A non-archimedean Ax-Lindemann theorem

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À Daniel Bertrand, en témoignage d’amitié

1. Introduction

1.1. — The classical Lindemann–Weierstrass theorem states that if algebraic numbers \( \alpha_1, \ldots, \alpha_n \) are \( \mathbb{Q} \)-linearly independent, then their exponentials \( \exp(\alpha_1), \ldots, \exp(\alpha_n) \) are algebraically independent over \( \mathbb{Q} \). More generally, if \( \alpha_1, \ldots, \alpha_n \) are complex numbers which are no longer assumed to be algebraic, Schanuel’s conjecture predicts that the field \( \mathbb{Q}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)) \) has transcendence degree at least \( n \) over \( \mathbb{Q} \). In [1], Ax established power series and differential field versions of Schanuel’s conjecture. In particular the part of Ax’s results corresponding to the Lindemann–Weierstrass theorem can be recasted into geometrical terms as follows:

**Theorem 1.2 (Exponential Ax-Lindemann).** — Let \( \exp : \mathbb{C}^n \to (\mathbb{C}^\times)^n \) be the morphism \( (z_1, \ldots, z_n) \mapsto (\exp(z_1), \ldots, \exp(z_n)) \). Let \( V \) be an irreducible algebraic subvariety of \( (\mathbb{C}^\times)^n \) and let \( W \) be an irreducible component of a maximal algebraic subvariety of \( \exp^{-1}(V) \). Then \( W \) is geodesic, that is, \( W \) is defined by a finite family of equations of the form \( \sum_{i=1}^n a_i z_i = b \) with \( a_i \in \mathbb{Q} \) and \( b \in \mathbb{C} \).

In the breakthrough paper [19], Pila succeeded in providing an unconditional proof of the André-Oort conjecture for products of modular curves. One of his main ingredients was to prove an hyperbolic version of the above Ax-Lindemann theorem, which we now state in a simplified version.

Let \( \mathbb{H} \) denote the complex upper half-plane and \( j : \mathbb{H} \to \mathbb{C} \) the elliptic modular function. By an algebraic subvariety of \( \mathbb{H}^n \) we shall mean the trace in \( \mathbb{H}^n \) of an algebraic subvariety of \( \mathbb{C}^n \). An algebraic subvariety of \( \mathbb{H}^n \) if said to be geodesic if it is defined by equations of the form \( z_i = c_i \) and \( z_k = g_{kl} z_l \), with \( c_i \in \mathbb{C} \) and \( g_{kl} \in \text{GL}_2^\times(\mathbb{Q}) \).
**Theorem 1.3 (Hyperbolic Ax-Lindemann).** — Let \( j : H^n \to C^n \) be the morphism \((z_1, \ldots, z_n) \mapsto (j(z_1), \ldots, j(z_n))\). Let \( V \) be an irreducible algebraic subvariety of \( C^n \) and let \( W \) be an irreducible component of a maximal algebraic subvariety of \( j^{-1}(V) \). Then \( W \) is geodesic.

Pila’s method to prove this Ax-Lindemann theorem is quite different from the differential approach of Ax. It follows a strategy initiated by Pila and Zannier in their new proof of the Manin-Mumford conjecture for abelian varieties [23]; that approach makes crucial use of the bound on the number of rational points of bounded height in the transcendental part of sets definable in an o-minimal structure obtained by Pila and Wilkie in [22]. Recently, still using the Pila and Zannier strategy, Klingler, Ullmo and Yafaev have succeeded in proving a very general form of the hyperbolic Ax-Lindemann theorem valid for any arithmetic variety ([14], see also [25] for the compact case).

1.4. — In the recent paper [5], Cluckers, Comte and Loeser established a non-archimedean analogue of the Pila-Wilkie theorem of [22] in its block version of [18]. The purpose of this paper is to use this result to prove a version of Ax-Lindemann for varieties admitting a non-archimedean uniformization. Our main result is a statement of Ax-Lindemann type for products of Shimura curves admitting \( p \)-adic uniformization à la Cherednik-Drinfeld (theorem 2.7). The basic strategy we use is strongly inspired by that of Pila [19] (see also [20]), though some new ideas are required in order to adapt it to the non-archimedean setting. Similarly as in Pila’s approach one starts by working on some neighborhood of the boundary of our space (now a product of Drinfeld — instead of Poincaré — half-planes). Analytic continuation and monodromy arguments are replaced by more algebraic ones and explicit matrix computations by group theory considerations. We also take advantage of the fact that Schottky groups are free and of the geometric description of their fundamental domains.

To conclude, let us note that there are cases where \( p \)-adic analogues of theorems in transcendental number theory seem to require other methods than those used to prove their complex counterparts. For instance, it is still an open problem to prove a \( p \)-adic analogue, for values of the \( p \)-adic exponential function, of the classical Lindemann-Weierstrass theorem.

Since his first works (see, for example, [2]), Daniel Bertrand has shown deep insight into \( p \)-adic transcendental number theory, and disseminated his vision within the mathematical community. We are glad that the Schwarzian derivative, which is so dear to his heart, plays a role here, and we are pleased to dedicate this paper to him.

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2. Statement of the theorem

2.1. Complex Shimura curves. — Let \( B \) be a quaternion division algebra with center \( \mathbb{Q} \); we assume that it is indefinite, namely \( B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \). Let \( H \) be the algebraic group of units of \( B \), modulo center, defined by \( H(R) = (B \otimes_{\mathbb{Q}} R)^	imes / Z((B \otimes_{\mathbb{Q}} R)^	imes) \) for every \( \mathbb{Q} \)-algebra \( R \). In particular, the group \( H(R) \) is isomorphic to \( \text{PGL}(2, \mathbb{R}) \); and we fix such an isomorphism. Then the group \( G(R) \) acts by homographies on the double Poincaré upper half-plane

\[
h^\pm = \mathbb{C} \setminus \mathbb{R}.
\]

Let then \( \mathcal{O}_B \) be a maximal order of \( B \), that is a maximal sub-algebra of \( B \) which is isomorphic to \( \mathbb{Z}^4 \) as a \( \mathbb{Z} \)-module. Let also \( \Delta \) be a congruence subgroup of \( H(\mathcal{O}_B) \), small enough so that the stabilizer of every point of \( h^\pm \) is trivial. The quotient \( h^\pm / \Delta \) has a natural structure of a compact Riemann surface and the projection \( p: h^\pm \rightarrow h^\pm / \Delta \) is an étale covering.

This curve parameterizes triples \( (V, \iota, \nu) \), where \( V \) is a complex two dimensional abelian variety, \( \iota: \mathcal{O}_B \rightarrow \text{End}(V) \) is a faithful action of \( \mathcal{O}_B \) on \( V \) and \( \nu \) is a level structure “of type \( \Delta \)” on \( A \). When \( \Delta \) is the kernel of \( H(\mathcal{O}_B) \) to \( H(\mathcal{O}_B/N) \), for some integer \( N \geq 1 \), such a level structure corresponds to an equivariant isomorphism of \( V_N \), the subgroup of \( N \)-torsion of \( V \), with \( \mathcal{O}_B/N \).

It admits a canonical structure of an algebraic curve \( S \) which can be defined over a number field \( E \) in \( \mathbb{C} \).

2.2. Non-archimedean analytic spaces. — Given a complete non-archimedean valued field \( F \), we shall consider in this paper \( F \)-analytic spaces in the sense of Berkovich. However, the statements, and essentially the proofs, can be carried on \textit{mutatis mutandis} in the rigid analytic setting. In this context, there is a notion of irreducible component (see [12], or [9] for the rigid analytic version).

If \( V \) is an algebraic variety over \( F \), we denote by \( V^{\text{an}} \) the corresponding \( F \)-analytic space. It canonically contains \( V(F) \) as a closed subset.

2.3. The Drinfeld upper half-plane. — Let \( F \) be complete extension of \( \mathbb{Q}_p \). We denote by \( \Omega \) the Drinfeld upper half-plane; it is the complement of \( \mathbb{P}^1(\mathbb{Q}_p) \) inside the analytic space \((\mathbb{P}^1)^{\text{an}} \). The group \( \text{PGL}(2, \mathbb{Q}_p) \) acts by homographies on \((\mathbb{P}^1)^{\text{an}} \), and this action preserves \( \Omega \).

We shall say that a closed analytic subspace \( W \) of \( \Omega^n \) is algebraic if there exists an \( F \)-algebraic subvariety \( Y \subset (\mathbb{P}^1)^n \) such that \( W = \Omega^n \cap Y^{\text{an}} \).
2.4. \textit{$p$-adic uniformization of Shimura curves.} — Let $p$ be a prime number at which $B$ ramifies, which means that $B \otimes \mathbb{Q} \mathbb{Q}_p$ is a division algebra. Let also $F$ be a completion of the field $E$ at a place dividing $p$; we denote by $C_p$ a $p$-adic completion of an algebraic closure of $F$.

According to the theorem of Cherednik-Drinfeld ([4, 11]; see also [3] for a detailed exposition), and up to replacing $F$ by a finite unramified extension, the $F$-analytic curve $S_{\text{an}}$ admits a “$p$-adic uniformization” which takes the form of a surjective analytic morphism

$$j : \Omega \to S_{\text{an}},$$

which identifies $S_{\text{an}}$ with the quotient of $\Omega$ by the action of a subgroup $\Gamma$ of $\text{PGL}(2, \mathbb{Q}_p)$. Up to replacing $\Delta$ by a smaller congruence subgroup, which replaces $S$ by a finite (possibly ramified) covering, we may also assume that $\Gamma$ is a $p$-adic Schottky subgroup acting freely on $\Omega$, and that $j$ is topologically étale.

Let us describe these subgroups. Let $A$ be the quaternion division algebra over $\mathbb{Q}$ with the same invariants that $B$, except for those invariants at $p$ and $\infty$ which are switched. In particular, $A \otimes \mathbb{Q} \mathbb{R}$ is Hamilton’s quaternion algebra, while $A \otimes \mathbb{Q} \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$. Let $G$ be the algebraic group of units of $A$, modulo center, defined by

$$G(R) = (A \otimes \mathbb{Q} R)^{\times}/Z((A \otimes \mathbb{Q} R)^{\times})$$

for every $\mathbb{Q}$-algebra $R$. In particular, $G(\mathbb{Q}_p) \simeq \text{PGL}(2, \mathbb{Q}_p)$ and the discrete subgroup $\Gamma$ is the intersection of $G(\mathbb{Q})$ with a compact open subgroup of $G(A_f)$, the adelic group associated with $G$ where the place at $\infty$ is omitted.

2.5. — More generally, we consider a finite family $(S_i)$ of Shimura curves as above, associated with indefinite quaternion $\mathbb{Q}$-algebras which are ramified at a common prime number $p$. There exists a finite extension $E$ of $\mathbb{Q}_p$ on which each of them can be defined and admits a $p$-adic uniformization as above. Let $S$ be their product. One obtains a surjective and étale uniformization morphism

$$j : \Omega^n \to \prod_{i=1}^n \Omega/\Gamma_i \simeq S_{\text{an}} = (S_1 \times \cdots \times S_n)^{\text{an}}.$$
**Theorem 2.7 (Non-archimedean Ax-Lindemann theorem)**

Let $F$ be a finite extension of $\mathbb{Q}_p$ and, as above, let $j : \Omega^n \to \mathcal{S}^n$ be the Cherednik-Drinfeld uniformization of a product of Shimura curves. Let $V$ be an irreducible algebraic subvariety of $S$ and let $W \subset \Omega^n$ be an irreducible component of a maximal algebraic subvariety of $j^{-1}(V^n)$. Then $W$ is geodesic.

The proof of this theorem is given in section 6; it follows the strategy of Pila–Zannier. In the archimedean setting, this strategy relies crucially on a theorem of Pila–Wilkie about rational points on definable sets; we recall in section 3 the non-archimedean analogue of this theorem, due to Cluckers, Comte and Loeser (see [5]), which is used here. In section 5, we recall the reader a few facts on $p$-adic Schottky groups and $p$-adic uniformization, essentially borrowed from the book [13].

### 3. Definability — A $p$-adic Pila-Wilkie theorem

3.1. — Let $H$ be the standard height function on $\overline{\mathbb{Q}}$; for $x \in \mathbb{Q}$, written as a fraction $a/b$ in lowest terms, one has $H(x) = \max(|a|, |b|)$. We also write $H$ for the height function on $\overline{\mathbb{Q}}^n$ defined by $H(x_1, \ldots, x_n) = \max_i(H(x_i))$. Viewing $GL(d, \mathbb{Q})$ as a subspace of $Q^{d^2}$, it defines a height function on $GL(d, \mathbb{Q})$. There exists a strictly positive real number $c$ such that $H(gg') \leq cH(g)H(g')$ for every $g, g' \in GL(d, \mathbb{Q})$, and $H(g^{-1}) \ll H(g)^c$ for every $g \in SL(d, \mathbb{Q})$. When $d = 2$, one even has $H(g^{-1}) = H(g)$.

By abuse of language, if $G$ is a linear algebraic $\mathbb{Q}$-group, we implicitly choose an embedding in some linear group, which furnishes a height function on $G(\mathbb{Q})$. The actual choice of this height function depends on the chosen embedding, any other height function $H'$ is equivalent, in the sense that there is a strictly positive real number $c$ such that $H(x)^{1/c} \ll H'(x) \ll H(x)^c$ for every $x \in G(\mathbb{Q})$.

3.2. — Let $Z$ be a subset of $\mathbb{Q}^n_p$. We write $Z(\mathbb{Q}) = Z \cap \mathbb{Q}^n$ (rational points of $Z$). For every real number $T$, we also define $Z(\mathbb{Q}; T) = \{x \in Z(\mathbb{Q}) ; H(x) \leq T\}$; this is a finite set. We say that $Z$ has many rational points if there exist strictly positive real numbers $c, \alpha$ such that

$$\text{Card} \left( Z(\mathbb{Q}; T) \right) \geq c T^{\alpha}$$

for all $T \geq 1$.

This notion only depends on the equivalence class of the height.

3.3. — There are two distinct notions of $p$-adic analytic geometry, one is “naïve”, and the other one is rigid analytic or Berkovich geometry. These two notions give rise to two notions of subanalytic sets, and we shall use both in this paper.

a) Semialgebraic and subanalytic subsets of $\mathbb{Q}^n_p$ are defined by Denef and van den Dries in [10]; see also [5, p. 26]. When $F$ is a finite extension of $\mathbb{Q}_p$, of degree $d$, the Weil restriction functor maps the affine line $A^1_F$ over $F$ to $A^d_{\mathbb{Q}_p}$, identifying $F^n$ with $\mathbb{Q}_p^{dn}$; this gives rise to a notion of semialgebraic or subanalytic subsets of $F^n$. 
b) In [15], Lipshitz defined a notion of rigid subanalytic subset of $\mathbb{C}^n_p$; we shall use in this paper the variant (see [7, 6]) where the coefficients of all polynomials and power series involved belong to a fixed finite extension $F$ of $\mathbb{Q}_p$.

Considering affine charts, all three classes of sets can be defined in algebraic varieties. They are stable under boolean operations and projections, admit cell decompositions and a natural notion of dimension (in fact, they are b-minimal in the sense of [8]).

We shall use the fact that if $Z$ is a rigid subanalytic subset of $\mathbb{C}^n_p$, then $Z(\mathbb{Q}_p) = Z \cap \mathbb{Q}^n_p$ is a subanalytic subset of $\mathbb{Q}_p^n$. Indeed, $Z$ can be defined by a quantifier-free formula of the above-mentioned variant of Lipshitz’s analytic language, and our claim follows from the very definition of this language.

3.4. — A block in $\mathbb{Q}_p^n$ is either empty, or a singleton, or a smooth subanalytic subset of pure dimension $d > 0$ which is contained in a smooth semialgebraic subset of dimension $d$.

A family of blocks in $\mathbb{Q}_p^n \times \mathbb{Q}_p^n$ is a subanalytic subset $W$ such that there exists an integer $t \geq 0$ and a semialgebraic set $Z \subset \mathbb{Q}_p^n \times \mathbb{Q}_p^n$ such that for every $\sigma \in \mathbb{Q}_p^n$, there exists $\tau \in \mathbb{Q}_p^n$ such that $W_\sigma$ and $Z_\tau$ are smooth of the same dimension, and $W_\sigma \subset Z_\tau$. (In particular, the sets $W_\sigma$, for $\sigma \in \mathbb{Q}_p^n$, are blocks in $\mathbb{Q}_p^n$.)

3.5. — In [5], Cluckers, Comte and Loeser established a $p$-adic analogue of a theorem of Pila-Wilkie [22] concerning the rational points of a definable set. We will use the following particular case of [5, Theorem 4.2.4].

**Theorem 3.6.** — Let $Z \subset \mathbb{Q}_p^n$ be a subanalytic subset. Let $\varepsilon > 0$. There exists $s \in \mathbb{N}$, $c \in \mathbb{R}$ and a family of blocks $W \subset \mathbb{Q}_p^n \times \mathbb{Q}_p^n$, such that for any $T > 1$, there exists a subset $S \subset \mathbb{Q}_p^n$ of cardinality $< cT^\varepsilon$ such that $Z(\mathbb{Q}_p^T) \subset \bigcup_{\sigma \in S} W_\sigma$.

The definition of a family of blocks that we have adopted here is slightly stronger than the one in [5]. However, all proofs go over without any modification.

### 4. Zariski closures and analytic functions

**Lemma 4.1.** — Let $F$ be a complete non-archimedean valued field. Let $V$ be a $F$-scheme of finite type which is geometrically connected (resp. geometrically irreducible) and let $K$ be a compact subset of $V_{\text{an}}$ consisting of rigid points. Then $V_{\text{an}} \setminus K$ is a geometrically connected (resp. geometrically irreducible) analytic space.

**Proof.** — We may assume that $F$ is algebraically closed. Let us assume that $V$ is connected and let us prove that $V_{\text{an}} \setminus K$ is connected as well. Let $x, y \in V(F) \setminus K$. By [16, p. 56], there exists an irreducible curve $C \subset V$ which passes through $x$ and $y$. Then $C_{\text{an}}$ is connected and it follows from the structure of analytic curves that $C_{\text{an}} \setminus (K \cap C_{\text{an}})$ is connected as well. Consequently, $V(F) \setminus K$ is connected. Since $V(F)$ is dense in $V_{\text{an}}$, $V(F) \setminus K$ is dense in $V_{\text{an}} \setminus K$, hence $V_{\text{an}} \setminus K$ is connected.
Let us now assume that \( V \) is irreducible. The normalization morphism \( p: W \to V \) is finite, and \( W \) is connected. Since \( p^{-1}(K) \) is a compact subset of \( W \) consisting of rigid points, it follows from the first part of the lemma that \( W_{\text{an}} \setminus p^{-1}(K) \) is connected. Since \( W_{\text{an}} \) is the normalization of \( V_{\text{an}} \), then \( W_{\text{an}} \setminus p^{-1}(K) = p^{-1}(V_{\text{an}} \setminus K) \) is the normalization of \( V_{\text{an}} \setminus K \). By theorem 5.17 of [12], this implies that \( V_{\text{an}} \) is irreducible.

**Corollary 4.2.** — Let \( F \) be a complete valued field, let \( V \) be a \( F \)-scheme of finite type and let \( K \) be a compact subset of \( V_{\text{an}} \) consisting of rigid points. The set of irreducible components of \( V_{\text{an}} \setminus K \) is finite.

**Proof.** — Let \( \Omega = V_{\text{an}} \setminus K \). Let \( E \) be the completion of the algebraic closure of \( F \). By lemma 4.1, \( \Omega_E \cap Z_{\text{an}} \) is irreducible, for every irreducible component \( Z \) of \( Y_E \), and the family of these intersections is the family of irreducible components of \( W_E \). The corollary then follows from [12, lemme 4.25].

**Proposition 4.3.** — Let \( F \) be a finite extension of \( \mathbb{Q}_p \). Let \( A \) be an affine scheme of finite type over \( F \) and let \( \Omega \subset A_{\text{an}} \) be the complement of a compact subset consisting of rigid points. Let \( X \) be a closed analytic subspace of \( \Omega \). Let \( V \) be a semi-algebraic subset of \( F^n \), contained in \( X(F) \), and let \( W \) be its Zariski-closure in \( X \). If \( W \) is irreducible, then \( W_{\text{an}} \cap \Omega \subset X \).

**Proof.** — The following proof is inspired by that of lemma 4.1 in [21]. Up to repeatedly replacing \( X \) by its singular locus \( X_{\text{sing}} \), we may assume that \( V(F) \not\subset X_{\text{sing}} \). Let \( o \in V(F) \) be a smooth point of \( X \).

The rational points of a non-empty and non-geometrically irreducible variety are not Zariski dense. Since \( W \) is irreducible, it is geometrically irreducible; by lemma 4.1, \( W_{\text{an}} \cap \Omega \) is an irreducible analytic space.

Let \( R \) the Weil restriction functor from \( F \) to \( \mathbb{Q}_p \); it is a right adjoint to the functor of extensions of scalars from \( \mathbb{Q}_p \) to \( F \). Recall that \( R(A_F) \simeq A_{\mathbb{Q}_p}^d \), where \( d = [F : \mathbb{Q}_p] \).

Let \( Z \) be the Zariski closure of \( V \) inside \( R(A) \). By Weil restriction, the inclusion \( Z \to R(A) \) gives rise to a morphism \( j: Z_F \to A \); one has \( W = \overline{j(Z_F)} \). Let \( Z'_F \) be the set of points of \( Z_F \) at which \( j \) is open; it contains a Zariski dense open subset of \( Z_F \). Let also \( Y = j^{-1}(X) \subset (Z_F)^{\text{an}} \); it is a closed analytic subset of \( (Z_F)^{\text{an}} \).

Let us consider a semi-algebraic cellular decomposition of \( R(A) \) which is adapted to \( V \) and to \( Z(\mathbb{Q}_p) \). Let \( C \) be a cell of dimension \( m = \dim(V) = \dim(Z(F)) \) which meets \( V \). Let \( (u_1, \ldots, u_m) \) be a family of regular functions on \( R(A) \) which parameterizes \( C \).

Every point \( x \in C \) is a smooth point of \( V \) (in the sense of \( \mathbb{Q}_p \)-analytic geometry) and of \( Z \) (in the sense of algebraic geometry). The tangent space \( T_xY \) at \( x \) is an \( F \)-vector subspace of \( T_xZ_F \simeq F^m \) which contains \( T_xV = \mathbb{Q}_p^m \); consequently, \( T_xY = T_xZ_F \) for every \( x \in C \). This implies that \( Y(F) \) is a neighborhood of \( x \) in \( Z_F(F) \), hence \( X(F) \) contains a semi-algebraic subset of \( W \) of maximal dimension.
Since $W^\text{an} \cap \Omega$ is irreducible, such a subset is dense for the Zariski topology of $W^\text{an} \cap \Omega$, hence $W^\text{an} \cap \Omega \subset X$. \hfill \qed

5. Complements on $p$-adic Schottky groups and uniformization

5.1. — We endow $\mathbf{P}_1(\mathbf{C}_p)$ with the distance given by

$$d(x, y) = \frac{|x - y|}{\max(1, |x|) \max(1, |y|)}$$

for $x, y \in \mathbf{C}_p$ — it is invariant under the action of $\text{PGL}(2, \mathcal{O}_{\mathbf{C}_p})$. Moreover, an elementary calculation shows that every element $g \in \text{PGL}(2, \mathbf{C}_p)$ is Lipschitz for this distance (see also Thm 1.1.1 of [24]).

5.2. — For $x \in \Omega$, let $\delta(x)$ be the distance of $x$ to $\mathbf{P}_1(\mathbf{Q}_p)$; if $|x| \leq 1$, this is the largest real number $r$ such that the disk $D(x, r)$ of radius $r$ centered at $x$ is contained in $\Omega$. For every $\gamma \in \text{PGL}(2, \mathbf{Q}_p)$, there exists a positive real number $c$ such that $\delta(\gamma \cdot z) \leq c\delta(z)$ for every $z \in \Omega$.

5.3. — Let $\Gamma$ be a discrete subgroup of $\text{PGL}(2, \mathbf{Q}_p)$. Recall (see [13], I.1, for more details) that the limit set $\mathcal{L}(\Gamma)$ of $\Gamma$ is the set of points $x \in \mathbf{P}_1(\mathbf{C}_p)$ for which there exists $x \in \mathbf{P}_1(\mathbf{C}_p)$ and a sequence $(\gamma_n)$ of distinct elements of $\Gamma$ such that $x = \lim \gamma_n \cdot x$. It is a $\Gamma$-invariant compact subset of $\mathbf{P}_1(\mathbf{Q}_p)$.

5.4. — One says that $\Gamma$ is a Schottky group if, moreover, it is finitely generated and torsion free. Then $\Gamma$ is a free group of finite rank.

5.5. — Every Schottky group admits a good fundamental domain $\tilde{\mathfrak{F}}$, in the following sense:

1. There exists a finite family $(B_1, \ldots, B_g, C_1, \ldots, C_g)$ of open disks in $\mathbf{P}_1$, with centers in $k$, such that $\tilde{\mathfrak{F}} = \mathbf{P}_1 \setminus \bigcup B_i \cup \bigcup C_i$.
2. The closed disks $B^+_1, \ldots, B^+_g, C^+_1, \ldots, C^+_g$ are pairwise disjoint; let then $\tilde{\mathfrak{F}}^\circ = \mathbf{P}_1 \setminus \bigcup B^+_i \cup \bigcup C^+_i$.
3. The group $\Gamma$ is generated by elements $\gamma_1, \ldots, \gamma_g$ such that $\gamma_i(\mathbf{P}_1 \setminus B_i) = C^+_i$ and $\gamma_i(\mathbf{P}_1 \setminus B^+_i) = C_i$ for every $i \in \{1, \ldots, g\}$.

Moreover, the following properties are satisfied:

4. One has $\bigcup_{\gamma \in \Gamma} \gamma \cdot \tilde{\mathfrak{F}} = \mathbf{P}_1 \setminus \mathcal{L}(\Gamma)$;
5. For $\gamma \in \tilde{\mathfrak{F}}$, one has $\tilde{\mathfrak{F}} \cap \gamma \cdot \tilde{\mathfrak{F}} \neq \emptyset$ if and only if $\gamma \in \{1, \gamma^+_1, \ldots, \gamma^+g\}$;
6. For every $\gamma \in \Gamma \setminus \{1\}$, one has $\tilde{\mathfrak{F}}^\circ \cap \gamma \cdot \tilde{\mathfrak{F}} = \emptyset$.

In this context, we denote by $\ell(g)$ the length of an element $g \in \Gamma$ with respect to the generating set $\{\gamma_1, \ldots, \gamma_g\}$.

Lemma 5.6. — Let $\Gamma$ be a Schottky group in $\text{PGL}(2, \mathbf{Q}_p)$.
(1) For every point $\xi \in \mathcal{L}(\Gamma)$ and every open neighborhood $U_i$ of $\xi$ in $\mathbb{P}_1$, there exists an affinoid domain $\mathfrak{F}$ contained in $U$ which is a good fundamental domain for the action of the group $\Gamma$.

(2) There exists positive real numbers $a, b$ such that for every $x \in \mathbb{P}_1 \setminus \mathcal{L}(\Gamma)$, there exists $\gamma \in \Gamma$ such that $\gamma x \in \mathfrak{F}$ and $\ell(\gamma) \leq a - b \log(\delta(x))$.

Proof. — If $\mathfrak{F}$ is good fundamental domain for $\Gamma$, then so is $\gamma \cdot \mathfrak{F}$ for every $\gamma \in \Gamma$. Moreover, it follows from the description given in [13, I, §4] that the domain $\gamma \cdot \mathfrak{F}$ is contained in a disk whose radius is $\ll c\ell(\gamma)$, for some positive real number $c > 1$. This implies the lemma. \[\square\]

5.7. — Let $A$ be a definite quaternion algebra over $\mathbb{Q}$ such that $A \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$. Let $G$ be the algebraic $\mathbb{Q}$-group of units of $A$, modulo center; in particular, $G(\mathbb{Q}_p) \simeq \text{PGL}(2, \mathbb{Q}_p)$. Let $\mathbb{A}_p^\gamma$ be the ring of adeles where the places $p$ and $\infty$ are omitted and let $K$ be a compact open subgroup of $G(\mathbb{A}_p^\gamma)$. Let finally $\Gamma = G(\mathbb{Q}) \cap K$. It follows from the hypothesis that $A \otimes \mathbb{R}$ is definite that $\Gamma$ is discrete.

Let $\Omega = \mathbb{P}_1 \setminus \mathbb{P}_1(\mathbb{Q}_p)$ be the Drinfeld upper half-plane.

Proposition 5.8. — (1) The group $\Gamma$ is a Schottky subgroup in $\text{PGL}(2, \mathbb{Q}_p)$, whose limit set is equal to $\mathbb{P}_1(\mathbb{Q}_p)$.

(2) Let $\mathfrak{F}$ be a good fundamental domain for the action of $\Gamma$ and let $H$ be a height function on $G(\mathbb{Q})$. There exists positive real numbers $a, b$ such that for every $x \in \Omega$, there exists $\gamma \in \Gamma$ such that $\gamma \cdot x \in \mathfrak{F}$ and $H(\gamma) \leq a/\delta(x)^b$.

Proof. — (1) The group $\Gamma$ is a discrete subgroup of $\text{PGL}(2, \mathbb{Q}_p)$ hence its limit set $\mathcal{L}(\Gamma)$ is a $\Gamma$-invariant subset of $\mathbb{P}_1(\mathbb{Q}_p)$. Consequently, $\Omega$ is an open subset of $\mathbb{P}_1^\text{an} \setminus \mathcal{L}(\Gamma)$. By the theory of Mumford curves, the analytic curve $(\mathbb{P}_1^\text{an} \setminus \mathcal{L}(\Gamma))/\Gamma$ is algebraic, and admits the analytic curve $S^\text{an} = \Omega/\Gamma$ as an open subset. According to the Cherednik-Drinfeld theorem, the curve $S^\text{an}$ is projective. This implies that $\Omega = \mathbb{P}_1^\text{an} \setminus \mathcal{L}(\Gamma)$, hence $\mathcal{L}(\Gamma) = \mathbb{P}_1(\mathbb{Q}_p)$.

(2) Let $(\gamma_1, \ldots, \gamma_g)$ be a generating family of minimal cardinality of $\Gamma$. Let $c_1$ be a positive real number such that $H(hh') \leq c_1 H(h)H(h')$ for every $h, h' \in G(\mathbb{Q})$. For every $\gamma \in \Gamma$, one proves by induction on $\ell(\gamma)$ that

$$H(\gamma) \leq c_1^{\ell(\gamma)-1} \sup(H(\gamma_1), \ldots, H(\gamma_g))^{\ell(\gamma)}.$$ 

Let $c = \sup(1, c_1, H(\gamma_1), \ldots, H(\gamma_g))$; then $H(\gamma) \leq c^{\ell(\gamma)}$ for every $\gamma \in \Gamma$. The second property thus follows from lemma 5.6, (2). \[\square\]

6. Proof of the theorem

6.1. — Let $W \subset j^{-1}(V) \subset \Omega^n$ be an irreducible component of a maximal algebraic subvariety of $j^{-1}(V)$, and let us prove that $W$ is geodesic. The proof requires intermediate steps and will be concluded in proposition 6.8.
Let $Y$ be the closure of $W$ in $(\mathbb{P}_1)^n$ and let $m$ be its dimension. We may assume that $m > 0$. We also assume that there does not exist $j \in \{1, \ldots, n\}$ such that the $j$th projection $q_j : (\mathbb{P}_1)^n \to \mathbb{P}_1$ is constant on $Y$.

One then has $p_1(Y) = \mathbb{P}_1$. Since $\mathbb{P}_1(\mathbb{Q}_p)$ is dense in $\mathbb{P}_1$ for the Zariski topology, there exists a smooth point $\xi \in Y$ such that $p_1(\xi) \in \mathbb{P}_1(\mathbb{Q}_p)$. Then there exists a finite subset $J$ of $\{1, \ldots, n\}$ containing 1 and of cardinality $m$ such that the projection $q_J : \mathbb{P}_1^m \to \mathbb{P}_1^J$ induces a generically étale map from $Y$ to $\mathbb{P}_1^J$. Up to reordering the coordinates, we assume that $J = \{1, \ldots, m\}$. There exists a smooth rigid point $\xi' \in W$ such that $p_1(\xi') \in \mathbb{P}_1(\mathbb{Q}_p)$ and such that $q_J|_Y$ is étale at $\xi'$; we replacing $\xi$ by $\xi'$, we henceforth assume that $q_J|_W$ is étale at $\xi$. Then there exists an open neighborhood $U$ of $\xi$ in $Y$, of the form $U_1 \times \cdots \times U_n$ such that $q_J|_U$ is étale on $U$ and finite to its image. Let $\varphi = (\varphi_1, \ldots, \varphi_n) : V \to Y$ be an analytic section, defined around $q_J(\xi)$. If $\varphi_j$ is not constant, then it takes some value which does not belong to $\mathbb{P}_1(\mathbb{Q}_p)$; shrinking $V$, we then assume that for each $j \geq 2$, either $\varphi_j$ is constant, or $\varphi_j$ avoids $\mathbb{P}_1(\mathbb{Q}_p)$.

**6.2.** — For every $i$, let $\mathfrak{F}_i$ be a good fundamental domain for the action of $\Gamma_i$; we also assume that $\mathfrak{F}_1$ is contained in $U_1$. Let $\mathfrak{F} = \mathfrak{F}_1 \times \cdots \mathfrak{F}_n$. Let $G$ be the $\mathbb{Q}$-algebraic group given by $G = G_1 \times \cdots G_n$, let $G_0$ be the algebraic subgroup of $G$ defined by

\[(g_1, \ldots, g_n) \in G_0 \iff g_2 = \cdots = g_m = 1\]

and let $R$ be the subset of $G_0(\mathbb{Q}_p)$ defined by

\[g \in R \iff \dim(gW \cap \mathfrak{F} \cap j^{-1}(V)) = m.\]

**Lemma 6.3.** — The set $R$ is a subanalytic subset of $G_0(\mathbb{Q}_p)$.

**Proof.** — The sets $V$ and $W$ are algebraic, hence rigid subanalytic. Since $\mathfrak{F}$ is affinoid, the function $j|_\mathfrak{F}$ is rigid subanalytic, so that $\mathfrak{F} \cap j^{-1}(V)$ is rigid subanalytic as well. Consequently, $(gW \cap \mathfrak{F} \cap j^{-1}(V))_g$ is a rigid subanalytic family of rigid subanalytic subsets of $\mathbb{Q}^n$, parameterized by $G_0(\mathbb{C}_p)$. By $b$-minimality, the set of points $g \in G_0(\mathbb{C}_p)$ such that $\dim(gW \cap \mathfrak{F} \cap j^{-1}(V)) = m$ is a rigid subanalytic subset of $G_0(\mathbb{C}_p)$. It then follows from the remark at the end of §3.3 that $R$ is a subanalytic subset of $G_0(\mathbb{Q}_p)$. \hfill \qedsymbol

**Lemma 6.4.** — Let $r$ be a positive real number and let $f \in \mathbb{C}_p[[z]]$ be a power series which converges on the closed disk $D(0, r)$. Let $K$ be a closed subset of $\mathbb{C}_p$ containing 0 such that $f^{-1}(K) \subset K$; for every $x \in \mathbb{C}_p$, let $\delta_K(x)$ be the distance of $x$ to $K$. Then there exists real numbers $m, c, s$ such that $0 < s < r$ and such that $\delta_K(f(x)) \geq c \delta_K(x)^m$ for every $x \in D(0, s)$.

**Proof.** — Write $f(z) = \sum c_n z^n$; for simplicity of notation, we assume that $r = 1$ and that $|c_n| \leq 1$ for all $n$; write $D = D(0, 1)$.

Let us first treat the case where $f(0) \notin K$. Then there exists a real number $u > 0$ such that $D(f(0), u) \cap K = \emptyset$. For $x \in D$ such that $|x| < u$, one has $|f(x) - f(0)| < u$, hence $\delta_K(f(x)) > u$. To
We now assume that \( f(0) \in K \); let \( m = \text{ord}_0(f - f(0)) \). Since \( f'(z) = \sum_{n\geq m} nc_n z^{n-1} \), there exists a real number \( s \) such that \( 0 < s \leq 1 \) and such that \(|f'(z)| = |mc_m| |z|^{m-1}\) provided \( |z| \leq s \). Moreover, \(|f^{(n)}(z)/n!| \leq 1\) for every \( n \geq 0 \) and any \( z \in D \). Considering the Taylor expansion \( f(y) = \sum_{n\geq 0} \frac{1}{n!} f^{(n)}(x)(y - x)^n \), we then see that there exists a real number \( s' \)

\[
f(D(x, u)) = D(f(x), |f'(x)| u)
\]

for every real number \( u \) such that \( 0 < u \leq s' \) and every \( x \in D \) such that \( 0 < |x| \leq s \). If \( u < \delta_K(x) \), then \( D(x, u) \cap K = \emptyset \), hence \( D(f(x), |f'(x)| u) \cap K = \emptyset \); consequently, \( \delta_K(f(x)) \geq |f'(x)| \delta_K(x) \). Since \( 0 \in K \), one has \( |x| \geq \delta_K(x) \). Consequently,

\[
\delta_K(f(x)) \geq |mc_m| \delta_K(x)^{m-1} \delta_K(x) \geq |mc_m| \delta_K(x)^m.
\]

This concludes the proof. \( \square \)

**Lemma 6.5.** — There exists a real number \( c > 0 \) such that \( \text{Card}(R(Q; T) \cap \Gamma) \geq T^c \) for \( T \) large enough.

**Proof.** — Let \( a = (a_1, \ldots, a_m) \) be a point of \( \Omega^m \cap (U_1 \times \cdots \times U_m) \).

Let \( q \) be the genus of \( S_1 \); it will be fundamental below that \( q \geq 2 \). We recall that the fundamental domain \( \mathfrak{F}_1 \) is the complement of a union of \( 2q \) open disks in \( P_1 \), with disjoint closures, say \( D_1, \ldots, D_q, D'_1, \ldots, D'_q \), such that \( \xi_1 \in D_1 \subset U_1 \subset P_1 \setminus D'_1 \). Moreover, there exist independent elements \( \alpha_1, \ldots, \alpha_q \) of \( \Gamma_1 \) such that \( \alpha_i \cdot (P_1 \setminus D'_i) \subset D_i \).

For a word \( \gamma_1 \) of large length in \( \alpha_1, \ldots, \alpha_q \), we will consider the point \( a(\gamma_1) = (\gamma_1 \cdot a_1, a_2, \ldots, a_m) \) of \( U_1 \times \cdots \times U_m \) and its image \( \varphi(a(\gamma_1)) \) under the section \( \varphi \).

We set \( \gamma_j = \cdots = \gamma_m = 1 \). For \( j > m \), either \( \varphi_j \) is constant, and we set \( \gamma_j = 1 \in \Gamma_j \), or \( \varphi_j(a(\gamma)) \) takes only values in \( \Omega \). By proposition 5.8, (3), there exists \( \gamma_j \in \Gamma_j \) such that \( \gamma_j \cdot \varphi_j(a(\gamma_1)) \in \mathfrak{F}_j \) and

\[
H(\gamma_j) \ll d(\varphi_j(a(\gamma_1)), P_1(Q_p))^{-\kappa},
\]

where \( \kappa \) is a positive real number, independent of \( \gamma_1 \). By lemma 6.4, one has \( d(\varphi_j(a(\gamma_1)), P_1(Q_p)) \gg d(\gamma_1 \cdot a_1, P_1(Q_p))^m \), for some positive integer \( m \). By §5.2, one has inequalities \( d(\alpha_i \cdot a_1, P_1(Q_p)) \gg d(a_1, P_1(Q_p)) \), uniformly in \( a_1 \), for every \( j \in \{1, \ldots, q\} \). If \( \ell(\gamma_1) \) denotes the length of \( \gamma_1 \) as a word in \( \alpha_1, \ldots, \alpha_q \), this implies that

\[
H(\gamma_j) \ll d(\gamma_1 \cdot a_1, P_1(Q_p))^{-mk} \ll c^{\ell(\gamma_1)},
\]

where \( c \) is a positive real number, independent of \( \gamma_1 \). Enlarging \( c \), we also assume that \( H(\gamma_1) \ll c^{\ell(\gamma_1)} \).

Let now \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \); by construction, \( \gamma \cdot W \cap \mathfrak{F} \cap j^{-1}(V) = \gamma \cdot W \cap \mathfrak{F} \) is of dimension \( m \), so that \( \gamma \in R \), and \( H(\gamma) \leq c^{\ell(\gamma_1)} \). The number of words of length \( \ell \) in \( \alpha_1, \ldots, \alpha_q \) is \( q^\ell \). Consequently, \( R(Q) \cap \Gamma \) contains at least \( q^\ell \) pairs of height \( \leq c^\ell \). This implies the lemma. \( \square \)

**Lemma 6.6.** — The stabilizer of \( W \) inside \( G_0 \cap \Gamma \) is infinite.
Proof. — By lemma 6.5, there exists a positive real number $c_1$ such that
\[ \text{Card}(R(Q;T) \cap \Gamma) \geq T^{c_1} \] for $T$ large enough. Since $R$ is subanalytic, we may apply the $p$-adic Pila-Wilkie theorem of [5], as stated in theorem 3.6. Let thus $s \in \mathbb{N}$, $c \in \mathbb{R}$, $e > 0$, and $B \in Q_p^n \times Q_p^e$ be a family of blocks such that for every $T > 1$, there exists a subset $\Sigma \in Q_p^s$ of cardinality $< cT^e$ such that $R(Q;T) \subset \bigcup_{\sigma \in \Sigma} B_\sigma$. Let also $t \in \mathbb{N}$ and $Z \subset Q_p^n \times Q_p^e$ be a semi-algebraic subset such that for every $\sigma \in Q_p^t$, there exists $\tau \in Q_p^t$ such that $B_\sigma \subset Z_\tau$ and $\dim(B_\sigma) = \dim(Z_\tau)$. Let $c'$ be an upper bound for the number irreducible components of the Zariski closures of the sets $Z_\tau$, for $\tau \in Q_p^t$.

By the pigeonhole principle, there exists $\sigma \in \Sigma$ such that
\[ \text{Card}(R(Q;T) \cap \Gamma \cap B_\sigma) \geq \frac{1}{c} T^{c_1-e}. \]
Moreover, the Zariski closure of $B_\sigma$ in $Q_p^n$ has at most $c'$ irreducible components. Consequently, we may choose such an irreducible component whose trace $M$ on $B_\sigma$ satisfies
\[ \text{Card}(R(Q;T) \cap \Gamma \cap M) \geq \frac{1}{cc'} T^{c_1-e}. \]
Observe that $M$ is the chosen irreducible component of $B_\sigma$.

Let $g \in M$. Let $W'$ be the Zariski closure of $g^{-1}M \cdot W$ in $A_k^n$. It is irreducible.

There exists a finite extension $F'$ of $F$ such that $W(F')$ is Zariski dense in $W_{F'}$. The Zariski closure of $g^{-1}M \cdot W(F')$ in $A_k^n$ is an irreducible component $W'_0$ of $W'_F$. By proposition 4.3, one has $(W'_0)^{an} \cap \Omega_{F'} \subset j^{-1}(V_{F'})$. Since $j$ is defined over $F$, this implies that $(W')^{an} \cap \Omega \subset j^{-1}(V)$.

The inclusions $W \subset W'$ and $(W')^{an} \cap \Omega \subset j^{-1}(V)$, the irreducibility of $W'$ and the maximality hypothesis on $W$ imply that $W' = W$. Consequently, $g^{-1}M$ stabilizes $W$.

We have shown that the stabilizer of $W$ inside $G_0 \cap \Gamma$ contains subsets of the form $R(Q;T) \cap \Gamma \cap M$, whose cardinality can be prescribed to be larger than any given integer. This concludes the proof of the lemma.

Proposition 6.7. — The subvariety $W$ is flat.

Proof. — We show that all components of the section $\varphi$ are either constant or given by homographies. Fix an integer $j$ such that $m < j \leq n$ and we consider $\varphi_j$ as a function of the first variable, all other variables being fixed.

For $2 \leq i \leq m$, fix an element $a_i \in \Omega \cap U_i$. Let us assume that the function $z \mapsto \varphi_j(z, a_2, \ldots, a_m)$ is not constant.

Recall that $Y$ is the Zariski closure of $W$ in $(P_1)^n$. By lemma 6.6, the number of irreducible components of $Y \cap \mathbb{C}$ is smaller than the cardinality of the stabilizer of $W$ inside $G_0 \cap \Gamma$. This furnishes functional equations of the form
\[ h_1 \varphi_j(g_1 z, a_2, \ldots, a_m) = h_2 \varphi_j(g_2 z, a_2, \ldots, a_m) \]
where $g_1 \neq g_2 \in \Gamma_1$, and $h_1, h_2 \in \Gamma_j$. 


Let \( g = g_2g_1^{-1} \) and \( h = h_2^{-1}h_1 \); then \( \varphi_j(gu,a_2,\ldots,a_m) = h\varphi_j(u,a_2,\ldots,a_m) \). Consequently, the Schwarzian derivative \( S\varphi_j \) of \( \varphi_j \) satisfies \( S\varphi_j(gu,a_2,\ldots,a_m) = S\varphi_j(u,a_2,\ldots,a_m) \), a relation which is valid for \( u \in g_1^{-1}U_1 \).

Since \( \varphi_j \) is algebraic, so is \( S\varphi_j \). By construction, \( g \) is a non-trivial element of the Schottky group \( \Gamma_1 \), hence \( g \) is not of finite order. Consequently, \( S\varphi_j \) is constant.

We now apply a remark of Nevanlinna ([17], p. 344–345) according to which the function \( \psi = (\varphi'_j)^{-1/2} \) satisfies the differential linear equation \( \psi'' + \frac{1}{2}(S\varphi_j)\psi = 0 \). If \( S\varphi_j \) were a non-zero constant, then the function \( \psi = 0 \) would be the only algebraic solution of this equation. This shows that \( S\varphi_j = 0 \).

This proves that for every \( a_2,\ldots,a_m \), the function \( z_1 \mapsto \varphi_j(z_1,a_2,\ldots,a_m) \) is a homography. By construction, it is defined over the finite extension \( F \) of \( \mathbb{Q}_p \). If we represent it as a matrix, its coefficients will be analytic functions of \( a_2,\ldots,a_m \) which belong to \( F \); necessarily they are constant hence \( z_1 \mapsto \varphi_j(z_1,a_2,\ldots,a_m) \) does not depend on \( a_2,\ldots,a_m \).

Let now identify \( \varphi_{m+1},\ldots,\varphi_n \) with these homographies. The image of the map from \((P_1)^m \) to \((P_1)^n \) given by \( (z_1,\ldots,z_m) \mapsto (z_1,\ldots,z_m,\varphi_{m+1}(z_1),\ldots,\varphi_n(z_1)) \) is then contained in \( Y \). Since \( Y \) is irreducible and \( m \)-dimensional, this concludes the proof. \( \square \)

**Proposition 6.8.** — The subvariety \( W \) is geodesic.

**Proof.** — According to proposition 6.7, we know that \( W \) is flat. Up to selecting those coordinates which move on \( W \) and to reordering them, we assume that for every \( j \in \{m+1,\ldots,n\} \), there exists an integer \( i_j \in \{1,\ldots,m\} \) and \( g_j \in \text{PGL}(2,\mathbb{Q}_p) \) such that

\[
W = \{(z_1,\ldots,z_m,g_{m+1} \cdot z_{m+1},\ldots,g_n \cdot z_n) ; z_1,\ldots,z_m \in \Omega \}.
\]

To establish the proposition, it remains to prove that for every \( j \in \{m+1,\ldots,n\} \), the groups \( \Gamma_j \) and \( g_j\Gamma_i g_j^{-1} \) are commensurable, a property which is equivalent to the finiteness of \( \Gamma_j \setminus \Gamma_i g_j \Gamma_i \).

Let thus argue by contradiction; let \( j \in \{m+1,\ldots,n\} \) and let \( i = i_j \) be such that \( \Gamma_j \setminus \Gamma_i g_j \Gamma_i \) is infinite. For simplicity of notation, we assume that \( j = m+1 \) and \( i = 1 \). Since \( \Gamma \setminus W \subset V \), the algebraic variety \( V \) contains the pairwise distinct points of the form

\[
(q_1(z_1),\ldots,q_m(z_m), q_{m+1}(g_{m+1} \cdot z_{m+1}), q_{m+1}(g_{m+2} \cdot z_{m+2}), \ldots, q_n(g_n \cdot z_n)) ,
\]

where \( \gamma \) ranges over an infinite set of representatives of \( \Gamma_j \setminus \Gamma_i g_j \Gamma_i \). Then \( g_{m+1} \gamma z_1 \) ranges over an infinite set modulo \( \Gamma_{m+1} \), so that the Zariski closure of this set contains all points of the form

\[
(q_1(z_1),\ldots,q_m(z_m), q_{m+1}(z_{m+1}), q_{m+2}(g_{m+2} \cdot z_{m+2}), \ldots, q_n(g_n \cdot z_n)) .
\]

In particular, this latter set is contained in \( V \) as well. Consequently, \( j^{-1}(V) \) contains the flat subvariety of dimension \( m+1 \) defined by the equations \( z_j = g_j \cdot z_j \), for \( j > m+1 \), and this contradicts the assumption that \( W \) is a maximal connected algebraic subvariety of \( j^{-1}(V) \). \( \square \)
References


