1

We give a brief sketch of the proof of Theorem 3.1, in the special case that  $X$  is a hypersurface in  $\mathbf{A}_k^d$  with equation  $f(x) = 0$ . Actually, here we will only explain why the image  $\widehat{P}(T)$  of  $P(T)$  in the ring of power series over  $\widehat{K}_0^{mot}(\mathrm{Var}_k)[\mathbf{L}^{-1}] \otimes \mathbf{Q}$  is rational. The rationality of  $P(T)$  requires additional work. Let  $\varphi(x, n)$  be the DVR-formula  $(\exists y)(f(y) = 0 \text{ and } \mathrm{ord}(x - y) \geq n)$ , with  $d$  free variables  $x$  running over the discrete valuation ring, and one free variable  $n$  running over the value group. That formula determines a definable family of definable subassignments  $h_{\varphi(-, n)}$  of  $h_{\mathcal{L}(\mathbf{A}_k^d)}$ . Since  $h_{\varphi(-, n)}$  is weakly stable at level  $n$ , unwinding our definitions yields that the arithmetic motivic volume on  $h_{\mathcal{L}(\mathbf{A}_k^d)}$  of  $h_{\varphi(-, n)}$  equals  $\mathbf{L}^{-(n+1)d}$  times the  $n$ -th coefficient of  $\widehat{P}(T)$ . To prove that  $\widehat{P}(T)$  is a rational power series we have to analyze how the arithmetic motivic volume of  $h_{\varphi(-, n)}$  depends on  $n$ . To study this, we use Theorem 4.1 (quantifier elimination of Pas) to replace the formula  $\varphi(x, n)$  by a DVR-formula  $\psi(x, n)$  with no quantifiers running over the valuation ring and no quantifiers over the value group. We take an embedded resolution of singularities  $\pi : Y \rightarrow \mathbf{A}_k^d$  of the union of the loci of the polynomials over the valuation ring, that appear in  $\psi(x, n)$ . Thus the pull-backs to  $Y$  of these polynomials, and the jacobian determinant of  $\pi$ , are locally a monomial times a unit. Thus the pull-back of the formula  $\psi(x, n)$  is easy to study, at least if one is not scared of complicated formula in residue field variables. The key idea is to calculate the arithmetic motivic volume of  $h_{\psi(-, n)}$ , by expressing it as a sum of arithmetic motivic volumes on  $h_{\mathcal{L}(Y)}$ , using the change of variables formula in Theorem 6.2. These volumes can be computed explicitly, and this yields the rationality of  $\widehat{P}(T)$ .

To prove that  $\widehat{P}(T)$  specializes to the Serre Poincaré series  $P_p(T)$  for  $p \gg 0$ , we repeat the above argument working with  $\mathbf{Z}_p^d$  instead of  $\mathcal{L}(\mathbf{A}_k^d)$ . The  $p$ -adic volume of the subset of  $\mathbf{Z}_p^d$  defined by the formula  $\varphi(x, n)$  equals  $p^{-(n+1)d}$  times the  $n$ -th coefficient of  $P_p(T)$ . Because of Theorem 4.1.(2), we can again replace  $\varphi(x, n)$  by the formula  $\psi(x, n)$  that we obtained already above. That  $p$ -adic volume can be calculated explicitly by pulling it back to the  $p$ -adic manifold  $Y(\mathbf{Z}_p)$ , and one verifies a posteriori that it is obtained by applying the operator  $N_p$  to the arithmetic motivic volume that we calculated above. This verification uses the last assertion in Theorem 2.1.

## 8. The general setting

We denote by  $\overline{\mathcal{M}}$  the image of  $K_0^{mot}(\mathrm{Var}_k)[\mathbf{L}^{-1}]$  in  $\widehat{K}_0^{mot}(\mathrm{Var}_k)[\mathbf{L}^{-1}]$ , and by  $\overline{\mathcal{M}}_{loc}$  the localization of  $\overline{\mathcal{M}} \otimes \mathbf{Q}$  obtained by inverting the elements  $\mathbf{L}^i - 1$ , for all  $i \geq 1$ .

One verifies that the operator  $N_p$  can be applied to any element of  $\overline{\mathcal{M}}_{loc}$ , for  $p \gg 0$ , yielding a rational number. The same holds for the Hodge-Deligne polynomial which now belongs to  $\mathbf{Q}(u, v)$ . By the method of section 7, we proved in [10] the following

**Theorem 8.1.** *Let  $X$  be an algebraic variety over a field  $k$  of characteristic zero, let  $h$  be a definable subassignment of  $h_{\mathcal{L}(X)}$ , and  $h_n$  a definable family of definable subassignments of  $h_{\mathcal{L}(X)}$ .*

(1) *The motivic volume  $\nu(h)$  is contained in  $\overline{\mathcal{M}}_{loc}$ .*

(2) *The power series  $\sum_n \nu(h_n)T^n \in \overline{\mathcal{M}}_{loc}[[T]]$  is rational, with denominator a product of factors of the form  $1 - \mathbf{L}^{-a}T^b$ , with  $a, b \in \mathbf{N}$ ,  $b \neq 0$ .*

Let  $X$  be a reduced separable scheme of finite type over  $\mathbf{Z}$ , and let  $A = (A_p)_{p \gg 0}$  be a definable family of subsets of  $X(\mathbf{Z}_p)$ , meaning that on each affine open, of a suitable finite affine covering of  $X$ ,  $A_p$  can be described by a DVR-formula over  $\mathbf{Z}$ . (Here  $p$  runs over all large enough primes.) To  $A$  we associate in a canonical way, its motivic volume  $\nu(h_A) \in \overline{\mathcal{M}}_{loc}$ , in the following way: Let  $h_A$  be a definable subassignment of  $h_{\mathcal{L}(X \otimes \mathbf{Q})}$ , given by DVR-formulas that define  $A$ . Because these formulas are not canonical, the subassignment  $h_A$  is not canonical. But by the Ax-Kochen-Ersov Principle (see 4.2), the set  $h_A(K)$  is canonical for each pseudo-finite field  $K$  containing  $\mathbf{Q}$ . Hence  $\nu(h_A) \in \overline{\mathcal{M}}_{loc}$  is canonical, by Theorem 6.2.(1). By the method of section 7, we proved in [10] the following comparison result:

**Theorem 8.2.** *With the above notation, for all large enough primes  $p$ ,  $N_p(\nu(h_A))$  equals the measure of  $A_p$  with respect to the canonical measure on  $X(\mathbf{Z}_p)$ .*

When  $X \otimes \mathbf{Q}$  is nonsingular and of dimension  $d$ , the canonical measure on  $X(\mathbf{Z}_p)$  is defined by requiring that each fiber of the map  $X(\mathbf{Z}_p) \rightarrow X(\mathbf{Z}_p/p^m)$  has measure  $p^{-md}$  whenever  $m \gg 0$ . For the definition of the canonical measure in the general case, we refer to [25].

The above theorem easily generalizes to integrals instead of measures, but this yields little more because quite general  $p$ -adic integrals (such as the orbital integrals appearing in the Langlands program) can be written as measures of the definable sets we consider. For example the  $p$ -adic integral  $\int |f(x)|dx$  on  $\mathbf{Z}_p^d$  equals the  $p$ -adic measure of  $\{(x, t) \in \mathbf{Z}_p^{d+1} : \text{ord}_p(f(x)) \leq \text{ord}_p(t)\}$ .

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