MOTIVIC HEIGHT ZETA FUNCTIONS

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Dedicated to the memory of Professor Jun-ichi Igusa.

Abstract. We consider a motivic analogue of the height zeta function for integral points of equivariant partial compactifications of affine spaces. We establish its rationality and determine its largest pole.

A major problem of Diophantine geometry is to understand the distribution of rational or integral points of algebraic varieties defined over number fields. For example, a well-studied question put forward by Manin in [1] is that of an asymptotic expansion for the number of rational/integral points of bounded height. A basic tool is the height zeta function which is a Dirichlet series.

Around 2000, E. Peyre suggested to consider the analogous problem over function fields, which has then an even more geometric flavor since it translates as a problem of enumerative geometry, namely counting algebraic curves of given degree and establishing properties of the corresponding generating series. In view of the developments of motivic integration by Kontsevich, Denef-Loeser [11], etc., it is natural to look at a more general generating series which not only counts the *number* of such algebraic curves, but takes into account the *space* they constitute in a suitable Hilbert scheme.

A natural coefficient ring for the generating series is the Grothendieck ring of varieties KVar_k: if k is a field, this ring is generated as a group by the isomorphism classes of k-schemes of finite type, the addition being subject to obvious cut-and-paste relations, and the product is induced by the product of k-varieties. One interest of this generalization is that it also makes sense in a purely geometric context, where no counting is available.

Such a situation has been first studied in a paper by Kapranov in [15], where the analogy with the mentioned diophantine problem is not pointed out. Later, Bourqui made some progress on the motivic analogue of Manin's problem, see [4], as well as his survey report [5].

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In this article, we consider the following situation:

SETTING 1. *Let k be an algebraically closed field of characteristic zero. Let C_0 be a quasi-projective smooth connected curve over k and let C be its smooth projective compactification; we let $S = C \setminus C_0$. Let $F = k(C)$ be the function field of C and g be its genus.*

Let X be a projective irreducible k-scheme together with a non-constant morphism $\pi \colon X \to C$. Let G and U be Zariski open subsets of X such that $G \subset U \subset X$. Let $\mathscr{L}$ be a line bundle on X; we assume that there exists an effective $\mathbf{Q}$-divisor D supported on $(X \setminus U)_F$ such that $\mathscr{L}(-D)$ is ample on X_F.

We are interested in sections $\sigma \colon C \to X$ of π such that $\sigma(C_F) \subset G_F$ and $\sigma(C_0) \subset U$. As in the Hasse principle, existence of *local* such sections is a necessary condition to the existence of global sections σ.

SETTING 2. *We assume that for every $v \in C_0$, $G(F_v) \cap U(\mathfrak{o}_v) \neq \varnothing$, where $\mathfrak{o}_v$ is the completion of $\mathscr{O}_{C,v}$ and F_v is its field of functions.*

We then want to study the family of such sections σ with prescribed degree $n = \deg \sigma^* \mathscr{L}$. This is a geometric/motivic analogue of the variant of Manin's problem for *integral points*.

By Proposition 2.2.2, these conditions define a constructible set $M_{U,n}$ (in some k-scheme); moreover, the hypothesis on $\mathscr{L}$ implies that there exists $n_0 \in \mathbf{Z}$ such that $M_{U,n}$ is empty for $n \leq n_0$. Considering the classes $[M_{U,n}]$ of these sets in KVar_k, we form the generating Laurent series

$$Z_U(T) = \sum_{n \in \mathbf{Z}} [M_{U,n}] T^n$$

and ask about its properties.

Precisely, we investigate in this paper the motivic counterpart of the situation studied recently in the paper [8] by Y. Tschinkel and the first author.

SETTING 3. *In this paper, we consider the particular case where G_F is the additive group $\mathbf{G}^n_{\mathrm{a},F}$, $U_F = G_F$ and X_F admits an action of G_F which extends the group action of G_F over itself. We also assume that the irreducible components of the divisor at infinity $\partial X = X \setminus G$ are smooth and meet transversally. Finally, we restrict ourselves to the case where the restriction of $\mathscr{L}$ to the generic fiber X_F is equal to $-K_{X_F}(\partial X_F)$, the log-anticanonical line bundle.*

(As explained at the end of Section 3.1, this line bundle satisfies the previous ampleness assumption.)

Let $\mathbf{L}$ be the class of the affine line $\mathbf{A}^1_k$ in KVar_k and let $\mathscr{M}_k$ be the localized ring of KVar_k with respect to the multiplicative subset S generated by $\mathbf{L}$ and the elements $\mathbf{L}^a - 1$, for $a \in \mathbf{N}_{>0}$. An element of KVar_k is said to be *effective* if it can be written as a sum of classes of algebraic varieties; similarly, an element of $\mathscr{M}_k$ is effective if its product by some element of S is the image of an effective element of KVar_k. For example, $1-\mathbf{L}^{-a} = \mathbf{L}^{-a}(\mathbf{L}^a - 1)$ is effective for every $a > 0$.

Let $\mathscr{M}_k\{T\}^\dagger$ and $\mathscr{M}_k\{T\}$ be the subrings of $\mathscr{M}_k[[T]][T^{-1}]$ generated by $\mathscr{M}_k[T,T^{-1}]$ and the inverses of the polynomials $1-\mathbf{L}^aT^b$, where $(a,b)\in\mathbf{N}\times\mathbf{N}_{>0}$ are integers such that $b>a$, respectively $b\geq a$. For $b>a\geq 0$, $1-\mathbf{L}^{a-b} = \mathbf{L}^{a-b}(\mathbf{L}^{b-a}-1)$ is invertible in $\mathscr{M}_k$, so that every element P of $\mathscr{M}_k\{T\}^\dagger$ has a value $P(\mathbf{L}^{-1})$ at $T=\mathbf{L}^{-1}$ which is an element of $\mathscr{M}_k$.

The following theorem is the main result of this paper.

THEOREM 1. *Assume the notation and hypotheses of Settings 1, 2, and 3, are in force.*

The Laurent series $Z_U(T)$ belongs to $\mathscr{M}_k\{T\}$. More precisely, there exists an integer $a\geq 1$, an element $P_U(T)\in\mathscr{M}_k\{T\}^\dagger$ such that $P_U(\mathbf{L}^{-1})$ is an effective non-zero element of $\mathscr{M}_k$, and a positive integer d such that

$$(1-\mathbf{L}^aT^a)^d Z_U(T) = P_U(T).$$

Any k-constructible set M can be written as a finite disjoint union of integral k-varieties; we let $\dim(M)$ be the maximal dimension of these varieties and $\kappa(M)$ be the number of such varieties of maximal dimension; they do not depend on the chosen partition.

COROLLARY 1. *For every integer $p\in\{0,\dots,a-1\}$, one of the following cases occur when n tends to infinity in the congruence class of p modulo a:*

(1) *Either* $\dim(M_{U,n}) = \mathrm{o}(n)$,

(2) *Or* $\dim(M_{U,n})-n$ *has a finite limit and* $\log(\kappa(M_{U,n}))/\log(n)$ *converges to some integer in* $\{0,\dots,d-1\}$.

Moreover, the second case happens at least for one integer p.

We observe that this condition on congruence classes is unavoidable in general. For example, if $\mathscr{L}$ is a multiple $\mathscr{L}_0^a$ of a class in $\mathrm{Pic}(\mathscr{X})$, then $M_{U,n}=\varnothing$ for $n\nmid a$.

In the arithmetic case, the corresponding question consists in establishing the analytic property of the height zeta function (holomorphy for $\Re(s)>1$, meromorphy on a larger half-plane, pole of order d at $s=1$) as well as showing that the number of points of height $\leq B$ grows as $B(\log(B))^{d-1}$. Its proof in [8] relies on the Poisson summation formula for the discrete cocompact subgroup $G(F)$ of the adelic group $G(\mathbb{A}_F)$. In the present work, we take advantage of the motivic Poisson formula recently established by E. Hrushovski and D. Kazhdan in [13] to prove new results in the geometric setting.

However, in its present form, this motivic Poisson formula suffers two limitations. Firstly, the functions it takes as input may only depend on finitely many places of the given function field. For this reason, the question we solve in this paper is a geometric analogue of Manin's problem for integral points, rather than for rational points. Secondly, the Poisson formula only applies to vector groups, and this is why our varieties are assumed to be equivariant compactifications of such groups.

The plan of the paper is the following.

We begin the paper by an exposition, in a self-contained geometric language, of the motivic Poisson formula of Hrushovski-Kazhdan. We then gather in Section 2 some preliminary results needed for the proof. In particular, we show in Proposition 2.1.3 that Corollary 1 is a consequence from Theorem 1. For eventual reference, we also prove there a general existence theorem for the moduli spaces which we study here, see Proposition 2.2.2. We end this Section by recalling some notation on Clemens complexes, and on functions on arc spaces with values in $\mathscr{M}_k$.

In Section 3, we lay out the foundations for the proof of Theorem 1. Its main goal consists in describing the moduli spaces as adelic subsets of the group G.

The core of the proof of Theorem 1 begins with Section 4. We first apply the motivic Poisson summation formula of Hrushovski and Kazhdan. We show that this formula gives an expression $Z(T)$ as a "sum" (in the sense of motivic integration) over $\xi \in G(F)$ of rational functions $Z(T,\xi)$ whose denominators are products of factors of the form $1-\mathbf{L}^aT^b$ for $b \geq a$. The point is that the term corresponding to the parameter $\xi = 0$ is the one which involves the largest number of such factors with $a = b$; intuitively, the "order of the pole of $Z(T,\xi)$ at $T = \mathbf{L}^{-1}$" is larger for $\xi = 0$ than for $\xi \neq 0$. Admitting these facts, it is therefore a simple matter to conclude the proof of Theorem 1.

The proof of these facts are the subject of Sections 5 and 6. In fact, once rewritten as a motivic integral, the Laurent series $Z(T,0)$ is a kind of "geometric" motivic Igusa zeta function. Its analysis, using embedded resolution of singularities, would be classical; in fact, our geometric setting is so strong that we even do not need to resolve singularities in this case. For general ξ, however, what we obtain is a sort of "motivic oscillatory integral". Such integrals are studied in a coordinate system in Section 5. Finally, in Section 6, we establish the three propositions that we had temporarily admitted in Section 4.

In this paper, an important role is played by variants of the local zeta functions that Igusa had introduced in [14] and which are studied by refining Igusa's initial analysis. We are honored to dedicate this work to the memory of late Professor Igusa. The second author had the privilege to first meet Professor Igusa more than thirty years ago. He would like to acknowledge the profound impact of Professor Igusa's vision on his own research during all these years.

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1. The motivic Poisson formula of Hrushovski-Kazhdan. For the convenience of the reader, we begin this paper with an exposition of Hrushovski-Kazhdan's motivic Poisson summation formula. We follow closely the relevant sections from [13], but adopt a self-contained geometric language. In the rest of the paper, we will make an essential use of the formalism recalled here.

To motivate the definitions, let us discuss rapidly the dictionary with the Poisson summation formula for the adele groups of global fields. So assume that F is a global field. Let $\mathbb{A}_F$ be the ring of adeles of F; it is the restricted product of the completions F_v at all places v of F and is endowed with a natural structure of a locally compact abelian group. The field F embeds diagonally in $\mathbb{A}_F$ and its image is a discrete cocompact subgroup. Fix a Haar measure μ on $\mathbb{A}_F$ as well as a non-trivial character $\psi\colon \mathbb{A}_F \to \mathbf{C}^*$. For every Schwartz-Bruhat function φ on $\mathbb{A}_F^n$, its Fourier transform is the function $\mathscr{F}\varphi$ on $\mathbb{A}_F^n$ defined by

$$\mathscr{F}\varphi(y) = \int_{\mathbb{A}_F^n} \varphi(x)\psi(xy)\,\mathrm{d}\mu(x);$$

it is again a Schwartz-Bruhat function. Moreover, the global Haar measure, additive character and Fourier transform can be written as products of similar local objects. Then, one has

$$\sum_{x\in F^n} \varphi(x) = \mu(\mathbb{A}_F/F)^{-n} \sum_{y\in F^n} \mathscr{F}\varphi(y).$$

The motivic Poisson summation formula provides an analogue of this formalism, when F is the function field of a curve C over an algebraically closed field. Integrals belong to the Grothendieck ring of varieties, more precisely, to a (suitably localized) variant "with exponentials" of this ring. They are constructed using *motivic integration* at the "local" level of completions F_v; here F_v is identified with the field $k((t))$ of Laurent series, so that F_v^n can be considered as an infinite dimensional k-variety, more precisely, an inductive limit of arc spaces $t^{-m}k[[t]]^n \simeq \mathscr{L}(\mathbf{A}_k^n)$. Motivic Schwartz-Bruhat functions are elements of relative Grothendieck rings. The possibility to define the "sum over F^n" of a motivic function follows from the fact that it is zero outside of a finite dimensional subvariety of this ind-arc space. The Poisson summation formula then appears as a reformulation of the Riemann–Roch theorem for curves combined with the Serre duality theorem, as formulated in [19].

1.1. The Grothendieck ring of varieties with exponentials.

1.1.1. Let k be a field. The *Grothendieck group of varieties* KVar_k is defined by generators and relations; generators are k-varieties X (=k-schemes of finite type); relations are the following:

$$X - Y,$$

whenever X and Y are isomorphic k-varieties;

$$X - Y - U,$$

whenever X is k-variety, Y a closed subscheme of X and $U = X \setminus Y$ is the complementary open subscheme. Every k-constructible set X has a class $[X]$ in the group KVar_k.

The *Grothendieck group of varieties with exponentials* $\mathrm{KExpVar}_k$ is defined by generators and relations (cf. [10, 13]). Generators are pairs (X, f), where X is a k-variety and $f\colon X \to \mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[T])$ is a morphism. Relations are the following:

$$(X, f) - (Y, f \circ u)$$

whenever X, Y are k-varieties, $f\colon X \to \mathbf{A}^1$ a morphism, and $u\colon Y \to X$ a k-isomorphism;

$$(X, f) - (Y, f|_Y) - (U, f|_U)$$

whenever X is a k-variety, $f\colon X \to \mathbf{A}^1$ a morphism, Y a closed subscheme of X and $U = X \setminus Y$ the complementary open subscheme;

$$(X \times_{\mathbf{Z}} \mathbf{A}^1, \mathrm{pr}_2)$$

where X is a k-variety and pr_2 is the second projection. We will write $[X, f]$ to denote the class in $\mathrm{KExpVar}_k$ of a pair (X, f).

There is a morphism of Abelian groups $\iota\colon \mathrm{KVar}_k \to \mathrm{KExpVar}_k$ which sends the class of X to the class $[X, 0]$.

Any pair (X, f) consisting of a constructible set X and of a piecewise morphism $f\colon X \to \mathbf{A}^1$ has a class $[X, f]$ in $\mathrm{KExpVar}_k$.

1.1.2. One endows KVar_k with a ring structure by setting

$$[X][Y] = [X \times_k Y]$$

whenever X and Y are k-varieties. The unit element is the class of the point $\mathrm{Spec}(k)$.

One endows $\mathrm{KExpVar}_k$ with a ring structure by setting

$$[X, f][Y, g] = [X \times_k Y, \mathrm{pr}_1^* f + \mathrm{pr}_2^* g],$$

whenever X and Y are k-varieties, $f\colon X \to \mathbf{A}^1$ and $g\colon Y \to \mathbf{A}^1$ are k-morphisms; $\mathrm{pr}_1^* f + \mathrm{pr}_2^* g$ is the morphism from $X \times_k Y$ to $\mathbf{A}^1$ sending (x, y) to $f(x) + g(y)$. The unit element for this ring structure is the class $[\mathrm{Spec}(k), 0] = \iota([\mathrm{Spec}(k)])$.

The morphism $\iota\colon \mathrm{KVar}_k \to \mathrm{KExpVar}_k$ is a morphism of rings.

One writes $\mathbf{L}$ for the class of $\mathbf{A}^1_k$ in KVar_k, or for the class of $(\mathbf{A}^1_k, 0)$ in $\mathrm{KExpVar}_k$. Let S be the multiplicative subset of KVar_k generated by $\mathbf{L}$ and the elements $\mathbf{L}^n - 1$, for $n \geq 1$. The localizations of the rings KVar_k and $\mathrm{KExpVar}_k$ with respect to S are denoted $\mathscr{M}_k$ and $\mathscr{E}xp\mathscr{M}_k$ respectively. There is a morphism of rings $\iota\colon \mathscr{M}_k \to \mathscr{E}xp\mathscr{M}_k$.

LEMMA 1.1.3. [10, Lemma 3.1.3] *The two ring morphisms* $\iota\colon \mathrm{KVar}_k \to \mathrm{KExpVar}_k$ *and* $\iota\colon \mathscr{M}_k \to \mathscr{E}xp\mathscr{M}_k$ *are injective.*

Proof. For $t \in \mathbf{A}^1_k(k)$, let j_t be the map that sends a pair (X,f) to the class in KVar_k of the k-variety $[f^{-1}(t)]$. One observes that $j_0 - j_1$ defines a morphism of groups $j\colon \mathrm{KExpVar}_k \to \mathrm{KVar}_k$. Indeed, for every $t \in \mathbf{A}^1_k(k)$, j_t maps the additivity relations in $\mathrm{KExpVar}_k$ to additivity relations in KVar_k. Moreover, $j_t(Y \times \mathbf{A}^1_k, \mathrm{pr}_2) = [Y]$ for every k-variety Y, so that $j((Y \times \mathbf{A}^1_k, \mathrm{pr}_2)) = 0$. This proves the existence of j. By construction, ι is a section of j, hence ι is injective. $\square$

LEMMA 1.1.4. *Let X be a k-variety with a* $\mathbf{G}_{\mathrm{a}}$*-action and let* $f\colon X \to \mathbf{A}^1$ *be a morphism. Let $\overline{k}$ be an algebraic closure of k. Assume that $f(t+x) = t + f(x)$ for every $t \in \mathbf{G}_{\mathrm{a}}(\overline{k})$ and every $x \in X(\overline{k})$. Then, the class of (X,f) is zero in* $\mathrm{KExpVar}_k$.

Proof. By a theorem of Rosenlicht [18], there exists a $\mathbf{G}_{\mathrm{a}}$-stable dense open subset U and a quotient map $U \to Y$ which is a $\mathbf{G}_{\mathrm{a}}$-torsor. Every such torsor is locally trivial for the Zariski topology. Consequently, up to shrinking U (and Y accordingly), this $\mathbf{G}_{\mathrm{a}}$-torsor is trivial, so that there exists a $\mathbf{G}_{\mathrm{a}}$-equivariant isomorphism $u\colon \mathbf{G}_{\mathrm{a}} \times Y \simeq U$. Let $g\colon Y \to \mathbf{A}^1$ be the morphism given by $y \mapsto f(u(0,y))$. For $y \in Y(\overline{k})$ and $t \in \mathbf{A}^1_k(\overline{k})$, one has $f(u(t,y)) = f(t + u(0,y)) = t + f(u(0,y))$. This shows that the class of $(Y, f\circ u)$ equals the product of the classes of $(\mathbf{A}^1_k, \mathrm{Id})$ and (Y,g). It is zero in $\mathrm{KExpVar}_k$, so that the class of $(U, f|_U)$ is zero too. One concludes the proof by Noetherian induction. $\square$

1.1.5. Relative variants. Let S be a k-variety. One can define relative analogues KVar_S, $\mathrm{KExpVar}_S$, $\mathscr{M}_S$ and $\mathscr{E}xp\mathscr{M}_S$ of the above rings by replacing k-varieties by S-varieties in the definitions. We write $[X,f]_S \in \mathrm{KExpVar}_S$ for the class of a pair (X,f), where X is an S-variety and $f\colon X \to \mathbf{A}^1$ is a morphism.

Any morphism $u\colon S \to T$ of k-varieties induces morphisms $u_!$ and u^* between the corresponding Grothendieck groups. The definitions are similar; let us explain the case of KExpVar.

Let X be an S-variety and let $f\colon X \to \mathbf{A}^1$ be a morphism. Via the morphism $u\colon S \to T$, we may view X as a T-variety, so that (X,f) gives rise to a class $[X,f]_T$ in $\mathrm{KExpVar}_T$. This induces a morphism of groups

$$u_!\colon \mathrm{KExpVar}_S \longrightarrow \mathrm{KExpVar}_T, \quad [X,f]_S \longmapsto [X,f]_T.$$

If u is an immersion, then $u_!$ is a morphism of rings.

In the other direction, there is a unique morphism of rings

$$u^*\colon \mathrm{KExpVar}_T \longrightarrow \mathrm{KExpVar}_S$$

such that $u^*([X,f]_T) = (X \times_T S, f \circ \mathrm{pr}_1)$ for every pair (X,f) consisting of a T-variety X and of a morphism $f\colon X \to \mathbf{A}^1$.

Remark 1.1.6. Let $A = \mathbf{Z}[T]$ and B be the localization of A with respect to the multiplicative subset generated by T and the $T^n - 1$, for $n \geq 1$. The unique ring morphism from A to KVar_k which sends T to $\mathbf{L}$ endowes KVar_k and $\mathrm{KExpVar}_k$ with structures of A-algebras. Moreover, $\mathscr{M}_k \simeq B \otimes_A \mathrm{KVar}_k$ and $\mathscr{E}xp\mathscr{M}_k \simeq B \otimes_A \mathrm{KExpVar}_k$.

More generally, for every k-variety S, KVar_S and $\mathrm{KExpVar}_S$ are A-algebras, and one has natural isomorphisms

$$\mathscr{M}_S \simeq B \otimes_A \mathrm{KVar}_S \simeq \mathscr{M}_k \otimes_{\mathrm{KVar}_k} \mathrm{KVar}_S$$

and

$$\mathscr{E}xp\mathscr{M}_S \simeq B \otimes_A \mathrm{KExpVar}_S \simeq \mathscr{E}xp\mathscr{M}_k \otimes_{\mathrm{KExpVar}_k} \mathrm{KExpVar}_S.$$

Thanks to this remark, we will often allow ourselves to write formulas or proofs at the level of $\mathrm{KExpVar}_S$, when the generalization to $\mathscr{E}xp\mathscr{M}_S$ follows directly by localization.

1.1.7. Functional interpretation of the relative Grothendieck rings. Elements of $\mathrm{KExpVar}_S$ can be thought of as *motivic functions* with source S. In particular, for $\varphi \in \mathrm{KExpVar}_S$ and a point $s \in S$, considered as a morphism $\mathrm{Spec}(k(s)) \to S$, one writes $\varphi(s)$ for the element $s^*\varphi$ of $\mathrm{KExpVar}_{k(s)}$. By Lemma 1.1.8 below, a motivic function is determined by its values.

Let $u\colon S \to T$ be a morphism of k-varieties. The ring morphism $u^*\colon \mathrm{KExpVar}_T \to \mathrm{KExpVar}_S$ then corresponds to composition of functions.

If u is an immersion, the morphism of rings $u_!\colon \mathrm{KExpVar}_S \to \mathrm{KExpVar}_T$ corresponds in this interpretation to extension by zero. In the general case, we shall see that it corresponds to "summation over rational points" in the fibers of u.

LEMMA 1.1.8. *Let* $\varphi \in \mathrm{KVar}_S$ *(resp.* $\mathscr{M}_S$*, resp.* $\mathrm{KExpVar}_S$*, resp.* $\mathscr{E}xp\mathscr{M}_S$*). If* $\varphi(s) = 0$ *for every* $s \in S$*, then* $\varphi = 0$.

As a corollary, Lemmas 1.1.3 and 1.1.4 hold for relative Grothendieck groups.

Proof. We give the proof for $\mathrm{KExpVar}_S$, the other three cases are similar. Let us fix a representative M of φ in $\mathbf{Z}[\mathrm{ExpVar}_S]$, the free Abelian group generated by pairs (X,f), where X is an S-scheme and $f\colon X \to \mathbf{A}^1$ is a morphism. Let s be a generic point of S; since $\varphi(s) = 0$, the object $M_{k(s)}$ is a linear combination of elementary relations. By spreading out the varieties and the morphisms expressing these relations, there exists a dense open subset U of S such that the object M_U in $\mathbf{Z}[\mathrm{ExpVar}_U]$ is a linear combination of the corresponding elementary relations, hence one has $[M_U] = 0$. On the other hand, we have $[M_T](s) = 0$ for every point s in $T = S \setminus U$. By Noetherian descending induction it follows that $[M_T] = 0$. Thus $[M] = 0$, and $\varphi = 0$. $\square$

1.1.9. Exponential sums. The class θ of a pair (X,f) in $\mathrm{KExpVar}_k$ can be thought of as an analogue of the exponential sum

$$\sum_{x\in X(k)} \psi(f(x)),$$

when k is a finite field and $\psi\colon k\to \mathbf{C}^*$ is a fixed non-trivial additive character. This justifies the *notation* $\sum_{x\in X}\psi(f(x))$ for the class $[X,f]$ in $\mathrm{KExpVar}_k$.

More generally, let S be a k-variety, let $\theta\in \mathscr{E}xp\mathscr{M}_S$ and let $u\colon S\to\mathbf{A}^1$ be a morphism. We *define*

$$\sum_{s\in S}\theta(s)\psi(u(s)) = \theta\cdot[S,u]_S, \tag{1.1.10}$$

the product being taken in $\mathscr{E}xp\mathscr{M}_S$, and its result being viewed in $\mathscr{E}xp\mathscr{M}_k$. Let us make this definition explicit, assuming that $\theta=[X,f]_S$, where X is a S-variety and $f\colon X\to\mathbf{A}^1$ is a morphism; in this case,

$$\sum_{s\in S}\theta(s)\psi(u(s)) = [X,f]_S[S,u]_S = [X\times_S S, f\circ \mathrm{pr}_1 + u\circ\mathrm{pr}_2]_S = [X, f+u\circ g].$$

To support this notation, observe that when k is a finite field and $s\in S(k)$, denoting by g the morphisms $X\to S$, one has

$$\theta(s) = \sum_{\substack{x\in X(k)\\ g(x)=s}} \psi(f(x)),$$

so that

$$\begin{aligned}\sum_{s\in S(k)}\theta(s)\psi(u(s)) &= \sum_{s\in S(k)}\left(\sum_{\substack{x\in X(k)\\ g(x)=s}}\psi(f(x))\right)\psi(u(s))\\ &= \sum_{x\in X(k)}\psi\big(f(x)+u(g(x))\big).\end{aligned}$$

Let $u\colon S\to T$ be a morphism of k-varieties. This notation of "summation over rational points" is consistent with the functional interpretation of the morphism $u_!\colon \mathrm{KExpVar}_S\to\mathrm{KExpVar}_T$. Indeed, for every $\varphi\in\mathrm{KExpVar}_S$ and every $t\in T$, one has

$$u_!\varphi(t) = \sum_{s\in u^{-1}(t)}\varphi(s),$$

with notation similar to (1.1.10).

LEMMA 1.1.11. *Let V be a finite dimensional k-vector space, let f be a linear form on V. Then,*

$$\sum_{x\in V}\psi(f(x))=\begin{cases}\mathbf{L}^{\dim(V)} & \text{if } f=0;\\ 0 & \text{otherwise}.\end{cases}$$

Proof. By definition, the left-hand side is the class of (V,f) in $\mathrm{KExpVar}_k$. This equals $[V]=\mathbf{L}^{\dim(V)}$ if $f=0$. Otherwise, let $a\in V$ be such that $f(a)=1$ and let us consider the action of the additive group $\mathbf{G}_{\mathrm{a}}$ on V given by $(t,v)\mapsto v+ta$. Since $f(v+ta)=f(v)+t$, it follows from Lemma 1.1.4 that $[V,f]=0$. □

1.2. Local Fourier transforms.

1.2.1. Schwartz-Bruhat functions of given level and their integral. Let F° be a complete discrete valuation ring, with field of fractions F and perfect residue field k; we write $\operatorname{ord}\colon F\to\mathbf{Z}$ for the (normalized) valuation on F. We assume that F and k have the same characteristic; let us fix a section of the morphism $F^\circ\to k$, so that F° is a k-algebra. Every local parameter in F, i.e., every element $t\in F$ of valuation 1, then gives rise to isomorphisms $k[\![t]\!]\simeq F^\circ$ and $k(\!(t)\!)\simeq F$.

Fix such a local parameter t. For every two integers $M\leq N$, we can identify the quotient set $\{x;\ \operatorname{ord}(x)\geq M\}/\{x;\ \operatorname{ord}(x)\geq N\}=t^MF^\circ/t^NF^\circ$ of the elements x in F satisfying $\operatorname{ord}(x)\geq M$, modulo those satisfying $\operatorname{ord}(x)\geq N$, with the k-rational points of the affine space $\mathbf{A}_k^{(M,N)}=\mathbf{A}_k^{N-M}$, via the formula

$$x=\sum_{j=M}^{N-1}x_it^i \pmod{t^N}\longmapsto(x_M,\dots,x_{N-1}).$$

For every integer $n\geq 0$, let $\mathscr{S}(F^n;M,N)$ be the ring $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{n(M,N)}}$; its elements are called *motivic Schwartz-Bruhat function* of level (M,N) on F^n. We define the integral of such a function $\varphi\in\mathscr{S}(F^n;M,N)$ by the formula

$$\int_{F^n}\varphi(x)\,\mathrm{d}x=\mathbf{L}^{-nN}\sum_{x\in\mathbf{A}_k^{n(M,N)}}\varphi(x). \tag{1.2.2}$$

1.2.3. Compatibilities. The natural injection $t^MF^\circ/t^NF^\circ\to t^{M-1}F^\circ/t^NF^\circ$ is turned into a closed immersion

$$\iota\colon\mathbf{A}_k^{(M,N)}\longrightarrow\mathbf{A}_k^{(M-1,N)},\quad(x_M,\dots,x_{N-1})\longmapsto(0,x_M,\dots,x_{N-1}).$$

This gives rise to ring morphisms $\iota^*\colon\mathscr{S}(F^n;M-1,N)\to\mathscr{S}(F^n;M,N)$ (*restriction*) and $\iota_*\colon\mathscr{S}(F^n;M,N)\to\mathscr{S}(F^n;M-1,N)$ (*extension by zero*). One has $\iota^*\iota_*=\mathrm{Id}$.

Similarly, the natural projection $t^M F^\circ/t^{N+1}F^\circ \to t^M F^\circ/t^N F^\circ$ induces a morphism

$$\pi\colon \mathbf{A}_k^{(M,N+1)} \longrightarrow \mathbf{A}_k^{(M,N)}, \quad (x_M,\dots,x_N) \longmapsto (x_M,\dots,x_{N-1})$$

which is a trivial fibration with fiber $\mathbf{A}_k^1$. This gives rise to a ring morphism $\pi^*\colon \mathscr{S}(F^n;M,N) \to \mathscr{S}(F^n;M,N+1)$ and to a group morphism $\pi_*\colon \mathscr{S}(F^n;M,N+1) \to \mathscr{S}(F^n;M,N)$ (*integration over the fiber*). One has $\pi_*\pi^*(\varphi) = \mathbf{L}^n\varphi$ for every $\varphi \in \mathscr{S}(F^n;M,N)$.

The space of *motivic smooth functions* on F^n is then defined by

$$\mathscr{D}(F^n) = \varprojlim_{M,\iota^*} \varinjlim_{N,\pi^*} \mathscr{S}(F^n;M,N), \tag{1.2.4}$$

while the space of *motivic Schwartz-Bruhat functions* on F^n is defined by

$$\mathscr{S}(F^n) = \varinjlim_{M,\iota_*} \varinjlim_{N,\pi^*} \mathscr{S}(F^n;M,N). \tag{1.2.5}$$

These spaces have a ring structure, but $\mathscr{S}(F^n)$ has no unit element; the natural injection $\mathscr{S}(F^n) \subset \mathscr{D}(F^n)$ is a morphism of rings. We denote by $\mathbf{1}_{(F^\circ)^n}$ the class in $\mathscr{S}(F^n)$ of the unit element of $\mathscr{S}(F^n;0,0)$.

Observe that ι_* commutes with the sum over points, while π^* only commutes up to multiplication by $\mathbf{L}^n$. Consequently, the integral of a Schwartz-Bruhat function does not depend on the choice of a level (M,N) at which it is defined. This gives rise to an additive map $\mathscr{S}(F^n) \to \mathscr{E}xp\mathscr{M}_k$, denoted $\varphi \mapsto \int_{F^n}\varphi$. For every subset W of F^n whose characteristic function $\mathbf{1}_W$ is a motivic Schwartz-Bruhat function, one also writes $\int_W \varphi = \int_{F^n}\varphi\mathbf{1}_W$.

1.2.6. The Fourier kernel. Let $r\colon F \to k$ a non-zero k-linear map which vanishes on $t^a F^\circ$ for some integer a. We define the conductor ν of r as the smallest integer a such that r vanishes on $t^a F^\circ$.

In the sequel, our main source of such a linear form will be given by residues of differential forms. Assume that F is the completion at a closed point s of a function field in one variable over k, and let $\mathrm{res}_s\colon \Omega_{F/k} \to k$ be the residue map at the closed point s [20, p. 154]. Then fix some non-zero meromorphic differential form $\omega \in \Omega_{F/k}$ and set $r_s\colon F \to k$, $x \mapsto \mathrm{res}_s(x\omega)$. In this case, the conductor of r is equal to the order of the pole of ω (Theorem 2 of [20]; see also Section 1.3.7 below).

The kernel of the Fourier transform is the element of $\mathscr{D}(F^2)$ informally written

$$(x,y) \longmapsto \psi(r(xy)) = \mathrm{e}(xy).$$

Let us make explicit this definition.

Let $x \in F$, let us write $x = \sum_n x_n t^n$, where $x_n = 0$ for $n < \mathrm{ord}(x)$. One has $r(x) = \sum_{n=\mathrm{ord}(x)}^{\nu-1} x_n r(t^n)$. Consequently, restricted to the subset of F consisting of elements x such that $\mathrm{ord}(x) \leq M$, r can be interpreted as a linear morphism $r^{(M,N)} \colon \mathbf{A}_k^{(M,N)} \to \mathbf{A}_k^1$, for every integer N such that $N \geq \nu$.

Let $N'' = M + M' + \min(N - M, N' - M') = \min(M' + N, M + N')$. The product map $F \times F \to F$ gives rise to a morphism

$$\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \longrightarrow \mathbf{A}_k^{(M+M',N'')}.$$

Let us assume that $N'' \geq \nu$. Composing with $r^{(M+M',N'')}$, we obtain a morphism

$$\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \longrightarrow \mathbf{A}_k^1,$$

hence an element of $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')}}$, whose class in $\mathscr{D}(F^2)$ is our kernel.

1.2.7. Fourier transformation. The Fourier transform of a Schwartz-Bruhat function $\varphi \in \mathscr{S}(F; M, N)$ is defined formally as

$$\mathscr{F}\varphi(y) = \int_F \varphi(x) \mathrm{e}(xy) \, \mathrm{d}x.$$

More generally, we write $\langle x, y \rangle = \sum_{j=1}^n x_j y_j$ for the self-pairing of F^n and define the Fourier transform of a Schwartz-Bruhat function $\varphi \in \mathscr{S}(F^n; M, N)$ by

$$\mathscr{F}\varphi(y) = \int_{F^n} \varphi(x) \mathrm{e}(\langle x, y \rangle) \, \mathrm{d}x,$$

where, we recall, $\mathrm{e}(\cdot)$ is a short-hand notation for $\psi(r(\cdot))$.

Observe that $\varphi \mapsto \mathscr{F}\varphi$ is $\mathscr{E}xp\mathscr{M}_k$-linear.

Let us make the definition explicit, assuming that $n = 1$, φ is of the form $[U, f]$, where (U, g) is a $\mathbf{A}_k^{(M,N)}$-variety and $f \colon U \to \mathbf{A}^1$ is a morphism. Then, $\mathscr{F}\varphi$ is represented by

$$\mathbf{L}^{-N} \Big[U \times_g \mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')}, f(u) + r(xy) \Big]$$

in the Grothendieck group $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{(M',N')}}$, where we define $U \times_g \mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')}$ as the fiber product of the $\mathbf{A}_k^{(M,N)}$-varieties (U, g) and $(\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')}, \mathrm{pr}_1)$, viewed as an $\mathbf{A}_k^{(M',N')}$-variety, the structural morphism

$$U \times_g \mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \longrightarrow \mathbf{A}_k^{(M',N')}$$

being the projection to the third factor. For this to make sense, we only need to take $M' \leq \nu - N$ and $N' \geq \nu - M$.

PROPOSITION 1.2.8. *Let ν be the conductor of r. Then for every $\varphi \in \mathscr{S}(F^n; M, N)$, one has $\mathscr{F}\varphi \in \mathscr{S}(F^n; \nu - N, \nu - M)$.*

THEOREM 1.2.9. (Fourier inversion) *Let ν be the conductor of r. Then for every $\varphi \in \mathscr{S}(F^n; M, N)$, one has $\mathscr{F}\mathscr{F}\varphi(x) = \mathbf{L}^{-n\nu}\varphi(-x)$.*

Proof. For simplicity of notation, we assume that $n = 1$. We may assume that φ is represented by $[U, f]$ as above. To compute $\mathscr{F}\mathscr{F}\varphi$, we may set $(M', N') = (\nu - N, \nu - M)$ and $(M'', N'') = (M, N)$, so that $\mathscr{F}\mathscr{F}\varphi$ is represented by

$$\mathbf{L}^{-N-N'}\Big[U \times_g \mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \times \mathbf{A}_k^{(M'',N'')}, f(u) + r(xy) + r(yz)\Big].$$

The contribution of the part where $x + z \neq 0$ is zero, because of Lemma 1.1.4. The part where $x + z = 0$ is equal to

$$\begin{aligned}\mathbf{L}^{-N-N'}\Big[U \times_g \mathbf{A}_k^{(M',N')} \times \mathbf{A}_k^{(M'',N'')}, f(u)\Big] &= \mathbf{L}^{-N-M'}\Big[U \times_g \mathbf{A}_k^{(M'',N'')}, f(u)\Big] \\ &= \mathbf{L}^{-\nu}[U, f],\end{aligned}$$

where U is viewed as an $\mathbf{A}_k^{(M'',N'')}$-variety via the morphism $-g$. This proves the theorem. ☐

1.2.10. This theory is extended in a straightforward way to products of local fields. Let $(F_s)_{s\in S}$ be a finite family of fields as above, fields of fractions of complete discrete valuation rings F_s°, with local parameters t_s and residue fields k_s. Assume that for each s, k_s is a finite extension of k. In practice, one will start from the function field F of a (projective, smooth, geometrically connected) k-curve C, S will be a set of closed points of C, and for every $s \in S$, the field F_s will be the completion of F at the point s.

For every $s \in S$, write $\mathrm{Res}_{k_s/k}$ for the functor of Weil restriction of scalars; one has $\mathrm{Res}_{k_s/k}(\mathbf{A}_{k_s}^m) \simeq \mathbf{A}_k^{m[k_s:k]}$. For every family $(M_s, N_s)_{s\in S}$ of integers such that $M_s \leq N_s$, one then sets

$$V(n(M_s, N_s)) = \prod_{s\in S} \mathrm{Res}_{k_s/k}\mathbf{A}_{k_s}^{n(N_s - M_s)},$$

and defines the space of Schwartz-Bruhat functions of levels (M_s, N_s) on $\prod_{s\in S} F_s^n$ by

$$\mathscr{S}\left(\prod_{s\in S} F_s^n; (M_s, N_s)\right) = \mathscr{E}xp\mathscr{M}_{V(n(M_s,N_s))}. \tag{1.2.11}$$

One then sets

$$\mathscr{S}\left(\prod_{s\in S} F_s^n\right) = \varinjlim_{(M_s,\iota_*)} \varinjlim_{(N_s,\pi^*)} \mathscr{S}\left(\prod_{s\in S} F_s^n;(M_s,N_s)\right). \tag{1.2.12}$$

There is a natural morphism of rings

$$\bigotimes_{s\in S}\mathscr{S}(F_s^n)\longrightarrow \mathscr{S}\left(\prod_{s\in S} F_s^n\right).$$

Contrary to the classical arithmetic case, it is not surjective in general.

The definition of the integral extends to a linear map $\mathscr{S}(\prod_{s\in S} F_s^n)\to \mathscr{E}xp\mathscr{M}_k$.

Let $(r_s\colon F_s\to k)$ be a family of non-trivial k-linearmaps, let ν_s be the conductor of r_s. Then the definition of the Fourier Transform $\mathscr{F}$ extends naturally to $\mathscr{S}(\prod_{s\in S} F_s^n)$. For every $\varphi\in\mathscr{S}(\prod_{s\in S} F_s^n)$, one sets

$$\mathscr{F}\varphi((y_s)) = \int_{\prod_{F_s^n}} \varphi(x)e\left(\sum\langle x_s,y_s\rangle\right)\prod \mathrm{d}x_s.$$

Fourier inversion still holds, with the same proof:

$$\mathscr{F}\mathscr{F}\varphi(x) = \mathbf{L}^{-n\sum[k_s:k]\nu_s}\varphi(-x).$$

1.3. Global Fourier transforms.

1.3.1. Let k be a perfect field, let C be a projective, geometrically connected, smooth curve over k, and let $F=k(C)$ be its field of functions. We fix a non-zero meromorphic differential form $\omega\in\Omega^1_{F/k}$.

One can interpret the field $F=k(C)$ as the k-points of an ind-k-variety. The simplest way to do so consists maybe in considering the family of all Riemann-Roch spaces $\mathscr{L}(D)=\mathrm{H}^0(C,\mathscr{O}(D))$, indexed by effective divisors D on C. Concretely, $\mathscr{L}(D)$ is the set of non-zero rational functions f on C such that $\operatorname{div}(f)+D\geq 0$, together with the 0 function. It is a finite dimensional k-vector space and we view it as a k-variety. The natural inclusions from $\mathscr{L}(D)$ to $\mathscr{L}(D')$, where D and D' are effective divisors such that $D'-D$ is effective, give this family the structure of an inductive system, the limit of which is interpreted as $k(C)$.

1.3.2. Global Schwartz-Bruhat functions. For every closed point $s\in C$, write ord_s for the corresponding normalized valuation on F, F_s for the its completion, and F_s° for the valuation ring of F_s; we also fix a local parameter t_s at s.

The adele ring $\mathbb{A}_F$ of F is the subring of $\prod_{s\in C}F_s$ consisting of families (x_s) such that $x_s\in F_s^\circ$ for all but finitely many s. (By abuse of notation, the condition "$s\in C$" means that s belongs to the set of closed points of C.)

In the classical arithmetic setting, the ring $\mathbb{A}_F$ has a locally compact totally disconnected topology, and the space of Schwartz-Bruhat functions on $\mathbb{A}_F^n$ is the ring of real valued locally constant with compact support on $\mathbb{A}_F^n$.

We now describe its geometric analogue $\mathscr{S}(\mathbb{A}_F^n)$.

Let S and S' be finite sets of closed points of C such that $S \subset S'$. There is a natural morphism of rings:

$$j_S^{S'} : \mathscr{S}\left(\prod_{s\in S} F_s^n\right) \longrightarrow \mathscr{S}\left(\prod_{s\in S'} F_s^n\right), \quad \varphi \longmapsto \varphi \otimes \bigotimes_{s\in S'\setminus S} \mathbf{1}_{(F_s^\circ)^n}.$$

The ring $\mathscr{S}(\mathbb{A}_F^n)$ of global motivic Schwartz-Bruhat function on $\mathbb{A}_F^n$ is defined by

$$\mathscr{S}\left(\mathbb{A}_F^n\right) = \varinjlim_{S\subset C, j_S^{S'}} \mathscr{S}\left(\prod_{s\in S} F_s^n\right).$$

It is important to observe that the global motivic Schwartz-Bruhat functions on $\mathbb{A}_F^n$ induce the characteristic function of $(F_s^\circ)^n$ at all but finitely many closed points $s \in C$. This is a notable difference with the arithmetic setting.

1.3.3. Simple functions. In the classical arithmetic case, simple functions are characteristic functions of a ball, or of products of balls. Let us describe their analogues in the motivic setting. Let S be a finite subset of closed points of C, let $a = (a_s)_{s\in S} \in \prod_{s\in S} F_s$, let $(M_s, N_s)_{s\in S}$ be a family of pairs of integers such that $\operatorname{ord}(a_s) \geq M_s$ for every $s \in S$. Let $W = \prod_{s\in S} \operatorname{Res}_{k_s/k} \mathbf{A}_{k_s}^{n(M_s,N_s)}$, let $W_a = \operatorname{Spec}(k)$ and let $W_a \to W$ be the canonical map induced, for every $s \in S$, by the t_s-adic expansion of a_s. The motivic function on W associated with the pair $(W_a \to W, 0)$ is called a *simple function*. The corresponding Schwartz-Bruhat function on $\prod_{s\in S} F_s^n$ represents the characteristic function of the product of the balls of centers a_s and radius N_s in F_s^n.

More generally, let us consider a k-variety Z and a morphism $u = (u_s)\colon Z \to W$; let $\varphi \in \mathscr{E}xp\mathscr{M}_{W\times_k Z}$ be the motivic function associated with $(Z, 0)$, where Z is considered as a $W \times_k Z$-variety through the morphism $u \times \operatorname{Id}_Z$. For each $z \in Z$, we write φ_z for the motivic function on $W_{k(z)}$ deduced from φ. When $z \in Z(k)$, the corresponding Schwartz-Bruhat function on $\prod_{s\in S} F_s^n$ represents the characteristic function of the product of the polydiscs of radius N_s and centers $u_s(z)$. Consequently, we call φ a *family of simple functions parameterized by the k-variety Z*.

Let $\chi \in \mathscr{E}xp\mathscr{M}_Z$ be a motivic function on Z, represented by $[X \xrightarrow{g} Z, f]_Z$, where X is a Z-variety and $f\colon X \to \mathbf{A}^1$ is a morphism. We then define the Schwartz-Bruhat function $\sum_{z\in Z} \chi(z)\varphi_z$ on $\mathbb{A}_F^n$ as the one represented by the pair $[X \xrightarrow{u\circ g} W, f]$. By linearity, this definition is extended to every element χ of $\mathscr{E}xp\mathscr{M}_Z$.

LEMMA 1.3.4. *Any global Schwartz-Bruhat function on $\mathbb{A}_F^n$ can be written in this way.*

Proof. Let Φ be a global Schwartz-Bruhat function on $\mathbb{A}_F^n$, represented by a pair $[Z,f]$, where Z is a variety over $W = \prod_s \mathrm{Res}_{k_s/k}\mathbf{A}_{k_s}^{n(M_s,N_s)}$, for some finite set S of closed points of C and integers (M_s,N_s), and $f\colon Z \to \mathbf{A}^1$. Let φ be the family of simple functions parameterized by W given by the pair $(W \xrightarrow{\mathrm{Id}} W, 0)$. One checks readily that $\Phi = \sum_{w\in W} \Phi\varphi_w$. □

1.3.5. Summation over rational points. Let φ be a global Schwartz-Bruhat function on $\mathbb{A}_F^n$, represented by a class φ_S in $\mathscr{E}xp\mathscr{M}_k\big(\prod_{s\in S}\mathrm{Res}_{k_s/k}\mathbf{A}_{k_s}^{n(M_s,N_s)}\big)$, for some finite set S of closed points of C and some family $(M_s,N_s)_{s\in S}$.

Consider the divisor $D = -\sum M_s[s]$ on C. For every $s\in S$, the natural embedding of $F = k(C)$ into the field F_s maps $\mathscr{L}(D)$ into $t^{M_s}F_s^\circ$. This gives rise to a morphism of algebraic varieties $\alpha\colon \mathscr{L}(D)^n \to \big(\prod_{s\in S}\mathrm{Res}_{k_s/k}\mathbf{A}_k^{(M_s,N_s)}\big)^n$. We then define $\sum_{x\in F^n}\varphi(x)$ as the image in $\mathscr{E}xp\mathscr{M}_k$ of the element $\alpha^*\varphi_S$ of $\mathscr{E}xp\mathscr{M}_{\mathscr{L}(D)^n}$. It does not depend on the choice of the set S nor on the choice of the integers (M_s,N_s) and of the class φ_S.

Let us give a more explicit formula, assuming that φ_S is of the form $[X,f]$, where $W = \prod_{s\in S}\mathrm{Res}_{k_s/k}\mathbf{A}_{k_s}^{n(M_s,N_s)}$, X is a W-variety and $f\colon X\to\mathbf{A}^1$ is a morphism. In that case, one has

$$\sum_{x\in F^n}\varphi(x) = [\mathscr{L}(D)^n \times_W X, f\circ \mathrm{pr}_2]. \tag{1.3.6}$$

1.3.7. Reminders on residues and duality for curves. We need to recall a few results concerning residues, duality and the Riemann-Roch theorem on smooth curves.

We fix a non-zero meromorphic differential form $\omega\in\Omega_{F/k}$. Let ν_s be the order of the pole, or minus the order of the zero, of ω at s, and let ν be the divisor $\sum\nu_s[s]$ on C. One has $\deg(\nu) = 2-2g$, where g is the genus of C.

For every closed point $s\in C$, we define a map $r_s\colon F_s\to k$ by $r_s(x) = \mathrm{res}_{C,s}(x\omega)$, where $\mathrm{res}_{C,s}\colon \Omega_{F/k}\to k$ is Tate's residue [20] on the curve C at s; since the field k is perfect, it is non-zero and its conductor is equal to ν_s. This follows from Theorem 2 of [20] if s is a rational point of the curve C; in the general case, one checks that

$$\mathrm{res}_{C,s}(\omega) = \mathrm{Tr}_{k(s)/k}(\mathrm{res}_{C_{k(s)},s}(\omega)),$$

where we indicated the curve as in index.

Let D be a divisor on C. Let $\mathscr{L}(D)$ be the set of rational functions $y\in F^\times$ such that $\mathrm{div}(y)+D\geq 0$ to which we adjoin 0; this is a finite dimensional k-vector space.

Let $\Omega(D)$ be the set of meromorphic forms $\alpha \in \Omega_{F/k}$ such that $\operatorname{div}(\alpha) \geq D$ (together with $\alpha = 0$). The map $y \mapsto y\omega$ from F to $\Omega^1_{F/k}$ induces an isomorphism $\mathscr{L}(\operatorname{div}(\omega) - D) \to \Omega(D)$.

We embed F diagonally in $\mathbb{A}_F$. For every divisor D, let $\mathbb{A}_F(D)$ be the subspace of $\mathbb{A}_F$ consisting of families (x_s) such that $\operatorname{div}(x_s) + \operatorname{ord}_s(D) \geq 0$ for every closed point $s \in C$. There is an isomorphism of k-vector spaces (see [19, Chapitre II, Section 5, proposition 3]; see also [20, p. 157])

$$\mathrm{H}^1(\mathscr{L}(D)) \simeq \mathbb{A}_F/(\mathbb{A}_F(D) + F).$$

According to Serre's duality theorem (see [19, Chapitre II, Section 8, théorème 2]; see also [20, Theorem 5]), the morphism

$$\theta\colon \Omega_{F/k} \longrightarrow \operatorname{Hom}(\mathbb{A}_F, k), \quad \alpha \longmapsto \left((x_s) \longmapsto \sum_s \operatorname{res}_s(x_s\alpha) \right)$$

identifies $\Omega(D)$ with the orthogonal of $\mathbb{A}_F(D) + F$ in $\operatorname{Hom}(\mathbb{A}_F, k)$, i.e., with the dual of $\mathrm{H}^1(\mathscr{L}(D))$. This contains the theorem of residues according to which

$$\sum_{s \in C} \operatorname{res}_s(x\omega) = 0$$

for every $x \in F$.

1.3.8. Global Fourier transformation. Observe that if s is any closed point of C such that $\nu_s = 0$, then $\mathbf{1}_{F_s^\circ}$ is its own Fourier transform. Consequently, we may define the Fourier transform $\mathscr{F}\varphi$ of every global Schwartz-Bruhat function $\varphi \in \mathscr{S}(\mathbb{A}_F^n)$ as the image in $\mathscr{S}(\mathbb{A}_F^n)$ of $\mathscr{F}\varphi_S$, where S is any finite set of places such that $\nu_s = 0$ for $s \notin S$, and $\varphi_S \in \mathscr{S}(\prod_{s \in S} F_s^n)$ is a representative of φ. By construction, $\mathscr{F}\varphi$ is itself a global Schwartz-Bruhat function on the "dual" space $\mathbb{A}_F^n$.

THEOREM 1.3.9. (Fourier inversion formula) *For every $\varphi \in \mathscr{S}(\mathbb{A}_F^n)$, one has*

$$\mathscr{F}\mathscr{F}\varphi(x) = \mathbf{L}^{n(2g-2)}\varphi(-x).$$

Proof. When φ is a simple function, this is nothing but the Fourier inversion formula 1.2.9. The general case follows from Lemma 1.3.4. Indeed, if φ is written as a sum of simple functions $\sum \varphi(z)\psi_z$, it follows from the definitions that $\mathscr{F}\varphi = \sum \varphi(z)\mathscr{F}\psi_z$, so that

$$\begin{aligned}\mathscr{F}\mathscr{F}\varphi(x) &= \sum \varphi(z)\mathscr{F}\mathscr{F}\psi_z(x) = \sum \varphi(z)\mathbf{L}^{n(2g-2)}\psi_z(-x) \\ &= \mathbf{L}^{n(2g-2)}\varphi(-x).\end{aligned}$$

□

THEOREM 1.3.10. (Motivic Poisson formula) *Let* $\varphi \in \mathscr{S}(\mathbb{A}_F^n)$. *Then,*

$$\sum_{x\in F^n} \varphi(x) = \mathbf{L}^{(1-g)n} \sum_{y\in F^n} \mathscr{F}\varphi(y).$$

Proof. For simplicity of notation, we assume that $n=1$. By Lemma 1.3.4, we may also assume that φ is a simple function $\otimes_{s\in S}\varphi_s$, where for each $s\in S$, φ_s is the characteristic function of the ball of center $a_s \in F_s$ and radius N_s. Let D be the divisor $\sum_{s\in S} N_s s$ on C.

For every $s\in S$, $\mathscr{F}\varphi_s$ is a Schwartz-Bruhat function on F_s and

$$\mathscr{F}\varphi(y_s) = \begin{cases} \psi\big(\operatorname{res}_s\big(a_s y_s \omega\big)\big)\mathbf{L}^{-N_s} & \text{if } \operatorname{ord}_s(y_s)+\operatorname{ord}_s(D)\geq 0; \\ 0 & \text{otherwise.}\end{cases}$$

Then $\mathscr{F}\varphi$ is a global Schwartz-Bruhat function on $\mathbb{A}_F$, represented by $\bigotimes_{s\in S}\mathscr{F}\varphi_s$ and

$$\mathscr{F}\varphi(y) = \begin{cases} \psi\left(\sum_{s\in S}\operatorname{res}_s\big(a_s y_s\omega\big)\right)\mathbf{L}^{-\deg(D)} & \text{if } \operatorname{div}(y\omega)+D\geq 0; \\ 0 & \text{otherwise.}\end{cases}$$

Recall that the map $y\mapsto y\omega$ identifies $\mathscr{L}(\operatorname{div}(\omega)+D)$ with $\Omega(-D)$. Let $f\colon \mathscr{L}(\operatorname{div}(\omega)+D)\to k$ be the linear map $y\mapsto \langle\theta(y\omega),(a_s)\rangle$; it is identically zero if and only if (a_s) belongs to the orthogonal $\Omega(-D)^\perp$ of $\Omega(-D)$ with respect to the Serre duality pairing. By Lemma 1.1.11, we thus have

$$\begin{aligned}\sum_{y\in F}\mathscr{F}\varphi(y) &= \mathbf{L}^{-\deg(D)} \sum_{y\in\mathscr{L}(\operatorname{div}(\omega)+D)} \psi(f(y)) \\ &= \begin{cases}\mathbf{L}^{-\deg(D)+\dim\mathscr{L}(\operatorname{div}(\omega)+D)} & \text{if } (a_s)\in\Omega(-D)^\perp, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Moreover, the Riemann-Roch formula asserts that

$$\begin{aligned}\dim\mathscr{L}(-D) &= \dim \mathrm{H}^1(C,-D)-\deg(D)+1-g \\ &= \dim\Omega(-D)-\deg(D)+1-g \\ &= \dim\mathscr{L}(\operatorname{div}(\omega)+D)-\deg(D)+1-g.\end{aligned}$$

Consequently,

$$\mathbf{L}^{1-g}\sum_{y\in F}\mathscr{F}\varphi(y) = \begin{cases}\mathbf{L}^{\dim\mathscr{L}(-D)} & \text{if } (a_s)\in\Omega(-D)^\perp, \\ 0 & \text{otherwise.}\end{cases}$$

Let us now compute the left-hand side of the Poisson formula. In the case where

$$(a_s) \in \Omega(-D)^{\perp} = \mathbb{A}_F(-D) + F,$$

there exists $a \in F$ such that $\mathrm{ord}_s(a - a_s) \geq N_s$ for all s. Then,

$$\varphi(x) = \begin{cases} 1 & \text{if } x - a \in \mathscr{L}(-D), \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\sum_{x \in F} \varphi(x) = \sum_{x \in F} \varphi(x - a) = \sum_{x \in \mathscr{L}(-D)} 1 = \mathbf{L}^{\dim \mathscr{L}(-D)}.$$

In the other case, there does not exist any $a \in k(C)$ such that $\mathrm{ord}_s(a - a_s) \geq N_s$ for all s. Then, $\varphi(x) = 0$ for all $x \in F$ and $\sum_{x \in F} \varphi(x) = 0$. In both cases, this concludes the proof of the motivic Poisson formula. □

Remark 1.3.11. By Fourier inversion, we have $\mathscr{F}\mathscr{F}\varphi(x) = \mathbf{L}^{-n\deg(\nu)}\varphi(-x) = \mathbf{L}^{(2-2g)n}\varphi(-x)$. Consequently, if we apply the Poisson formula to $\mathscr{F}\varphi$, we obtain

$$\sum_{y \in F^n} \mathscr{F}\varphi(y) = \mathbf{L}^{(1-g)n} \sum_{x \in F^n} \mathscr{F}\mathscr{F}\varphi(x) = \mathbf{L}^{(g-1)n} \sum_{x \in F^n} \varphi(x),$$

as expected.

2. Further preliminaries.

2.1. Motivic invariants. Let k be a field. For every $m \geq 0$, let $\mathrm{KVar}_k^{\leq m}$ be the subgroup of KVar_k generated by classes of varieties of dimension $\leq m$. If $x \in \mathrm{KVar}_k^{\leq m}$ and $y \in \mathrm{KVar}_k^{\leq n}$, then $xy \in \mathrm{KVar}_k^{\leq m+n}$. Let $(\mathscr{M}_k^{\leq m})_{m \in \mathbf{Z}}$ be the similar filtration on $\mathscr{M}_k$; explicitly, $\mathscr{M}_k^{\leq m}$ is generated by fractions $[X][Y]^{-1}$ where X is a k-variety, Y is a product of varieties of the form $\mathbf{A}^1$, $\mathbf{A}^a \setminus \{0\}$ (for $a \geq 1$), and $\dim(X) - \dim(Y) \leq m$. For every class $x \in \mathscr{M}_k$, let $\dim(x) \in \mathbf{Z} \cup \{-\infty\}$ be the infimum of the integers $m \in \mathbf{Z}$ such that $x \in \mathscr{M}_k^{\leq m}$. For every $x, y \in \mathscr{M}_k$, one has $\dim(x + y) \leq \max(\dim(x), \dim(y))$ and $\dim(xy) \leq \dim(x) + \dim(y)$; moreover, $\dim(x\mathbf{L}^n) = \dim(x) + n$ for every $n \in \mathbf{Z}$.

Assume that k is algebraically closed. For every k-variety X, we denote by $\mathrm{H}^p(X)$ (resp. $\mathrm{H}^p_{\mathrm{c}}(X)$) its pth singular cohomology group (resp. with compact support) and $\mathbf{Q}$-coefficients (if $k = \mathbf{C}$), or its pth étale cohomology group (resp. with proper support) and $\mathbf{Q}_\ell$-coefficients (for some fixed prime number ℓ distinct from the characteristic of k). There is a unique ring morphism $\boxdot$ from KVar_k to the polynomial ring $\mathbf{Z}[t]$ such that for every variety X, $\boxdot([X])$ is the Poincaré polynomial of X. Its definition relies on the weight filtration on the cohomology groups with compact support of X. If k has characteristic zero (which will be the case below), the morphism $\boxdot$ is characterized by its values on projective smooth varieties: for every such X, one has

$$\boxdot([X]) = \boxdot_X(t) = \sum_{p=0}^{2\dim(X)} \dim(\mathrm{H}^p(X))t^p.$$

This implies that for every variety X, the leading term of $\boxdot([X])$ is given by $\kappa(X)t^{2\dim(X)}$, where $\kappa(X)$ is the number of irreducible components of X of dimension $\dim(X)$.

One has $\boxdot(\mathbf{L}) = \boxdot([\mathbf{P}^1]) - 1 = t^2$; for every $a \geq 1$, $\boxdot(\mathbf{L}^a - 1) = t^{2a} - 1 = t^{2a}(1 - t^{-2a})$ is invertible in the ring $\mathbf{Z}[\![t^{-1}]\!][t]$, with inverse

$$\sum_{m\geq 1} t^{-2ma}.$$

Consequently, the morphism $\boxdot$ extends uniquely to a ring morphism from $\mathscr{M}_k$ to the ring $\mathbf{Z}[\![t^{-1}]\!][t]$. For every element $x \in \mathscr{M}_k$, one has

$$\dim(x) \geq \frac{1}{2}\deg(\boxdot(x)).$$

LEMMA 2.1.1. *Let A be a ring and let $P_1, \dots, P_r \in A[T]$ be polynomials with coefficients in A. Assume that for every i, the leading coefficient of P_i is a unit in A, and that for every distinct i, j, the resultant of P_i and P_j is a unit in A. Then, for every polynomial $P \in A[T]$ and every family $(n_1, \dots, n_r)$ of nonnegative integers, there exists a unique family $(Q_{i,j})$ of polynomials in $A[T]$, indexed by pairs of integers (i, j) such that $1 \leq i \leq r$ and $1 \leq j \leq n_i$, and a unique polynomial $Q \in A[T]$ such that* $\deg(Q_{i,j}) \leq \deg(P_i) - 1$ *for all i, j and*

$$P(T) = Q(T)\prod_{i=1}^{r} P_i(T)^{n_i} + \sum_{i=1}^{r}\sum_{j=1}^{n_i} Q_{i,j}(T)P_i(T)^{n_i-j}\prod_{k\neq i} P_k(T)^{n_k}.$$

Since the leading coefficient of P_i is a unit, P_i is not a zero divisor in $A[T]$. Observe that the last equality is a decomposition into partial fractions

$$\frac{P(T)}{\prod_{i=1}^{r} P_i(T)^{n_i}} = Q(T) + \sum_{i=1}^{r}\sum_{j=1}^{n_i} \frac{Q_{i,j}(T)}{P_i(T)^j}$$

in the total ring of fractions of $A[T]$.

Proof. First assume that $r = 1$. In this case, the desired assertion follows from considering the Euclidean divisions by P_1 of the polynomial P, of its quotient, etc.

$$\begin{aligned} P(T) &= Q_1(T)P_1(T) + R_1(T) \\ &= Q_2(T)P_1(T)^2 + R_2(T)P_1(T) + R_1(T) \\ &= \cdots \\ &= Q_{n_1}(T)P_1(T)^{n_1} + R_{n_1}(T)P_1(T)^{n_1-1} + \cdots + R_1(T), \end{aligned}$$

where $Q_1, \ldots, Q_{n_1} \in A[T]$ and $R_1, \ldots, R_{n_1}$ are polynomials of degrees $\leq \deg(P_1) - 1$.

Now assume $r \geq 2$. Let i, j be distinct integers in $\{1, \ldots, r\}$. By the assumption and basic properties of the resultant, there exist polynomials $U, V \in A[T]$ such that $1 = UP_i + VP_j$. Consequently, the ideals (P_i) and (P_j) generate the unit ideal of $A[T]$. By induction on $n_1, \ldots, n_r$, it follows that the ideals $(\prod_{k \neq i} P_k^{n_k})$, for $1 \leq i \leq r$, are comaximal in $A[T]$. Therefore, there exist polynomials $U_1, \ldots, U_r \in A[T]$ such that

$$1 = \sum_{i=1}^{r} U_i(T) \prod_{k \neq i} P_k(T)^{n_k}.$$

By the case $r = 1$ applied to the polynomials $U_i(T)$ and $P_i(T)$, we obtain the desired decomposition.

Uniqueness is left to the reader. □

LEMMA 2.1.2. *Let a, a' be nonnegative integers and b, b' be positive integers. Let $d = \gcd(b, b')$. Then,*

$$\operatorname{Res}(1 - \mathbf{L}^a T^b, 1 - \mathbf{L}^{a'} T^{b'}) = (-1)^{b'} \mathbf{L}^{ab'} (1 - \mathbf{L}^{(a'b - ab')/d})^d.$$

In particular, this resultant is a unit in $\mathscr{M}_k$ if (a, b) and (a', b') are not proportional.

Proof. It is sufficient to prove this formula when the ring $\mathscr{M}_k$ is replaced by the ring $A = \mathbf{C}[\mathbf{L}^{\pm 1/bb'}]$ of Laurent polynomials in an indeterminate $\mathbf{L}^{1/bb'}$. Then, the polynomial $1 - \mathbf{L}^a T^b$ is split in A; this leads to the explicit elementary computation

$$\begin{aligned} \operatorname{Res}(1 - \mathbf{L}^a T^b, 1 - \mathbf{L}^{a'} T^{b'}) &= (-\mathbf{L}^a)^{b'} \prod_{\zeta^b = 1} (1 - \zeta^{b'} \mathbf{L}^{a' - ab'/b}) \\ &= (-1)^{b'} \mathbf{L}^{ab'} \prod_{\xi^{b/d} = 1} (1 - \xi \mathbf{L}^{a' - ab'/b})^d \\ &= (-1)^{b'} \mathbf{L}^{ab'} (1 - \mathbf{L}^{(a' - ab'/b)b/d})^d \\ &= (-1)^{b'} \mathbf{L}^{ab'} (1 - \mathbf{L}^{(a'b - ab')/d})^d. \end{aligned}$$ □

PROPOSITION 2.1.3. *Let $Z(T)=\sum_{n\in\mathbf{Z}}[M_n]T^n\in \mathrm{KVar}_k^+[\![T]\!][T^{-1}]$ be a Laurent series with effective coefficients in* KVar_k.

Let a and d be positive integers and let $P(T)=(1-\mathbf{L}^aT^a)^dZ(T)$. Assume that $P(T)$ belongs to $\mathscr{M}_k\{T\}^\dagger$ and that $P(\mathbf{L}^{-1})$ is an effective non-zero element of $\mathscr{M}_k$. Then, for every $p\in\{0,\dots,a-1\}$, one of the following cases occur when n tends to infinity in the congruence class of p modulo a:

(1) *Either* $\dim(M_n)=\mathrm{o}(n)$,

(2) *Or* $\dim(M_n)-n$ *has a finite limit and* $\frac{\log(\kappa(M_n))}{\log(n)}$ *converges to some integer in* $\{0,\dots,d-1\}$.

Moreover, the second case happens at least once.

Proof. Without lack of generality, we assume that $Z(T)$ is a power series. Set $a_1=b_1=a$ and $d_1=d$. By assumption, there exist a finite family $(a_i,b_i)_{2\le i\le r}$ of integers such that $0\le a_i<b_i$ for all $i\ge 2$, and integers d_i such that

$$Q(T)=Z(T)\prod_{i=1}^r(1-\mathbf{L}^{a_i}T^{b_i})^{d_i}$$

is a polynomial in $\mathscr{M}_k[T]$. Using the fact that $1-\mathbf{L}^mT^n$ divides $1-\mathbf{L}^{mp}T^{np}$ for every positive integer p, we may assume that no two pairs (a_i,b_i) and (a_j,b_j) are proportional.

For every $i\in\{1,\dots,r\}$, set $P_i(T)=1-\mathbf{L}^{a_i}T^{b_i}$; its leading coefficient is invertible in $\mathscr{M}_k$. Moreover, for i and j such that $1\le i<j\le r$, it follows from Lemma 2.1.2 that the resultant of P_i and P_j is a unit in $\mathscr{M}_k$. Thus, by decomposition in partial fractions (Lemma 2.1.1), there exist polynomials Q_0 and $Q_{i,j}$ in $\mathscr{M}_k[T]$ such that

$$Z(T)=Q_0(T)+\sum_{i=1}^r\sum_{j=1}^{d_i}\frac{Q_{i,j}(T)}{(1-\mathbf{L}^{a_i}T^{b_i})^j} \tag{2.1.4}$$

and $\deg(Q_{i,j})\le b_i-1$ for every $i\in\{1,\dots,r\}$.

For $i\in\{1,\dots,r\}$ and $j\in\{1,\dots,d_i\}$, write $Q_{i,j}=\sum_{n=0}^{b_i-1}q_{i,j,n}T^n$, for some elements $q_{i,j,n}\in\mathscr{M}_k$. This leads to the following power expansion in $\mathscr{M}_k[\![T]\!]$:

$$\begin{aligned}Z(T)&=Q_0(T)+\sum_{i=1}^r\sum_{j=1}^{d_i}\sum_{n=0}^{b_i-1}q_{i,j,n}T^n\sum_{m=0}^\infty\binom{j+m-1}{j-1}\mathbf{L}^{a_im}T^{b_im}\\&=Q_0(T)+\sum_{n=0}^\infty\left(\sum_{i=1}^r\sum_{j=1}^{d_i}\binom{j+\lfloor n/b_i\rfloor-1}{j-1}q_{i,j,n \bmod b_i}\mathbf{L}^{a_i\lfloor n/b_i\rfloor}\right)T^n,\end{aligned}$$

so that for every $n > \deg(Q_0)$, one has

$$[M_n] = \sum_{i=1}^{r} \sum_{j=1}^{d_i} \binom{j + \lfloor n/b_i \rfloor - 1}{j-1} q_{i,j,n \bmod b_i} \mathbf{L}^{a_i \lfloor n/b_i \rfloor}. \tag{2.1.5}$$

For every i, j, define

$$[M_n]^{i,j} = \binom{j + \lfloor n/b_i \rfloor - 1}{j-1} q_{i,j,n \bmod b_i} \mathbf{L}^{a_i \lfloor n/b_i \rfloor} \tag{2.1.6}$$

and

$$[M_n]^{i} = \sum_{j=1}^{d_i} [M_n]^{i,j}, \tag{2.1.7}$$

so that

$$[M_n] = \sum_{i=1}^{r} [M_n]^{i}. \tag{2.1.8}$$

It follows directly from the definitions that for every $i \geq 1$ and every $n \geq 1$, $\dim([M_n]^i) \leq (a_i/b_i)n + \mathrm{O}(1)$. Since $a_i \leq b_i$ for all i, this implies $\dim([M_n]) \leq n + \mathrm{O}(1)$. We will now show that when n belongs to appropriate congruence classes modulo a, one has the equality $\dim([M_n]^1) = n + \mathrm{O}(1)$. Since $a_i < b_i$ for $i \geq 2$ and $a_1 = b_1 = a$, this will imply the relations $\dim([M_n]) = n + \mathrm{O}(1)$ and $\kappa([M_n]) = \kappa([M_n]^1)$ (for n large enough in this congruence class).

Let n be any integer $> \deg(Q_0)$, let $n = am + \overline{n}$ be the Euclidean division of n by a. It follows from the definition of $[M_n]^1$ that

$$[M_n]^1 \mathbf{L}^{-n} = \sum_{j=1}^{d} \binom{j+m-1}{j-1} q_{1,j,\overline{n}} \mathbf{L}^{am-n} = \sum_{j=1}^{d} \binom{j+m-1}{j-1} q_{1,j,\overline{n}} \mathbf{L}^{-\overline{n}}.$$

It follows from Equation (2.1.4) that

$$P(\mathbf{L}^{-1}) = \Big[Z(T)(1 - \mathbf{L}^a T^a)^d \Big]_{T=\mathbf{L}^{-1}} = Q_{1,d}(\mathbf{L}^{-1}) = \sum_{p=0}^{a-1} q_{1,d,p} \mathbf{L}^{-p}.$$

Since $P(\mathbf{L}^{-1})$ is effective and non-zero, its Poincaré polynomial $\boxdot(P(\mathbf{L}^{-1}))$ is non-zero. Consequently, there must exist an integer $p \in \{0, \ldots, a-1\}$ such that $\dim(q_{1,d,p}) \neq -\infty$. We now restrict the analysis to integers n congruent to p modulo a. Set

$$d_p = \max_{1 \leq j \leq d} \dim(q_{1,j,p}) - p$$

and let j_p be the largest integer j such that $d_p = \dim(q_{1,j,p}) - p$. Looking at Poincaré polynomials and using that for $j \neq j_p$, either $\dim(q_{1,j,p}) < \dim(q_{1,j_p,p})$, or the binomial coefficient $\binom{j_p+m-1}{j_p-1}$ goes to infinity faster than $\binom{j+m-1}{j-1}$ when $m \to \infty$, we get the following asymptotic expansions

$$\dim([M_n]^1\mathbf{L}^{-n}) = d_p \quad \text{and} \quad \kappa([M_n]) \sim \binom{j_p+m-1}{j_p-1}\kappa(q_{1,j_p,p})$$

for n large enough and congruent to p modulo a. In particular,

$$\dim([M_n]) = n + \mathrm{O}(1) \quad \text{and} \quad \frac{\log(\kappa([M_n]))}{\log(n)} \longrightarrow j_p - 1.$$

This concludes the proof of the proposition. □

2.2. Existence of the moduli spaces. In this section, we prove a general proposition that asserts existence of moduli schemes of sections of bounded height in a general context.

Let k be a field, let C be an irreducible projective smooth k-curve; let η be its generic point and let $F = k(C)$ be the function field of C. Let C_0 be a non-empty Zariski open subset of C.

Let X be an irreducible projective k-variety together with a surjective flat morphism $\pi\colon X \to C$. Let G be a Zariski open subset of X_F, assumed to be affine. Let U be a Zariski open subset of X such that $G \subset U_F$ and $\pi(U) \supset C_0$.

Let $(D_\alpha)_{\alpha\in\mathscr{A}}$ be a finite family of Cartier divisors on X such that, for each α, the restriction of D_α to X_F is effective and $X_F \setminus G = \bigcup |D_\alpha|_F$. For each α, we also let $\mathscr{L}_\alpha$ be the line bundle $\mathscr{O}_X(D_\alpha)$. Finally, we assume that there exists a linear combination with positive coefficients $\mathscr{L} = \sum \lambda_\alpha \mathscr{L}_\alpha$, as well as a Cartier $\mathbf{Q}$-divisor D on X such that D_F is effective, supported by $(X\setminus G)_F$ and such that $\mathscr{L}(-D)$ is ample.

Remark 2.2.1. Let $\mathscr{L}$ be a line bundle on X and let $f \in \Gamma(X_F, \mathscr{L})$ be a non-zero global section. Let us show that there exists an integer m such that for every section $\sigma\colon C \to X$ of π satisfying $\sigma(\eta) \notin \mathrm{div}(f)$, one has $\deg(\sigma^*\mathscr{L}) \geq -m$.

There exists an effective Cartier divisor E on C such that f extends to a global section of $\mathscr{L} \otimes \pi^*(E)$. Indeed, viewing f as a meromorphic section of $\mathscr{L}$ on X, let us decompose its divisor as the sum $H + V$ of its horizontal (i.e., faithfully flat over C) and vertical (mapping to a point) irreducible components. By construction, the components of H are the Zariski closures of the components of the divisor of f, viewed as a section of $\mathscr{L}$ on X_F; consequently, H is effective by hypothesis. Still by definition, V is a linear combination of irreducible components of closed fibers. Consequently, there exists an effective divisor E on C such that $V \geq -\pi^*E$; then f extends to a global section of $\mathscr{L} \otimes \pi^*(E)$.

In particular, for every section $\sigma\colon C\to X$ of π satisfying $\sigma(\eta)\notin \operatorname{div}(f)$, one has $\deg(\sigma^*\mathscr{L})\geq -\deg(\sigma^*\pi^*E)=-\deg(E)$.

PROPOSITION 2.2.2. *For every* $\mathbf{n}$ *in* $\mathbf{Z}^{\mathscr{A}}$*, there is a quasi-projective* k*-scheme* $M_{G,\mathbf{n}}$ *parameterizing sections* $\sigma\colon C\to X$ *of* π *satisfying the following properties:*

- *under* σ*, the generic point* η *of* C *is mapped to a point of* G*;*
- *for each* $\alpha\in\mathscr{A}$*,* $\deg_C \sigma^*\mathscr{L}_\alpha = n_\alpha$*.*

In that scheme, the sections σ *such that* $\sigma(C_0)\subset U$ *constitute a constructible set* $M_{U,\mathbf{n}}$*. Moreover, there exists* $n_0\in\mathbf{Z}$ *such that* $M_{U,\mathbf{n}}$ *is empty if* $n_\alpha<n_0$ *for some* $\alpha\in\mathscr{A}$*.*

Proof. As a standard consequence of the existence of Hilbert schemes, there exists a k-scheme $M_{X,\mathbf{n}}$ which parameterizes sections $\sigma\colon C\to X$ such that $\deg_C\sigma^*\mathscr{L}_\alpha=n_\alpha$ for each α. Indeed, the functor of sections $\sigma\colon C\to X$ is represented by the open subscheme of the Hilbert scheme Hilb_X which parameterizes the closed subschemes of X which are mapped isomorphically by π. By flatness, each of the condition $\deg_C\sigma^*\mathscr{L}_\alpha=n_\alpha$ is open and closed in the Hilbert scheme.

The condition that the generic point of C is mapped to a point of G means that $\sigma(C)\not\subset |X\setminus G|$, while the condition $\sigma(C)\subset|X\setminus G|$ defines an closed subscheme of $M_{X,\mathbf{n}}$. Let $M_{G,\mathbf{n}}$ be its complement. By construction, this scheme represents the given functor, and we have to prove that it is quasi-projective.

First of all, since the restriction to X_F of the divisor D_α is effective and disjoint from G, Remark 2.2.1 asserts that there exists an integer m such that $\deg(\sigma^*\mathscr{L}_\alpha)\geq -m$ for every $\mathbf{n}$ and every section σ in $M_{G,\mathbf{n}}$.

Let $\mathscr{M}$ be an ample $\mathbf{Q}$-line bundle on X of the form $\mathscr{L}(-D)$, where D is a Cartier $\mathbf{Q}$-divisor such that D_F is effective and disjoint from G. For every $\sigma\in M_{G,\mathbf{n}}$, one has

$$\deg\sigma^*(\mathscr{M}+\mathscr{O}(D))=\deg\sigma^*\mathscr{M}+\deg\sigma^*\mathscr{O}(D).$$

By Remark 2.2.1, there exists an integer m' such that

$$\deg\sigma^*\mathscr{O}(D)\geq -m'$$

for all sections $\sigma\in M_{G,\mathbf{n}}$. Therefore, $\deg\sigma^*\mathscr{M}\leq m+\sum\lambda_\alpha n_\alpha$ for all $\sigma\in M_{G,\mathbf{n}}$. By a theorem of Chow [2, XIII, Cor. 6.11], this gives only finitely many possibilities for the Hilbert polynomial (relative to $\mathscr{M}$) of the image of a section σ which belongs to $M_{G,\mathbf{n}}$. It is well known that the subschemes of X with given Hilbert polynomial with respect to the ample line bundle $\mathscr{M}$ form a closed and open subscheme of Hilb_X, which is projective as a scheme. Consequently, $M_{G,\mathbf{n}}$ is quasi-projective.

It remains to prove that the condition "$\sigma(C_0)\subset U$" defines a constructible subset of $M_{G,\mathbf{n}}$. Indeed, let T be a scheme and let $\sigma\colon C\times T\to X$ be a morphism. Let $V=\sigma^{-1}(U)$ and let Z be the complement of $V\cap(C_0\times T)$ in $C_0\times T$; this is a closed subset of $C_0 \times T$. The set of points $t \in T$ such that $\sigma(C_0 \times \{t\}) \not\subset U$ is equal to the projection in T of Z, so is constructible, as claimed. □

Remark 2.2.3. Assume that for each $\alpha \in \mathscr{A}$ the divisor D_α is effective; then $M_{U,\mathbf{n}}$ is actually an *open and closed subscheme* of $M_{G,\mathbf{n}}$. Indeed, let us write $\deg \sigma^*\mathscr{L}_\alpha$ as the intersection number of σ_*C with D_α. By definition of $M_{U,\mathbf{n}}$, this is a sum of local contributions $(\sigma_*C, D_\alpha)_v$ at all points of $C \setminus C_0$. Since D_α is effective, each of these contributions is lower semi-continuous as a function of σ (it may increase on closed subsets), while their sum is the constant n_α on $M_{G,\mathbf{n}}$. This decomposes $M_{U,\mathbf{n}}$ as a disjoint union of open and closed subschemes defined by prescribing the possible values for $(\sigma_*C, D_\alpha)_v$.

2.3. Clemens complexes. Let X be a smooth algebraic variety over a field K and let D be an effective divisor with strict normal crossings on X; in other words, the support of$D$ is the union of its irreducible components which are themselves smooth and meet transversally.

The Clemens complex $\mathrm{Cl}(X,D)$ of (X,D) is the simplicial complex whose points are irreducible components of D, edges are irreducible components of intersections of two distinct irreducible components, etc. By the normal crossing assumption, all of these schemes are smooth. Thus, the dimension of the Clemens complex is the maximal number of irreducible components of D whose intersection is non-empty, minus 1. For every integer d, we also write $\mathrm{Cl}^d(X,D)$ for the set of simplices (also called faces) of dimension d of $\mathrm{Cl}(X,D)$.

The analytic Clemens complex $\mathrm{Cl}^{\mathrm{an}}(X,D)$ is the subcomplex of $\mathrm{Cl}(X,D)$ consisting of those simplices $Z \in \mathrm{Cl}(X,D)$ such that $Z(K) \neq \varnothing$. One writes $\mathrm{Cl}^{\mathrm{an,max}}(X,D)$ for the set of maximal faces of $\mathrm{Cl}^{\mathrm{an}}(X,D)$ and $\mathrm{Cl}^{\mathrm{an},d}(X,D)$ for the set of faces of dimension d of $\mathrm{Cl}^{\mathrm{an}}(X,D)$.

If L is an extension of K, the divisor D_L on X_L still has strict normal crossings and one writes $\mathrm{Cl}_L(X,D) = \mathrm{Cl}(X_L, D_L)$ and $\mathrm{Cl}^{\mathrm{an}}_L(X,D) = \mathrm{Cl}^{\mathrm{an}}(X_L, D_L)$.

2.4. Motivic residual functions on arc spaces. Let k be an algebraically closed field of characteristic zero, let R be the complete discrete valuation ring $k[[t]]$, and let $K = k((t))$ be its field of fractions.

Let $\mathscr{X}$ be a flat R-scheme of finite type, equidimensional of relative dimension n.

For every integer $m \geq 0$, we write $\mathscr{L}_m(\mathscr{X})$ or $\mathscr{X}(m)$ for the mth Greenberg space of $\mathscr{X}$, see Section 2.3 of [16] for the precise general definition. Let us simply recall that $\mathscr{X}(m)$ is the algebraic variety over k which represents the functor $\ell \mapsto \mathscr{X}(\ell[[t]]/(t^{m+1}))$ on the category of k-algebras. There are natural affine morphisms $p_m^{m+1} \colon \mathscr{X}(m+1) \to \mathscr{X}(m)$; consequently, the projective limit $\mathscr{L}(\mathscr{X}) = \varprojlim_m \mathscr{L}_m(\mathscr{X})$ exists as a k-scheme. Let $p_m \colon \mathscr{L}(\mathscr{X}) \to \mathscr{L}_m(\mathscr{X})$ be the canonical projection. When $\mathscr{X} = X \otimes_k R$, for some k-variety X, then $\mathscr{L}_m(\mathscr{X})$ is the space of m-jets of X, and $\mathscr{L}(\mathscr{X})$ is the arc space of X.

The paper [9] introduces a general definition of constructible motivic functions on arc spaces. In this paper, we shall mostly consider the following more restrictive class: We define a motivic residual function h on $\mathscr{L}(\mathscr{X})$ to be an element of the inductive limit of all relative Grothendieck groups $\mathscr{M}_{\mathscr{X}(m)}$. Recall that $\mathscr{M}_{\mathscr{X}(m)}$ is a localization of the Grothendieck ring $\mathrm{KVar}_{\mathscr{X}(m)}$; in particular, a motivic residual function comes from the latter ring if it is given by a formal linear combination of varieties $H \to \mathscr{X}(m)$; in addition to the cut-and-paste relations at the heart of the definition of the Grothendieck groups, we identify the diagrams $H \to \mathscr{X}(m)$ and $H \times_{\mathscr{X}(m)} \mathscr{X}(m+1) \to \mathscr{X}(m+1)$. The fiber product structure of varieties gives rise to a ring structure on the set of motivic residual functions on $\mathscr{L}(\mathscr{X})$.

An example of such a motivic residual function is the characteristic function of a constructible subset W of $\mathscr{L}(\mathscr{X})$: such a W is of the form $p_m^{-1}(W_m)$ for a constructible subset W_m of $\mathscr{L}_m(\mathscr{X})$ and $\mathbf{1}_W$ is given by the obvious diagram $W_m \to \mathscr{X}(m)$. Let A be an algebraic variety over k, and let a be its class in $\mathscr{M}_k$; then the motivic residual function $a\mathbf{1}_W$ is the diagram $A \times W_m \to \mathscr{X}(m)$, the map being the second projection composed by the inclusion of W_m into $\mathscr{X}(m)$. Motivic residual functions on $\mathscr{L}(\mathbf{A}^n)$ are examples of Schwartz-Bruhat motivic functions in K^n with support in R^n (see Section 1.2.3).

3. Setup and notation. In this Section, we fix the notation that will be used for the rest of the paper. Compared with the introduction, we denote varieties fibered over the base curve by script letters, and use capital letters for their generic fiber. This reflects the fact that, even if models are given in the statement of Theorem 1, its proof requires us to adjust them somewhat.

3.1. Algebraic geometry. Let k be an algebraically closed field of characteristic zero, let C_0 be a smooth quasi-projective connected curve over k, let C be its smooth projective compactification and let $S = C \setminus C_0$. Let $F = k(C) = k(C_0)$ be the function field of C; let η_C be its generic point.

Let G be the group scheme $\mathbf{G}_{\mathrm{a}}^n$ and let X be a smooth projective equivariant compactification of G_F. In other words, X is a smooth projective F-scheme containing G_F as a dense open subset, and the group law $G_F \times G_F \to G_F$ extends as a group action $G_F \times X \to X$. The boundary $X \setminus G_F$ of G_F in X is a divisor. In this paper, we make the hypothesis that this divisor has strict normal crossings. More precisely, we assume that its irreducible components are geometrically irreducible, smooth and meet transversally, so that for every p, the intersection of any p of those components is either empty or smooth of dimension $n-p$. This is a slightly stronger assumption that the one done in the arithmetic case [8], where we only made this hypothesis after base change to $\overline{F}$. The general case can be treated in a similar way, by constructing appropriate weak Néron models; we leave it to the interested reader.

We write $D = X \setminus G_F$ and $(D_\alpha)_{\alpha \in \mathscr{A}}$ for the family of its irreducible components. The divisors D_α form a basis of the group $\mathrm{Pic}(X)$, and a basis of the monoid

$\Lambda_{\text{eff}}(X)$ of effective divisors in $\text{Pic}(X)$. We will freely identify line bundles on X with divisors whose support is contained in the boundary, and with their classes in the Picard group.

Up to multiplication by a scalar, there is a unique G_F-invariant meromorphic differential form ω_X on X; its restriction to G_F is proportional to the form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. Its divisor, or its class, is the canonical class K_X of X. The divisor $-\operatorname{div}(\omega_X)$ can be written as $\sum \rho_\alpha D_\alpha$ for some integers $\rho_\alpha \geq 2$ (see [12], Theorem 2.7). In particular, the anticanonical class K_X^{-1} is effective.

The log-canonical class of the pair (X,D) in $\text{Pic}(X)$ is the class of $K'_X = K_X + D$. Its opposite, the log-anticanonical class, is given by $\sum \rho'_\alpha D_\alpha$ with $\rho'_\alpha = \rho_\alpha - 1$ for all α. Since $\rho_\alpha \geq 2$ for all $\alpha \in \mathscr{A}$, the divisor $-K'_X$ can be written as the sum of an ample line divisor and of an effective divisor (in other words, it is big), as claimed in the introduction.

We also recall that $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for every integer $i > 0$.

3.2. Models and heights. A *model* of X over C is a projective flat scheme $\pi\colon \mathscr{X} \to C$ whose generic fiber is equal to X. If, moreover, $\mathscr{X}$ is regular and if the sum of the non-smooth fibers of $\mathscr{X}$ and the closures $\mathscr{D}_\alpha$ of the divisors D_α is a divisor with strict normal crossings on $\mathscr{X}$, then we will say that $\mathscr{X}$ is a *good model*. One defines analogously good models of X over C_0, or even over local rings whose field of fractions contains $k(C)$.

Embedded resolution of singularities in characteristic zero implies that good models exist.

We choose a good model $\pi\colon \mathscr{X} \to C$ of X over C.

For every point $v \in C(k)$, we write $\mathscr{B}_v$ for the set of irreducible components of $\pi^{-1}(v)$; for $\beta \in \mathscr{B}_v$, let E_β be the corresponding component and μ_β be its multiplicty in the special fibre of $\mathscr{X}$ at v. Let $\mathscr{B}$ be the disjoint union of all $\mathscr{B}_v$, for $v \in C(k)$. Let $\mathscr{B}_1$ be the subset of $\mathscr{B}$ consisting of those β for which the multiplicity μ_β equals 1; let $\mathscr{B}_{1,v} = \mathscr{B}_1 \cap \mathscr{B}_v$.

The complement $\mathscr{X}_1$ in $\mathscr{X}$ of the union of the components E_β, for $\beta \in \mathscr{B} \setminus \mathscr{B}_1$ and of the intersections of distinct vertical components, is a smooth scheme over C.

LEMMA 3.2.1. *The C-scheme $\mathscr{X}_1$ is a weak Néron model of X: for every smooth C-scheme $\mathscr{Z}$, the canonical map from* $\text{Hom}_C(\mathscr{Z}, \mathscr{X}_1)$ *to* $\text{Hom}_F(\mathscr{Z}_F, X)$ *is a bijection.*

Proof. This follows from the fact that the C-scheme $\mathscr{X}_1$ is the smooth locus of the proper map $\pi\colon \mathscr{X} \to C$, and that $\mathscr{X}$ is regular. See [3] for details, especially p. 61. $\square$

For every $\alpha \in \mathscr{A}$, we assume given a divisor $\mathscr{L}_\alpha$ on $\mathscr{X}$ which extends D_α. There exists a family of integers $(e_{\alpha,\beta})$, all but finitely many of them being equal to 0, indexed by $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$ such that

$$\mathscr{L}_\alpha = \mathscr{D}_\alpha + \sum_{\beta \in \mathscr{B}} e_{\alpha,\beta} E_\beta. \tag{3.2.2}$$

We also define integers ρ_β, for $\beta \in \mathscr{B}$, by the formula

$$-\operatorname{div}(\omega_X) = \sum_{\alpha \in \mathscr{A}} \rho_\alpha \mathscr{D}_\alpha + \sum_{\beta \in \mathscr{B}} \rho_\beta E_\beta, \tag{3.2.3}$$

where ω_X is viewed as a meromorphic section of the line bundle $K_{\mathscr{X}/C}$.

Since π is proper and C is a smooth curve, the map $\sigma \mapsto \sigma(\eta_C)$ is a bijection between the set of sections $\sigma\colon C \to \mathscr{X}$ of π and the set of rational points $X(F)$ of X. For every line bundle $\mathscr{L}$ on $\mathscr{X}$ and every section $\sigma\colon C \to \mathscr{X}$, the degree $\deg_C \sigma^*\mathscr{L}$ is the geometric analogue of the *height* of the corresponding rational point.

3.3. Local descriptions. Let $v \in C(k)$. We write F_v for the completion of $F = k(C)$ at v. If t is a local parameter of C at v, then $F_v \simeq k((t))$ and $\widehat{\mathscr{O}}_{C,v} \simeq k[[t]]$. Writing an element x of $\widehat{\mathscr{O}}_{C,v}$ as a power series $x_0 + x_1 t + \cdots$, we consider $k[[t]]$ as the set of k-points of the scheme $\operatorname{Spec}(k[x_0, x_1, \ldots])$; writing an element of F_v as a Laurent series $x_{-m}t^{-m} + \cdots + x_0 + x_1 t + \cdots$, we view $k((t))$ as the set of k-points of the ind-scheme whose mth term is $\operatorname{Spec}(k[x_{-m}, \ldots, x_0, x_1, \ldots])$. Fixing an isomorphism $G \simeq \mathbf{G}_{\mathrm{a}}^n$, we have an identification $G(F_v) \simeq k((t))^n$ of $G(F_v)$ with the k-points of an ind-k-scheme. We will say that a subset of $G(F_v)$ is definable if it can be defined in the language $\mathscr{L}_{\mathrm{DP,P}}$ of Denef-Pas (see [9, Section 2.1], for more details). In particular, the set of k-points of a constructible subset of a finite level of this ind-scheme is definable.

For every point $g \in G(F_v)$, one can attach local intersection degrees $(g, \mathscr{D}_\alpha)_v$, for $\alpha \in \mathscr{A}$, defined as follows. By the valuative criterion of properness, the map $g\colon \operatorname{Spec}(F_v) \to G_F$ extends to a morphism $\tilde{g}\colon \operatorname{Spec}(\widehat{\mathscr{O}}_{C,v}) \to \mathscr{X}$ and we can consider the pull-back $\tilde{g}^*\mathscr{D}_\alpha$ of $\mathscr{D}_\alpha$ as an effective Cartier divisor on $\operatorname{Spec}(\widehat{\mathscr{O}}_{C,v})$. We define $(g, \mathscr{D}_\alpha)_v \in \mathbf{N}$ by the formula $\tilde{g}^*\mathscr{D}_\alpha = (g, \mathscr{D}_\alpha)_v [v]$. For $\beta \in \mathscr{B}_v$, we define an integer $(g, E_\beta)_v \in \{0, 1\}$ similarly, considering the pull-back of E_β.

Observe also that $\sum_{\beta \in \mathscr{B}_v} \mu_\beta (g, E_\beta)_v = 1$. In particular, for every $g \in G(F_v)$, there is exactly one index $\beta \in \mathscr{B}_v$ for which $(g, E_\beta)_v = 1$ and one has $\mu_\beta = 1$.

By the valuative criterion properness, every point $g \in G(F)$ extends canonically to a section $\sigma_g\colon C \to \mathscr{X}$.

LEMMA 3.3.1. *For every $g \in G(F)$ and every $\alpha \in \mathscr{A}$, one has*

$$\deg_C(\sigma_g^*(\mathscr{D}_\alpha)) = \sum_{v \in C(k)} (g, \mathscr{D}_\alpha)_v.$$

Proof. Since $g \in G(F)$, the Cartier divisor $\sigma_g^*(\mathscr{D}_\alpha)$ on C is well-defined and represents the inverse image by σ_g of the line bundle $\mathscr{O}_{\mathscr{X}}(\mathscr{D}_\alpha)$. The given formula asserts that its degree is the sum of its multiplicities at all closed points of C. □

For $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$, we define the subset $G(\mathbf{m})_v$ (also denoted $G(\mathbf{m})$ if no confusion can arise concerning the point v) of $G(F_v)$ as the set of all points g such that $(g, \mathscr{D}_\alpha)_v = m_\alpha$ for all $\alpha \in \mathscr{A}$. For $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and $\beta \in \mathscr{B}_v$, the subset $G(\mathbf{m}, \beta)$ of $G(\mathbf{m})_v$ consists of points g such that $(g, E_\beta)_v = 1$ (hence $(g, E_{\beta'})_v = 0$ for all $\beta' \in \mathscr{B}_v$ such that $\beta' \neq \beta$). When $\mathscr{B}_v$ has a single element, we often call it β_v.

LEMMA 3.3.2. *For every $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and every $\beta \in \mathscr{B}_v$, the sets $G(\mathbf{m})_v$ and $G(\mathbf{m}, \beta)$ are bounded definable subsets of $G(F_v)$ and $G(F_v) = \bigcup_{\mathbf{m} \in \mathbf{N}^{\mathscr{A}}} G(\mathbf{m})_v = \bigcup_{\substack{\mathbf{m} \in \mathbf{N}^{\mathscr{A}} \\ \beta \in \mathscr{B}_v}} G(\mathbf{m}, \beta)$ (disjoint unions).*

Proof. Since G is affine, $X \setminus G_F$ contains the support of an ample line bundle. We thus see that the valuations of the coordinates of the points of $G(\mathbf{m}, \beta)$ are bounded from below. Since $\mathscr{D}_\alpha$ (resp. E_β) is effective, the condition $(g, \mathscr{D}_\alpha)_v \geq n_\alpha$ (resp. the condition $(g, E_\beta)_v \geq 1$) defines a definable subset. Taking differences, one gets that the sets $G(\mathbf{m})_v$ and $G(\mathbf{m}, \beta)$ are bounded definable. The last assertion is obvious. □

LEMMA 3.3.3. *There exists a dense open subset C_1 of C_0 such that for every closed point $v \in C_1$, the following properties hold:*

(1) *One has $\mathscr{B}_v = \mathscr{B}_{1,v} = \{\beta_v\}$;*

(2) *The set $G(\mathbf{0})_v = G(\mathfrak{o}_v)$ is a subgroup of $G(F_v)$;*

(3) *For every $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and every $\beta \in \mathscr{B}_v$, the set $G(\mathbf{m}, \beta)$ is invariant under the action of $G(\mathbf{0})_v$.*

Proof. By assumption, X is a smooth equivariant compactification of the F-group scheme G_F. By spreading-out, there exists a dense Zariski open subset C_1 of C such that $\mathscr{X}_{C_1}$ is a smooth equivariant compactification of the C_1-group scheme G_{C_1}, more precisely, such that the following properties hold:

- The morphism $\mathscr{X}_{C_1} \to C_1$ is proper and smooth, with geometrically integral fibers;
- The action $G_F \times X \to X$ of G_F on X extends to an action $m \colon G_{C_1} \times \mathscr{X}_{C_1} \to \mathscr{X}_{C_1}$;
- The image of the section $\sigma_0 \in \mathscr{X}(C_1)$ extending the point $0 \in G(F)$ is disjoint from all $\mathscr{D}_\alpha$;
- The morphism $g \mapsto m(g, \sigma_0)$ is an isomorphism from G_{C_1} to an open dense subscheme of $\mathscr{X}_{C_1}$;
- The Cartier divisors $m^*\mathscr{D}_\alpha - \mathrm{pr}_2^*\mathscr{D}_\alpha$ on $G_{C_1} \times \mathscr{X}_{C_1}$ are trivial, so that $\mathscr{X}_{C_1} \setminus G_{C_1}$ is the union of the divisors $\mathscr{D}_{\alpha, C_1}$.

This open set C_1 satisfies the requirements of the Lemma. □

LEMMA 3.3.4. *Let $v \in C(k)$. For every integer r, let $G(\mathfrak{m}_v^r)$ be the bounded definable subgroup of $G(F_v)$ consisting of points g such that, in the identification $G = \mathbf{G}_{\mathrm{a}}^n$, $\mathrm{ord}_v(g_i) \geq r$ for $i \in \{1,\dots,n\}$. For every $v \in C(k)$, there exists an integer r_v such that, for every $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and every $\beta \in \mathscr{B}_v$, $G(\mathbf{m},\beta)$ is invariant under $G(\mathfrak{m}_v^{r_v})$. Moreover, one can take $r_v = 0$ for all but finitely $v \in C(k)$.*

Proof. When v belongs to the open subset C_1 constructed by Lemma 3.3.3, one may take $r_v = 0$, hence the last claim.

In the remaining of the proof, we fix $v \in C(k)$. Fix $\alpha \in \mathscr{A}$ and let f_α be the canonical global section of $\mathscr{O}_{\mathscr{X}}(\mathscr{D}_\alpha)$ whose zero-divisor is $\mathscr{D}_\alpha$. We need to prove that there exists an integer r_v such that $(gg',\mathscr{D}_\alpha)_v = (g',\mathscr{D}_\alpha)_v$ and $(gg',E_\beta)_v = (g',E_\beta)_v$, for every $g \in G(\mathfrak{m}_v^{r_v})$, every $g' \in G(F) \subset \mathscr{X}(F)$, every $\alpha \in \mathscr{A}$, and every $\beta \in \mathscr{B}_v$.

Since G_F fixes $\mathscr{D}_\alpha$ on the generic fiber, the line bundles $m^*\mathscr{O}_{\mathscr{X}}(\mathscr{D}_\alpha)$ and $\mathrm{pr}_2^*\mathscr{O}_{\mathscr{X}}(\mathscr{D}_\alpha)$ are isomorphic on $G_F \times \mathscr{X}_F$ and $u = m^*f_\alpha/\mathrm{pr}_2^*f_\alpha$ is a rational function on $G_F \times X$. The domain of definition of u contains $G_F \times \mathscr{X}_F$. Since $\mathscr{X}$ is proper over C, there exists a closed subset Z of G_C disjoint from G_F such that u is defined on the complement of $\mathrm{pr}_1^{-1}(Z)$. Moreover, $u(0,x) = 1$ on X. Cover $\mathscr{X}$ by finitely many affine open subsets $\mathrm{Spec}(A_i)$. Then u defines a rational function on $\mathbf{G}_{\mathrm{a}}^n \times \mathrm{Spec}(A_i) = \mathrm{Spec}(A_i[\mathbf{T}])$. Since u is defined on $\mathrm{Spec}(A_i[\mathbf{T}][1/\varpi_v])$, there exists an integer m such that $\varpi_v^m u \in A_i[\mathbf{T}]$ for all i, ϖ_v denoting an uniformizer of $\mathfrak{o}_v$.

It is now clear that if $\mathrm{ord}_v(g_i) > m$ for $i \in \{1,\dots,n\}$, then $\mathrm{ord}_v(u(g,x)) = 0$ for every integral point of $\mathrm{Spec}(A_i)$. Since every rational point of X extends to an integral point of some $\mathrm{Spec}(A_i)$, we obtain the desired conclusion for $\mathscr{D}_\alpha$.

Now fix $\beta \in \mathscr{B}_v$. Since E_β is vertical, $E_\beta \otimes_C F = \varnothing$ and E_β is fixed by G_F on the generic fiber. Then, the proof is identical to the one for $\mathscr{D}_\alpha$. $\square$

COROLLARY 3.3.5. *For every $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and every $\beta \in \mathscr{B}_v$, the characteristic function of $G(\mathbf{m},\beta)$ is a motivic Schwartz-Bruhat function on $G(F_v)$ in the sense of Section 1.2.3.*

3.4. Integral points.

LEMMA 3.4.1. *Let $\mathscr{U}$ be a flat model of G_F over $C_0 = C \setminus S$, let $\mathscr{X}$ be a flat model of X over C. There exists a good model $\mathscr{X}'$ of X over C whose projection $\pi'\colon \mathscr{X}' \to C$ factors through $\mathscr{X}$, and a open subset $\mathscr{U}'$ of $\mathscr{X}' \times_C C_0$ such that for every point $v \in C_0$, the intersection $G(F_v) \cap \mathscr{U}(\mathfrak{o}_v)$ (taken in $\mathscr{U}(F_v)$) coincides with the intersection $G(F_v) \cap \mathscr{U}'(\mathfrak{o}_v)$ taken in $\mathscr{X}'(\mathfrak{o}_v)$. We may also assume that $\mathscr{U}'$ is the complement to a divisor with strict normal crossings in $\mathscr{X}'$. Moreover, $G(F_v) \cap \mathscr{U}(\mathfrak{o}_v)$ is non-empty if and only if $\mathscr{U}'(\mathfrak{o}_v)$ is non-empty.*

Proof. Up to replacing $\mathscr{U}$ by an adequate blow-up, we may assume that the open immersion $i\colon G_F \hookrightarrow X$ extends to a morphism $p\colon \mathscr{U} \to \mathscr{X}$. Then, replacing

$\mathscr{X}$ by some blow-up $\mathscr{X}'$ and $\mathscr{U}$ by its strict transform $\mathscr{U}'$, we may assume that p is flat ([17], Théorème 5.2.2); it is then a open immersion. A further blowing-up allows to assume that $\mathscr{X}' \setminus \mathscr{U}'$ is a divisor. Applying embedded resolution of singularities, we may also assume that $\mathscr{X}'$ is smooth over k, that the fibers of its projection to C are divisors with strict normal crossings, as well as $\mathscr{X}' \setminus \mathscr{U}'$.

Finally, if $G(F_v) \cap \mathscr{U}(\mathfrak{o}_v)$ is non-empty, then $\mathscr{U}'(\mathfrak{o}_v)$ is non-empty as well. Conversely, assume that $\mathscr{U}'(\mathfrak{o}_v)$ is non-empty. Then $\mathscr{U}'$ meets the smooth locus of $\mathscr{X}' \to C$, so that $\mathscr{U}'(\mathfrak{o}_v)$ has non-empty interior; in particular, $G(F_v) \cap \mathscr{U}'(\mathfrak{o}_v)$ is non-empty. $\square$

LEMMA 3.4.2. *Let $\mathscr{U}$ be a flat model of G_F over C_0. For every $v \in C_0(k)$, $\mathscr{U}(\mathfrak{o}_v)$ is a bounded definable subset of $G(F_v)$. For almost all $v \in C_0(k)$, one has even $\mathscr{U}(\mathfrak{o}_v) = G(\mathbf{0})_v$.*

Proof. We may assume that $\mathscr{U}$ is an open subset of $\mathscr{X}$; it is then clear that $\mathscr{U}(\mathfrak{o}_v)$ is definable in $G(F_v)$ and that it equals $\mathbf{G}_{\mathrm{a}}^n(\mathfrak{o}_v)$ for almost all $v \in C_0(k)$ (Lemma 3.3.3). Let us now prove its boundedness.

We view the n coordinate functions on $G_F = \mathbf{G}^n_{\mathrm{a},F}$ as rational functions $f_1, \dots, f_n$ on $\mathscr{U}$, regular over its generic fiber $\mathscr{U}_F = G_F$. Up to resolving the indeterminacies of the f_i (which replaces $\mathscr{U}$ by some other scheme $\mathscr{U}'$ but does not change the sets $\mathscr{U}(\mathfrak{o}_v)$), we view the f_i as regular morphisms from $\mathscr{U}$ to $\mathbf{P}^1_C$, such that $f_i^*(\{\infty\}) \cap \mathscr{U}_F = \varnothing$.

Cover $\mathscr{U}$ by finitely many affine open subsets $\operatorname{Spec}(A_j)$. There exists an integer r such that $\varpi_v^r f_i \in A_j \otimes \mathfrak{o}_v$ for all i and j. For every point $g \in \mathscr{U}(\mathfrak{o}_v)$, there exists j such that the morphism $g\colon \operatorname{Spec}(\mathfrak{o}_v) \to \mathscr{U}$ restricts to a morphism $\operatorname{Spec}(\mathfrak{o}_v) \to \operatorname{Spec}(A_j)$, because $\mathfrak{o}_v$ is a local ring. Then, $\operatorname{ord}_v(f_i(g)) \geq -r$, so that $\mathscr{U}(\mathfrak{o}_v)$ is bounded in $G(F_v)$.

The last assertion follows from the fact that the equality $\mathscr{U}_F = G_F$ extends to an isomorphism over a dense open subset of C_0. $\square$

3.5. Height zeta functions. Let (λ_α) be a family of positive integers and let $\mathscr{L}$ be the line bundle $\sum_{\alpha \in \mathscr{A}} \lambda_\alpha \mathscr{L}_\alpha$ on the chosen good model $\mathscr{X}$. Let $\mathscr{U}$ be a flat model of G_F over the affine curve $C_0 = C \setminus S$. For every integer $n \in \mathbf{Z}$, let M_n be the moduli space of sections $\sigma\colon C \to \mathscr{X}$ such that $\sigma(\eta_C) \in G(F)$, $\sigma(C_0) \subset \mathscr{U}$ and $\deg_C(\sigma^* \mathscr{L}) = n$. By Proposition 2.2.2, this moduli space exists as a quasi-projective k-scheme, and is empty for $n \ll 0$. The geometric analogue of Manin's *height zeta function* is the formal Laurent series in one variable T with coefficients in $\mathscr{M}_k$ given by

$$Z_\lambda(T) = \sum_{n \in \mathbf{Z}} [M_n] T^n \in \mathscr{M}_k[[T]][T^{-1}]. \tag{3.5.1}$$

As was already the case in number theory, it is convenient to separate the roles of the various divisors $\mathscr{D}_\alpha$ and to introduce a multivariable height zeta function. So, for every $\mathbf{n} = (n_\alpha) \in \mathbf{Z}^{\mathscr{A}}$, let $M_{\mathbf{n}}$ be the moduli space of sections $\sigma\colon C \to \mathscr{X}$ such that $\sigma(\eta_C) \in G(F)$, $\sigma(C_0) \subset \mathscr{U}$ and $\deg_C(\sigma^*\mathscr{L}_\alpha) = n_\alpha$ for every $\alpha \in \mathscr{A}$. Again by Proposition 2.2.2, this moduli space exists as a quasi-projective k-scheme $M_{\mathbf{n}}$; moreover, there exists an integer m such that $M_{\mathbf{n}} = \varnothing$ if $n_\alpha < -m$ for some $\alpha \in \mathscr{A}$. One then defines the generating series

$$Z(\mathbf{T}) = \sum_{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}} [M_{\mathbf{n}}]\mathbf{T}^{\mathbf{n}} \in \mathscr{M}_k[\![(T_\alpha)]\!]\left[\prod_\alpha T_\alpha^{-1}\right]. \tag{3.5.2}$$

By definition of $\mathscr{L}$, we have

$$Z_\lambda(T) = Z((T^{\lambda_\alpha})) = \sum_{m\in\mathbf{Z}}\left(\sum_{\substack{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}\\ \lambda\cdot\mathbf{n}=m}} [M_{\mathbf{n}}]\right) T^m \in \mathscr{M}_k[\![T]\!][T^{-1}]. \tag{3.5.3}$$

3.5.4. In the sequel, we assume that $\mathscr{U}$ is an open subset of $\mathscr{X}$. Its complement consists of the union of the divisors $\mathscr{D}_\alpha$, and of the vertical components E_β, for β in a finite subset $\mathscr{B}^0$ of $\mathscr{B}$. By Lemma 3.4.1, this does not restrict the generality. We then set $\mathscr{B}_v^0 = \mathscr{B}^0 \cap \mathscr{B}_v$ for every $v \in C(k)$, and define

$$\mathscr{B}_0 = \mathscr{B}_1 \setminus \left(\bigcup_{v\in C_0} \mathscr{B}_v^0\right);$$

set also $\mathscr{B}_{0,v} = \mathscr{B}_0 \cap \mathscr{B}_v$. Let $\mathbf{m}_v \in \mathbf{N}^{\mathscr{A}}$ and $\beta_v \in \mathscr{B}_v$. We say that the pair $(\mathbf{m}_v, \beta_v)$ *is v-integral* if either $v \notin C_0$, or if $v \in C_0$, $\beta_v \in \mathscr{B}_0$ and $m_{\alpha,v} = 0$ for every α. In other words, the union of the sets $G(\mathbf{m}_v, \beta_v)$ for all v-integral pairs $(\mathbf{m}_v, \beta_v)$ is equal to $\mathscr{U}(\mathfrak{o}_v)$ if $v \in C_0$, and to $G(F_v)$ otherwise.

3.6. Adelic descriptions. For every subset W of $G(\mathbb{A}_F)$ whose characteristic function is an adelic motivic Schwartz-Bruhat function, the intersection $G(F) \cap W$ is represented by a constructible set $[W]$ over k. Our goal now is to describe a family of adelic sets $G(\mathbf{m}, \beta)$ which will allow us to recover the constructible sets $M_{\mathbf{n}}$.

Let $\mathbf{m} = (\mathbf{m}_v)_v$ and $\beta = (\beta_v)_v$ be families indexed by $v \in C(k)$, where $\mathbf{m}_v = (m_{\alpha,v}) \in \mathbf{N}^{\mathscr{A}}$ and $\beta_v \in \mathscr{B}_v$ for all v. We say that $(\mathbf{m}, \beta)$ *is integral* if $(\mathbf{m}_v, \beta_v)$ is v-integral for every v. For each family $(\mathbf{m}, \beta)$, define a set

$$G(\mathbf{m}, \beta) = \prod_{v\in C(k)} G(\mathbf{m}_v, \beta_v)$$

in the product of all $G(F_v)$. If $(\mathbf{m}, \beta)$ is integral, then the characteristic function of $G(\mathbf{m}, \beta)$ is an adelic motivic Schwartz-Bruhat function, because then $G(\mathbf{m}_v, \beta_v) \subset G(\mathbf{0})_v = \mathbf{G}_{\mathrm{a}}^n(\mathfrak{o}_v)$ for almost all $v \in C_0(k)$ (Lemma 3.3.3).

For every $g \in G(F) \cap G(\mathbf{m}, \beta)$, one has

$$\deg_C \sigma_g^*(\mathscr{D}_\alpha) = \sum_{v \in C(k)} m_{\alpha,v},$$

and

$$\deg_C \sigma_g^*(\mathscr{L}_\alpha) = \sum_{v \in C(k)} \left(m_{\alpha,v} + e_{\alpha,\beta_v}\right).$$

Such a point g defines an integral point of $\mathscr{U}(C_0)$ if and only if $(\mathbf{m}, \beta)$ is integral.

To shorten the notation, define, for every $v \in C(k)$, every $\alpha \in \mathscr{A}$, every $\mathbf{m}_v \in \mathbf{N}^{\mathscr{A}}$ and every $\beta_v \in \mathscr{B}_v$ such that $(\mathbf{m}_v, \beta_v)$ is v-integral,

$$\|\mathbf{m}_v, \beta_v\|_\alpha = m_{\alpha,v} + e_{\alpha,\beta_v} \quad \text{and} \quad \mathbf{T}^{\|\mathbf{m}_v,\beta_v\|} = \prod_{\alpha \in \mathscr{A}} T_\alpha^{\|\mathbf{m}_v,\beta_v\|_\alpha}. \tag{3.6.1}$$

Similarly, for every $\mathbf{m} = (\mathbf{m}_v)_{v \in C}$ and $\beta = (\beta_v)$ such that $(\mathbf{m}, \beta)$ is integral, set

$$\|\mathbf{m}, \beta\|_\alpha = \sum_v \|\mathbf{m}_v, \beta_v\|_\alpha \quad \text{and} \quad \mathbf{T}^{\|\mathbf{m},\beta\|} = \prod_{\alpha \in \mathscr{A}} T_\alpha^{\|\mathbf{m},\beta\|_\alpha}. \tag{3.6.2}$$

For every subset W of $G(\mathbb{A}_F)$ whose characteristic function is an adelic motivic Schwartz-Bruhat function, such as the sets $G(\mathbf{m}, \beta)$, the intersection $G(F) \cap W$ is represented by a constructible set $[W]$ over k. Consequently, one has the following adelic description of the height zeta function $Z(\mathbf{T})$ defined by (3.5.2):

$$Z(\mathbf{T}) = \sum_{(\mathbf{m},\beta) \text{ integral}} [G(\mathbf{m}, \beta)] \mathbf{T}^{\|\mathbf{m},\beta\|}. \tag{3.6.3}$$

We shall prove our main theorem in the next section by applying the motivic Poisson summation formula (Theorem 1.3.10) to each term $[G(\mathbf{m}, \beta)]$, assuming the analysis of the local Fourier transforms of the sets $G(\mathbf{m}_v, \beta_v)$ in $G(F_v)$. This local analysis is postponed to Section 6 and will use computations of "motivic oscillatory integrals" which are the topic of Section 5.

4. Proof of the theorem.

4.1. Application of the motivic Poisson summation formula. Let W be any subset of $G(\mathbb{A}_F)$ whose characteristic function $\mathbf{1}_W$ is an adelic Schwartz-Bruhat function. The motivic Fourier transform of $\mathbf{1}_W$, denoted $\mathscr{F}(\mathbf{1}_W, \cdot)$ is also a Schwartz-Bruhat function on the "dual" group $G(\mathbb{A}_F)$. Using Hrushovski-Kazhdan's suggestive notation of "sum over F-rational points", the motivic Poisson summation formula (Theorem 1.3.10) is the equality

$$[W] = \sum_{x \in \mathbf{G}_{\mathrm{a}}^n(F)} \mathbf{1}_W(x) = \mathbf{L}^{(1-g)n} \sum_{\xi \in \mathbf{G}_{\mathrm{a}}^n(F)} \mathscr{F}(\mathbf{1}_W, \xi). \tag{4.1.1}$$

(Recall that g is the genus of C.) Recall also that when W is of the form $\prod W_v$, the Fourier transform $\mathscr{F}(\mathbf{1}_W, \cdot)$ can be written as a product of local Fourier transforms at all points v of C,

$$\mathscr{F}(\mathbf{1}_W, \cdot) = \bigotimes_{v \in C} \mathscr{F}_v(\mathbf{1}_{W_v}, \cdot);$$

in this expression, almost all factors are equal to 1.

We apply this formula to each of the adelic sets $G(\mathbf{m}, \beta)$, where $(\mathbf{m}, \beta)$ is integral. From Equation (3.6.3), we thus get

$$\begin{aligned} Z(\mathbf{T}) &= \sum_{(\mathbf{m},\beta) \text{ integral}} [G(\mathbf{m}, \beta)] \mathbf{T}^{\|\mathbf{m},\beta\|} \\ &= \sum_{(\mathbf{m},\beta) \text{ integral}} \sum_{x \in G(F)} \mathbf{1}_{G(\mathbf{m},\beta)}(x) \mathbf{T}^{\|\mathbf{m},\beta\|} \\ &= \mathbf{L}^{(1-g)n} \sum_{(\mathbf{m},\beta) \text{ integral}} \sum_{\xi \in G(F)} \mathscr{F}(\mathbf{1}_{G(\mathbf{m},\beta)}, \xi) \mathbf{T}^{\|\mathbf{m},\beta\|}. \end{aligned}$$

Let us define a Laurent series $Z(\mathbf{T}, \cdot)$ whose coefficients are adelic Schwartz-Bruhat function by the formula

$$Z(\mathbf{T}, \xi) = \sum_{(\mathbf{m},\beta) \text{ integral}} \mathscr{F}(\mathbf{1}_{G(\mathbf{m},\beta)}, \xi) \mathbf{T}^{\|\mathbf{m},\beta\|}. \tag{4.1.2}$$

With this notation, the height zeta function (3.5.2) can be rewritten as

$$Z(\mathbf{T}) = \mathbf{L}^{(1-g)n} \sum_{\xi \in G(F)} Z(\mathbf{T}, \xi). \tag{4.1.3}$$

In this formula, "summation over F-rational points" of a Laurent series has to be understood termwise.

4.2. Restriction of the summation domain. The following lemma shows that the coefficients of the Laurent series $Z(\mathbf{T}, \xi)$ given by Equation (4.1.2) are "uniformly" adelic Schwartz-Bruhat functions.

LEMMA 4.2.1. *There exists a finite dimensional k-vector space E, a linear F-morphism $\mathbf{a}\colon E_F \to G_F$, and a finite subset $\Sigma \subset C(k)$ containing S and satisfying the following properties: for every integral $(\mathbf{m}, \beta)$ and every $\xi \in G(F)$,*

- *If $\xi \notin \mathbf{a}(E(k))$, then there exists $v \in C$ such that $\mathscr{F}_v(1_{G(\mathbf{m},\beta)}, \xi) = 0$;*
- *If $\xi \in \mathbf{a}(E(k))$ and $v \notin \Sigma$, then $\mathscr{F}_v(1_{G(\mathbf{m},\beta)}, \xi) = 1$.*

Proof. With the notation from Lemma 3.3.4, there is, for every point $v \in C(k)$, an integer r_v such that the characteristic function of the definable set $G(\mathbf{m}_v, \beta_v)$ in $G(F_v)$ is invariant under the action of the subgroup $G(\mathfrak{m}_v^{r_v})$. Consequently, its Fourier transform vanishes outside of the orthogonal of this subgroup. Let $\sum a_v[v]$

be the divisor of the global differential form in $\Omega_{F/k}$ that has been used to define the global Fourier transform. For almost all points v, one has $a_v=0$. Moreover, the orthogonal of $G(\mathfrak{m}_v^{r_v})$ contains $G(\mathfrak{m}_v^{-r_v+a_v})$. For every $v\in C(k)$, set $s_v=-r_v+a_v$; one has $s_v=0$ for all but finitely many points $v\in C(k)$. By the Riemann-Roch theorem, the space E of points $\xi\in G(F)$ such that $\xi_v\in G(\mathfrak{m}_v^{s_v})$ for all v is a finite dimensional k-vector space. This proves the first part of the Lemma.

Moreover, for every $(\mathbf{m},\beta)$ and every $v\in C_0(k)$ such that $\mathbf{m}_v=0$ and $\mathscr{B}_v$ is a singleton, then the subset $G(\mathbf{m}_v,\beta_v)$ of $G(F_v)$ identifies with $G(\mathfrak{o}_v)$; if, moreover, $a_v=0$, then the characteristic function of $G(\mathfrak{o}_v)$ is self-dual. Up to enlarging the set Σ, this implies the second assertion. $\square$

This suggests to introduce, for every place $v\in\Sigma$, a Laurent series whose coefficients are motivic Schwartz-Bruhat functions on $G(F_v)$ by

$$Z_v(\mathbf{T},\cdot)=\sum_{(\mathbf{m}_v,\beta_v)\,\text{integral}}\mathscr{F}_v(\mathbf{1}_{G(\mathbf{m}_v,\beta_v)},\cdot)\mathbf{T}^{\|\mathbf{m}_v,\beta_v\|}. \tag{4.2.2}$$

By Lemma 4.2.1, one has $\mathscr{F}(1_{G(\mathbf{m},\beta)},\xi)=0$ if $\xi\notin\mathbf{a}(E(k))$, while $\mathscr{F}(1_{G(\mathbf{m},\beta)},\xi)=\prod_{v\in\Sigma}\mathscr{F}_v(1_{G(\mathbf{m},\beta)},\xi)$ otherwise. Consequently, one has

$$Z(\mathbf{T})=\mathbf{L}^{(1-g)n}\sum_{\xi\in\mathbf{a}(E(k))}\prod_{v\in\Sigma}Z_v(\mathbf{T},\cdot). \tag{4.2.3}$$

4.3. Local results. In all of this section, we fix a point $v\in\Sigma$ and state the properties of the Laurent series $Z_v(\mathbf{T},\cdot)$, and of its specialization $Z_{\lambda,v}(T,\cdot)=Z((T^{\lambda_\alpha}),\cdot)$. They will be proved in Section 6. We fix a finite dimensional k-vector space E and a linear F-morphism $\mathbf{a}\colon E_F\to G_F$ satisfying the conditions of Lemma 4.2.1. Recall also that $\mathscr{U}$ is the good model of G_F over C of which we study the integral sections of bounded height.

PROPOSITION 4.3.1. *Assume that $v\in\Sigma\cap C_0$. Then $Z_v(\mathbf{T},\cdot)$ is a polynomial in $\mathbf{T}$. Moreover, $Z_{\lambda,v}(\mathbf{L}^{-1},0)$ is a non-zero effective element of $\mathscr{M}_k$.*

The following result is a motivic analogue of Proposition 4.6 of [7]. In that paper, some formalism of "residue measures" was introduced, which is useful for *describing* the kind of integrals that appear in the right-hand side. Observe indeed that this is a sum of motivic integrals on arc spaces $\mathscr{L}_v(\mathscr{D}_A)$ attached to the faces of dimension d of the analytic Clemens complex of (X,D) at the place v.

PROPOSITION 4.3.2. *Assume that $v\in C\setminus C_0$. Then the Laurent series $Z_v(\mathbf{T},0)$ is a rational function. More precisely, there exists a family $(P_{v,A})$ of Laurent polynomials with coefficients in $\mathscr{M}_k$, a family $(u_{v,A})$ of motivic, integer valued functions, indexed by the set of maximal faces A of the analytic Clemens complex*

$\mathrm{Cl}_v^{\mathrm{an}}(X,D)$ *such that*

$$Z_v(\mathbf{T},0) = \sum_{A\in\mathrm{Cl}_v^{\mathrm{an,max}}(X,D)} P_{v,A}(\mathbf{T}) \prod_{\alpha\in A} \frac{1}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}$$

and

$$P_{v,A}(\mathbf{T}) \equiv (1-\mathbf{L}^{-1})^{\mathrm{Card}(A)} \int_{\mathscr{L}_v(\mathscr{D}_A)} \mathbf{L}^{u_{v,A}(x)}\,\mathrm{d}x$$

modulo the ideal generated by the polynomials $1-\mathbf{L}^{\rho_\alpha-1}T_\alpha$, *for* $\alpha\in A$.

COROLLARY 4.3.3. *Assume that* $v\in C\setminus C_0$ *and that* $\lambda=(\rho_\alpha-1)_\alpha$. *Let* $d_v = 1+\dim\mathrm{Cl}_v^{\mathrm{an}}(X,D)$. *The Laurent series* $Z_{\lambda,v}(T,0)$ *in the variable* T *is a rational function. More precisely, for every non-zero common multiple* a *of the integers* $\rho_\alpha-1$, *for* $\alpha\in\mathscr{A}$, *then* $P_{\lambda,v}(T)=(1-\mathbf{L}^aT^a)^{d_v}Z_{\lambda,v}(T,0)$ *belongs to* $\mathscr{M}_k[T,T^{-1}]$ *and satisfies*

$$P_{\lambda,v}(\mathbf{L}^{-1}) = (1-\mathbf{L}^{-1})^{d_v} \sum_{\substack{A\in\mathrm{Cl}^{\mathrm{an,max}}(X,D)\\ \mathrm{Card}(A)=d_v}} \prod_{\alpha\in A} \frac{a}{\rho_\alpha-1} \int_{\mathscr{L}_v(\mathscr{D}_A)} \mathbf{L}^{u_{v,A}(x)}\,\mathrm{d}x.$$

PROPOSITION 4.3.4. *Let* $v\in C\setminus C_0$ *and let* $d_v=1+\dim\mathrm{Cl}_v^{\mathrm{an}}(X,D)$. *There exists a constructible partition* $(U_{v,i})$ *of* $E\setminus\{0\}$ *and, for every* i, *an element* $P_{v,i}\in \mathscr{E}xp\mathscr{M}_{U_{v,i}}[\mathbf{T},\mathbf{T}^{-1}]$ *and finite families* $(a_{v,i,j})$, $(b_{v,i,j})$ *where* $a_{v,i,j}\in\mathbf{N}$, $b_{v,i,j}\in\mathbf{N}^{\mathscr{A}}$, *such that the restriction to* $U_{v,i}$ *of* $Z_v(\mathbf{T},\mathbf{a}(\cdot))$ *equals*

$$\prod_j (1-\mathbf{L}^{a_{v,i,j}}\mathbf{T}^{b_{v,i,j}})^{-1} P_{v,i}(\mathbf{T};\cdot).$$

Moreover, assuming that $\lambda=(\rho_\alpha-1)_\alpha$, *there exist integers* $a_{v,i}\geq 1$ *and* $d_{v,i}\in[0,d_v-1]$ *such that the restriction to* $U_{i,v}$ *of* $(1-(\mathbf{L}T)^{a_{v,i}})^{d_{v,i}}Z_{\lambda,v}(T,\mathbf{a}(\cdot))$ *belongs to* $\mathscr{E}xp\mathscr{M}_{U_{v,i}}\{T\}^\dagger$.

For a moment, we take these three propositions for granted and complete the proof of Theorem 1.

4.4. Conclusion: Proof of Theorem 1. Recall from Equation (4.2.3) that our goal is to evaluate the sum

$$Z(\mathbf{T}) = \mathbf{L}^{(1-g)n} \sum_{\xi\in\mathbf{a}(E(k))} \prod_{v\in\Sigma} Z_v(\mathbf{T},\cdot).$$

For every $v\in C\setminus C_0$, let $d_v=1+\dim\mathrm{Cl}_v^{\mathrm{an}}(X,D)$; let also $d=\sum_{v\in C\setminus C_0} d_v$. Propositions 4.3.1, 4.3.2, and 4.3.4 show that for every $\xi\in\mathbf{a}(E(k))$, the Laurent series $Z(\mathbf{T},\xi)=\prod_{v\in C} Z_v(\mathbf{T},\xi)$ with coefficients in $\mathscr{E}xp\mathscr{M}_k$ is a rational function of $\mathbf{T}$, and admits a denominator of the form $\prod(1-\mathbf{L}^a\mathbf{T}^b)$.

We set

$$\mathrm{Cl}_{\infty}^{\mathrm{an,max}}(X,D)=\prod_{v\in C\setminus C_0}\mathrm{Cl}_{v}^{\mathrm{an,max}}(X,D).$$

For $\xi=0$, with the notation of Proposition 4.3.2, one has

$$Z(\mathbf{T},0)=\sum_{A=(A_v)\in \mathrm{Cl}_{\infty}^{\mathrm{an,max}}(X,D)}\prod_{v\in C\setminus C_0}\prod_{\alpha\in A_v}\frac{1}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}P_{v,A_v}(\mathbf{T})\prod_{v\in C_0}Z_v(\mathbf{T},0).$$

In particular, if $\lambda=(\rho_\alpha-1)_\alpha$, one has

$$\begin{aligned}Z_\lambda(T,0)&=Z\big((T^{\rho_\alpha-1}),0\big)\\&=\sum_{A=(A_v)\in \mathrm{Cl}_{\infty}^{\mathrm{an,max}}(X,D)}\prod_{v\in C\setminus C_0}\prod_{\alpha\in A_v}\frac{1}{1-(\mathbf{L}T)^{\rho_\alpha-1}}\\&\qquad\times P_{v,A_v}\big((T^{\rho_\alpha-1})\big)\prod_{v\in C_0}Z_v\big((T^{\rho_\alpha-1}),0\big)\\&=\sum_{A=(A_v)\in \mathrm{Cl}_{\infty}^{\mathrm{an,max}}(X,D)}P_A(T)\prod_{v\in C\setminus C_0}\prod_{\alpha\in A_v}\frac{1}{1-(\mathbf{L}T)^{\rho_\alpha-1}}\end{aligned}$$

where the polynomial $P_A\in\mathscr{M}_k[T]$ is defined by

$$P_A(T)=\prod_{v\in C\setminus C_0}P_{v,A_v}((T^{\rho_\alpha-1}))\prod_{v\in C_0}Z_v((T^{\rho_\alpha-1}),0).$$

Consequently, $Z_\lambda(T,0)$ is both a rational function, and an element of $\mathscr{M}_k\{T\}$; moreover,

$$\begin{aligned}&(1-\mathbf{L}^aT^a)^dZ_\lambda(T,0)\\&=\sum_{A=(A_v)\in \mathrm{Cl}_{\infty}^{\mathrm{an,max}}(X,D)}P_A(T)\prod_{v\in C\setminus C_0}(1-(\mathbf{L}T)^a)^{d_v-\mathrm{Card}(A_v)}\prod_{\alpha\in A_v}\frac{1-(\mathbf{L}T)^a}{1-(\mathbf{L}T)^{\rho_\alpha-1}}.\end{aligned}$$

The right-hand side of the preceding formula is a polynomial in T with coefficients in $\mathscr{M}_k$; when one sets $T=\mathbf{L}^{-1}$, only the terms remain for which $\mathrm{Card}(A_v)=d_v$ for every v, and one gets

$$\sum_{\substack{A=(A_v)\in \mathrm{Cl}_{\infty}^{\mathrm{an,max}}(X,D)\\ \mathrm{Card}(A_v)=d_v}}P_A(\mathbf{L}^{-1})\prod_{v\in C\setminus C_0}\prod_{\alpha\in A_v}\frac{a}{\rho_\alpha-1}. \tag{4.4.1}$$

It then follows from Propositions 4.3.1 and 4.3.2 that this is an effective element of $\mathscr{M}_k$, which is non-zero since, by assumption, $\mathscr{U}(\mathfrak{o}_v)$ is non-empty for every $v\in C_0$. In this case, one concludes that $Z_\lambda(T,0)$ has a pole of order exactly d at $T=\mathbf{L}^{-1}$.

For $\xi\neq 0$, one deduces in a similar way from Proposition 4.3.4 that the Laurent series $Z(\mathbf{T},\xi)$ is rational, as well as its specializations. By uniformity, the same property holds when one takes the sum over F-rational points, so that the height zeta function $Z_\lambda(T)$ is a rational function which belongs to $\mathscr{E}xp\mathscr{M}_k\{T\}$.

Since $\mathbf{L}$ belongs to $\mathscr{M}_k$ and the natural map from $\mathscr{M}_k$ to $\mathscr{E}xp\mathscr{M}_k$ is injective (Lemma 1.1.3), the power series $Z(\mathbf{T})$ is rational when viewed as a Laurent series with coefficients in $\mathscr{M}_k$. In particular, the specialization $Z_\lambda(T)$ is a rational function too.

For every $\xi \neq 0$, the specialization $Z_\lambda(T,\xi) = Z((T^{\rho_\alpha-1}),\xi)$ is a rational function, and an element of $\mathscr{E}xp\mathscr{M}_k\{T\}$. Moreover, Proposition 4.3.4 asserts that the order of its pole at $T = \mathbf{L}^{-1}$ is strictly smaller than d. Taking for the integer a any common multiple of the $\rho_\alpha - 1$ and of the integers $a_{v,i}$ appearing in the statement of Proposition 4.3.4, and summing over rational points $\xi \in G(F)$, we obtain that

$$(1-(\mathbf{L}T)^a)^d Z_\lambda(T) \in \mathscr{E}xp\mathscr{M}_k\{T\}^\dagger$$

and its value at $T = \mathbf{L}^{-1}$ is given by Equation (4.4.1), multiplied by $\mathbf{L}^{(1-g)n}$.

This concludes the proof of Theorem 1.

5. Motivic oscillatory integrals. In this section, we consider a field k of characteristic zero and let K be the local field $k((t))$. We write ord for the valuation of K, normalized by $\operatorname{ord}(t) = 1$, R for the valuation ring of K and $\mathfrak{m}$ for its maximal ideal. The angular component map $\operatorname{ac}\colon K \to k$ is the unique multiplicative map which is trivial on $1 + k[\![t]\!]$, on t, and maps constants $a \in k$ to themselves. We also fix a real number $q > 1$ and set $|x| = q^{-\operatorname{ord}(x)}$.

With the notation of Section 1, let $r\colon K \to k$ be the linear map, given by $r(1) = 1$ for $n = 0$ and $r(t^n) = 0$ otherwise, so that $r(a) = \operatorname{res}_0(a\,\mathrm{d}t/t)$. Set $\mathrm{e}(\cdot) = \psi(r(\cdot))$; it is an analogue of a non-trivial character of $R/\mathfrak{m}$.

5.1. Decay of motivic integrals.

LEMMA 5.1.1. *Let d be a positive integer and let $\xi \in K$ be such that $|\xi| = 1$. Then, for every $a \in K$ and every $n \in \mathbf{N}$ such that* $\operatorname{ord}(a) + n \leq 0 < \operatorname{ord}(a) + 2n$*, one has*

$$\int_{\xi + t^n R} \mathrm{e}(ax^d)\mathrm{d}x = 0.$$

Proof. We follow the arguments of Lemma 2.3.1 in [8]. One can write

$$\int_{\xi+t^nR} \mathrm{e}(ax^d)\,\mathrm{d}x = \mathbf{L}^{-n} \int_R \mathrm{e}(a\xi^d(1+t^nu)^d)\,\mathrm{d}u.$$

For $u \in R$, all terms starting from the third one in the binomial expansion

$$a\xi^d(1+t^nu)^d = a\xi^d + \binom{d}{1} a\xi^d t^n u + \binom{d}{2} a\xi^d t^{2n} u^2 + \cdots + \binom{d}{d} a\xi^d t^{dn} u^d$$

belong to $\mathfrak{m}$, since $\operatorname{ord}(a) > -2n$ and $\operatorname{ord}(\xi) = 0$. Therefore

$$r(a\xi^d(1+t^n u)^d) = r(a\xi^d) + dr(a\xi^d t^n u)$$

and

$$\int_R \mathrm{e}(a\xi^d(1+t^n u)^d)\,\mathrm{d}u = [\operatorname{Spec}(k), r(a\xi^d)] \int_R \mathrm{e}(da\xi^d t^n u)\,\mathrm{d}u.$$

Since $\operatorname{ord}(a\xi^d t^n) = \operatorname{ord}(a) + n \leq 0$, Proposition 1.2.8 implies that

$$\int_R \mathrm{e}(da\xi^d t^n u)\,\mathrm{d}u = 0,$$

and the lemma follows. $\square$

For $m \in \mathbf{Z}$, let C_m be the annulus defined by $\operatorname{ord}(x) = m$. For $d \in \mathbf{Z}$, $d \neq 0$, and $a \in K^*$, set

$$I(m,d,a) = \int_{C_m} \mathrm{e}(ax^d)\,\mathrm{d}x \tag{5.1.2}$$

in $\mathscr{M}_k$.

LEMMA 5.1.3. *The integrals $I(m,d,a)$ satisfy the following properties:*

(1) *Let $m \in \mathbf{Z}$ and $d \in \mathbf{Z}$, $d \neq 0$. Let $a, b \in K^*$ be such that* $\operatorname{ord}(b) = \operatorname{ord}(a) + md$ *and* $\operatorname{ac}(b) = \operatorname{ac}(a) \pmod{(k^*)^d}$. *Then $I(m,d,a) = \mathbf{L}^{-m} I(0,d,b)$.*

(2) *Assume that k is algebraically closed. Then*

$$I(m,d,a) = \mathbf{L}^{-m} I(0,d,t^{\operatorname{ord}(a)+md}).$$

In particular, $I(m,d,a)$ depends only on m, d, and $\operatorname{ord}(a)$.

(3) *If* $\operatorname{ord}(a) + md < 0$, *then $I(m,d,a) = 0$.*

(4) *If* $\operatorname{ord}(a) + md > 0$, *then $I(m,d,a) = \mathbf{L}^{-m}(\mathbf{L}-1)/\mathbf{L}$.*

Proof. (1) Let $u \in k^*$ be such that $\operatorname{ac}(a)u^d = \operatorname{ac}(b)$. By assumption, there exists $v_1 \in 1 + tk[[t]]$ such that $b = au^d t^{md} v_1$; since k has characteristic zero, there exists $v \in 1 + tk[[t]]$ such that $v_1 = v^d$. Let us make the change of variables $x = uvt^m y$. This gives

$$\begin{aligned} I(m,d,a) = \int_{C_m} \mathrm{e}(ax^d)\,\mathrm{d}x &= \mathbf{L}^{-m} \int_{C_0} \mathrm{e}(au^d v^d t^{md} y^d)\,\mathrm{d}y \\ &= \mathbf{L}^{-m} \int_{C_0} \mathrm{e}(by^d)\,\mathrm{d}y = I(0,d,b). \end{aligned}$$

Assertion (2) follows at once.

Let us prove (3). Since $I(m,d,a) = \mathbf{L}^{-m} I(0,d,at^{md})$ we only need to prove that $I(0,d,a) = 0$ for $\operatorname{ord}(a) < 0$. Let $n = -\operatorname{ord}(a)$. Observe that

$$I(0,d,a) = \int_{C_0} \mathrm{e}(ax^d)\,\mathrm{d}x = \int_{C_0}\int_R \mathrm{e}(a(x+t^n y)^d)\,\mathrm{d}y\,\mathrm{d}x = 0.$$

Since $\operatorname{ord}(a) < 0$, $\operatorname{ord}(a) + 2n = -\operatorname{ord}(a) > 0$, hence by Lemma 5.1.1, $\int_{x+t^nR} \mathrm{e}(ay^d)\mathrm{d}y = 0$ for every $x \in K$ such that $\operatorname{ord}(x) = 0$. The statement follows.

(4) It suffices to prove that $I(0,d,a) = (\mathbf{L}-1)/\mathbf{L}$ for $\operatorname{ord}(a) > 0$. In this case, one has $r(ax^d) = 0$ for every $x \in R^*$, hence the claim. ☐

Let $u \in K((x))$ be a Laurent series of positive radius of convergence; write $u = \sum u_n x^n$. Let $\mu \in \mathbf{Z}$ be such that u converges on the closed disk D_μ defined by the inequality $\operatorname{ord}(x) \geq \mu$ deprived from 0; in other words, μ is such that $\operatorname{ord}(u_n) + n\mu \to +\infty$ when $n \to +\infty$. Let m be an integer such that $m \geq \mu$; let $\nu \geq 0$ be such that $\operatorname{ord}(u_n) + nm > 0$ for $n > \nu$. By construction, $\operatorname{ord}(u_n x^n) = \operatorname{ord}(u_n) + n\operatorname{ord}(x) > 0$ for $n > \nu$ and $\operatorname{ord}(x) = m$, so that $r(u(x)) = r(u^\nu(x))$, for $x \in D_\mu$ such that $\operatorname{ord}(x) = m$, where $u^\nu(x) = \sum_{n\leq\nu} u_n x^n$. Therefore, for $m \geq \mu$, we can define the motivic integrals $\int_{C_m} \mathrm{e}(u(x))\,\mathrm{d}x$ as given by $\int_{C_m} \mathrm{e}(u^\nu(x))\,\mathrm{d}x$ in $\mathscr{M}_k$. More generally, for every definable subset W of D_μ, one can define $\int_W \mathrm{e}(u(x))\,\mathrm{d}x$ as an element of a suitable completion of $\mathscr{M}_k$, and as an element of $\mathscr{M}_k$ itself if $\operatorname{ord}(x)$ is bounded from above on W.

PROPOSITION 5.1.4. *Let $u \in K((x))$ be a Laurent series of positive radius of convergence; let $d = -\operatorname{ord}_x(u)$ and $a = \lim_{x\to 0} u(x)x^d$. Assume that $d > 0$. The motivic integrals*

$$\int_{C_m} \mathrm{e}(u(x))\,\mathrm{d}x$$

vanish for every large enough integer m; more precisely, it suffices that u converges and has no root in the punctured disk defined by $\operatorname{ord}(x) \geq m$*, and that* $\operatorname{ord}(a) < md$.

Proof. Since K has characteristic zero, there exists $a \in K^*$ and a power series $v \in K[\![x]\!]$ such that $v(0) = 1$ and $u(x) = ax^{-d}v(x)^{-d}$. Let $m_0 \in \mathbf{N}$ be a large enough integer such that $x^d u$ converges on the disk $\{\operatorname{ord}(x) \geq m_0\}$ and does not vanish on this disk. If one writes $u = \sum u_n x^n$, we thus have the following properties:

- one has $u_{-d} = a$ and $u_n = 0$ for $n < -d$;
- for $n > -d$, $\operatorname{ord}(u_n) + nm_0 > \operatorname{ord}(a)$;
- when $n \to +\infty$, one has $\operatorname{ord}(u_n) + nm_0 \to +\infty$.

Writing $v = \sum_{n\geq 0} v_n x^n$, it follows that $\operatorname{ord}(v_n) + nm_0 > 0$ for every $n \in \mathbf{N}_{>0}$. Consequently, the change of variables $y = xv(x)$ maps the annuli C_m to themselves, for $m \geq m_0$, and preserves the motivic measure. Therefore, for $m \geq m_0$, one has

$$\int_{C_m} \mathrm{e}(u(x))\,\mathrm{d}x = \int_{C_m} \mathrm{e}(ax^{-d}v(x)^{-d})\,\mathrm{d}x = \int_{C_m} \mathrm{e}(ay^{-d})\,\mathrm{d}y.$$

According to Lemma 5.1.3(3), this integral vanishes if $\operatorname{ord}(a) - md < 0$. This concludes the proof of the proposition. $\square$

5.2. Motivic Igusa integrals with exponentials—the regular case.

5.2.1. Setup. Let $\mathscr{X}$ be a flat R-scheme of finite type, equidimensional of relative dimension n, let $\mathscr{D}$ be a relative divisor on $\mathscr{X}$. We assume that $\mathscr{X}$ is smooth, everywhere of relative dimension n, and that $\mathscr{D}$ has strict normal crossings over R. Let also $X = \mathscr{X}_k$ and $D = \mathscr{D}_k$ be their special fibers. Let $\mathscr{A}$ be the set of irreducible components of $\mathscr{D}$; for $\alpha \in \mathscr{A}$, let $\mathscr{D}_\alpha$ be the corresponding irreducible component, and let D_α be its special fiber. For every $A \subset \mathscr{A}$, let $\mathscr{D}_A = \bigcap_{\alpha\in A} \mathscr{D}_\alpha$ and let $\mathscr{D}_A^\circ = \mathscr{D}_A \setminus \bigcup_{\alpha\notin A} \mathscr{D}_\alpha$; one defines D_A and D_A° in a similar way. By definition of a divisor with normal crossings, every irreducible component of D_A has codimension $\operatorname{Card}(A)$.

For every constructible subset W of X, let $\mathscr{L}(\mathscr{X};W)$ be the constructible subset of $\mathscr{L}(\mathscr{X})$ parameterizing arcs $x \in \mathscr{X}(R)$ whose origin lies in W. For every $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$, we write $W(\mathbf{m})$ for the constructible subset of $\mathscr{L}(\mathscr{X})$ consisting of arcs x such that $\operatorname{ord}_{\mathscr{D}_\alpha}(x) = m_\alpha$ for every $\alpha \in \mathscr{A}$.

Let h be a motivic residual function on $\mathscr{L}(\mathscr{X})$. Let f be a meromorphic function on $\mathscr{X}$ such that the polar divisor $\operatorname{div}_\infty(f_K)$ of the restriction f_K to $\mathscr{X}_K$ is contained in the union $\bigcup_{\alpha\in\mathscr{A}} \mathscr{D}_{\alpha,K}$. Let $(d_\alpha)_{\alpha\in\mathscr{A}}$ be nonnegative integers such that on $\mathscr{X}_K$,

$$\operatorname{div}_\infty(f_K) = \sum_{\alpha\in\mathscr{A}} d_\alpha \mathscr{D}_{\alpha,K}. \tag{5.2.2}$$

For a family $\mathbf{T} = (T_\alpha)_{\alpha\in\mathscr{A}}$ of indeterminates, define the *motivic Igusa integral with exponentials*

$$Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) = \int_{\mathscr{L}(\mathscr{X})} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\operatorname{ord}_{\mathscr{D}_\alpha}(x)} h(x)\,\mathrm{e}(f(x))\,\mathrm{d}x, \tag{5.2.3}$$

a power series in $\mathbf{T}$ with coefficients in $\mathscr{E}xp\mathscr{M}_k$. Although f is only a rational function on $\mathscr{X}$, note that $r(f)$ is a well defined residual function on $W(\mathbf{m})$ for each $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$, so that we have

$$Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) = \sum_{\mathbf{m}\in\mathbf{N}^{\mathscr{A}}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{m_\alpha} \int_{W(\mathbf{m})} h(x)\mathrm{e}(f(x))\,\mathrm{d}x. \tag{5.2.4}$$

This power series is an analogue of the classical motivic Igusa zeta integrals which would correspond to the case $f = 0$.

More generally, for every subset $A \subset \mathscr{A}$, let

$$Z_A(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) = \int_{\mathscr{L}(\mathscr{X}; D_A^\circ)} \prod_{\alpha \in \mathscr{A}} T_\alpha^{\mathrm{ord}_{\mathscr{D}_\alpha}(x)} h(x) \, \mathrm{e}(f(x)) \, \mathrm{d}x. \tag{5.2.5}$$

When A runs among $\mathrm{Cl}^{\mathrm{an}}(X, D)$, the subsets $\mathscr{L}(\mathscr{X}; D_A^\circ)$ form a partition of $\mathscr{L}(\mathscr{X})$ into constructible subsets and we decompose the motivic integral defining $Z(X, h\mathrm{e}(f); \mathbf{T})$ as the sum of motivic integrals over each of them, so that

$$Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) = \sum_{A \subset \mathscr{A}} Z_A(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}).$$

For every $\mathbf{m} \in \mathbf{N}_{>0}^A$, let $W_A(\mathbf{m})$ be the constructible subset of $\mathscr{L}(\mathscr{X}; D_A^\circ)$ defined by the conditions $\mathrm{ord}_{\mathscr{D}_\alpha}(x) = m_\alpha$ for $\alpha \in A$ and $\mathrm{ord}_{\mathscr{D}_\alpha}(x) = 0$ for $\alpha \notin A$. With this notation, one has

$$Z_A(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) = \sum_{\mathbf{m} \in \mathbf{N}_{>0}^A} \prod_{\alpha \in A} T_\alpha^{m_\alpha} \int_{W_A(\mathbf{m})} h(x) \mathrm{e}(f(x)) \, \mathrm{d}x.$$

LEMMA 5.2.6. *Let A be a subset of $\mathscr{A}$ and let B be a set of cardinality equal to $n - \mathrm{Card}(A)$. There exists a measure-preserving definable isomorphism θ from $D_A^\circ \times \mathscr{L}(\mathbf{A}^1; 0)^A \times \mathscr{L}(\mathbf{A}^1)^B$, with coordinates x_α (for $\alpha \in A$) and y_β (for $\beta \in B$), to $\mathscr{L}(\mathscr{X}; D_A^\circ)$ such that $\mathrm{ord}_{\mathscr{D}_\alpha}(\theta(x)) = \mathrm{ord}(x_\alpha)$ for $\alpha \in A$, and $\mathrm{ord}_{\mathscr{D}_\alpha}(\theta(x)) = 0$ for $\alpha \notin A$.*

Proof. This is a standard fact in the theory of motivic zeta functions. We may assume that $\mathscr{D}_\alpha = \varnothing$ for $\alpha \notin A$, and that there exist regular functions u_α (for $\alpha \in A$) on $\mathscr{X}$ such that $\mathrm{div}(u_\alpha) = \mathscr{D}_\alpha$. By definition of a divisor with strict normal crossings, the morphism $u = (u_\alpha) \colon \mathscr{X} \to (\mathbf{A}^1)^A$ is then smooth. Hence we may assume that there exists regular functions v_β (for $\beta \in B$) in $\mathscr{X}$ such that the morphism $(u, v) = ((u_\alpha); (v_\beta))$ from $\mathscr{X}$ to $(\mathbf{A}^1)^A \times (\mathbf{A}^1)^B$ is étale. Both of these assumptions are only valid up to replacing $\mathscr{X}$ by a Zariski dense open subset containing any given point of the special fibre. Since we only seek for a definable isomorphism, they do not restrict the generality.

It then follows from the definition of an étale morphism that the induced morphism $\mathscr{L}(\mathscr{X}; D_A^\circ) \to D_A^\circ \times \mathscr{L}(\mathbf{A}^1; 0)^A \times \mathscr{L}(\mathbf{A}^1)^B$ is an isomorphism. It preserves the motivic measure by construction of latter. Moreover, denoting the standard coordinates on $\mathscr{L}(\mathbf{A}^1; 0)$ by x_α, for $\alpha \in A$, this isomorphism maps the definable function $\mathrm{ord}_{\mathscr{D}_\alpha}$ to the function $\mathrm{ord}(x_\alpha)$. $\square$

In this section and the next one, we study the rationality and the poles of the Igusa integral with exponentials. We first consider the special case where f is regular on the generic fiber $\mathscr{X}_K$.

PROPOSITION 5.2.7. *Assume that f_K is regular on the generic fiber $\mathscr{X}_K$. Let A be a subset of $\mathscr{A}$. The power series $Z_A(\mathscr{X}, h\mathrm{e}(f);\mathbf{T})$ is a rational function. More precisely, there exists a polynomial Q_A with coefficients in $\mathscr{E}xp\mathscr{M}_k$, such that*

$$Z_A(\mathscr{X}, h\mathrm{e}(f);\mathbf{T}) = Q_A(\mathbf{T}) \prod_{\alpha\in A} \frac{1}{1-\mathbf{L}^{-1}T_\alpha}$$

and such that

$$Q_A(\mathbf{T}) - (1-\mathbf{L}^{-1})^{\mathrm{Card}(A)} \int_{\mathscr{L}(\mathscr{D}_A^\circ)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x \tag{5.2.8}$$

belongs to the ideal generated by the polynomials $1-\mathbf{L}^{-1}T_\alpha$, for $\alpha \in A$.

Proof. By Lemma 5.2.6, there is a measure-preserving definable isomorphism from $\mathscr{L}(\mathscr{X}; D_A^\circ)$ to $D_A^\circ \times \mathscr{L}(\mathbf{A}^1;0)^A \times \mathscr{L}(\mathbf{A}^1)^B$, where B is some set of cardinality $n-\mathrm{Card}(A)$, with coordinates x_α (for $\alpha \in A$), y_β (for $\beta \in B$) under which $\mathrm{ord}_{\mathscr{D}_\alpha}(x) = \mathrm{ord}(x_\alpha)$ for $\alpha \in A$, and $\mathrm{ord}_{\mathscr{D}_\alpha}(x) = 0$ for $\alpha \notin A$. In the sequel, we identify a point $x \in \mathscr{L}(\mathscr{X}; D_A^\circ)$ with a triple (ξ, x, y), where $\xi \in D_A^\circ$, $x \in \mathscr{L}(\mathbf{A}^1;0)^A$ and $y \in \mathscr{L}(\mathbf{A}^1)^B$.

Fix an integer a and a regular function g on $\mathscr{X}$ such that $f = t^a g$. On $\mathscr{L}(\mathscr{X}; D_A^\circ)$, we can expand the function g as a power series

$$g_A(x,y) = \sum_{\substack{p\in\mathbf{N}^A\\ q\in\mathbf{N}^B}} g_{p,q} x^p y^q,$$

where $g_{p,q} \in \mathscr{O}(D_A^\circ)[[t]]$. Then

$$\mathrm{ord}(g_A(x,y) - g_A(0,y)) \geq \min_{\alpha\in A} \mathrm{ord}(x_\alpha).$$

In particular, we see that $r(t^a g_A(x,y)) = r(t^a g_A(0,y))$ if $a + \min_{\alpha\in A}\mathrm{ord}(x_\alpha) > 0$. Let μ be a positive integer such that $\mu > -a$ and such that the Schwartz-Bruhat function h factors through $\mathscr{L}_\mu(\mathscr{X})$.

If one has $m_\alpha \geq \mu$ for every $\alpha \in A$, it then follows that

$$\begin{aligned}
&\int_{W_A(\mathbf{m})} h(x)\mathrm{e}(f(x))\,\mathrm{d}x \\
&\quad = \left(\prod_{\alpha\in A} \mathbf{L}^{-m_\alpha}\right) \int_{\substack{\mathrm{ord}(x'_\alpha)=0\\ x_\alpha = t^{m_\alpha}x'_\alpha}} h(\xi,x,y)\mathrm{e}(t^a g_A(x,y))\,\mathrm{d}x'\mathrm{d}y \\
&\quad = \left(\prod_{\alpha\in A} \mathbf{L}^{-m_\alpha}\right) \int_{\substack{\mathrm{ord}(x'_\alpha)=0\\ x_\alpha = t^{m_\alpha}x'_\alpha}} h(\xi,0,y)\mathrm{e}(t^a g_A(0,y))\,\mathrm{d}x'\mathrm{d}y
\end{aligned}$$

$$= \prod_{\alpha\in A} \left(\mathbf{L}^{-m_\alpha}(1-\mathbf{L}^{-1})\right) \int_{D_A^\circ \times \mathscr{L}(\mathbf{A}^1)^B} h(\xi,0,y)\mathrm{e}(t^a g_A(0,y))\,\mathrm{d}y$$
$$= \prod_{\alpha\in A} \left(\mathbf{L}^{-m_\alpha}(1-\mathbf{L}^{-1})\right) \int_{\mathscr{L}(\mathscr{D}_A; D_A^\circ)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x.$$

In general, let

$$A_1 = \left\{\alpha \in A;\ m_\alpha < \mu\right\} \quad \text{and} \quad A_2 = \left\{\alpha \in A;\ m_\alpha \geq \mu\right\},$$

so that A is the disjoint union $A = A_1 \cup A_2$. Write $x = (x_1, x_2)$, where $x_1 = (x_\alpha)_{\alpha\in A_1}$ and $x_2 = (x_\alpha)_{\alpha\in A_2}$, and split $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2)$ accordingly. Analogously, one has

$$\int_{W_A(\mathbf{m})} h(x)\mathrm{e}(f(x))\,\mathrm{d}x = \prod_{\alpha\in A_2} \left(\mathbf{L}^{-m_\alpha}(1-\mathbf{L}^{-1})\right) \int_{W'_{A_2}(\mathbf{m}_1)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x,$$

where $W'_{A_2}(\mathbf{m}_1)$ is the definable subset of $\mathscr{L}(\mathscr{D}_{A_2})$ consisting of arcs x on $\mathscr{D}_{A_2}$ with origin on D_{A_2} and such that $\mathrm{ord}_{\mathscr{D}_\alpha}(x) = m_\alpha$ for $\alpha \in A_1$. We can then write

$$\begin{aligned}
Z_A(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) &= \sum_{\mathbf{m}\in\mathbf{N}^A_{>0}} \prod_{\alpha\in A} T_\alpha^{m_\alpha} \int_{W_A(\mathbf{m})} h(x)\mathrm{e}(f(x))\,\mathrm{d}x \\
&= \sum_{\substack{A_1\subset A\\ A_2 = A\setminus A_1}} \sum_{\mathbf{m}_1\in(0,\mu)^{A_1}} \prod_{\alpha\in A_1} T_\alpha^{m_\alpha} \\
&\quad \times \sum_{\mathbf{m}_2\in[\mu,\infty)^{A_2}} \prod_{\alpha\in A_2} (1-\mathbf{L}^{-1})(\mathbf{L}^{-1}T_\alpha)^{m_\alpha} \int_{W'_{A_2}(\mathbf{m}_1)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x \\
&= \sum_{\substack{A_1\subset A\\ A_2 = A\setminus A_1}} \sum_{\mathbf{m}_1\in(0,\mu)^{A_1}} \left(\int_{W'_{A_2}(\mathbf{m}_1)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x\right) \prod_{\alpha\in A_1} T_\alpha^{m_\alpha} \\
&\quad \times \prod_{\alpha\in A_2} \frac{(1-\mathbf{L}^{-1})(\mathbf{L}^{-1}T_\alpha)^{\mu}}{1-\mathbf{L}^{-1}T_\alpha}.
\end{aligned}$$

It follows from this computation that the power series $Q_A(\mathbf{T})$ defined by

$$Q_A(\mathbf{T}) = Z_A(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) \prod_{\alpha\in A} (1-\mathbf{L}^{-1}T_\alpha)$$

is in fact a polynomial, which establishes the first assertion of the proposition. Moreover, if we compute $Q_A(\mathbf{T})$ modulo the ideal generated by the polynomials $1-\mathbf{L}^{-1}T_\alpha$ for $\alpha \in A$, only the term corresponding to $A_1 = \varnothing$ and $A_2 = A$ survives in the sum. In this case, $W'_{A_2}(\mathbf{m}_1) = \mathscr{L}(\mathscr{D}_A^\circ)$, and we get

$$\begin{aligned} Q_A(\mathbf{T}) &\equiv \left(\int_{\mathscr{L}(\mathscr{D}_A^\circ)} h(x) \mathrm{e}(f(x))\,\mathrm{d}x \right) \prod_{\alpha\in A} (1-\mathbf{L}^{-1})(\mathbf{L}^{-1}T_\alpha)^{\mu} \\ &\equiv (1-\mathbf{L}^{-1})^{\mathrm{Card}(A)} \left(\int_{\mathscr{L}(\mathscr{D}_A^\circ)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x \right), \end{aligned}$$

as claimed. □

COROLLARY 5.2.9. *Assume that f_K is a regular function on the generic fiber $\mathscr{X}_K$. There exists a family (P_A) of polynomials with coefficients in $\mathscr{E}xp\mathscr{M}_k$, indexed by $\mathrm{Cl}^{\mathrm{an},\max}(X,D)$, such that*

$$Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) = \sum_{A\in \mathrm{Cl}^{\mathrm{an},\max}(X,D)} P_A(\mathbf{T}) \prod_{\alpha\in A} \frac{1}{1-\mathbf{L}^{-1}T_\alpha} \tag{5.2.10}$$

and such that for each $A \in \mathrm{Cl}^{\mathrm{an},\max}(X,D)$,

$$P_A(\mathbf{T}) - (1-\mathbf{L}^{-1})^{\mathrm{Card}(A)} \int_{\mathscr{L}(\mathscr{D}_A)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x \tag{5.2.11}$$

belongs to the ideal generated by the polynomials $1-\mathbf{L}^{-1}T_\alpha$, for $\alpha \in A$.

Proof. By definition, one has

$$Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) = \int_{\mathscr{L}(\mathscr{X})} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\mathrm{ord}_{\mathscr{D}_\alpha}(x)} h(x)\,\mathrm{e}(f(x))\,\mathrm{d}x = \sum_{A\subset\mathscr{A}} Z_A(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}).$$

For each $A \subset \mathscr{A}$ such that $D_A^\circ(k) \neq \varnothing$, choose a maximal subset $A' \in \mathrm{Cl}^{\mathrm{an},\max}(X,D)$ such that $A \subset A'$. In the previous formula for $Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T})$, we can collect terms according to the chosen maximal subset. Applying Proposition 5.2.7, we obtain

$$\begin{aligned} Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) &= \sum_{A\in \mathrm{Cl}^{\mathrm{an},\max}(X,D)} \sum_{\substack{B\subset A \\ B'=A}} Q_B(\mathbf{T}) \left(\prod_{\alpha\in B} \frac{1}{1-\mathbf{L}^{-1}T_\alpha} \right) \\ &= \sum_{A\in \mathrm{Cl}^{\mathrm{an},\max}(X,D)} \prod_{\alpha\in A} \frac{1}{1-\mathbf{L}^{-1}T_\alpha} \sum_{\substack{B\subset A \\ B'=A}} Q_B(\mathbf{T}) \prod_{\alpha\in A\setminus B} (1-\mathbf{L}^{-1}T_\alpha). \end{aligned}$$

For every $A \in \mathrm{Cl}^{\mathrm{an},\max}(X,D)$, we set

$$P_A(\mathbf{T}) = \sum_{\substack{B\subset A \\ B'=A}} Q_B(\mathbf{T}) \prod_{\alpha\in A\setminus B} (1-\mathbf{L}^{-1}T_\alpha).$$

Then we have

$$Z(\mathscr{X},h\mathrm{e}(f);\mathbf{T}) = \sum_{A\in \mathrm{Cl}^{\mathrm{an,max}}(X,D)} P_A(\mathbf{T}) \prod_{\alpha\in A} \frac{1}{1-\mathbf{L}^{-1}T_\alpha}.$$

Moreover, modulo the ideal generated by the polynomials $1-\mathbf{L}^{-1}T_\alpha$ (for $\alpha\in A$), $P_A(\mathbf{T})$ is congruent to $Q_A(\mathbf{T})$, which is itself congruent to

$$(1-\mathbf{L}^{-1})^{\mathrm{Card}(A)} \int_{\mathscr{L}(\mathscr{D}_A^\circ)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x.$$

This proves the corollary since $\mathscr{D}_A^\circ = \mathscr{D}_A$ for $A\in \mathrm{Cl}^{\mathrm{an,max}}(X,D)$. □

5.3. Motivic Igusa integrals with exponentials—the general case. We keep the setup and notation as described in Section 5.2.1.

In the previous section, we assumed that f was regular on $\mathscr{X}_K$. In the general case where, on $\mathscr{X}_K$, the polar divisor $\mathrm{div}_\infty(f)$ of f is contained in the union $\bigcup_{\alpha\in\mathscr{A}} \mathscr{D}_{\alpha,K}$, we shall prove in Proposition 5.3.4 that the motivic Igusa integral with exponentials is a rational function. Under the additional condition that f_K extends to a regular morphism from $\mathscr{X}_K$ to $\mathbf{P}^1_K$, we have the following more precise result.

PROPOSITION 5.3.1. *Assume that f_K extends to a regular map from $\mathscr{X}_K$ to $\mathbf{P}^1_K$. Let $\mathrm{Cl}^{\mathrm{an}}(X,D)_0$ be the subcomplex of $\mathrm{Cl}^{\mathrm{an}}(X,D)$ where we only keep vertices $\alpha\in\mathscr{A}$ such that $d_\alpha=0$. Then there is a family (P_A) of polynomials with coefficients in $\mathscr{E}xp\mathscr{M}_k$, indexed by the set $\mathrm{Cl}^{\mathrm{an,max}}(X,D)_0$ of maximal faces of $\mathrm{Cl}^{\mathrm{an}}(X,D)_0$, such that*

$$Z(\mathscr{X},h\mathrm{e}(f);\mathbf{T}) = \sum_{A\in \mathrm{Cl}^{\mathrm{an,max}}(X,D)_0} P_A(\mathbf{T}) \prod_{\alpha\in A} \frac{1}{1-\mathbf{L}^{-1}T_\alpha}. \tag{5.3.2}$$

Proof. Let us first assume that f extends to a regular map from $\mathscr{X}$ to $\mathbf{P}^1_R$.

The special case $\mathrm{Cl}^{\mathrm{an}}(X,D)_0 = \mathrm{Cl}^{\mathrm{an}}(X,D)$ is treated by Proposition 5.2.9. As in the previous section, we write

$$Z(\mathscr{X},h\mathrm{e}(f);\mathbf{T}) = \sum_{A\in \mathrm{Cl}^{\mathrm{an}}(X,D)} Z_A(\mathscr{X},h\mathrm{e}(f);\mathbf{T}). \tag{5.3.3}$$

For every $A\in\mathrm{Cl}^{\mathrm{an}}(X,D)_0$, $Z_A(\mathscr{X},h\mathrm{e}(f);\mathbf{T})$ can be evaluated by the same computation as the one that we performed in the proof of Proposition 5.2.7: there is a polynomial $Q_A(\mathbf{T})$ with coefficients in $\mathscr{E}xp\mathscr{M}_k$ such that

$$\int_{\mathscr{L}(\mathscr{X};D_A^\circ)} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\mathrm{ord}_{\mathscr{D}_\alpha}(x)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x = Q_A(\mathbf{T})(1-\mathbf{L}^{-1})^{\mathrm{Card}(A)} \prod_{\alpha\in A} \frac{1}{1-\mathbf{L}^{-1}T_\alpha}$$

and such that $Q_A(\mathbf{T})$ is congruent to

$$\int_{\mathscr{L}(\mathscr{D}_A;D_A^\circ)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x$$

modulo the ideal generated by the polynomials $1-\mathbf{L}^{-1}T_\alpha$.

The general case is treated by adapting the arguments given in the proof of Proposition 5.2.7. Let indeed $A\subset\mathscr{A}$, let $A_0=\{\alpha\in A;\ d_\alpha=0\}$ and let $A_1=A\setminus A_0$. Define a function g_A on $\mathscr{L}(\mathscr{X};D_A^\circ)$ by

$$f=g_A\prod_{\alpha\in A}x_\alpha^{-d_\alpha}.$$

Since f extends to a regular map to $\mathbf{P}^1$, the divisors of zeroes and of poles of f do not meet. Consequently, one has $d_\alpha>0$ for every $\alpha\in A_1$, and $\mathrm{ord}\circ g_A$ is bounded from above. As in the proof of Proposition 5.2.7, g_A can be expanded as a converging power series. Then, applying Proposition 5.1.4, we observe that there exists an integer m such that the integral

$$\int_{\mathscr{L}(\mathscr{X};D_A^\circ)}\prod_{\alpha\in\mathscr{A}}T_\alpha^{\mathrm{ord}_{\mathscr{D}_\alpha(x)}}h(x)\mathrm{e}(f(x))\,\mathrm{d}x$$

is equal to the analogous integral but restricted to the subset defined by the inequalities $\mathrm{ord}(x_\alpha)\leq m$ for all $\alpha\in A_1$. By a similar argument to the proof of Proposition 5.2.7, we conclude that the power series $Q_A(\mathbf{T})$ defined by

$$Q_A(\mathbf{T})=Z_A(\mathscr{X},h\mathrm{e}(f);\mathbf{T})\prod_{\alpha\in A_0}(1-\mathbf{L}^{-1}T_\alpha)$$

is in fact a polynomial.

As in the proof of Corollary 5.2.9, the proposition follows by choosing, for every subset $A\subset\mathscr{A}$ some maximal subset $A'\in\mathrm{Cl}^{\mathrm{an}}(X,D)_0$ such that $A_0\subset A'$ and regrouping the terms according to the chosen subset.

This concludes the proof when f extends to a regular morphism from $\mathscr{X}$ to $\mathbf{P}^1_R$. To treat the general case, recall that there exists a proper birational morphism $\pi\colon\mathscr{Y}\to\mathscr{X}$ which is a composition of blowing-ups with smooth centers contained in the special fiber such that π^*f extends to a regular map from $\mathscr{Y}$ to $\mathbf{P}^1_R$. Since π induces an isomorphism on the generic fiber, it does not modify the set $\mathscr{A}$, nor the analytic Clemens complexes $\mathrm{Cl}^{\mathrm{an}}(X,D)$ and $\mathrm{Cl}^{\mathrm{an}}(X,D)_0$.

Let $\mathscr{D}'_\alpha$ be the strict transform of $\mathscr{D}_\alpha$. Then $h_\alpha=\mathrm{ord}_{\mathscr{D}_\alpha}-\mathrm{ord}_{\mathscr{D}'_\alpha}$ is a constructible function on $\mathscr{L}(\mathscr{Y})$ which takes only finitely many values. By the change of variable formulas, one has

$$\begin{aligned}
Z(\mathscr{X}, h\mathrm{e}(f), \mathbf{T}) &= \int_{\mathscr{X}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\operatorname{ord}_{\mathscr{D}_\alpha}(x)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x \\
&= \int_{\mathscr{Y}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\operatorname{ord}_{\mathscr{D}_\alpha}(\pi(y))} \pi^* h(y)\mathrm{e}(\pi^* f(y))\mathbf{L}^{\operatorname{ord} J_\pi(y)}\,\mathrm{d}y \\
&= \int_{\mathscr{Y}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\operatorname{ord}_{\mathscr{D}'_\alpha}(y)} \prod_{\alpha\in\mathscr{A}} T_\alpha^{h_\alpha(x)} \pi^* h(y)\mathrm{e}(\pi^* f(y))\mathbf{L}^{\operatorname{ord} J_\pi(y)}\,\mathrm{d}y.
\end{aligned}$$

We then decompose this integral according to the values of h_α and $\operatorname{ord} J_\pi$ and compute each individual part as in the first part of the proof. Combining the various contributions implies the proposition. □

PROPOSITION 5.3.4. *The power series $Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T})$ is a rational function.*

Proof. Let $\pi\colon \mathscr{Y} \to \mathscr{X}$ be a proper birational morphism such that the rational map $\pi^* f$ extends to a regular morphism from $\mathscr{Y}_K$ to $\mathbf{P}^1_K$ and such that the horizontal part of $\pi^*\mathscr{D}$ is a relative divisor with strict normal crossings. For every $\alpha \in \mathscr{A}$, we write $\mathscr{D}'_\alpha$ for the strict transform of $\mathscr{D}_\alpha$; let $(\mathscr{E}_\beta)_{\beta\in\mathscr{B}}$ be the family of horizontal exceptional divisors.

Let $\alpha \in \mathscr{A}$. There exists a family $(m_{\alpha,\beta})_{\beta\in\mathscr{B}}$ of nonnegative integers such that

$$\pi^*\mathscr{D}_{\alpha,K} = \mathscr{D}'_{\alpha,K} + \sum_{\beta\in\mathscr{B}} m_{\alpha,\beta}\mathscr{E}_{\beta,K}.$$

Consequently, there exists a bounded constructible function u_α on $\mathscr{L}(\mathscr{Y})$ such that

$$\pi^* \operatorname{ord}_{\mathscr{D}_\alpha} = \operatorname{ord}_{\mathscr{D}'_\alpha} + \sum_{\beta\in\mathscr{B}} m_{\alpha,\beta} \operatorname{ord}_{\mathscr{E}_\beta} + u_\alpha. \tag{5.3.5}$$

Let also $(\nu_\beta)_{\beta\in\mathscr{B}}$ be the family of positive integers such that

$$K_{\mathscr{Y}_K/\mathscr{X}_K} = \sum_{\beta\in\mathscr{B}} (\nu_\beta - 1)\mathscr{E}_{\beta,K}.$$

This implies that there exists a bounded constructible function v on $\mathscr{L}(\mathscr{Y})$ such that

$$\operatorname{ord}_{K_{\mathscr{Y}/\mathscr{X}}} = \sum_{\beta\in\mathscr{B}} (\nu_\beta - 1) \operatorname{ord}_{\mathscr{E}_\beta} + v.$$

Then, using the change of variables formula (Theorem 13.2.2 in [9]), we find

$$\begin{aligned} Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) &= \int_{\mathscr{L}(\mathscr{X})} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\operatorname{ord}_{\mathscr{D}_\alpha}(x)} h(x)\mathrm{e}(f(x))\,\mathrm{d}x \\ &= \int_{\mathscr{L}(\mathscr{Y})} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\operatorname{ord}_{\mathscr{D}'_\alpha}(y)+\sum_\beta m_{\alpha,\beta}\operatorname{ord}_{\mathscr{E}_\beta}(y)+u_\alpha(y)} \\ &\qquad\qquad \times \mathbf{L}^{v(y)+\sum_\beta(\nu_\beta-1)\operatorname{ord}_{\mathscr{E}_\beta}(y)} \pi^* h(y)\mathrm{e}(\pi^* f(y))\,\mathrm{d}y \\ &= \int_{\mathscr{L}(\mathscr{Y})} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\operatorname{ord}_{\mathscr{D}'_\alpha}(y)+u_\alpha(y)} \\ &\quad \times \prod_{\beta\in\mathscr{B}} P_\beta(\mathbf{T})^{\operatorname{ord}_{\mathscr{E}_\beta}(y)} \mathbf{L}^{v(y)} \pi^* h(y)\mathrm{e}(\pi^* f(y))\,\mathrm{d}y, \end{aligned}$$

where, for every $\beta\in\mathscr{B}$, we have set

$$P_\beta(\mathbf{T}) = \mathbf{L}^{\nu_\beta-1}\prod_{\alpha\in\mathscr{A}} T_\alpha^{m_{\alpha,\beta}}. \tag{5.3.6}$$

Let us introduce a family $\mathbf{S}=(S_\beta)_{\beta\in\mathscr{B}}$ of indeterminates. For every motivic residual function w on $\mathscr{L}(\mathscr{Y})$ and every rational function g on $\mathscr{Y}_K$, let us also define, analogously to Equation (5.2.3), a generating series

$$Z(\mathscr{Y}, w\mathrm{e}(g); (\mathbf{T},\mathbf{S})) = \int_{\mathscr{L}(\mathscr{Y})} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\operatorname{ord}_{\mathscr{D}'_\alpha}(y)} \prod_{\beta\in\mathscr{B}} S_\beta^{\operatorname{ord}_{\mathscr{E}_\beta}(y)} w(y)\mathrm{e}(g(y))\,\mathrm{d}y.$$

If g induces a morphism from $\mathscr{Y}_K$ to $\mathbf{P}^1_K$, it follows from Proposition 5.3.1 and the proof of Proposition 5.3.4 that $Z(\mathscr{Y}, w\mathrm{e}(g); (\mathbf{T},\mathbf{S}))$ is a rational function of $(\mathbf{T},\mathbf{S})$, for every w.

For every $p\in\mathbf{Z}^{\mathscr{A}}$, let W_p be the constructible subset of $\mathscr{L}(\mathscr{Y})$ on which the family (u_α) equals p. These subsets form a finite partition of $\mathscr{L}(\mathscr{Y})$ and one has

$$Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T}) = \sum_p \prod_{\alpha\in\mathscr{A}} T_\alpha^{p_\alpha} Z\big(\mathscr{Y}, \mathbf{1}_{W_p}\mathbf{L}^v \pi^* h\mathrm{e}(\pi^* f); (\mathbf{T}, (P_\beta(\mathbf{T})))\big),$$

where the polynomials P_β are defined in (5.3.6). Consequently, $Z(\mathscr{X}, h\mathrm{e}(f); \mathbf{T})$ is a rational function of $\mathbf{T}$. □

6. Local Fourier transforms. In this section, we finally prove the propositions stated in Section 4.3.

Fix a place $v\in C(k)$. We have to study the Laurent series (4.2.2) given by

$$Z_v(\mathbf{T},\xi) = \sum_{(\mathbf{m}_v,\beta_v)\ \text{integral}} \mathscr{F}(\mathbf{1}_{G(\mathbf{m}_v,\beta_v)}, \xi) \prod_{\alpha\in\mathscr{A}} T_\alpha^{\|\mathbf{m}_v,\beta_v\|_\alpha}$$

$$= \sum_{(\mathbf{m}_v,\beta_v)\text{ integral}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{\|\mathbf{m}_v,\beta_v\|_\alpha} \int_{G(\mathbf{m}_v,\beta_v)} \mathrm{e}(\langle g,\xi\rangle)\,\mathrm{d}g.$$

Our first step will be to replace the motivic Haar integral with respect to $\mathrm{d}g$ by a motivic integral with respect to the motivic measure on the arc space $\mathscr{L}(\mathscr{X})$. Then, we will split the motivic integral according to the natural stratification of the special fibre.

Since the place v is fixed, we often omit the index v from the notation, writing $K \simeq k((t))$ for $F_v = k(C)_v$ and $R \simeq k[[t]]$ for the ring of integers of F_v.

6.1. The Haar integral as a motivic measure. View the invariant top-differential form $\mathrm{d}g$ on G_F as a meromorphic top-differential form ω_X on $\mathscr{X}$. Since $\mathscr{X}$ is proper over C, one has $\mathscr{X}(k[[t]]) = X(K)$. The order of contact of an arc with the divisor of ω induces an order function $\mathrm{ord}_\omega\colon X(K) \to \mathbf{Z}\cup\{\infty\}$, which takes finite values on $G(K)$.

The Poisson summation formula involves (motivic) integrals on $k((t))^n = G(F)$. The injection $G(F) \subset \mathscr{X}(k[[t]])$ allows to view any Schwartz-Bruhat function Φ on $G(F)$ as an exponential motivic function on the arc space $\mathscr{L}(\mathscr{X})$ of $\mathscr{X}$. The following lemma shows how both motivic integrals are related. It is a motivic analogue of the standard fact that the Lebesgue mesure on the real line $\mathbf{R}$ is the volume-form associated with the differential form $\mathrm{d}x$, or with the corresponding singular differential form on $\mathbf{P}^1(\mathbf{R})$.

LEMMA 6.1.1. *Let $\Phi \in \mathscr{S}(F^n)$. Then the motivic integral $\int_{G(K)} \Phi(g)\,\mathrm{d}g$ can be rewritten as*

$$\int_{\mathscr{L}(\mathscr{X})} \Phi(x)\mathbf{L}^{-\mathrm{ord}_\omega(x)}\,\mathrm{d}x$$

where $\mathrm{d}x$ denotes the motivic measure on the arc space $\mathscr{L}(\mathscr{X})$.

Proof. Let us begin with a remark. Let Z be a smooth projective K-scheme and let ω be a meromorphic differential form on Z. Let f be a motivic function on Z. By this, we mean that one is given a proper flat R-model $\mathscr{Z}$ of Z, an integer m, and a class $\varphi \in \mathscr{E}xp\mathscr{M}_{\mathscr{L}_m(\mathscr{Z})}$. We identify the function associated with a triple $(\mathscr{Z},m,\varphi)$ and the function associated with the triple $(\mathscr{Z},m+1,\pi^*\varphi)$, where π is the canonical morphism from $\mathscr{E}xp\mathscr{M}_{\mathscr{L}_m(\mathscr{Z})}$ to $\mathscr{E}xp\mathscr{M}_{\mathscr{L}_{m+1}(\mathscr{Z})}$ defined by base change; similarly, if $p\colon \mathscr{Z}' \to \mathscr{Z}$ is a morphism of models, we identify the functions associated with triples $(\mathscr{Z},m,\varphi)$ and $(\mathscr{Z}',m,p^*\varphi)$. Then one defines the integral $\int_{Z(K)} f\,|\omega|$ of the motivic function f with respect to ω by the formula

$$\int_{Z(K)} f\,|\omega| = \int_{\mathscr{L}(\mathscr{Z})} \varphi(z)\mathbf{L}^{-\mathrm{ord}_\omega(z)}\,\mathrm{d}z,$$

where $\mathrm{d}z$ is the motivic measure on the arc space $\mathscr{L}(\mathscr{Z})$. We may even assume that the smooth locus $\mathscr{Z}^0$ of $\mathscr{Z}$ is a weak Néron model of Z and restrict the integral over $\mathscr{L}(\mathscr{Z}^0)$. The definition of motivic integration and the change of variables formula (Theorem 13.2.2 in [9]) shows that this integral is independent of the choice of the triple $(\mathscr{Z}, m, \varphi)$ that defines f.

Let us show how this remark implies our lemma.

Let $\mathscr{P} = \mathbf{P}^n = \operatorname{Proj} R[x_0, \dots, x_n]$ be the natural compactification of $G_R = \mathbf{A}^n = \operatorname{Spec}(R[x_1, \dots, x_n])$, let $P = \mathscr{P}_K$. Let ω be the differential form $\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$ on G_R; we will write ω_P or ω_X according to ω being viewed as a meromorphic differential form on $\mathscr{P}$ or on $\mathscr{X}$. Since $\mathscr{X}$ is regular, one has

$$\int_{X(K)} \Phi(x)\,|\omega_X| = \int_{\mathscr{L}(\mathscr{X})} \Phi(x) \mathbf{L}^{-\operatorname{ord}_{\omega_X}(x)}\,\mathrm{d}x.$$

However, this integral can be computed starting from the model $\mathscr{P}$, and one has

$$\int_{X(K)} \Phi(x)\,|\omega_X| = \int_{P(K)} \Phi(p)\,|\omega_P|\,.$$

Since $\operatorname{div}(\omega_P) = (n+1)\operatorname{div}(x_0)$, $\operatorname{ord}_{\omega_P}(x) = (n+1)\min(0, \operatorname{ord}(x_1), \dots, \operatorname{ord}(x_n))$ for every $(x_1, \dots, x_n) \in K^n$. Consequently, by the very definition of the motivic integral on $G(K)$, this last integral equals $\int_{G(K)} \Phi(g)\,\mathrm{d}g$. It suffices to show this equality for simple functions, as in Section 1.3.3. Let $(m_1, \dots, m_n)$ be a family of integers and let Ω be the set of points in K^n such that $\operatorname{ord}(x_i) = m_i$ for every $i \in \{1, \dots, n\}$. Set $m_0 = 0$, let $m = \min(m_0, \dots, m_n)$, and let $j \in \{0, \dots, n\}$ be such that $m = m_j$. We view Ω in the affine chart $\{x_j = 1\}$ of $\mathscr{P}$ and identify it with $\prod_{\substack{0 \le i \le n \\ i \neq j}} t^{m_i - m} R^\times$. Therefore, its $\mathrm{d}p$-measure (that is, its motivic volume with respect to the arc space of $\mathscr{P}$) equals

$$\int_\Omega \mathrm{d}p = \mathbf{L}^{\sum_{i \neq j}(m_i - m)} (1 - \mathbf{L}^{-1})^n = (1 - \mathbf{L}^{-1})^n \mathbf{L}^{\sum m_i} \mathbf{L}^{-(n+1)m}.$$

On the other hand, Ω is also viewed as the set $\prod_{i=1}^n t^{-m_i} R^\times$ in K^n hence its $\mathrm{d}g$-integral is given by

$$\int_\Omega \mathrm{d}g = \mathbf{L}^{\sum m_i} (1 - \mathbf{L}^{-1})^n.$$

Consequently,

$$\int_\Omega \mathrm{d}g = \int_\Omega \mathbf{L}^{-\operatorname{ord}_\omega(p)}\,\mathrm{d}p.$$

This concludes the proof of the lemma. $\square$

6.2. Partitions of unity. Let $\mathscr{B}_1$ be the subset of $\mathscr{B}_v$ consisting of those β for which the multiplicity μ_β equals 1. Let $\mathscr{X}_1$ be the complement in $\mathscr{X}$ of the union of the components E_β, for $\beta \in \mathscr{B}_v \setminus \mathscr{B}_1$, and of the intersections of distinct vertical components. As we know from Lemma 3.2.1, this is a weak Néron model of X over $\operatorname{Spec}(R)$.

For every subset $A \subset \mathscr{A}$ and every $\beta \in \mathscr{B}_1$, let $\Delta(A,\beta)$ be the locally closed subset of the special fibre $\mathscr{X}_v$ corresponding to points $\tilde{x}$ which belong to the horizontal divisors $\mathscr{D}_\alpha$, for $\alpha \in A$, and to no other, as well as to the vertical divisor E_β, but no other. Let $\Omega(A,\beta)$ be its preimage in $\mathscr{L}(\mathscr{X})$ by the specialization morphism $\mathscr{L}(\mathscr{X}) \to \mathscr{X}_v$.

Recall that we have defined in Section 3.5.4 a subset $\mathscr{B}_0$ of $\mathscr{B}$ such that for every point $x \in G(F_v)$, the integrality condition "$x \in \mathscr{U}(\mathfrak{o}_{F_v})$ if $v \in C_0$" holds if and only if there exists $\beta \in \mathscr{B}_0$ such that $x \in \Omega(\varnothing,\beta)$.

Let $\mathscr{L}(\mathbf{A}^1) \simeq \operatorname{Spec}(k[a_0,a_1,\dots])$ be the arc space of the affine line, and let $\mathscr{L}(\mathbf{A}^1;0)$ be its closed subspace consisting of arcs based at the origin. Let A be a subset of $\mathscr{A}$. By Lemma 5.2.6, there is a definable isomorphism which preserves the motivic measure from $\Omega(A,\beta)$ to the product of $\mathscr{L}(\mathbf{A}^1;0)^A \times_k \Delta(A,\beta)$ with $\mathscr{L}(\mathbf{A}^1)^{n-\operatorname{Card}(A)}$. Moreover, this isomorphism induces the following equalities, for $g \in G(F)$:

(6.2.1)
$$(g,\mathscr{D}_\alpha)_v = \begin{cases} \operatorname{ord}_v(x_\alpha) & \text{if } \alpha \in A \\ 0 & \text{otherwise,} \end{cases}$$

(where, as in Lemma 5.2.6, x_α is the α-component of the image of g) and

(6.2.2)
$$(g,E_\gamma)_v = \begin{cases} 1 & \text{if } \gamma = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Recall that in Equation (3.2.3), we had defined integers ρ_β such that

$$-\operatorname{div}(\omega_X) = \sum_{\alpha \in \mathscr{A}} \rho_\alpha \mathscr{D}_\alpha + \sum_{\beta \in \mathscr{B}} \rho_\beta E_\beta.$$

This implies

(6.2.3)
$$-\operatorname{ord}_\omega(x) = \rho_\beta + \sum_{\alpha \in A} \rho_\alpha \operatorname{ord}_v(x_\alpha).$$

Recall also the definition of integers $e_{\alpha,\beta}$ in Equation 3.2.2:

$$\mathscr{L}_\alpha = \mathscr{D}_\alpha + \sum_{\beta \in \mathscr{B}} e_{\alpha,\beta} E_\beta.$$

To shorten the notation below, we then let

$$\rho(A,\beta) = \rho_\beta + \sum_{\alpha\in\mathscr{A}} \rho_\alpha e_{\alpha,\beta}. \tag{6.2.4}$$

We also define e_α to be the constructible function

$$e_\alpha(\cdot) = \sum_{\beta\in\mathscr{B}} e_{\alpha,\beta} \operatorname{ord}_{E_\beta}(\cdot) \tag{6.2.5}$$

on $\mathscr{L}(\mathscr{X})$ so that $e_\alpha(g) = e_{\alpha,\beta}$ for every $g \in G(F_v)$ such that $(g,E_\beta)_v = 1$.

Using this notation and applying Lemma 6.1.1, we can rewrite the motivic Fourier transforms $Z_v(\mathbf{T},\xi)$ as sums of motivic integrals over arc spaces $\Omega(A,\beta)$.

LEMMA 6.2.6. *For every motivic residual function h on $\mathscr{L}(\mathscr{X})$ and every $\xi \in G(F_v)$, one has*

$$\begin{aligned} &\int_{G(F_v)} \prod_{\alpha\in\mathscr{A}} T_\alpha^{(g,\mathscr{L}_\alpha)_v} h(g) \mathrm{e}(\langle g,\xi\rangle)\, \mathrm{d}g \\ &= \sum_{\substack{A\subset\mathscr{A}\\ \beta\in\mathscr{B}_1}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}} \mathbf{L}^{\rho_\beta} \int_{\Omega(A,\beta)} \prod_{\alpha\in A} (\mathbf{L}^{\rho_\alpha} T_\alpha)^{\operatorname{ord}(x_\alpha)} h(x) \mathrm{e}(\langle x,\xi\rangle)\, \mathrm{d}x. \end{aligned} \tag{6.2.7}$$

6.3. Places in C_0: Proof of Proposition 4.3.1. Let v be a place in C_0. In this case, $Z(\mathbf{T},\xi)$ is given by Lemma 6.2.6, taking for motivic function h the characteristic function of the set $\mathscr{U}(\mathfrak{o})$ within $G(F_v)$. In other words, one has $h \equiv 0$ on $\Omega(A,\beta)$ if $A \neq \varnothing$ or $\beta \notin \mathscr{B}_0$, and $h \equiv 1$ otherwise. Consequently,

$$Z(\mathbf{T},\xi) = \sum_{\beta\in\mathscr{B}_0} \prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}} \mathbf{L}^{\rho_\beta} \int_{\Omega(\varnothing,\beta)} \mathrm{e}(\langle x,\xi\rangle)\, \mathrm{d}x.$$

We see in particular that it is a polynomial. Assume that $\xi = 0$. Then, the factor $\mathrm{e}(\langle x,\xi\rangle)$ equals 1, so that

$$Z(\mathbf{T},0) = \sum_{\beta\in\mathscr{B}_0} \prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}} \mathbf{L}^{\rho_\beta} [\Delta(\varnothing,\beta)] \mathbf{L}^{-n}.$$

In particular,

$$Z_\lambda(\mathbf{L}^{-1},0) = Z((\mathbf{L}^{1-\rho_\alpha}),0) = \sum_{\beta\in\mathscr{B}_0} \prod_{\alpha\in\mathscr{A}} \mathbf{L}^{(1-\rho_\alpha)e_{\alpha,\beta}} \mathbf{L}^{\rho_\beta} [\Delta(\varnothing,\beta)] \mathbf{L}^{-n}.$$

This is an effective element of $\mathscr{M}_k$, non-zero unless $\mathscr{U}(\mathfrak{o}) = \varnothing$. By the last assertion of Lemma 3.4.1 and the hypothesis of Setting 2, this concludes the proof of Proposition 4.3.1.

6.4. Trivial character (places in $C\setminus C_0$): Proof of Proposition 4.3.2. Assume that $v\in C\setminus C_0$. Then, $Z(\mathbf{T},0)$ is given by Lemma 6.2.6, applied with $h\equiv 1$. This leads to the following computation:

$$\begin{aligned}Z(\mathbf{T},0) &= \sum_{\substack{A\subset\mathscr{A}\\ \beta\in\mathscr{B}_1}}\prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}\int_{\Omega(A,\beta)}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}T_\alpha)^{\operatorname{ord}(x_\alpha)}\,\mathrm{d}x\\ &= \sum_{\substack{A\subset\mathscr{A}\\ \beta\in\mathscr{B}_1}}\prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}[\Delta(A,\beta)]\mathbf{L}^{-n+\operatorname{Card}(A)}\prod_{\alpha\in A}\int_{\mathscr{L}(\mathbf{A}^1;0)}(\mathbf{L}^{\rho_\alpha}T_\alpha)^{\operatorname{ord}(x)}\,\mathrm{d}x.\end{aligned}$$

In the last formula, the integral over $\mathscr{L}(\mathbf{A}^1;0)$ is given by a geometric series, familiar in the theory of motivic Igusa functions. Indeed, for every $\alpha\in A$,

$$\begin{aligned}\int_{\mathscr{L}(\mathbf{A}^1;0)}(\mathbf{L}^{\rho_\alpha}T_\alpha)^{\operatorname{ord}(x)}\,\mathrm{d}x &= \sum_{m=1}^{\infty}(\mathbf{L}^{\rho_\alpha}T_\alpha)^m\int_{\operatorname{ord}(x)=m}\mathrm{d}x\\ &= \sum_{m=1}^{\infty}(\mathbf{L}^{\rho_\alpha}T_\alpha)^m\mathbf{L}^{-m}(1-\mathbf{L}^{-1})\\ &= (1-\mathbf{L}^{-1})\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}.\end{aligned}$$

Consequently,

$$\begin{aligned}Z(\mathbf{T},0) = &\sum_{\substack{A\subset\mathscr{A}\\ \beta\in\mathscr{B}_1}}\prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}[\Delta(A,\beta)]\\ &\times\mathbf{L}^{-n+\operatorname{Card}(A)}(1-\mathbf{L}^{-1})^{\operatorname{Card}(A)}\prod_{\alpha\in A}\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}.\end{aligned}\tag{6.4.1}$$

For every pair (A,β) such that $\Delta(A,\beta)\neq\varnothing$, fix a maximal subset A_0 of $\mathscr{A}$ such that $A\subset A_0$ and $\Delta(A_0,\beta)\neq\varnothing$. Let us fix such a maximal set A_0 and let us then collect the terms of Equation (6.4.1) corresponding to pairs (A,β) that are associated with A_0. The corresponding subseries of $Z(\mathbf{T},0)$ is then given by

$$\begin{aligned}Z_{A_0}(\mathbf{T},0) = &\sum_{\substack{\beta\in\mathscr{B}_1\\ (A,\beta)\mapsto A_0}}\prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}[\Delta(A,\beta)]\\ &\times\mathbf{L}^{-n+\operatorname{Card}(A)}(1-\mathbf{L}^{-1})^{\operatorname{Card}(A)}\prod_{\alpha\in A}\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}.\end{aligned}$$

Consequently, there exists a Laurent polynomial $P_{A_0}(\mathbf{T})$ such that

$$Z_{A_0}(\mathbf{T},0)\prod_{\alpha\in A_0}(1-\mathbf{L}^{\rho_\alpha-1}T_\alpha)=P_{A_0}(\mathbf{T}).$$

From this expression, we also see that modulo the ideal generated by the polynomials $1-\mathbf{L}^{\rho_\alpha-1}T_\alpha$, for $\alpha\in\mathscr{A}$, only the terms corresponding to $A=A_0$ remain, so that we have

$$P_{A_0}(\mathbf{T})\equiv\sum_{\beta\in\mathscr{B}_1}\prod_{\alpha\in\mathscr{A}}\mathbf{L}^{(1-\rho_\alpha)e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}[\Delta(A_0,\beta)]\mathbf{L}^{-n+\mathrm{Card}(A_0)}(1-\mathbf{L}^{-1})^{\mathrm{Card}(A_0)}.$$

Let u_{A_0} be the motivic residual function on $\mathscr{L}(\mathscr{X})$ which is given by

$$\rho_\beta+\sum_{\alpha\in\mathscr{A}}(1-\rho_\alpha)e_{\alpha,\beta}$$

on $\Omega(A_0,\beta)$. By definition of motivic integration, one has

$$\begin{aligned}\int_{\mathscr{L}(\mathscr{D}_{A_0})}\mathbf{L}^{u_{A_0}}&=\sum_{\beta\in\mathscr{B}_1}\int_{\Omega(A_0,\beta)}\mathbf{L}^{\rho_\beta}\prod_{\alpha\in\mathscr{A}}\mathbf{L}^{(1-\rho_\alpha)e_{\alpha,\beta}}\\&=\sum_{\beta\in\mathscr{B}_1}\mathbf{L}^{\rho_\beta}\prod_{\alpha\in\mathscr{A}}\mathbf{L}^{(1-\rho_\alpha)e_{\alpha,\beta}}\mathbf{L}^{-n+\mathrm{Card}(A_0)}[\Delta(A,\beta)]\end{aligned}$$

so that

$$P_{A_0}(\mathbf{T})\equiv(1-\mathbf{L}^{-1})^{\mathrm{Card}(A_0)}\int_{\mathscr{L}(\mathscr{D}_{A_0})}\mathbf{L}^{u_{A_0}}.$$

The right-hand side of the preceding congruence being a non-zero effective element of $\mathscr{M}_k$, this concludes the proof of Proposition 4.3.2.

COROLLARY 6.4.2. *Let $d=1+\dim\mathrm{Cl}^{\mathrm{an}}(X,D)$ and let a be a non-zero multiple of the integers $\rho_\alpha-1$, for $\alpha\in\mathscr{A}$. The Laurent series $Z_\lambda(T,0)$ in one variable T is a rational function which belongs to $\mathscr{M}_k\{T\}$. Moreover, $(1-\mathbf{L}^aT^a)^dZ_\lambda(T,0)$ belongs to $\mathscr{M}_k[T,T^{-1}]$ and*

$$\begin{aligned}&(1-\mathbf{L}^aT^a)^dZ_\lambda(T,0)\Big|_{T=\mathbf{L}^{-1}}\\&\qquad=(1-\mathbf{L}^{-1})^d\sum_{A\in\mathrm{Cl}^{\mathrm{an},d}(X,D)}\prod_{\alpha\in A}\frac{a}{\rho_\alpha-1}\int_{\mathscr{L}(\mathscr{D}_A)}\mathbf{L}^{u_A(x)}\,\mathrm{d}x.\end{aligned}$$

6.5. Non-trivial characters (places in $C\setminus C_0$): Proof of Proposition 4.3.4. Assume that v belongs to $C\setminus C_0$. In this Section, we establish the behavior of the Fourier transform $Z_v(\mathbf{T},\mathbf{a}(\xi))$ for non-zero $\xi\in E$. Since the place v is fixed, the motivic Igusa integrals $Z_v(\mathbf{T},\cdot)$ and $Z_{\lambda,v}(T,\cdot)$ are denoted $Z(\mathbf{T},\cdot)$ and $Z_\lambda(T,\cdot)$ respectively.

By Proposition 5.3.4, we already know that for each ξ, $Z(\mathbf{T},(\xi))$ is rational, with denominator given by products of polynomials of the form $1-\mathbf{L}^n\mathbf{T}^{\mathbf{m}}$ for some $n\in\mathbf{N}$ and $\mathbf{m}\in\mathbf{N}^{\mathscr{A}}$ which are described through some Clemens complex. To prove Proposition 4.3.4, we have to prove two more properties: first, that this rationality property holds uniformly on the strata of some constructible partition (U_i), and second, that for each i, there exists an integer $e \in [0, d-1]$ and an integer $a \geq 1$ such that the restriction to U_i of $(1 - \mathbf{L}^a T^a)^e Z_\lambda(T, \cdot) = (1 - \mathbf{L}^a T^a)^e Z((T_\alpha^{\lambda_\alpha}), \cdot)$ belongs to $\mathscr{E}xp\mathscr{M}_{U_i}\{T\}^\dagger$.

The following analysis thus refines the proof of Proposition 5.3.4. It is very close to the one of [8], except for the replacement of p-adic oscillatory integrals by the motivic integrals of Section 5.3.

Recall that any point $\xi \in G(F_v)$ gives rise to a linear form $f_\xi = \langle \xi, \cdot \rangle$ on G_{F_v}, hence to a rational function on X, or even on $\mathscr{X}$. More generally, the morphism $\mathbf{a}\colon E_{F_v} \to G$ induces a regular function f_E on $G \times_{F_v} E_{F_v}$, hence a rational function on $\mathscr{X} \times_k E$. We view f_E as a family of regular functions on G, resp. as a family of rational functions on $\mathscr{X}$, both indexed by E and study the variation, for $p \in E$, of the divisors $\operatorname{div}(f_{\mathbf{a}(p)})$.

LEMMA 6.5.1. *Let P be a reduced k-scheme of finite type and let $\mathbf{a}\colon P_{F_v} \to G$ be an F_v-morphism. Let f_P be the associated rational function on $\mathscr{X} \times_k P$. There exists a decomposition of P as a disjoint union of smooth locally closed subsets P_i, and for each i, a map $\pi_i\colon \mathscr{Y}_i \to \mathscr{X} \times_k P_i$ which is a composition of blowing-ups whose centers are smooth over P_i, with generic fiber invariant under the action of $G_{F_v} \times_k P_i$, and do not meet $G_{F_v} \times_k P_i$ such that the rational function $\pi_i^* f_P$ on $\mathscr{Y}_i$ defines a regular morphism from $\mathscr{Y}_i \times_k P_i$ to $\mathbf{P}^1_R$ and such that the horizontal part of $\pi_i^*(\mathscr{D} \times_k P_i)$ is a relative divisor with strict normal crossings.*

Proof. The arguments of Lemma 3.4.1 of [8] furnish a partition (P_i) of P by smooth locally closed subsets and morphisms $\pi_i\colon Y_{i,F_v} \to X \times_{F_v} P_{i,F_v}$ which are compositions of blowing-ups whose centers are smooth over P_{i,F_v} and do not meet $G_{F_v} \times_{F_v} P_{i,F_v}$ such that the rational function $\pi_i^* f_P$ defines a regular morphism $Y_{i,F_v} \times_{F_v} P_{i,F_v} \to \mathbf{P}^1_{F_v}$.

Let us fix such a stratum P_i and call it P; similarly, we write π for π_i and Y for Y_i. Up to replacing P by a dense open subset P', and applying embedded resolution of singularities to the Zariski closures of the centers of the blowing-ups which define π, we obtain a morphism $\pi'\colon \mathscr{Y} \to \mathscr{X} \times_k P'$, composition of blowing-ups with smooth centers over P', such that $\pi'_{F_v}\colon \mathscr{Y}_{F_v} \to \mathscr{X}_{F_v} \times_{F_v} P'_{F_v}$ satisfies the preceding assumptions. We then view the morphism f_P as a rational map $\mathscr{Y} \times_k P' \dashrightarrow \mathbf{P}^1$. Above a dense open subset P'' of P' we resolve its indeterminacies by a further composition of blowing-ups whose centers are smooth over P' and do not meet the generic fiber. We can now repeat these arguments for $P \setminus P''$, so that the lemma follows by Noetherian induction. □

We apply the preceding lemma to $E \setminus \{0\}$ and fix such a stratum, which we call P. Set $Y = \mathscr{Y}_{F_v}$. For $p \in P$, let $\mathscr{Y}_p$ be the fiber of $\mathscr{Y}$ above p under the composition $\mathscr{Y} \to \mathscr{X} \times_k P \to P$; let $Y_p = (\mathscr{Y}_p)_{F_v}$.

For $p \in P$, by the change of variable formula (Theorem 13.2.2 in [9]), both $Z(\mathbf{T}, 0)$ and $Z(\mathbf{T}, \mathbf{a}(p))$ can be computed as motivic integrals on $\mathscr{L}(\mathscr{Y}_p)$. Since the rational function $f_{\mathbf{a}}$ extends to a regular morphism from $\mathscr{Y}_{F_v} \times P_{F_v}$ to $\mathbf{P}^1_{F_v}$, Proposition 5.3.1 gives an explicit form for $Z(\mathbf{T}, \mathbf{a}(p))$ as a rational function, whose denominator is controlled in terms of the Jacobian divisor of Y/X and the sub-complex $\mathrm{Cl}^{\mathrm{an}}(Y_p, Y_p \setminus G)_0$ of $\mathrm{Cl}^{\mathrm{an}}(Y_p, Y_p \setminus G)$ corresponding to the irreducible components of $Y \setminus G$ along which $f_{\mathbf{a}(p)}$ has no pole. Since the sub-complex $\mathrm{Cl}^{\mathrm{an}}(Y_p, Y_p \setminus G)_0$ of $\mathrm{Cl}^{\mathrm{an}}(Y_p, Y_p \setminus G)$ depends constructibly on $p \in P$, this implies the first part of Proposition 4.3.4.

The final part of the proof follows by applying the geometric arguments leading to the proof of Proposition 3.4.4 of [8]. For every $p \in P$, the Clemens complex of $(Y_p, Y_p \setminus G)$ has distinguished vertices, those coming from X which we denote by D'_α, and exceptional ones, corresponding to divisors contracted by π_p, which we denote by E'_β. Since $Y_p \to X$ is G-equivariant, the rational function $f_{\mathbf{a}(p)}$ has a divisor $\mathrm{div}_X(f_{\mathbf{a}(p)})$ on X whose strict transform dominates the divisor of $f_{\mathbf{a}(p)}$ on Y_p: their difference is an effective linear combination of the divisors E'_β (Lemma 1.4 of [6]). Moreover, the relative Jacobian divisor of Y_p/X_p is a linear combination of these divisors, with positive coefficients.

As in Section 3.4 of [8], these geometric facts imply that only the distinguished vertices of $\mathrm{Cl}^{\mathrm{an}}(Y_p, Y_p \setminus G)$, that is, those of $\mathrm{Cl}^{\mathrm{an}}(X, X \setminus G)$, intervene. Moreover, letting $d_p = 1 + \dim(\mathrm{Cl}^{\mathrm{an}}(X, X \setminus G)_0)$ and $d = 1 + \dim(\mathrm{Cl}^{\mathrm{an}}(X, X \setminus G))$, then one has $d_p < d$ (cf. Lemmas 3.4.5 and 3.5.4 of [8]).

Applying Proposition 5.3.1, we conclude that there exists an integer a such that $(1 - \mathbf{L}^a T^a)^{d_p} Z_\lambda(T, \mathbf{a}(p)) \in \mathscr{M}_k\{T\}^\dagger$ for every $p \in P$.

This concludes the proof of Proposition 4.3.4.

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