NASH STRUCTURE OF CURVES OVER A VALUED FIELD AND TAME HENSELIAN RATIONALITY

by

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To the memory of Zoé Chatzidakis

Abstract. — We provide a new proof, model-theoretic and geometric in nature, of the tame henselian rationality theorem of Kuhlmann. The proof takes place within the framework of stable completions of algebraic varieties over a valued field. It relies on the proof of two main results of independent interest: a Nash structure theorem for the stable completion of an algebraic curve and a tame descent theorem for abstract polydiscs.

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0. Introduction

Motivation. — The initial goal of this work was to answer a question asked by Franziska Jahnke and Franz-Viktor Kuhlmann to the second author after a series of talks he gave in Singapore during Summer 2025. Among various results, he had presented a new proof, written down in [**DHLY24**, Appendix], of a classical result of valuation theory due to Kuhlmann [**Kuh10**]: *every Abhyankar extension of a defectless valued field is still defectless.* This new proof relies on model theory, and more precisely on the theory of *stable completions* (the model-theoretic version of Berkovich geometry) developped by Hrushovski and the second author [**HL16**] for studying the homotopy type of algebraic Berkovich spaces. The question of Jahnke and Kuhlmann was whether one can use the framework of stable completions for getting an alternative proof of another important result of valuation theory, namely the *tame henselian rationality theorem* of Kuhlmann [**Kuh19**].

Let us first say a few words about this theorem. A valued field is called tame if it is henselian and all its finite extensions are tamely ramified; this amounts to requiring that it be defectless with perfect residue field and p-divisible value group, for p the residue characteristic exponent; every henselian valued field with residue characteristic zero is tame. Tame henselian fields were first introduced and studied by Kuhlmann in [**Kuh16**] where he initiated the investigation of their algebraic and model theoretic properties, showing in particular an Ax-Kochen-Ershov principle for this class of fields. More recently, tame fields have played a crucial role in the groundbreaking work of Jahnke and Kartas on the model theory of perfectoid fields and tilting equivalence [**JK25**], see also [**Ans26**], the reason for this being that a continuous non-principal ultraproduct of perfectoid fields is a tame field. Both [**Kuh16**] and [**JK25**] use in a crucial way a certain relative embedding property, which in turn relies on the following tame henselian rationality theorem from Kuhlmann [**Kuh19**], that can be stated as follows:

Let L be a finitely generated extension of transcendence degree 1 of a tame valued field K, equipped with an immediate extension v of the structure valuation of K. Then there exists an element t of L transcendental over K such that the henselizations of $K(t)$ and L with respect to v are equal.

Let us mention that even the case where K is algebraically closed is difficult (and it was handled by Kuhlmann as a first step toward the general statement).

Our results. — In this paper, we are indeed able to use the formalism of stable completions to provide a new proof of the tame henselian rationality theorem (Theorem 10.4) with a model-theoretic and geometric flavour. In our approach, this theorem essentially relies on two results:

- ◇ A "Nash structure theorem" for the stable completion $\widehat{X}$ of a K-algebraic curve X (Theorem 9.15). This structure theorem is an analogue of a structure theorem for Berkovich analytic curves, which is equivalent to the semi-stable reduction theorem and can be stated and proved in several ways; the presentation we mimic here is the one provided by the first author in [**Duc24**].
- ◇ A tame descent theorem for "abstract polydiscs" (Theorem 6.2), which we use only in dimension 1 and which enables us to deduce the tame case from the algebraically closed one. This descent theorem is an analogue of the tame descent theorem for analytic polydiscs proved by the first author in [**Duc13**].

Let us say a few words about the reference to Nash. The theory of stable completions provides a very robust model-theoretic avatar of the underlying topological space of a Berkovich space, but it lacks a substitute for the analytic structure. And we definitely needed such a substitute: it was both technically essential for adapting the proof of the descent theorem of [**Duc13**] to our setting and conceptually helpful for stating, proving and understanding our results on the structure of stable completions of curves. This substitute is introduced in section 4, where we define *Nash isomorphisms* (Definition 4.8) and *Nash functions* (4.11.2); we have chosen this terminology since our definitions are reminiscent of those of Nash isomorphisms and Nash functions in the theory of real closed fields, in which they also are a good susbtitute for real-analytic isomorphisms and real-analytic functions.

Structure of this manuscript. — Section 1 is devoted to stating the conventions we use and to providing some basic reminders on the model theory of valued fields, with an emphasis on the sort RV which is used systematically throughout this article. We have chosen to use the denomination RV (which is standard in model theory) for the sake of consistency with the general viewpoint of this text, but we could have used equivalently the more algebraic language of *graded reduction* in the sense of Temkin [**Tem04**]; for similar reasons we borrow the terminology of *generalized varieties* from [**HK06**] but we could have called them *graded schemes* instead.

In Sections 2 and 3 we describe and investigate the structure of the stable completions of curves and of related spaces: the stable completion $\widehat{X}$ of a curve X over a valued field is the analogue of its Berkovich analytification, and we also consider the analogue $\widetilde{X}$ of its Huber analytification (mainly as an auxiliary tool for understanding $\widehat{X}$). Section 2 is essentially a recollection of statements from [**HL16**], the main one being the existence for any curve X of an *admissible K-skeleton* Σ on $\widehat{X}$, that is, a K-skeleton (i.e., a K-definable finite graph) which is the target of a K-definable deformation retraction of $\widehat{X}$. In Section 3 some new results are established concerning the complement in $\widehat{X}$ of a closed skeleton Σ: we prove that $\pi_0(\widehat{X}\backslash\Sigma)$ is definable and that each connected component of $\widehat{X}\backslash\Sigma$ is the stable completion of a definable subset of X (Proposition 3.17). The most important case, to which the general one reduces, and for which one has a very nice description of the set of components and of the components themselves (using $\widetilde{X}$) is when Σ is an admissible skeleton containing all branches at its simple points (Definition 3.5, Lemma 3.9 and 3.12.2).

In Section 4 we introduce the notion of Nash isomorphisms and Nash functions. Let X be a K-algebraic variety and let U be a k-definable subset of X. If Y is another K-algebraic variety and V is a K-definable subset of Y, a K-algebraic map $f\colon Y\to X$ is said to induce a Nash isomorphism between V and U it V is contained in the étale locus of f, if f induces a bijection $V\simeq U$, and if the induced map $\widehat{V}\to\widehat{U}$ is a homeomorphism. If this is the case any K-regular function on Y gives rise via f to a K-definable function on U, and K-definable functions on U arising this way for some (Y,V,f) as above are called Nash functions; we denote by $K\langle U\rangle$ the algebra of such functions. Under a boundedness assumption on U we can also define an RV-reduction $\mathsf{RV}(K)\langle U\rangle$ of the algebra of K-definable Nash function on U; this is a (generalized) algebra over $\mathsf{RV}(K)$. We show that when the ground field K is henselian $L\langle U\rangle = L\otimes_K K\langle U\rangle$ for every finite extension L of K (Lemma 4.12) and that $\mathsf{RV}(L)\langle U\rangle = \mathsf{RV}(L) \otimes_{\mathsf{RV}(K)} \mathsf{RV}(K)\langle U\rangle$ if L is moreover assumed to be tamely ramified over K (Corollary 4.14); this latter fact relies heavily on the characterization of tame ramification in terms of RV [**Duc13**].

In section 5 we assume that K is henselian and we study $K\langle U\rangle$ for U a K-definable abstract polydisc, that is, a K-definable subset of a variety Nash-isomorphic through some n-uple of K-regular functions $(f_1, \ldots, f_n)$ to an open polydisc centered at the origin, say of polyradius r. Any Nash function on U has a well-defined *formal* Taylor development $\sum a_I f^I$ at the unique point u of U at which all f_i vanish (because Nash isomorphisms preserve the henselizations of local rings, and a fortiori their completions), enabling us to see elements of $K\langle U\rangle$ as formal power series in the f_i. But be aware that the valued fields we deal with are not assumed to be complete in any sense, and can be of arbitrary height, so that there is no general theory here allowing for the evaluation of arbitrary power series in f at points of U, even under suitable growth conditions on the coefficients. We are nonetheless able to prove that Nash functions on U viewed as power series behave as expected (but this requires some work!). For instance we prove that if $g = \sum a_I f^I$ is such a function, if s is some polyradius $< r$ and if and if u_s denotes the unique pre-image under f of the Gauss point η_s, then

$$\sup_{\{|f| \leqslant s\}} |g| = |g(u_s)| = \max_I |a_I| \cdot s^I,$$

this maximum being achieved only for finitely many indices (Theorem 5.10); Corollary 5.11 of this theorem then asserts that $\sup_U |g| = \max_I |a_I| \cdot r^I$ (but here the maximum can be achieved for infinitely many indices). Theorem 5.10 is proved by induction on dimension, so the core is the proof in dimension 1, which uses the topology of curves investigated in Sections 2 and 3. And we also prove that if $n \geqslant 1$ and J denotes the set of multi-expoents I with last coordinate 0 (otherwise said the set of those I such that f_n does not occur in the monomial f^I) then $\sum_{I \in J} a_I f^I$ is (the Taylor developement of) a Nash function h on U which can be described as the composition of a "projection" onto the locus $\{f_n = 0\}$ and of a Nash function on the latter (Proposition 5.6). We warn the reader of the following: by definition of a Nash function, g is defined on some variety Y equipped with an étale map to X, but for defining h a further étale refinement $Z \to Y$ is likely unavoidable in general, see Remark 5.9.

In Section 6 we still assume that K is henselian and describe $\mathsf{RV}(K)\langle U\rangle$ for U a K-definable abstract polydisc, relying upon the results on $K\langle U\rangle$ proved in Section 5. If $(f_1, \ldots, f_n)$ are as above, we show that $\mathsf{RV}(K)\langle U\rangle$ is a local (generalized) sub-algebra of the (generalized) ring of (generalized) formal power series $\mathsf{RV}[\![\mathsf{rv}(f_1), \ldots, \mathsf{rv}(f_n)]\!]$ and that a family $(g_1, \ldots, g_n)$ of Nash functions on U induces a Nash isomorphism with an open polydisc if and only if the $\mathsf{rv}(g_i)$ generate the maximal ideal of $\mathsf{RV}(K)\langle U\rangle$ (Theorem 6.1). We can then prove our tame descent theorem for open polydiscs (Theorem 6.2): if L is a tamely ramified finite extension of K and if U is a K-definable subset of a K-variety X such that there exists $(g_1, \ldots, g_n) \in L\langle U\rangle$ inducing a Nash isomorphism between U and an open polydisc, then there exist $(f_1, \ldots, f_n) \in K\langle U\rangle$ inducing a Nash isomorphism between U and an open polydisc; the proof of this Theorem is completely parallel to that of [**Duc13**, Théorème 3.5]. Let us mention that if $n = 1$ one can do slightly better: in this case if g_1 is an L-regular function on some L-Zariski open neighborhood of U in X, one can choose f_1 in the ring of K-regular functions on some K-Zariski open neighborhood of U in X (no serious étale refinement is needed in this case).

The purpose of Section 7 is to construct "generic" abstract open polydiscs. More precisely we assume that the ground field K is algebraically closed. We start from a K-variety (say, integral) of dimension n and a K-Zariski generic point x of X such that the valued field $K(x)$ is Abhyankar over K. Let $f_1, \ldots, f_m$ be non-zero K-rational functions on X such that the $\mathsf{rv}(f_i(x))$ generate $\mathsf{RV}(K(x))$. By generic smoothness it is of course very easy to exhibit a $K(x)$-definable abstract open polydisc in X containing x, but we get a better and more useful result: indeed, we prove (Theorem 7.3) that there exists an abstract open polydisc $U \subset X$ containing x which is definable over $K \cup \{\mathsf{rv}(f(x))\}$ and which is moreover *atomic* (i.e., it has no proper non-empty definable subsets) over this set of parameters. The proof relies on generic smoothness but only at the level of the residue field, and makes a crucial use of the fact that if c is a point of RV^n with at least one coordinate transcendental over $\mathsf{RV}(K)$, the open polydisc $\mathsf{rv}^{-1}(c) \subset \mathbf{A}^n$ is atomic over $K \cup \{c\}$ and even over its model-theoretic algebraic closure ([**HK06**]; see the comments in 7.3.3).

Sections 8 and 9 are devoted to the Nash structure of the stable completion $\widehat{X}$ of a K-algebraic curve X. We prove more precisely (Theorem 9.15) that $\widehat{X}$ admits a K-*triangulation*. This is essentially a finite K-definable subset $\mathscr{V}$ such that there exists a finite set E of $\overline{K}$-regular functions (each of them being defined on a $\overline{K}$-Zariski open subset of X) fulfilling the following property: every connected component of $\widehat{X}\backslash\mathscr{V}$ is of the form $\widehat{U}$ for some definable subset U of X which is Nash-isomorphic via some $f \in E$ to an open disc or an open annulus, the second case occuring only for finitely many components (if the component U is Nash-isomorphic to an annulus, we also require that its two orientations be individually inter-definable with U; otherwise said if U is defined over some finite extension L of K, so are its two orientations). Such a triangulation gives rise to an admissible K-skeleton Σ, obtained by taking the union of $\mathscr{V}$ and of the skeletons of the finitely many connected components of $\widehat{X}\backslash\Sigma$ which are abstract annuli; then every connected component of $\widehat{X}\backslash\Sigma$ is of the form $\widehat{U}$ for U a definable subset of X which is Nash-isomorphic to an open disc via some $f \in E$ (we call such skeletons *Nash-admissible*). The proof of Theorem 9.15 reduces through easy Galois descent to the case where K is algebraically closed, in which case it is deduced from a first structure theorem (Theorem 8.5) asserting that $\widehat{X}$ can be covered by the stable completions of finitely many *nice charts* (Definition 8.1), which itself relies in an essential way on the existence of atomic open (poly)discs proved in Section 7. Let us mention that the existence of triangulations in this sense and in this setting is the exact analogue of the existence of triangulations on Berkovich analytic curves [**Duc24**, Théorème 5.1.14].

Section 10 is devoted to some applications of our former results. The first one (Theorem 10.1) still concerns the structure of curves: we deduce from the existence of triangulations that the pro-definable set $\widetilde{X}$ is actually definable, and also that if Σ is a closed skeleton of $\widehat{X}$, the definable set $\pi_0(\widehat{X}\backslash\Sigma)$ is RV-internal. The second one explains how our theorem on the existence of triangulations implies the corresponding theorem in the Berkovich setting, and the semi-stable reduction theorem over arbitrary height 1 valuation rings (Theorem 10.3). The third one is the tame henselian rationality Theorem (Theorem 10.4), which is proved as follows. If L is an immediate one-dimensional finitely generated extension of a tame valued field K, we write $L = K(x)$ where x is some K-Zariski generic point of a K-curve X. Choose a K-triangulation on $\widehat{X}$ and denote by Σ the corresponding Nash-admissible skeleton. We view x as a simple point of $\widehat{X}$; it then belongs to a connected component of $\widehat{X}\backslash\Sigma$, which is of the form $\widehat{U}$ for some definable subset U of X. As the set of connected components of $\widehat{X}\backslash\Sigma$ is RV-internal, $\widehat{U}$ is definable over $\overline{K}\mathsf{RV}(K(x)))$, but the fact that K is tame and that generalized residue fields behave well with respect to tamely ramified extensions enables us to get rid of $\overline{K}$ here, so $\widehat{U}$ is definable over $\mathsf{RV}(K(x)) = \mathsf{RV}(K)$. Since Σ is Nash-admissible there exists a Nash isomorphism between the K-definable set U and an open disc which is induced by an F-regular function on an F-Zariski open neigborhood of U for some finite extension F of K. As K is tame, F is tamely ramified over K, so by our tame descent theorem for abstract open discs (and more precisely by its stronger version specific to dimension 1) we can assume that $F = K$; then f induces an isomorphism between the henselizations of $K(f(x))$ and $K(x)$, and we are done.

Let us end this description of our work by a remark: Sections 4 to 7 go far beyond what was strictly needed for the structure of curves (Sections 8 and 9) and tame henselian rationality (Section 10.4), if only because they apply to varieties of arbitrary dimension. But we hope that the resuts therein are intrinsically interesting, and that they will be useful for future works in geometry over valued fields.

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We dedicate this paper to the memory of our friend and colleague Zoé Chatzidakis, who passed away on January 22, 2025. She was always keen on seeing new applications of model theory to algebra and geometry. We hope that this paper will serve as a fitting, if modest, tribute to her memory.

1. Model theory of valued fields: reminders and basic facts

1.1. — Throughout this paper we shall use the multiplicative notation for valued fields: a valuation on a field F is a map $|\cdot|\colon F \to G_0$ where G is a multiplicative ordered abelian group (with unit element 1) and $G_0 = G \cup \{0\}$ is the monoid obtained by formally adjoining to G a smallest absorbing element.

1.2. — We will use the multi-sorted language of valued fields introduced in [**HHM06**]. Besides the three standard sorts, namely, the valued field itself, the residue field and the value group, it encompasses a countable family of so-called *geometric sorts*. We will not use the latter explicitly but adjoining these geometric sorts ensures that the theory ACVF of algebraically closed (non-trivially) valued fields eliminates imaginaries [**HHM06**, Theorem 3.4.10], so that objects which are merely interpretable in the classical setting will be definable, like RV (see below) or the *stable completion* of an algebraic curve defined by Hrushovski and the second author in their work [**HL16**].

1.3. — In this text, definable "sets" will be considered as functors on the category whose objects are models of ACVF and whose arrows are isometric embeddings, and definable "maps" as natural transformations. Basic examples will be the value group Γ that maps F to $|F^\times|$, its variant $\Gamma_0 = \Gamma \cup \{0\}$ that maps F to $|F|$, and the residue field k that maps F to $\{z \in F, |z| \leqslant 1\}/\{z \in F, |F| < 1\}$.

We shall also need the functor

$$\mathsf{RV}\colon F \longmapsto \coprod_{\gamma \in |F^\times|} \{z \in F, |z| \leqslant \gamma\}/\{z \in F, |z| < \gamma\}.$$

Note that $\mathsf{RV}(F)$ inherits a multiplication, as well as a partially defined addition: one can only add up two terms belonging to the same summand, and each summand is an abelian group (with its own zero-element); the summand of degree one is nothing but the residue field. Therefore $\mathsf{RV}(F)$ appears as kind of a a generalized ring, and even as a generalized field since 1 is non-zero and its group $\mathsf{RV}^\times(F)$ of invertible elements coincides with its subset of non-zero elements.

The image of an element $x \in F^\times$ in the group $\mathsf{RV}^\times(F)$ will be denoted by $\mathsf{rv}(x)$. One has a natural exact sequence

$$1 \longrightarrow \mathsf{k}^\times \longrightarrow \mathsf{RV}^\times \xrightarrow{\quad |\cdot| \quad} \Gamma \longrightarrow 1 \ .$$

Most notions of commutative algebra have straighforward analogues to this context of generalized rings (or fields). We will use them freely, and refer the reader to the first section of [**Duc21**] for more details. For instance there is a notion of a generalized polynomial ring over $\mathsf{RV}(F)$, but the indeterminates have to be assigned a degree. This gives rise to a theory of generalized algebraic extensions, transcendence degree, etc.

1.4. — Let K be a valued field. A K-variety X (i.e., a separated K-scheme of finite type) will be here mainly considered as a functor on the category of models of ACVF "containing" K (i.e., equipped with an isometric embedding from K), or sometimes even implicitly containing a given valued extension of K. By a point of X we will mean an element of $X(F)$ for such a model F, which will in general not be specified, and could be enlarged if needed. (The reader who likes the model-theoretic version of Weil's viewpoint might also consider that one has fixed a huge "monster model"

$\mathscr{U}$ of ACVF, that all objects considered in this paper are embedded in $\mathscr{U}$, and that points of X are simply elements of $X(\mathscr{U})$.) If we need to consider scheme-theoretic points or types in ACVF rather than "naive" points living on some model, we will say so explicitly.

The ring of regular functions on the K-scheme X will usually be denoted by $K[X]$; if the scheme X is integral, its field of rational functions will be denoted by $K(X)$.

1.5. — For seing definable sets as functors one needs in general to restrict the evaluation to models of ACVF, to ensure that one only deals with elementary embedding of valued fields. But definable sets coming with a canonical quantifier-free description can be seen as functors on the category of all valued fields, or valued extensions of a given ground field. For instance k, Γ and RV extend to the category of all valued fields, and if X is a K-variety, it can be seen as a functor on the category of all valued extensions of K.

1.6. — In general, our sets of parameters for definable subsets of a K-variety X will consist of elements of one of the following sorts: the ground field, the value group or RV (remind that the latter contains the residue field). This means that such a set A will be contained in $F \coprod |F| \coprod \mathsf{RV}(F)$ for some (not necessarily specified) valued extension F of K, and we shall say for short that a valued extension M of K contains A if it contains the smallest subfield E of F containing K and such that $A \subset E \coprod |E| \coprod \mathsf{RV}(E)$. For such an A, an A-definable subset D of X will be understood as a functor on the category of algebraically closed, non-trivially valued extensions of K containing A – or on all valued extensions of K containing A if D admits a canonical quantifier-free description.

An A-definable subset E of X will be called *A-atomic* if the only A-definable subsets of E are $\varnothing$ and E.

1.7. — Let K be a valued field and let F be a valued extension of K. Then $\mathsf{RV}(F)$ is a (generalized) field extension of $\mathsf{RV}(K)$ and

$$[\mathsf{RV}(F) : \mathsf{RV}(K)] = (|F^\times| : |K^\times|) \cdot [\mathsf{k}(F) : \mathsf{k}(K)],$$

and F is an immediate extension of K if and only if $\mathsf{RV}(F) = \mathsf{RV}(K)$. If (a_i) is a family of elements of $F^\times$, the $\mathsf{rv}(a_i)$ are linearly independent over $\mathsf{RV}(K)$ if and only if $|\sum \lambda_i a_i| = \sup_i |\lambda_i| \cdot |a_i|$ for all (finitely supported) families (λ_i) of elements of K. If this is the case, the a_i are linearly independent over K. We thus see that $[\mathsf{RV}(F) : \mathsf{RV}(K)] \leqslant [F : K]$.

Now denote by L a finite extension of K and by $L_1, \ldots, L_r$ all the valued extensions of K with underlying field L. By the above we have for all i the inequality

$$[\mathsf{RV}(L_i) : \mathsf{RV}(K)] = [\mathsf{RV}(L_i^{\mathrm{h}}) : \mathsf{RV}(K^{\mathrm{h}})] \leqslant [L_i^{\mathrm{h}} : K^{\mathrm{h}}]$$

where $(\cdot)^{\mathrm{h}}$ stands for the henselization. It follows that

$$\sum_i [\mathsf{RV}(L_i) : \mathsf{RV}(K)] \leqslant \sum_i [L_i^{\mathrm{h}} : K^{\mathrm{h}}] = [(L \otimes_K K^{\mathrm{h}}) : K^{\mathrm{h}}] = [L : K].$$

The extension $K \hookrightarrow L$ is called *defectless* if $\sum_i [\mathsf{RV}(L_i) : \mathsf{RV}(K)] = [L : K]$, and a valued field is called defectless [(1)] if all its finite extensions are defectless. Algebraically closed valued fields, valued fields with residue characteristic zero, complete discretely valued fields, function fields of normal irreducible varieties equipped with the discrete valuation associated to a divisor, are defectless. A valued field is defectless if and only if its henselization is defectless.

1.8. — Let $K \hookrightarrow F$ be an extension of valued field. If (a_i) is a family of elements of $F^\times$, the $\mathsf{rv}(a_i)$ are algebraically independent over $\mathsf{RV}(K)$ if and only if $|\sum \lambda_I a^I| = \sup_i |\lambda_I| \cdot |a|^I$ for all polynomials $\sum \lambda_I T^I$ in the multi-variable $T = (T_i)$. If this is the case, the a_i are algebraically independent over K. We thus see that $\operatorname{trdeg}(\mathsf{RV}(F)/\mathsf{RV}(K)) \leqslant \operatorname{trdeg}(F/K)$. We say that F is an *Abhyankar extension* of K if it is finitely generated over K and $\operatorname{trdeg}(\mathsf{RV}(F)/\mathsf{RV}(K)) = \operatorname{trdeg}(F/K)$. If this is the case, an *Abhyankar basis* of F over K will be any family (a_i) of elements of $F^\times$ such that the $\mathsf{rv}(a_i)$ are a transcendence basis of $\mathsf{RV}(F)$ over $\mathsf{RV}(K)$. Otherwise said, this is a family (a_i) of elements of $F^\times$ whose cardinality is equal to $\operatorname{trdeg}(F/K)$ and such that $|\sum \lambda_I a^I| = \sup_i |\lambda_I| \cdot |a|^I$ for all polynomials $\sum \lambda_I T^I$.

An important result of valuation theory, which plays a key role in this manuscript, asserts that if K is a defectless field, every Abhyankar valued extension of K is still defectless. Its first proof was given by Franz-Viktor Kuhlmann [**Kuh10**]; a purely model-theoretic proof can be found in the Appendix of [**DHLY24**].

1.9. Example. — Let K be a valued field, let G be an ordered abelian group containing $|K^\times|$, and let $g = (g_1, \dots, g_n) \in G^n$. Let η_g be the valuation on $K(T_1, \dots, T_n)$ that maps $\sum a_I T^I$ to $\max |a_I| g^I$. Then $(K(T), \eta_g)$ is an Abhyankar extension of K, and $\mathsf{RV}(K(T), \eta_g)$ is the rational function field over $\mathsf{RV}(K)$ generated by he $\mathsf{rv}(T_i)$, which are algebraically independent over K. We call η_g the *Gauss valuation* with parameter g.

1.10. Generalized varieties. — We shall need the notion of a *generalized algebraic variety*, see [**HK06**, §4.1]; this is essentially the RV (or graded) version of a classical variery over the residue field. In particular for every finite tuple $\gamma = (\gamma_1, \dots, \gamma_e)$ of elements of Γ one can define the generalized affine space $\mathbf{A}^\gamma_{\mathsf{RV}}$; this is simply the γ-definable subset of RV^e consisting of those elements $(x_1, \dots, x_e)$ with each x_i of degree γ_i. There is a projective version of this construction: $\mathbf{P}^\gamma_{\mathsf{RV}}$ is the set of tuples $(x_0, \dots, x_e)$ of elements of RV with x_0 of degre 1 and x_i of degree γ_i for $i > 0$, modulo multiplication by $\mathsf{k}(K)^\times$. Note that if $\gamma = (1, \dots, 1)$ then $\mathbf{A}^\gamma_{\mathsf{RV}}$ is nothing but the classical affine space $\mathbf{A}^e_{\mathsf{k}}$ over the residue field; more generally if $\gamma_i = |\lambda_i|$ for some family λ_i of elements of a valued field K then $\mathbf{A}^\gamma_{\mathsf{RV}}$ is K-definably isomorphic to $\mathbf{A}^e_{\mathsf{k}}$ using the multiplication by $(\mathsf{rv}(\lambda_i^{-1})_i$.

Using the notion of a generalized polynomial we can define generalized algebraic functions from a generalized algebraic variety to RV.

1. Such valued fields are also often called *stable*, but to avoid any confusion with the model-theoretic notion of stability, we prefer to use “defectless” in this article.

1.11. Stable completions. — We will use freely the theory of *stable completions*, developped by Hrushovski and the second author in [**HL16**]. We will need it only for curves, for which it is far simpler than for higher dimensional varieties: indeed if K is a valued field and V is a K-definable subset of some algebraic K-variety, the stable completion $\widehat{V}$ (which is a functor on the category of all models of ACVF containing K) is K-definable as soon as $\dim V \leqslant 1$, while it is merely pro-definable if $\dim V > 1$. More precisely, the references inside [**HL16**] are §3.1 and Theorem 3.1.1 for the definition and the (strict) pro-definability of $\widehat{V}$, Theorem 7.1.1 for the definability in dimension $\leqslant 1$ (which ultimately relies on Riemann-Roch Theorem), and Remark 7.1.3 for the fact that one cannot hope for definability in dimension > 1.

In fact $\widehat{V}$ is not only a pro-definable set but a pro-definable topological space: for every model M of ACVF containing K the set $\widehat{V}(M)$ is equipped with a natural topology admitting a basis of relatively M-definable open subsets, inducing the valuative topology on $V(M)$, and for which strict inequalities give rise to open subsets. It therefore makes sense to say that some pro-definable subset of $\widehat{V}$ is definably closed (resp. open, resp. compact, resp. connected). We shall also use the relative version of definable compactness, that of definable properness: a pro-definable map between pro-definable topological spaces will be called definably proper if the pre-image of every definably compact subset is definably compact.

Since we shall always use the definable variants of the usual topological notions in this setting, we will allow ourselves to omit "definably" when refering to them, so we will say compact for definably compact, proper for definably proper, etc.

2. Topology of curves: admissible skeletons

We fix a valued field K and an algebraic closure $\overline{K}$ of K, endowed with an extension of the valuation of K.

2.1. — Let X be an algebraic K-variety. We shall denote by $\check{X}$ the functor that maps a model M of ACVF containing $\overline{K}$ to the set of M-definable types over M on X. Note that $\check{X}$ contains $\widehat{X}$ by definition. More precisely, $\widehat{X}(M)$ is the set of $x \in \check{X}(M)$ such that for every M-Zariski open subset U of X with $x \in \check{U}(M)$ and every M-regular function f on U there exists $\lambda \in M$ with $|f(x)| = |\lambda|$. There is a natural intermediate set $\widetilde{X}$ between $\widehat{X}$ and $\check{X}$, that of *bounded* definable types: $\widetilde{X}(M)$ is the set of $x \in \check{X}(M)$ such that for every M-Zariski open subset U of X with $x \in \check{U}(M)$ and every M-regular function f on U there exists $\lambda \in M$ with $|f(x)| \leqslant |\lambda|$. While $X \mapsto \widehat{X}$ is a model-theoretic avatar of Berkovich's analytification, $X \mapsto \widetilde{X}$ is that of Huber's. It follows from [**CKHY26**, Theorem 7.4.3] that $\check{X}$ and $\widetilde{X}$ are strictly pro-definable.

2.1.1. — The embedding $\widehat{X} \subset \widetilde{X}$ admits a retraction ϖ. If one sees a point of $\check{X}(M)$ as a scheme-theoretic point ξ on the M-scheme X_M together with an M-valuation v on the field $M(\xi)$, then ϖ maps (ξ, v) to (ξ, v') where v' is the coarsening of v obtained by modding out $v(M(\xi)^\times)$ by its convex subgroup consisting of the elements infinitesimally close to 1 with respect to $|M^\times|$.

2.1.2. — The map ϖ is pro-definable. Indeed, this follows from the general “explicit” description of a set of uniformly definable types as a pro-definable set [**CKY21**, Prop. 4.1], and the following facts (for M a model of ACVF containing K):

- ◇ A type x on X over M (in ACVF) is entirely determined once one knows the set of all M-Zariski open subsets U of X such that x lies on U and, for every such U, the set of all M-regular functions f on U such that $|f(x)| \leqslant 1$ and that of all M-regular functions f on U such that $|f(x)| < 1$.
- ◇ Let U be an M-Zariski open subset of X and let x be a point of $\widetilde{X}(M)$ with image x' on $\widehat{X}(M)$. The point x lies on U if and only if x' lies on U. If this is the case then for every M-regular function f on U one has:
  - $|f(x')| \leqslant 1$ if and only if $|\lambda f(x)| < 1$ for all $\lambda \in M$ with $|\lambda| < 1$;
  - $|f(x')| < 1$ if and only if there exists $\lambda \in M$ with $|\lambda| > 1$ and $|\lambda f(x)| < 1$.

2.1.3. — If U is a definable subset of X, we shall denote by $\breve{U}, \widetilde{U}$ and $\widehat{U}$ the subsets of $\breve{X}, \widehat{X}$ respectively consisting of those types that lie on U. We warn the reader that there is no general inclusion relation between $\widetilde{U}$ and $\varpi^{-1}(\widehat{U})$ in $\widetilde{X}$. Indeed, let us take for X the affine line, for U the open unit ball $|T| < 1$ and for V its complement $|T| \geqslant 1$. Then using the notation introduced at the beginning of 2.2 below $\eta_{0,1^-}$ belongs to $\widetilde{U}$ but $\varpi(\eta_{0,1^-}) = \eta_{0,1}$ does not lie on $\widehat{U}$; so $\eta_{0,1^-}$ does not belong to $\widetilde{V}$ but $\varpi(\eta_{0,1^-})$ belongs to $\widehat{V}$.

2.2. The case of the projective line. — Our purpose is to show explicitly that the pro-definable sets $\breve{\mathbf{P}}^1$ and $\widetilde{\mathbf{P}^1}$ are actually definable, like $\widehat{\mathbf{P}^1}$. We fix a model M of ACVF containing K, we denote by R the valuation ring of M and by $\mathfrak{m}$ its maximal ideal, and we set $E = \{a + bT\}_{a,b \in M} \subset M[T]$.

Let a be an element of M. For every r in an ordered abelian group containing $|M^\times|$ we denote by $\eta_{a,r}$ the type over M lying on $\mathbf{P}^1$ and given by the Gauss norm $\sum a_i(T-a)^i \mapsto \max |a_i| r^i$ on $M[T]$. One has $\eta_{a,r} = \eta_{a,s}$ if and only if r and s have the same type over $|M^\times|$ and $|a-b| \leqslant r$. The type $\eta_{a,r}$ is M-definable if and only if the type of r over $|M^\times|$ is $|M^\times|$-definable. This occurs if and only if one (and only one) of the following hold:

- ◇ $r \in |M^\times|$;
- ◇ r is infinitesimally lower (with respect to $|M^\times|$) than some $\gamma \in |M^\times|$; then we also write η_{a,γ^-} for the type $\eta_{a,r}$;
- ◇ r is infinitesimally larger than some $\gamma \in |M^\times|$; then we also write η_{a,γ^+} for the type $\eta_{a,r}$;
- ◇ r is infinitely small; then we also write $\eta_{a,0^+}$ for the type $\eta_{a,r}$;
- ◇ r is infinitely large; then we also write η_{a,∞^-} for the type $\eta_{a,r}$ (note that η_{a,∞^-} is equal to η_{0,∞^-}).

2.2.1. *Reminders on* $\widehat{\mathbf{P}^1}$. — Let us first remind the explicit description of $\widehat{\mathbf{P}^1}$ as a definable set: there is a natural bijection from $\widehat{\mathbf{P}}^1(M)$ to the definable set Λ of M-definable sub-modules E of the form $R \oplus S(T-a)/\lambda$ with $S = R$ or $S = M$, $a \in M$ and $\lambda \in M^\times$ (for the definability of Λ to make sense, one needs to use the so-called geometric sorts, at least S_2). This natural bijection maps $x \in \widehat{\mathbf{P}^1}(M)$ to $L_x := \{f \in E, |f(x)| \leqslant 1\}$; the converse bijection maps L_x to $\eta_{a,|\lambda|}$ if L_x is a lattice and $(1,(T-a)/\lambda)$ is a basis of L_x, to the simple point given by the equation $T = a$ if $M(T-a) \subset L_x$ (this determines a uniquely) and to ∞ if $L_x = R$.

2.2.2. *First description of* $\breve{\mathbf{P}}^1$ *and* $\widetilde{\mathbf{P}^1}$. — An M-type x lying on $\mathbf{P}^1$ which does not belong to $\widehat{\mathbf{P}}^1(M)$ is either realized by some element of $M'\backslash M$ for M' an immediate extension of M, either of the form $\eta_{a,r}$ for some r in $G\backslash|M^\times|$ for some ordered group $G \supset |M^\times|$; but in the first case the type x is not definable (for the set of $f \in E$ such that $|f(x)| \leqslant 1$ is not definable). We thus see that $\breve{\mathbf{P}}^1(M)\backslash\widehat{\mathbf{P}^1}(M)$ consists of the types of the form $\eta_{a,\gamma^-}, \eta_{a,\gamma^+}, \eta_{a,0^+}$ and η_{0,∞^-} for $a \in M$ and $\gamma \in |M^\times|$; among these types, only those of the form η_{a,γ^-} and η_{a,γ^+} are bounded, so $\widetilde{\mathbf{P}^1}(M)\backslash\widehat{\mathbf{P}^1}(M)$ consists of all types of the form η_{a,γ^-} and η_{a,γ^+}.

2.2.3. *Definability of* $\breve{\mathbf{P}}^1$ *and* $\widetilde{\mathbf{P}^1}$. — Let $x \in \breve{\mathbf{P}}^1(M)$. Let L_x be the sub-R-module $\{f \in E, |\lambda f(x)| < 1 \text{ for all } \lambda \in M \text{ with } |\lambda| < 1\}$ of E. Let us describe L_x more explicitly.

- ◇ Assume that $x \in \widehat{\mathbf{P}^1}(M)$. Then L_x coincides with the module L_x defined in 2.2.1.
- ◇ If $x = \eta_{a,\gamma^-}$ or η_{a,γ^+} then $L_x = R \oplus R(T-a)/\lambda$ where λ is any element of M with $|\lambda| = \gamma$ (this is also the lattice associated to $\eta_{a,\gamma}$). In this case denote by L'_x the subset of L_x consisting of those f such that $|f(x)| < 1$ and $|\lambda f(x)| > 1$ for every $\lambda \in M$ with $|\lambda| > 1$. Then $L'_x = R(T-a)/\lambda + \mathfrak{m}_x L_x$ if $x = \eta_{a,\gamma^-}$, and $L'_x = \varnothing$ if $r = \eta_{a,\gamma^+}$.
- ◇ If $x = \eta_{a,0^+}$ then $L_x = R \oplus M(T-a)$ (this is also the lattice associated to the simple point $T = a$);
- ◇ If $x = \eta_{0,\infty^-}$ then $L_x = R$ (this is also the lattice associated to ∞).

The discussion above furnishes a pro-definable partition $\breve{\mathbf{P}}^1 = \widehat{\mathbf{P}^1} \coprod \Omega \coprod \Omega'$ where:

- ◇ $\Omega(M)$ is the set of types of the form η_{a,γ^-} or η_{a,γ^+}. It is in pro-definable bijection with the definable set of pairs (L,D) where L is a lattice of E admitting R as a direct summand and D is an element of $\mathbb{P}(L/\mathfrak{m}L)$ (note that for this set of pairs to be definable, one needs the geometric sorts S_2 and T_2). The bijection maps x to $(L_x, L'_x/\mathfrak{m}L_x)$ if $L'_x \neq \varnothing$ and to $(L_x,(R+\mathfrak{m}L_x)/\mathfrak{m}L_x)$ otherwise. The converse bijection maps a pair (L,D) to $\eta_{a,|\lambda|^-}$ if $D \neq (R+\mathfrak{m}L_x)/\mathfrak{m}L_x$ and $(T-a)/\lambda$ lifts a generator of D, and to $\eta_{a,|\lambda|^+}$ if $D = (R+\mathfrak{m}L_x)\mathfrak{m}L_x$ and $(1,(T-a)/\lambda)$ is a basis of L_x.
- ◇ $\Omega'(M)$ is the set of types of the form $\eta_{a,0^+}$ or η_{0,∞^-}. It is in pro-definable bijection with the definable set $\mathbf{P}^1(M)$ by the map $\eta_{a,0^+} \mapsto a$ and $\eta_{0,\infty^-} \mapsto \infty$.

It follows that the strictly pro-definable set $\breve{\mathbf{P}}^1$ is actually definable, and so is $\widetilde{\mathbf{P}^1} = \widehat{\mathbf{P}^1} \coprod \Omega$. The natural retraction from $\widetilde{\mathbf{P}^1}$ to $\widehat{\mathbf{P}^1}$ is then definable. Explicitly if $x \in \widehat{\mathbf{P}^1}(M)$ it maps x to x; and if $x \in \Omega(M)$ it maps x to the unique point x' of $\widehat{\mathbf{P}^1(M)}$ such that $L_{x'} = L_x$, which can be rephrased by saying that it maps $\eta_{a,\gamma^\pm}$ to $\eta_{a,\gamma}$.

2.2.4. *The* RV*-skeleton of* $\widetilde{\mathbf{P}^1}$. — We define $\Sigma(M)$ as the subset of $\widehat{\mathbf{P}^1}(M)$ consisting of the origin and all points of the form $\eta_{0,\gamma}$ with $\gamma \in |M^\times|$. The "skeleton" Σ is a definable subfunctor of $\widehat{\mathbf{P}^1}$ which is Γ_0-internal; in fact $\gamma \mapsto \eta_{0,\gamma}$ (with the convention that $\eta_{0,0} = 0$ and $\eta_{0,\infty=}\infty$) establishes a definable homeomorphism $[0, \infty] \simeq \Sigma$.

Let Σ_{RV} be the subset of $\widetilde{\mathbf{P}^1}$ consisting of points of the form $\eta_{a,|a|^-}$ where a lies in $\mathbf{G}_{\mathrm{m}} = \mathbf{P}^1 \backslash \{0, \infty\}$. Note that for a and b in $\mathbf{G}_{\mathrm{m}}$ we have $\eta_{a,|a|^-} = \eta_{b,|b|^-}$ if and only if $|a| = |b|$ and $|a - b| < |a|$, so we have a natural K-definable bijection $\Sigma_{\mathsf{RV}} \simeq \mathsf{RV}^\times$. Note also that if $s \in \Sigma\mathsf{RV}$ then s does not belong to $\widehat{\mathbf{P}^1}$ and $\varpi(s)$ belongs to Σ: if $s = \eta_{a,|a^-|}$ then $\varpi(s) = \eta_{a,|a|}$.

Let I be the interval $[0, 1]$. Let $\breve{I}$ be the set of all definable types on I (it contains I, 0^+, 1^-, and r^+ and r^- for all $0 < r < 1$) and $\widetilde{I}$ that of "bounded" such types, which is merely $\breve{I} \backslash \{0^+\}$.

Let $h \colon \mathbf{P}^1 \times [0, 1] \to \widehat{\mathbf{P}^1}$ be the K-definable map $(a, t) \mapsto= \eta_{a,t|a|}$. By applying h on realizations of types we obtain maps

$$\widehat{h} \colon \widehat{\mathbf{P}^1} \times I \to \widehat{\mathbf{P}^1},\ \widetilde{h} \colon \widetilde{\mathbf{P}^1} \times \widetilde{I} \to \widetilde{\mathbf{P}^1} \ \text{ and } \breve{h} \colon \breve{\mathbf{P}}^1 \times \breve{I} \to \breve{\mathbf{P}}^1.$$

The definable map $\widehat{h}$ is the unique continuous pro-definable extension of h to $\widehat{\mathbf{P}^1} \times I$ (see [**HL16**], Lemma 3.8.5) and this is by a direct computation a strong deformation retraction from $\widehat{\mathbf{P}^1}$ to Σ. We set $\rho(a) = h(a, 1)$ for every $a \in \mathbf{P}^1$, so $\rho(0) = 0, \rho(\infty) = \infty$ and $\rho(a) = \eta_{a,|a|}$ for $a \in \mathbf{G}_{\mathrm{m}}$. By applying ρ on realization of types we get maps $\widehat{\rho}$ (resp. $\widetilde{\rho}$, resp. $\breve{\rho}$) from $\widehat{\mathbf{P}^1}$ (resp. $\widetilde{\mathbf{P}^1}$, resp. $\breve{\mathbf{P}}^1$) to itself; one has $\widehat{\rho}(a) = \widehat{h}(a, 1)$ for all $a \in \widehat{\mathbf{P}^1}$, and the analogous formulas for $\widetilde{\mathbf{P}^1}$ and $\breve{\mathbf{P}}^1$. The image of $\widehat{\rho}$ is still equal to Σ; the image of $\tilde{\rho}$ is equal to $\widetilde{\Sigma} := S \cup \{\eta_{0,\gamma^-}, \eta_{0,\gamma^+}\}_{\gamma \in \Gamma}$ and thet of $\breve{\rho}$ is equal to $\breve{\Sigma} := \Sigma \cup \{\eta_{0,\gamma^-}, \eta_{0,\gamma^+}\}_{\gamma \in \Gamma} \cup \{\eta_{0,0^+}, \eta_{0,\infty^-}\}$.

There is another way to describe $\widehat{\rho}$: for every $x \in \widehat{\mathbf{P}^1}$ there is a unique (up to definable homeomorphism) continuous injective path $p \colon I \to \widehat{\mathbf{P}^1}$ where $I = [o, e]$ is a generalized interval in the sense of [**HL16**, 3.9] such that $p(o) = x, p(e) \in S$, and $p(t) \notin \Sigma$ for $t \neq e$; then $\widehat{\rho}(x) = e$.

Now let $\mathfrak{r}$ be the definable map from $\mathbf{P}^1$ to $\widetilde{\mathbf{P}^1}$ that maps a to $h(a, 1^-)$. Then $\mathfrak{r}(0) = 0$ and $\mathfrak{r}(a) = \eta_{a,|a|^-}$ for every $a \in \mathbf{G}_{\mathrm{m}}$. The image $\mathfrak{r}(\mathbf{G}_{\mathrm{m}}) = \mathfrak{r}(\mathbf{P}^1 \backslash \Sigma)$ is thus equal to Σ_{RV}; observe that modulo the aforementionned bijection $\Sigma_{\mathsf{RV}} \simeq \mathsf{RV}^\times$ the map $\mathfrak{r}$ maps any $a \in \mathbf{G}_{\mathrm{m}}$ to $\mathsf{rv}(a)$. If $s \in \Sigma_{\mathsf{RV}} \backslash \{0, \infty\}$ the fiber $\mathfrak{r}^{-1}(s)$ is an open disc; more precisely if $s = \eta_{a,|a|^-}$ then $\mathfrak{r}^{-1}(s)$ is the open disc with center a and radius $|a|$; this is also $\mathsf{rv}^{-1}(\mathsf{rv}(a))$.

By applying $\mathfrak{r}$ to realizations of types, we get maps

$$\widehat{\mathfrak{r}}\colon \widehat{\mathbf{P}^1} \to \widehat{\mathbf{P}^1}, \widetilde{\mathfrak{r}}\colon \widetilde{\mathbf{P}^1} \to \widetilde{\mathbf{P}^1}, \text{ and } \breve{\mathfrak{r}}\colon \breve{\mathbf{P}}^1 \to \breve{\mathbf{P}}^1.$$

Note that $\widehat{\mathfrak{r}}(\widehat{\mathbf{P}^1}\backslash\Sigma)$ is still equal to Σ_{RV}, but $\widehat{\mathfrak{r}}(\widehat{\mathbf{P}^1})$ is equal to $\Sigma_{\mathsf{RV}} \cup \Sigma$, while $\widetilde{\mathfrak{r}}(\widetilde{\mathbf{P}^1})$ and $\breve{\mathfrak{r}}(\breve{\mathbf{P}}^1)$ are respectively equal to $\Sigma_{\mathsf{RV}} \cup \widetilde{\Sigma}$ and $\Sigma_{\mathsf{RV}} \cup \breve{\Sigma}$.

For defining $\mathfrak{r}$ we have used an explicit formula involving a specific deformation retraction from $\widehat{\mathbf{P}^1}$ to Σ, but it could have been done in a more intrinsic way (though less convenient for seeing that $\mathfrak{r}$ is definable) analogous to the direct definition of ρ given above. Indeed let $x \in \mathbf{P}^1$ and let $p\colon [o,e] \to \widehat{\mathbf{P}^1}$ be the unique injective path (up to re-parametrization) such that $p(o) = x, p(e) \in \Sigma$, and $p(t) \notin \Sigma$ for $t \neq e$. Then $\mathfrak{r}(x) = x$ if $o = e$, i.e., if $x \in \Sigma$, and $\mathfrak{r}(x) = p(e^-)$ otherwise.

2.3. Definition. — Let X be a K-algebraic curve (i.e., a K-algebraic variety of pure dimension 1) and let U be a K-definable subset of X. A *skeleton* of $\widehat{U}$ is a Γ_0-internal subset of $\widehat{U}$; a K-skeleton is a K-definable skeleton. A K-skeleton Σ of $\widehat{U}$ is called *admissible* if there exists a K-definable strong deformation retraction $\widehat{h}\colon \widehat{U} \times I \to \widehat{U}$ with $\widehat{h}(\widehat{U}, e) = \Sigma$ for e the right endpoint of I, induced (through the realization of types) by its restriction $h\colon U \times I \to \widehat{U}$ to simple points (we will call for short such an $\widehat{h}$ an admissible K-homotopy with final image Σ).

2.4. Remark. — Admissibility is a property of K-skeletons, and its definition involves K-definability. So if L is a valued extension of K, the property for a K-skeleton of being admissible as an L-skeleton is a priori weaker than being admissible as an K-skeleton; but it turns out that both are in fact equivalent, see Remark 2.6 below.

2.5. Existence of admissible skeletons and consequences. — Let X and U be as in the above definition. The following facts follow from [**HL16**, Chapter 7] (they are extended to higher dimensional varieties in op. cit., Theorem 11.1.1): every K-skeleton T of $\widehat{U}$ is contained in some admissible K-skeleton Σ; one can morever require that Σ be purely 1-dimensional, that it be the target of a *topologically proper* admissible K-homotopy h, and that finitely many given K-definable maps from U to Γ_0 be constant along the trajectories of h.

2.5.1. — The existence of admissible K-skeletons ensures that $\widehat{U}$ is locally path-connected, and even locally contractible. It also ensures that $\widehat{U}$ has finitely many (path-)connected components, each of which is the stable completion of some $\overline{K}$-definable subset of U; more precisely, if $\widehat{h}\colon \widehat{U} \times I \to \widehat{U}$ is an admissible K-homotopy with target Σ, and if e is the right endpoint of I, the connected components of $\widehat{U}$ are exactly the sets fo the form $\widehat{h}(\cdot, e)^{-1}(\mathrm{T})$ for T a connected component of Σ, and we conclude by observing that $\widehat{h}(\cdot, e)^{-1}(\mathrm{T})$ is the stable completion of $h(\cdot, e)^{-1}(\mathrm{T})$ (and as $\pi_0(\Sigma)$ is finite, each connected component of Σ is $\overline{K}$-definable).

2.5.2. — Every skeleton on $\widehat{U}$ is of dimension $\leqslant 1$; this prevents any (definable) loop on $\widehat{U}$ from moving along any homotopy, so that an admissible K-skeleton necessarily contains all loops of $\widehat{U}$ (there are therefore only finitey many such loops). Also note that any admissible K-skeleton of $\widehat{U}$ is closed in $\widehat{U}$ (as the set of points that are fixed at every time under a K-admissible retraction) and that its intersection with every connected component of $\widehat{U}$ is non-empty and connected.

Conversely, let Σ be a closed K-skeleton of $\widehat{U}$ containing all loops on $\widehat{U}$ and whose intersection with every connected component of $\widehat{U}$ is non-empty and connected. Choose an admissible K-skeleton T of $\widehat{U}$ containing Σ. Then Σ is closed in T, it contains all the loops of T, and the intersection of Σ with every connected component of T is non-empty and connected. This implies that there exists a K-definable strong deformation retraction from T to Σ (first build a $\overline{K}$-definable such retraction $h(\cdot,\cdot)$ and then define $h'(x,t)$ as the the closest point to x among all $\sigma^{-1}(h(\sigma(x),t))$ for $\sigma \in \mathrm{Aut}(\overline{K}/K)$); it follows that Σ itself is admissible (just retract $\widehat{X}$ to T, and then T to Σ).

It follows that if Σ is an admissible K-skeleton of $\widehat{U}$, a K-definable skeleton T of $\widehat{U}$ containing Σ is admissible if and only if the natural map form $\pi_0(\Sigma)$ to $\pi_0(\mathrm{T})$ is bijective. As Σ contains all loops of $\widehat{U}$ this amounts to requiring that T be built from Σ by successive applications of the operation consisting in adding to a $\overline{K}$-definable skeleton Υ a $\overline{K}$-definable interval $[x,y]$ with $[x,y] \cap \Upsilon = \{y\}$ (we insist that even if Σ and T are K-definable, the successive edges one adds might be merely $\overline{K}$-definable).

2.5.3. — An admissible K-skeleton Σ of $\widehat{U}$ has a strong convexity property: for all $(x,y) \in \Sigma^2$ and all compact intervals I on $\widehat{U}$ with endpoints x and y one has $I \subset \Sigma$. Indeed, if such an interval exists then x and y are on the same connected component of $\widehat{U}$, and therefore of Σ, so there is at least one such interval entirely contained in Σ; but then all of them are conained in Σ, since Σ contain all the loops drawn on $\widehat{U}$.

2.5.4. — Let us also mention that some GAGA principle holds here: if X is geometrically connected as an algebraic K-curve, $\widehat{X}$ is connected ([**HL16**], Theorem 10/4.2; this is stated and shown for an arbitrary X, but the proof almost immediately reduces through Bertini's theorem to the case of a curve).

2.6. Remark. — Let Σ be a K-skeleton and let L be an arbitrary valued extension of K. It follows from the characterization of admissibility given in 2.5.2 that Σ is admissible as a K-skeleton if and only if it is admissible as an L-skeleton (though the latter condition seems weaker, for it involves merely an L-definable retraction).

2.7. The canonical map to a K-admissible skeleton. — Let Σ be an admissible K-skeleton on $\widehat{U}$.

2.7.1. — Let $x \in \widehat{U}$. The existence of an admissible K-homotopy with final image Σ ensures the existence of a generalized interval $[x,y]$ on $\widehat{U}$ with $[x,y] \cap \Sigma = \{y\}$, and this interval is unique because the intersection of Σ with the connected component of x is connected and because all loops of $\widehat{U}$ are contained in Σ.

The latter property in fact characterizes the admissibility. Indeed, let T be a closed K-skeleton of $\widehat{U}$ such that for every $x \in \widehat{U}$ there exists a unique interval $[x,y]$ on $\widehat{U}$ with $[x,y] \cap \mathrm{T} = \{y\}$ it is easily seen that the intersection of T with every connected component of $\widehat{U}$ is connected and non-empty, and that T contains all loops drawn on $\widehat{U}$, so T is admissible.

2.7.2. — By the above there exists for every $x \in \widehat{U}$ a unique segment $[x, \widehat{\rho}_{U,\Sigma}(x)]$ on $\widehat{U}$ such that $[x, \widehat{\rho}_{U,\Sigma}(x)] \cap \Sigma = \{\widehat{\rho}_{U,\Sigma}(x)\}$. The map $\widehat{\rho}_{U,\Sigma}$ is K-definable: indeed, this is the final map associated to any admissible K-homotopy with final image Σ; this also implies that $\widehat{\rho}_{U,\Sigma}$ is induced by its restriction $\rho_{U,\Sigma}$ to U. As a consequence, for every K-definable subset D of Σ, the pre-image $\widehat{\rho}_{U,\Sigma}^{-1}(\Sigma)$ is the stable completion of the K-definable set $\rho_{U,\Sigma}^{-1}(D)$.

2.7.2.1. — Let $s \in \Sigma$ and let x be a point of $\widehat{\rho}_{U,\Sigma}^{-1}(s)\backslash\Sigma = \widehat{\rho}_{U,\Sigma}^{-1}(s)\backslash\{s\}$. Let Ω be a path-connected neighborood of x in $\widehat{U}$ that does not meet Σ. Let $y \in \Omega$. There is an interval $[x, y]$ on Ω (which is unique since Σ contains all loops on $\widehat{U}$). Then $[x, y] \cup [x, s]$ is connected and meets Σ only on s. It follows that $\widehat{\rho}_{U,\Sigma}(y) = s$; hence $\widehat{\rho}_{U,\Sigma}^{-1}(s)\backslash\{s\}$ is open in $\widehat{U}$.

2.7.2.2. — Let T be an admissible K-skeleton of $\widehat{U}$ containing Σ and let $x \in \widehat{U}$. It follows from the construction that $\widehat{\rho}_{U,\mathrm{T}}(x) \in [x, \widehat{\rho}_{U,\Sigma}(x)]$, and $[\widehat{\rho}_{U,\mathrm{T}}(x), \rho_{U,\Sigma}(x)]$ is contained in T since T is admissible. So $\widehat{\rho}_{U,\Sigma} = \widehat{\rho}_{U,\Sigma}|_{\mathrm{T}} \circ \widehat{\rho}_{U,\mathrm{T}}$; note that $\widehat{\rho}_{U,\Sigma}|_{\mathrm{T}}$ can be described purely in terms of T and Σ, without any reference to U: for each $t \in \mathrm{T}$ the interval $[t, \widehat{\rho}_{U,\Sigma}(t)]$ is contained in T, its intersection with Σ is equal to $\{\widehat{\rho}_{U,\Sigma}(t)\}$, and this determines uniquely $\widehat{\rho}_{U,\Sigma}(t)$.

2.8. Definition. — Let V be a K-definable subset of U. An admissible K-skeleton Σ of $\widehat{U}$ is *adapted to* V if there exists a K-definable subset D of the skeleton Σ such that $V = \rho_{U,\Sigma}^{-1}(D)$, which amounts to saying that $\widehat{V} = \widehat{\rho}_{U,\Sigma}^{-1}(D)$.

2.9. Remark. — Assume that Σ is adapted to V, and let D be as in the definition. The set D is uniquely determined: it is necessarily equal to $\rho_{U,\Sigma}(V) = \widehat{\rho}_{U,\Sigma}(\widehat{V})$, and $\widehat{V}$ is stable under any admissible K-homotopy of $\widehat{U}$ with final image Σ, so D is an admissible K-skeleton of $\widehat{V}$.

Note also that any admissible K-skeleton T of $\widehat{U}$ containing Σ is still adapted to V: indeed, for such a T one has $\rho_{U,\Sigma} = \widehat{\rho}_{U,\Sigma}|_{\mathrm{T}} \circ \rho_{U,\mathrm{T}}$ (by 2.7.2.1), so if $V = \rho_{U,\Sigma}^{-1}(D)$ then $V = \rho_{U,\mathrm{T}}^{-1}(\widehat{\rho}_{U,\Sigma}^{-1}(D) \cap \mathrm{T})$.

2.10. Lemma. — *Let (V_i) be a finite family of K-definable subsets of U and let Υ be K-skeleton of $\widehat{U}$. There exists an admissible K-skeleton Σ of $\widehat{U}$ containing Υ and adapted to each V_i.*

Proof. — For each i denote by χ_i the characteristic function of V_i in U, viewed as a map from U to Γ_0. Then there exists an admissible K-skeleton Σ of $\widehat{U}$ containing Υ and an admissible K-homotpy h from $\widehat{U}$ to Σ such that each χ_i is constant along the trajectories of h; then Σ is adapted by design to every V_i. $\square$

2.11. Corollary. — *Let V be a K-definable subset of U. The topological boundary of $\widehat{V}$ in $\widehat{U}$ is finite; it coincides with the topological boundary of $\rho_{U,\Sigma}(V)$ in Σ for any admissible K-skeleton Σ of $\widehat{U}$ adapted to V.*

Proof. — By the lemma above there exists an admissible K-skeleton of $\widehat{U}$ adapted to V. Set $D = \rho_{U,\Sigma}(V)$. We then have $V = \rho_{U,\Sigma}^{-1}(D)$ and $\widehat{V} = \widehat{\rho}_{U,\Sigma}^{-1}(D)$ (so in particular $D = \widehat{V} \cap \Sigma$).

Let ∂D and $\mathring{D}$ denote the topological boundary of D and the topological interior of D in Σ; note that ∂D is finite. It is tautological that ∂D is contained in the topological boundary $\partial\widehat{V}$ of $\widehat{V}$ in $\widehat{U}$; moreover $\widehat{\rho}_{U,\Sigma}^{-1}(\mathring{D})$ is contained in the topological interior of $\widehat{U}$ in $\widehat{V}$, and so is $\widehat{\rho}_{U,\Sigma}^{-1}(d)\backslash\{d\}$ for every $d \in D$ by 2.7.2.1 above, so $\partial\widehat{V} = \partial D$. □

3. Topology of curves: the complement of an admissible skeleton

3.1. Germs of paths and residue curves. — Let X be a K-algebraic curve and let U be a K-definable open subset of X.

3.1.1. — Let M be a model of ACVF containing K. Let $f: (a,b) \to \widehat{U}$ be a continuous M-definable map where $a < b$ are two different elements of $|M|$ with $f(a) = x$. Let a^+ be the M-definable type on (a,b) that lies on (a,c) for all $c \in |M|$ with $a < c < b$. Then $f(a^+)$ is an element of $\breve{U}(M)$ that only depends on the "left germ" of the parametrized path f, by which we refer to the relation identifying $f: (a,b) \to \widehat{U}$ and $f': (a',b') \to \widehat{U}$ if there are $a < c \leqslant b$ and $a' < c' \leqslant b'$ such that $f|_{(a,c)}$ and $f'|_{(a',c')}$ either are constant with the same value or agree up to M-definable homeomorphic reparametrization. It follows from [**HL16**, Theorem 2.1.15] that every element of $\breve{U}(M)$ arises in this way.

3.1.2. — If M and f are as above, one is necessarily in one the following three cases.

3.1.2.1. *The constant case.* — If the left-germ of f is constant (i.e., it admits a constant representative, allowing to assume that f itself is constant), then $f(a^+)$ belongs to $\widehat{U}(M)$; it is equal to the constant value of f.

3.1.2.2. *The unbounded case.* — If the left-germ of f is not constant (allowing ourselves to assume that f is injective) and $a = 0$ then $f(a^+)$ is unbounded, i.e., it belongs to $\breve{U}(M)\backslash\widetilde{U}(M)$; if $\overline{X}$ denotes a projective compactification of X then $f(a^+)$ is the composition of the discrete valuation associated to an M-point ξ of the normalizaiton of $\overline{X}$ with the valuation of M. Let x be the image of ξ in $\overline{X}(M)$; the type $f(a^+)$ has limit x in $\overline{X}(M)$; otherwise said, f has a continuous extension to $[a,b)$ with $f(a) = x$.

3.1.2.3. *The bounded case.* — If the left-germ of f is not constant and $a \neq 0$ then $f(a^+)$ is an element of $\widetilde{U}$; if we set $x = \varpi(f(a^+))$ (the map ϖ has been defined in 2.1.1) the point x is not simple and $f(a^+)$ has limit x; otherwise said, f has a continuous extension to $[a,b)$ with $f(a) = x$.

3.1.3. *The residue curve: the classical and the generalized viewpoints.* — We still denote by M a model of ACVF containing K. Let x be a non-simple point of $\widehat{X}(M)$.

The type x is induced by a valuation v (extending that of M) on the function field of some (reduced) irreducible component of the scheme X_M.

3.1.3.1. — The value group of v is $|M^\times|$ and it residue field is is $\mathsf{k}(M)(C_x)$ for some canonical smooth projective C_x defined over $\mathsf{k}(M)$, which we call the *residue curve* at x.

3.1.3.2. — As the value group of v is equal to that of M, the generalized residue field of v is then merely $\mathsf{RV}(M)\otimes_{\mathsf{k}(M)}\mathsf{k}(M)(C_x)$. We can also write it $\mathsf{RV}(M)(C_x)$, if we to implicitly interpret C_x as a generalized variety over $\mathsf{RV}(M)$ (1.10). This is harmless since this generalized variety coincides with C_x as a geometric object, and this is often more natural (see already Example 3.2 just below), especially if one wants to deal with families $(C_x)_x$ of residue curves when x runs through a skeleton. This viewpoint will also be convenient for the followig reason: if f is an M-regular function invertible at x then $|f(x)|$ belongs to $|M^\times|$, so we can choose $a \in M^\times$ such that $|f(x)| = |a|$ and then consider the element $\mathsf{res}(f(x)/a)$ of $\mathsf{k}(M)(C_x)^\times$, interpreting it as a rational function on the classical algebraic curve C_x. But it depends on the choice of a, so this is more natural to deal with $\mathsf{rv}(f(x)) \in \mathsf{RV}(M)(C_x)$ and interpret it as a generalized rational function on C_x viewed as a generalized algebraic curve.

3.2. Example. — Let $r \in |M^\times|$. If a denotes any element of M with $|a| = r$, the classical residue field at $\eta_{0,r}$ (over M) can be identified to the rational function field $\mathsf{k}(M)(\theta)$ with $\theta = \mathsf{res}((T/a)(\eta_{0,r}))$, so the residue curve C at $\eta_{0,r}$ is isomorphic to the projective line over $\mathsf{k}(M)$ and one has

$$C(M) = C(\mathsf{k}(M)) \simeq \{0\} \cup \{\mathsf{res}(\lambda)\}_{\lambda\in M, |\lambda|=1} \cup \{\infty\}\ ;$$

but this isomorphism depends on the choice of the rescaling factor a. It is therefore more natural to deal with the generalized residue field at $\eta_{0,r}$ which is equal to $\mathsf{RV}(M)(\tau)$ with $\tau = \mathsf{rv}(T(\eta_{0,r}))$, so the residue curve C at $\eta_{0,r}$ seen as a generalized variety is isomorphic to the generalized projective line $\mathbf{P}^r_{\mathsf{RV}}$ (see 1.10 for a definition) and one has

$$C(M) = C(\mathsf{RV}(M))) \simeq \{0\} \cup \{\mathsf{rv}(\lambda)\}_{\lambda\in M, |\lambda|=r} \cup \{\infty\}\ ;$$

of course one goes from one description of $C(M)$ to the other one by using $\mu \mapsto \mu/\mathsf{rv}(a)$.

3.3. — We fix for the remaining part of section 3 a K-algebraic curve and a K-definable open subset U of X.

3.3.1. — Let x be a non-simple point of $\widehat{U}$. By the above there is a natural bijection between the set $\mathsf{P}_{x,\widehat{X}}(M)$ of germs of M-definable paths on $\widehat{X}$ emanating from x and $\varpi^{-1}(x)$. The latter is itself in bijection with the set of M-refinements of v. Such a refinement is given by a valuation on the classical residue field $\mathsf{k}(M)(C_x)$ trivial on $\mathsf{k}(M)$, and there are two kind of such valuations: the trivial one (giving rise to the trivial refinement, i.e., v itself) and the discrete valuations associated to the $\mathsf{k}(M)$-point of C_x (viewed as a classical algebraic curve), that corresponds to the proper refinements of v. Of course for handling C_x as a generalized curve we will rather write $C_x(M)$ than $C_x(\mathsf{k}(M))$.

We thus eventually get a bijection between $\mathsf{P}_{x,\widehat{X}}(M)$ and a set obtained by adjoining to $C_x(M)$ a single element η; the most natural way to see $C_x(M \cup \{\eta\}$ in this setting is to identify it with $\widehat{C}_x(M)$ where $\widehat{C}_x$ is the set of types on C_x over the generalized residue field. Under this bijection, the constant path is mapped to the generic type η, and every (germ of) injective path to a simple point belonging to $C_x(M)$.

3.3.2. — By unraveling the constructions, we see that the bijection can be described as follows. Let $p\colon [a,b) \to \widehat{X}$ be an element of $\mathsf{P}_{x,\widehat{X}}(M)$ (with $p(a) = x$) and let f be an M-regular function defined on an M-Zariski neighborhood of x and such that $f(x) \neq 0$. Let $\xi \in \widehat{C_x}(M)$ the point corresponding to I. Then $|f| \circ p$ is $< |f(x)|$, resp. equal to $|f(x)|$, resp. $> |f(x)|$ on some sub-interval (a,c) of (a,b) if and only if the generalized rational function $\mathsf{rv}(f(x))$ on C_x vanishes at ξ, resp. is well-defined and invertible at ξ, resp. has a pole at ξ.

3.3.3. — Now assume that the point x above is contained in $\widehat{U}$; choose an admissible K-skeleton Σ of $\widehat{X}$ adapted to U and containing x (Definition 2.8), Lemma 2.10). Set $D = \rho_{X,\Sigma}(U)$; this is an M-definable subset of Σ and $\widehat{U} = \widehat{\rho}_{U,\Sigma}^{-1}(D)$. Let F be the finite set of M-points of C_x corresponding to the elements of $\mathsf{P}_{x,\widehat{X}}$ which "are" (i.e., have a representative which is) contained in Σ and not contained in D. Let I be any non-constant element of $\mathsf{P}_{x,\widehat{X}}(M)$ and let ζ be the corresponding M-point of C_x. Suppose that I is contained in $\widehat{U}$. Then if I is contained in Σ it is contained in D, since $\widehat{U} \cap \Sigma = D$ by construction; as a consequence, $\zeta \notin F$. Conversely, assume that $\zeta \notin F$. Then if I is contained in Σ, it is contained in D, hence in $\widehat{U}$. And if I is not contained in Σ there exists some $y \in \widehat{X}\backslash\Sigma$ such that $I = (y,x)$ (as a germ) and $(y,x) \cap \Sigma = \varnothing$. Then $x = \widehat{\rho}(U,\Sigma)(z)$ for every $z \in (y,x)$, so $(y,x) \in \widehat{\rho}(U,\Sigma)^{-1}(x)$; and since x belongs to $\widehat{U} \cap \Sigma$ it belongs to D, so $(y,x) \in \widehat{U}$. Otherwise said, if we set $C_{x,\widehat{U}} = C_x\backslash F$, the natural bijection between $\mathsf{P}_{x,\widehat{X}}$ and $\widehat{C}_x$ induces a bijection between $\mathsf{P}_{x,\widehat{U}}$ and $\widehat{C}_{x,\widehat{U}}$. Note that the latter property shows that the dense M-Zariski open subset $C_{x,\widehat{U}}$ of C_x actually only depends on U, and not on the choice of a particular skeleton adapted to U. It follows from Corollary 2.11 that the curve $C_{x,\widehat{U}}$ is equal to the whole of C_x if and only if x belongs to the topological interior of x in $\widehat{U}$. More generally, if V is a K-definable subset of U with $x \in \widehat{V}$, then $C_{x,\widehat{V}} \subset C_{x,\widehat{U}}$ with equality if and only if x belongs to the topological interior of $\widehat{V}$ in $\widehat{U}$: one sees this by considering an admissible K-skeleton adapted to both U and V.

3.4. Warning. — Even if the point x is K-definable, there is no reason that the curve C_x itself be K-definable: indeed, the formation of the generalized residue field does not commute to ground field extensions. All we can say in general is that the curve C_x is M-definable if M is a model of ACVF (or even an algebraically closed valued field, the trivial valuation does not cause any trouble here) such that $x \in \widehat{X}(M)$.

3.5. Definition. — Let Σ be an admissible K-skeleton of $\widehat{U}$. We shall say that Σ *contains all U-branches at its simple points* if for every simple point x of Σ the two following equivalent condition are fulfilled:

(i) every germ of interval on $\widehat{U}$ emanating from x is contained in Σ;

(ii) $\widehat{\rho}_{U,\Sigma}^{-1}(x) = \{x\}$.

3.6. Remark. — Let x be a point of X. it follows from 3.1.1 that the set of germs of intervals on $\widehat{X}$ emanating from x is in bijection with the set of elements of $\check{X}$ that belong to $\check{U}$ for every definable subset U of X such that $\widehat{U}$ is an open neighborhood of x in $\widehat{X}$, which is itself in bijection with the set of pre-images of x on the (geometric) normalization of X (to such a pre-image, associate the type obtained by composing the corresponding discrete valuation with the valuation of the ground field). In particular this set is always finite, and is a singleton as soon as x is a smooth point.

3.7. Lemma. — *Let Σ be a K-skeleton of $\widehat{U}$. There exists an admissible K-skeleton of $\widehat{U}$ containing all U-branches at its simple points and containing Σ.*

Proof. — We can replace K by its perfection, and X by the associated reduced (hence generically smooth) curve. Let X' be the normalization of X, let U' be the pre-image of U on X' and Σ' be the pre-image of Σ on $\widehat{U'}$. Let E be the set of singular points of X and let E' be its pre-image on X'.

Let T be an admissible K-skeleton of $\widehat{U'}$ containing Σ' and $E' \cap U'$. The image Υ of T on $\widehat{U}$ is a K-skeleton. Let x be a simple point of Υ. Let I be a compact interval on $\widehat{U}$ having x as an endpoint and which does not contain any simple point other than x. Let I' be the pre-image of I on $\widehat{U'}$. Then I' consists of a connected component J homeomorphic to I, containing a unique pre-image x' of x, and of a finite set (possibly empty) of isolated pre-images of x. The existence of J ensures that x' is not isolated in $\widehat{U'}$. Moreover x' belongs to T : indeed this is clear if x is a smooth point, for x' is then the unique pre-image of x in $\widehat{X'}$, and x has a pre-image in T by definition of Υ; and if x is not smooth then $x' \in E'$, so $x' \in \mathrm{T}$. Now as x' is not an isolated point of $\widehat{U'}$ and as T is admissible, x' is not isolated in T, so up to shrinking I we can assume that $J \subset \mathrm{T}$ (as X' is normal, there is a unique germ of interval on $\widehat{X'}$ emanating from x', see Remark 3.6). Then $I \subset \Upsilon$.

It remains to show that Υ (which obviously contain Σ) is admissible. But T is admisssible, so there exists an admissible K-homotopy on $\widehat{U'}$ with target T. As T contains $E' \cap U'$ and as $X' \to X$ is an isomorphism outside E, this homotopy descends to $\widehat{U}$ on which it induces an admissible K-homotopy with target Υ. □

3.8. The map $\widehat{\mathfrak{r}}$. — From now on we denote by Σ an admissible skeleton of $\widehat{U}$ containing all U-branches at its simple points.

3.8.1. *Construction of* $\widehat{\mathfrak{r}}$. — Let u be a point of $\widehat{U}$.

3.8.1.1. — If $u \in \Sigma$ we set $\widehat{\mathfrak{r}}_{U,\Sigma}(u) = u$.

3.8.1.2. — If $u \notin \Sigma$ then $[u, \widehat{\rho}_{U,\Sigma}(u))$ defines a germ of injective path emanating from $\widehat{\rho}_{U,\Sigma}(u)$ and not contained in Σ, and in turn defines a point $\widehat{\mathfrak{r}}_{U,\Sigma}(u)$ of $\check{U}$ by 3.1.2. Since Σ contains all U-branches at its simple points, the point $\widehat{\rho}_{U,\Sigma}(u)$ is not simple, so one is in the bounded case described in 3.1.2.3: the point $\widehat{\mathfrak{r}}_{U,\Sigma}(u)$ actually belongs to $\widetilde{U}$, and $\varpi(\widehat{\mathfrak{r}}_{U,\Sigma}(u)) = \widehat{\rho}_{U,\Sigma}(u)$ (we remind that ϖ denotes the natural map from $\widetilde{X}$ to $\widehat{X}$, see 2.1.1).

3.8.2. — We denote by $\mathfrak{r}_{U,\Sigma}$ the restriction of $\widehat{\mathfrak{r}}_{U,\Sigma}$ to U. The map $\widehat{\mathfrak{r}}_{U,\Sigma}$ can be reconstructed from $\mathfrak{r}_{U,\Sigma}$ by realizing types. The maps $\mathfrak{r}_{U,\Sigma}$ and $\widehat{\mathfrak{r}}_{U,\Sigma}$ are clearly pro-K-definable. As $\widetilde{U}$ is strictly pro-K-definable, it follows that the images $\mathfrak{r}_{U,\Sigma}(U)$ and $\widehat{\mathfrak{r}}_{U,\Sigma}(\widehat{U})$ are K-definable.

We set $\Sigma_{\mathsf{RV},U} = \widehat{\mathfrak{r}}_{U,\Sigma}(\widehat{U}\backslash\Sigma)$. It follows from 3.8.1.2 that $\varpi(\Sigma_{\mathsf{RV},U})$ is contained in Σ and only consists of non-simple points, and that $\widehat{\rho}_{U,\Sigma} = \varpi \circ \widehat{\mathfrak{r}}_{U,\Sigma}$ on $\widehat{U}\backslash\Sigma$.

3.9. Lemma. — *The fibers of $\widehat{\mathfrak{r}}$ are exactly the connected components of $\widehat{U}\backslash\Sigma$.*

Proof. — Let x and y be two points of $\widehat{U}\backslash\Sigma$.

If $\widehat{\mathfrak{r}}_{U,\Sigma}(x) = \widehat{\mathfrak{r}}_{U,\Sigma}(y)$ then x and y have the same image, say z, under the map $\widehat{\rho}_{U,\Sigma}$; moreover $[x, z]$ and $[y, z]$ define the same germ of interval emanating from z, so there is some $t \in [x, z) \cap [y, z)$ such that $[x, z] \cap [y, z] = [t, z]$. Then $[x, z) \cup [y, z)$ is connected, so there is an interval $[x, y]$ on $\widehat{U}\backslash\Sigma$ (which is unique since Σ contains all loops on $\widehat{U}$), so x and y lie in the same connected component of $\widehat{U}\backslash\Sigma$.

Conversely, assume that x and y lie in the same connected component of $\widehat{U}\backslash\Sigma$. Since $\widehat{U}$ is locally path-connected, there is some interval $[x, y]$ on $\widehat{U}\backslash\Sigma$. Then the union $[x, y] \cup [y, \widehat{\rho}_{U,\Sigma}(y)]$ is connected, and its intersection with Σ is equal to $\{\widehat{\rho}_{U,\Sigma}(y)\}$. It follows that $[x, y] \cup [y, \widehat{\rho}_{U,\Sigma}(y)]$ contains an interval $[x, \widehat{\rho}_{U,\Sigma}(y)]$ whose intersection with Σ is equal to $\{\widehat{\rho}_{U,\Sigma}(y)\}$, and which coincides with $[y, \widehat{\rho}_{U,\Sigma}(y)]$ near $\widehat{\rho}_{U,\Sigma}(y)$. As a consequence, $\widehat{\rho}_{U,\Sigma}(x) = \widehat{\rho}_{U,\Sigma}(y)$ and $\widehat{\mathfrak{r}}_{U,\Sigma}(x) = \widehat{\mathfrak{r}}_{U,\Sigma}(y)$. □

3.10. Corollary. — *One has $\Sigma_{\mathsf{RV},U} = \mathfrak{r}(U\backslash\Sigma)$, so $\Sigma_{\mathsf{RV},U} = \mathfrak{r}(U)$ if Σ has no simple points.*

Proof. — The inclusion $\mathfrak{r}(U\backslash\Sigma) \subset \Sigma_{\mathsf{RV},U}$ is obvious, so it remains to show the converse inclusion. Let $x \in \widehat{U}\backslash\Sigma$, say $x \in \widehat{U}(M)$ for some model M of ACVF. By 2.5.1, there exists some M-definable subset V of U such that $\widehat{V}$ is a connected open neighborhood of x in $\widehat{U}$. By Lemma 3.9 above one has then $\widehat{\mathfrak{r}}_{U,\Sigma}(y) = \widehat{\mathfrak{r}}_{U,\Sigma}(x)$ for every $y \in \widehat{V}$; since $V(M) \neq \varnothing$ (as $\widehat{V}$ contains x, so V is non-empty) we get the existence of $y \in V(M)$ with $\widehat{\mathfrak{r}}_{U,\Sigma}(y) = \widehat{\mathfrak{r}}_{U,\Sigma}(x)$. □

3.11. Remark. — Lemma 3.9 shows that lying in the same connected component of $\widehat{U}\backslash\Sigma$ is a K-definable equivalence relation (with quotient $\Sigma_{\mathsf{RV},U}$), which did not seem a priori obvious.

3.12. The fibers of $\varpi\colon \Sigma_{\mathsf{RV},U} \to \Sigma$. — Our purpose is now to investigate the fibers of the map $\varpi\colon \Sigma_{\mathsf{RV},U} \to \Sigma$. In fact one has $\varpi(s) = s$ for every $s \in \Sigma$ (even for every s in $\widehat{U}$) and it will be to some extent more natural to study the fibers of $\varpi\colon \Sigma_{\mathsf{RV},U} \cup \Sigma \to \Sigma$.

3.12.1. — Let s be a point of Σ.

3.12.1.1. — If s is simple, its only pre-image in $\Sigma_{\mathsf{RV},U} \cup \Sigma$ under ϖ is s itself.

3.12.1.2. — Assume that s is not simple. By the very construction of $\widehat{\mathfrak{r}}_{U,\Sigma}$, the set of points ζ of $\Sigma_{\mathsf{RV},U}$ such that $\varpi(\zeta) = s$ can be identified with the set of germs of injective paths emanating from s, contained in $\widehat{U}$ and not contained in Σ; hence $\varpi^{-1}(s) \cap \Sigma_{\mathsf{RV},U}$ is in bijection with the dense Zariski-open subset $C_{s,\widehat{U}\backslash\Sigma}$ of $C_{s,\widehat{U}}$ obtained by removing the (finitely many) points corresponding to germs of edges of Σ emanating from s. It follows that $\varpi^{-1}(s) \cap (\Sigma_{\mathsf{RV},U} \cup \Sigma)$ is in natural bijection with $\widehat{C}_{s,\widehat{U}\backslash\Sigma}$ (with s corresponding to the generic type).

3.12.2. — Let s be a non-simple point of Σ and let D be a Zariski-constructible subset of $C_{s,\widehat{U}\backslash\Sigma}$. We observe that the stable completion $\widehat{D}$ of D is then equal to D if D is finite, and to $D \cup \{\eta\}$ otherwise (with η the generic type). Let us view $\widehat{C}_{s,\widehat{U}\backslash\Sigma}$ as contained in $(\Sigma_{\mathsf{RV},U} \cup \Sigma) \subset \widetilde{U}$. Since $\widehat{\mathfrak{r}}_{U,\Sigma}$ is induced by $\mathfrak{r}_{U,\Sigma}$ through realization of types, the pre-image $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\widehat{D})$ is the stable completion of $\mathfrak{r}_{U,\Sigma}^{-1}(D)$. In particular, if ζ is a point of $C_{s,\widehat{U}\backslash\Sigma}$ then $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$, which is a connected component of $\widehat{U}\backslash\Sigma$ (Lemma 3.9), is equal to the stable completion of $\mathfrak{r}_{U,\Sigma}^{-1}(\zeta)$.

3.13. Example. — A particular case of the construction above had already been carried out directly in 2.2 for $\mathbf{P}^1$ and its standard K-skeleton $\Sigma = [0, \infty]$ (what we denoted there by Σ_{RV} and $\mathfrak{r}$ should be denoted by $S_{\mathsf{RV},\mathbf{P}^1}$ and $\mathfrak{r}_{\mathbf{P}^1,\Sigma}$ according to our current notation, but we will keep the simpler notation of 2.2 in what folllows).

We saw in 2.2 that there is a K-definable isomorphism between Σ_{RV} and RV, modulo which $\mathfrak{r}\colon (\mathbf{P}^1\backslash\Sigma) = \mathbf{G}_{\mathrm{m}} \to \Sigma_{\mathsf{RV}}$ is simply $a \mapsto \mathsf{rv}(a)$, and modulo which the map $\varpi\colon \Sigma_{\mathsf{RV}} \to \Sigma$ is simply $\mathsf{rv}(a) \mapsto \eta_{0,|a|}$. If ζ is an element of Σ_{RV} identified to $\mathsf{rv}(a)$ for some $a \neq 0$ then $\mathfrak{r}^{-1}(\zeta)$ is the open disc $\mathsf{rv}^{-1}(\mathsf{rv}(a))$ of center a and radius $|a|$ and the connected component $\widehat{\mathfrak{r}}^{-1}(\zeta)$ of $\widehat{\mathbf{P}^1}\backslash\Sigma$ is the stable completion of this open disc.

For any $r \in \Gamma$ the fiber $\varpi^{-1}(\eta_{0,r}) \cap \Sigma_{\mathsf{RV}}$ is equal, modulo the bijection $\Sigma_{\mathsf{RV}} \simeq \mathsf{RV}$, to $\{\mathsf{rv}(a)\}_{|a|=r}$, so this is precisely $C_{\eta_{0,r},\widehat{\mathbf{P}^1}\backslash\Sigma} = \mathbf{P}^r_{\mathsf{RV}}\backslash\{0,\infty\}$ *when* $C_{\eta_{0,r}}$ *is seen as the generalized curve* $\mathbf{P}^r_{\mathsf{RV}}$, see Example 3.2 (the two germs of intervals emanating from $\eta_{0,r}$ and contained in Σ correspond precisely to the points 0 and ∞ of $\mathbf{P}^r_{\mathsf{RV}}$). It is thus definitely more natural in this setting to deal with generalized residue curves.

3.14. A variant: the map $\widetilde{\mathfrak{r}}$. — The map $\widehat{\mathfrak{r}}_{U,\Sigma}$ can be deduced from $\mathfrak{r}_{U,\Sigma}$ through the realization of types. By the same procedure we get a map $\widetilde{\mathfrak{r}}_{U,\Sigma}\colon \widetilde{U} \to \widetilde{U}$. But any point of $\widetilde{U}$ can be seen as a definable type on $\widehat{U}$ by the procedure described in 3.1.1, and with this viewpoint $\widetilde{\mathfrak{r}}_{U,\Sigma}$ is the extension of $\widehat{\mathfrak{r}}_{U,\Sigma}$ through the realization of types.

3.14.1. — Let M be a model of ACVF containing K and let $u \in \widetilde{U}(M)$. Let us describe it like in 3.1.1 as $f(a^+)$ for some germ of path $f\colon (a,b) \to \widehat{U}$ with a and b in $|M|$. We can in fact always assume that $a \neq 0$: indeed, since u belongs to $\widetilde{U}(M)$ it is bounded with respect to M, so if $a = 0$ the germ f is constant near a and thus can be replaced by any other constant germ of path with the same value. We can also assume that the image I of f either avoids Σ or is entirely contained in Σ. We choose a model N of ACVF containing M and such that the type a^+ is realized over M by some $\alpha \in |N|$. Set $v = f(\alpha) \in \widehat{U}(N)$, and we distinguish two cases.

3.14.1.1. — Assume first that I avoids Σ. Then I is contained in some component of $\widehat{U}\backslash\Sigma$, so $\widehat{\mathfrak{r}}_{U,\Sigma}$ is constant on I and its value is an element of $\Sigma_{\mathsf{RV},U}(M)$, i.e., it can be identified to some $\zeta \in C_{s,\widehat{U}\backslash\Sigma}(M)$ for some non-simple point $s \in \Sigma(M)$. Then $\widehat{\mathfrak{r}}_{U,\Sigma}(v) = \zeta$ as well, so $\widetilde{\mathfrak{r}}_{U,\Sigma}(u) = \zeta$.

3.14.1.2. — Assume now that $I \subset \Sigma$. This is the case where there is some $s \in \Sigma(M)$ such that u is either equal to s or can be identified to ζ for some $\zeta \in C_{s,\widehat{U}}$ corresponding to an edge of Σ emanating from s (the one provided by I). Then as $f(\alpha) \in \Sigma$ we have $\widehat{\mathfrak{r}}_{U,\Sigma}(v) = v$, so $\widetilde{\mathfrak{r}}_{U,\Sigma}(u) = u$.

3.14.2. — Let s be a non-simple point of $\Sigma(M)$ and let ζ be a point of $C_{s,\widehat{U}\backslash\Sigma}$. We can see ζ as a point of $\widetilde{U}$ satisfying $\varpi(\zeta) = s$. Then by considering a germ of interval emanating from s and defining ζ and applying 3.14.1.1 we see that $\widetilde{\mathfrak{r}}_{U,\Sigma}(\zeta) = \zeta$.

3.14.3. *Recapitulation.* — The image $\widetilde{\mathfrak{r}}_{U,\Sigma}(\widetilde{U})$ is fibered through ϖ over Σ; its fiber over a simple point s is $\{s\}$, and its fiber over a non-simple point s can be identified to $\widehat{C}_{s,\widehat{U}}$ (with s corresponding to the generic type). Let us use this identification, and choose $\zeta \in \widehat{C}_{s,\widehat{U}}$. Then $\widetilde{\mathfrak{r}}^{-1}_{U,\Sigma}(\zeta)$ contains ζ; it is reduced to $\{\zeta\}$ if ζ is the generic type or a point corresponding to a germ of interval emanating from s and contained in Σ, and it is equal to $\widetilde{\mathfrak{r}^{-1}_{U,\Sigma}(\zeta)}$ otherwise.

3.15. Sub-skeletons. — Let T be a K-definable subset of Σ and set $V = \rho^{-1}_{U,\Sigma}(\mathrm{T})$. Then $\widehat{V} = \widehat{\rho}^{-1}_{U,\Sigma}(\mathrm{T})$ and T is an admissible K-skeleton of $\widehat{V}$. It follows from the construction that T contains all V-branches at its simple points, that for every non-simple point s of T one has $C_{s,\widehat{V}\backslash\mathrm{T}} = C_{s,\widehat{U}\backslash\Sigma}$, that $\mathrm{T}_{\mathsf{RV},V} = \Sigma_{\mathsf{RV},U} \times_{\Sigma} \mathrm{T}$, and that $\mathfrak{r}_{V,\mathrm{T}}$, resp. $\widehat{\mathfrak{r}}_{V,\mathrm{T}}$, resp. $\widetilde{\mathfrak{r}}_{V,\mathrm{T}}$ is the restriction of $\mathfrak{r}_{U,\Sigma}$, resp. $\widehat{\mathfrak{r}}_{U,\Sigma}$, resp. $\widetilde{\mathfrak{r}}_{U,\Sigma}$ to V, resp. $\widehat{V}$, resp. $\widetilde{V}$.

3.16. More topology : branches at infinity, trees. — Let U be a K-definable subset of a K-algebraic curve X.

3.16.1. — A *branch at infinity* of $\widehat{U}$ is an equivalence class of closed definable subsets of $\widehat{U}$ which are (definably) homeomorphic to a semi-open interval, for the relation identifying two such subsets if their intersection is still a semi-open interval.

The definable set $\widehat{U}$ has only finitely many branches at infinity: if Σ is an admissible K-skeleton of $\widehat{U}$ such that $\rho_{U,\Sigma}$ is topologically proper, the branches at infinity of $\widehat{U}$ are exactly those of Σ.

3.16.2. — We shall say that $\widehat{U}$ is a *tree* if for all x, y in $\widehat{U}$ there is a unique K-definable interval on $\widehat{U}$ with endpoints x and y. If U is non-empty then $\widehat{U}$ is a tree if and only if it is (K-definably) contractible, and also if and only if it is connected and contains no loop. If U is a Zariski open subset of $\mathbf{P}^1$, resp. an open or a closed disc, resp. an open, closed or semi-open annulus, $\widehat{U}$ is a tree.

If $\widehat{U}$ is a tree and if $b_1, \dots, b_n$ are its branches at infinity, $\widehat{U}$ admits a canonical definable compactification, which is still a tree, and is obtained by taking for each j a representative I_j of b_j and by adding a point at the open end of the interval I_j.

3.16.3. — Let Σ be an admissible K-skeleton of $\widehat{U}$ containing all U-branches at its simple points. Let Ω be a connected component of $\widehat{U}\backslash\Sigma$. It is equal to $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ for some point $\zeta \in \Sigma_{\mathsf{RV},U}$ and we have $\Omega = \widehat{V}$ where V is the definable subset $\mathfrak{r}_{U,\Sigma}^{-1}(\zeta)$ of U (Lemma 3.9,3.12.2). Set $s = \varpi(\zeta)$. For every $\omega \in \Omega$ there is a unique interval of the form $[\omega, t]$ with $[\omega, t] \cap \Sigma = \{t\}$, and one has necessarily $t = s = \rho_{U,\Sigma}(\omega)$; moreover, the germ of the interval $[\omega, s]$ at s does not depend on ω. It follows that Ω is a tree, and that there exists a unique branch at infinity on Ω having a limit in $\widehat{U}$ (this branch is the class of $[\omega, s)$ for any $\omega \in \Omega$, and its limit is s); the topological boundary of Ω in $\widehat{U}$ is equal to $\{s\}$.

The map $\rho_{U,\Sigma}$ is topologically proper if and only every connected component Ω of $\widehat{U}\backslash\Sigma$ has only one branch at infinity; this amounts to require that every such component have compact closure in $\widehat{U}$.

3.17. Proposition. — *Let Σ be any closed K-skeleton of $\widehat{U}$. Let T be an admissible K-skeleton that contains all U-branches at its simple points and that contains Σ (note that such a T exists by Lemma 3.7). Let Π be the finite set of connected components of $\mathrm{T}\backslash\Sigma$.*

(1) *The family of connected component of $\widehat{U}\backslash\Sigma$ is*
$$(\widehat{\rho}_{U,\mathrm{T}}^{-1}(\Theta))_{\Theta\in\Pi} \coprod (\widehat{\mathfrak{r}}_{U,\mathrm{T}}^{-1}(\zeta))_{s\in\Sigma\backslash U, \zeta\in C_{s,\widehat{U}\backslash\mathrm{T}}}.$$

(2) *Lying on the same connected component of $\widehat{U}\backslash\Sigma$ is a K-definable equivalence relation with quotient set $\Pi \coprod (\varpi^{-1}(\Sigma) \cap \mathrm{T}_{\mathsf{RV},U})$.*

(3) *Each connected component of $\widehat{U}\backslash\Sigma$ is the stable completion of a definable subset of U.*

Proof. — The set $\widehat{U}\backslash\Sigma$ is the disjoint union of its intersection with $\widehat{\rho}_{U,\mathrm{T}}^{-1}(\mathrm{T}\backslash\Sigma)$ and $\widehat{\rho}_{U,\mathrm{T}}^{-1}(\Sigma)$, which can be respectively written $\coprod_{\Theta\in\Pi} \widehat{\rho}_{U,\mathrm{T}}^{-1}(\Theta)$ and $\coprod_{t\in\Sigma\backslash U, \zeta\in C_{t,\widehat{U}\backslash\mathrm{T}}} \widehat{\mathfrak{r}}_{U,\mathrm{T}}^{-1}(\zeta)$.

Now for each $\Theta \in \Pi$ the summand $\widehat{\rho}_{U,\mathrm{T}}^{-1}(\Theta)$ is the stable completion of the K-definable subset $\rho_{U,\mathrm{T}}^{-1}(\Theta)$ of U, and it is open in $\widehat{U}$, connected and non-empty; for each $t \in \Sigma\backslash U$ and each $\zeta \in C_{t,\widehat{U}\backslash\mathrm{T}}$, the pre-image $\widehat{\mathfrak{r}}_{U,\mathrm{T}}^{-1}(\zeta)$ is the stable completion of the $K\zeta$-definable subset $\mathfrak{r}_{U,\mathrm{T}}^{-1}(\zeta)$ of U, and it is open in $\widehat{U}$ (and even in $\widehat{X}$), connected and non-empty. This ends the proof of (1) and (3), and (2) is an obvious consequence of (1). □

3.18. Lemma. — *Let X be an algebraic K-curve, let U be a K-definable subset of X and let Σ be an admissible K-skeleton of $\widehat{U}$. Let V be a K-definable subset of U such that $\widehat{V}$ is a non-empty connected open subset of $\widehat{U}$ and such that Σ contains the topological boundary $\partial\widehat{V}$ of $\widehat{V}$ in $\widehat{U}$. Then the following holds:*

(1) *If $\widehat{V} \cap \Sigma = \varnothing$ then $\widehat{V}$ is a connected component of $\widehat{U}\backslash\Sigma$.*

(2) *If* $\Sigma \cap \widehat{V} \neq \varnothing$ *then:*

- $\diamond$ $\Sigma \cap \widehat{V}$ *is an admissible* K*-skeleton of* $\widehat{V}$*;*
- $\diamond$ $\widehat{\rho}_{V,\Sigma\cap\widehat{V}} = \widehat{\rho}_{U,\Sigma}|_{\widehat{V}}$*;*
- $\diamond$ $\widehat{V} = \widehat{\rho}_{U,\Sigma}^{-1}(\widehat{V} \cap \Sigma)$.

Proof. — Assume that $\widehat{V}$ does not intersect Σ. Then $\widehat{V}$ is an open, non-empty and connected subset of $\widehat{U}\backslash\Sigma$, which is moreover closed in $\widehat{U}\backslash\Sigma$, for the boundary of $\widehat{V}$ in $\widehat{U}$ is contained in Σ. So $\widehat{V}$ is a connected component of $\widehat{U}\backslash\Sigma$.

Assume now that $\Sigma \cap \widehat{V}$ is non-empty. Choose a point $v \in \Sigma \cap \widehat{V}$. As $\widehat{V}$ is connected, its closure is contained in a connected component Ω of $\widehat{U}$, and $\Sigma \cap \Omega$ is connected since Σ is K-admissible. Let $w \in \partial\widehat{V}$; it belongs to Σ by assumption, so by connectedness of $\Sigma \cap \Omega$ there is a path joining w to v on Σ; and as Σ contains all loops drawn on $\widehat{U}$, every interval joining w to v is contained in Σ. By connectedness of $\widehat{V}$, every branch at infinity of $\widehat{V}$ with limit w is contained in an interval joining w to v all of whose points but w belong to $\widehat{V}$. Therefore $\Sigma \cap \widehat{V}$ contains all branches at infinity of $\widehat{V}$ with limit w; as w is arbitrary, $\Sigma \cap \widehat{V}$ contains all branches at infinity of $\widehat{V}$ having a limit in $\widehat{U}$.

Let $x \in \widehat{V}$. In order to prove that $\Sigma \cap \widehat{V}$ is an admissible K-skeleton of $\widehat{V}$ and $\widehat{\rho}_{V,\Sigma\cap\widehat{V}} = \widehat{\rho}_{U,\Sigma}|_{\widehat{V}}$, it suffices to show that $[x, \widehat{\rho}_{U,\Sigma}(x)] \subset \widehat{V}$. We argue by contradiction, so we assume that this is not the case. Then $[x, \widehat{\rho}_{U,\Sigma}(x)] \cap \widehat{V}$ is a proper open subset of $[x, \widehat{\rho}_{U,\Sigma}(x)]$, and the connected component of x in the latter open subset if of the form $[x, y)$ with $y \in \partial\widehat{V}$. But we have seen that any branch at infinity of $\widehat{V}$ having a limit in $\widehat{U}$ is contained in Σ, so there exists $z \in [x, y)$ with $[z, y) \subset \Sigma$, contradicting the fact that $[x, \widehat{\rho}_{U,\Sigma}(x))$ does not intersect Σ.

It remains to show that $\widehat{V} = \widehat{\rho}_{U,\Sigma}^{-1}(\widehat{V} \cap \Sigma)$. It follows from the above that $\widehat{V}$ is contained in $\widehat{\rho}_{U,\Sigma}^{-1}(\widehat{V} \cap \Sigma)$. Conversely, let x be a point of $\widehat{U}$ such that $\widehat{\rho}_{U,\Sigma}(x) \in \widehat{V}$. If $x \notin \widehat{V}$ then there exists $y \in [x, \widehat{\rho}_{U,\Sigma}(x))$ with $y \in \partial\widehat{V}$; but then $y \in \Sigma$, contradicting the definition of $\widehat{\rho}_{U,\Sigma}$. □

4. Generalities on Nash functions

We start this section by proving that a flat map induces an open map at the level of stable completions. We shall need it only for étale maps, for which it could be deduced (through base-change by the normalization of X) from [**HL16**, Cor. 9.7.2]. But as its proof is quite short, it seems to us to be of independent interest to handle the flat case.

4.1. Proposition. — *Let* $f\colon Y \to X$ *be a flat map between algebraic* K*-varieties. The induced map* $\widehat{Y} \to \widehat{X}$ *is open.*

Proof. — We can assume that K is a model of ACVF. Let V be a K-definable subset of Y such that $\widehat{V}$ is open in $\widehat{Y}$; set $U = f(V)$. We are going to prove that $\widehat{U}$ is open in $\widehat{X}$. Let u be a point of $\widehat{U}(K)$ and let p be a type over K on X such that u belongs to the K-Zariski closure of p and for every pair (f,g) of K-regular functions defined on a K-Zariski neighborhood of u such that $|f(u)| < |g(u)|$ one has $|f(p)| < |g(p)|$; we are going to show that p lies on U, which will end the proof.

Since $f(\widehat{V}) = \widehat{U}$ there exists some $v \in \widehat{V}(K)$ lying above u. From the remaining part of the proof we will see X and Y as K-schemes, and use the corresponding language, which is more convenient for exploiting flatness; so *points* will be *scheme-theoretic points* and we will use contructible compactness instead of logical compactness. We see u and p as elements of the valuation spectrum of X; i.e., pairs consisting of a point of X and a valuation on its residue field extending that of K. Let θ, resp. ξ, be the points of X supporting u, resp. p. As the type u belongs to $\widehat{U}(K)$, it is given by a valuation $|\cdot|$ on $\kappa(\theta)$, extending that of K and with value group $|K^\times|$. The assumption on the pair (u,p) implies that ξ specializes to θ (since $|f(u)| > 0 \Rightarrow |f(p)| > 0$) and that p is of the form $|\cdot|' \circ \nu$ where ν is a valuation on $\kappa(\xi)$ trivial on K and centered at θ and where $|\cdot|'$ is a valuation on the residue field of ν whose restriction to $\kappa(\theta)$ refines $|\cdot|$ (as $|f(u)| < 1 \Rightarrow |f(p)| < 1$); but note that $|\cdot|'$ and $|\cdot|$ coincide on K.

For proving the result, we can replace the scheme X with the reduced closure of ξ (and all other data by the corresponding fiber products) so we can assume that X is integral and ξ is its generic point. We denote by ω the point of Y supporting v.

Our purpose is now to show the existence of some q on the valuation spectrum of Y that lies above p and has the property that its support generalizes ω and that $|f(q)| < |g(q)|$ for all pairs (f,g) of K-regular functions on Y around ω such that $|f(v)| < |g(v)|$. By openness of $\widehat{V}$ in $\widehat{Y}$ this will ensure that q lies on V, so that $p = f(q)$ will lie on U – we will then be done.

Let Z be any blow-up of X along a proper closed subscheme C. By flatness $Y \times_X Z$ is the blow-up of Y along $Y \times_X C$. Let ζ be the center of μ on Z and let E_Z be the set of points of $Y \times_X Z$ lying above ω on Y and ζ on Z; it is pro-constructible (in the sense of scheme theory) and non-empty, as ω and ζ both lie over θ. The collection (E_Z) for Z running through all possible blow-ups of X along a proper closed subscheme is a filtering system of non-empty compact sets (for the constructible topology), so it has a non-empty limit. Let ν be an element of this limit. For every (Z,C) as above, $Y \times_Z C$ is nowhere dense in Y by flatness of $Y \to Z$, so the set of irreducible components of $Y \times_X Z$ is in natural bijection with the set I of irreducible components of Y (to an irreducible component T of $Y \times_X Z$ one associates its image S on Y; one recovers T from S by taking its strict transform). Then ν defines a projective system of non-empty subsets of I, which necessarily stabilizes. This implies that there exists an irreducible component S of Y such that ν induces an element of the limit of a filtering family of blow-ups of S along proper centers, which can be lifted (again by constructible compactness) to an element of the limit of *all* blow-ups of S along proper centers; we still call this element ν. By construction, ν induces an element of the valuative spectrum Y, supported at the generic point of S and centered at ω, which we still denote by ν. The valuation $|\cdot|$ on $\kappa(\omega)$ defined by v restricts to the valuation $|\cdot|$ of $\kappa(\theta)$ defined by u; then the refinement $|\cdot|'$ of $|\cdot|$ on $\kappa(\theta)$ extends to a refinement $|\cdot|'$ of $|\cdot|$ on $\kappa(\omega)$, and in turn to a valuation $|\cdot|'$ on the residue field of ν. Now the valuation $q = |\cdot|' \circ \nu$ satisfies the required properties. □

4.2. ***Definition.*** — Let K be a valued field, let X be a K-variety and let U be a K-definable subset of X. A *nice K-definable exhaustion* of U is a non-decreasing K-definable family $(U_t)_{0\leqslant t<a}$ of subsets of U, where a is an element of $|K^\times|^{\mathbf{Q}}$, such that U is the union of the U_t and each $\widehat{U_t}$ is compact.

4.3. ***Example.*** — Let $D \subset \mathbf{A}^n$ be an open polydisc of polyradius r. For every $t \in [0,1)$, let D_t be the closed polydisc of polyradius tr. Then (D_t) is a nice K-definable exhaustion of D.

4.4. ***Remark.*** — Let X be a K-algebraic variety and let U be a K-definable subset of X. Assume that there exists a finite extension L of K such that U has a nice L-definable exhaustion $(U_t)_t$. Then U has nice K-definable exhaustion.

Indeed, for seeing this we can replace K by K^{h} and L by L^{h}, so we can assume that K is henselian. Then we can enlarge L and assume it is normal over K, and now $(\bigcup_{\sigma\in\mathrm{Aut}(L/K)}\sigma(U_t))$ is a nice K-definable exhaustion of U.

More generally, assume that there exists a valued extension M of K such that U admits a nice M-definable exhaustion *and that K is non-trivially valued*; then U admits a nice K-definable exhaustion. Indeed, we can enlarge M and assume that it is a model of ACVF. The algebraic closure $\overline{K}$ of K in M is then a model of ACVF (as K is non-trivially valued), so U has a nice $\overline{K}$-definable exhaustion, necessarily defined over some finite extension of K. Then by the above U has a nice K-definable exhaustion. Note that one cannot drop the assumption that K is non-trivially valued: a positive dimensional open unit disc over a trivially valued fields does not admit any nice K-definable exhaustion, but of course it admits one over any non-trivially valued extension of K.

4.5. ***Lemma.*** — *Let K be a valued field, let X be a K-variety and let Y be a Zariski-closed subset of X. There exists a K-definable function $\delta\colon X \to \Gamma_0$ inducing a continuous map $\widehat{X} \to \Gamma_0$ and whose vanishing locus is Y.*

Proof. — Let $\mathscr{X}$ be any proper model of X over the ring of integers $\mathscr{O}_K$ of K (one can build such a model by applying Nagata's compactification over some finitely generated $\mathbf{Z}$-subalgbera of $\mathscr{O}_K$ over which X is defined). Let $\mathscr{Y}$ be the schematic closure of Y in $\mathscr{X}$, let $\mathscr{X}'$ be the blow-up of $\mathscr{X}$ along $\mathscr{Y}$, and let $\mathscr{Y}'$ be the exceptional divisor of the blow-up; let Y' and X' denote the generic fibers of $\mathscr{X}'$ and $\mathscr{Y}'$ respectively. As $X' \to X$ is an isomorphism outside of Z and as $\widehat{X'} \to \widehat{X}$ is topologically proper, we can replace (X,Y) by (X',Y'); hence we can assume that X has a proper model $\mathscr{X}$ over $\mathscr{O}_K$ and that Y is the generic fiber of an effective Cartier divisor $\mathscr{Y}$ of $\mathscr{X}$. Let $\mathscr{L}$ be the line bundle on $\mathscr{X}$ associated to $\mathscr{Y}$; it comes equipped with a canonical section s having $\mathscr{Y}$ as its zero-locus. Let L be the generic fiber of $\mathscr{L}$; it is endowed with a natural model metric $|\cdot|$ induced by $\mathscr{L}$, and $\delta := x \mapsto |s(x)|$ is a K-definable map taking values in Γ_0 that vanishes exactly on Z; being locally given by norms of regular functions, it induces a continuous definable map on $\widehat{X}$. □

4.6. Lemma. — *Let K be a valued field, let X be a K-variety and let U be a K-definable subset of X admitting a nice K-definable exhaustion. Then $\widehat{U}$ has compact closure in $\widehat{X}$.*

Proof. — Fix a nice exhaustion $(U_t)_{0\leqslant t<a}$ of U. As X is separated (by our very definition of a variety) there exists a proper K-variety Y equipped with a dense open immersion $i\colon X \hookrightarrow Y$ (defined over K). We use i to identify X with an open subset of Y.

Let Z be the complement of X in Y. By the preceding lemma there exists a K-definable function $\delta\colon Y \to \Gamma_0$ that vanishes exactly on Z and extends to a continuous definable from $\widehat{Y}$ to Γ_0.

Our purpose is to prove that δ does not vanish on the closure of $\widehat{U}$ in $\widehat{Y}$; this will ensure that the latter closure, which is compact by properness of Y, is contained in $\widehat{X}$, and therefore that it coincides with the closure of $\widehat{U}$ in $\widehat{X}$, which will end the proof.

Let $\psi\colon [0,a) \to \Gamma_0$ be the map that sends t to the infimum of δ on U_t. It is K-definable, non-increasing, and it does not vanish: indeed if one had $\psi(t) = 0$ from some t then δ would vanish on U, contradicting the fact that $U \cap Z = \varnothing$ since $U \subset X$. Then by o-minimality there exists some $\varepsilon < 1$, some $b \in |K^\times|^{\mathbf{Q}}$ and some non-positive rational number r such that $\psi(t) = bt^r$ for all $t \in (\varepsilon, 1)$. But then $\psi \geqslant ba^r$ on the whole of U, which implies that $\delta \geqslant ba^r$ on the whole of U, and this also holds on the whole of $\widehat{U}$ (just by realizing types); by continuity of δ one still has $\delta \geqslant ba^r$ on the closure of $\widehat{U}$ in $\widehat{X}$, and we are done. ☐

4.7. Lemma. — *Let $f\colon Y \to X$ be a quasi-finite morphism of K-varieties. Assume that, for every $y \in Y$, the field $K(y)$ is separable over $K(f(y))$ (e.g., f is étale, or merely unramified). Let D be a K definable subset of Y such that $f|_D$ is injective. One has then $K(y)^{\mathrm{h}} = K(f(y))^{\mathrm{h}}$ for every $y \in D$.*

Proof. — Set $E = f(D)$. This is a K-definable subset of X, and f induces a K-definable bijection $D \simeq E$. Let σ be the inverse bijection. This is a K-definable map. By construction one has $y = \sigma(f(y))$ for every $y \in D$, so that y is definable over $K(f(y))$. As $K(y)$ is a finite separable extension of $K(f(y))$ by assumption, this means that $K(y) \subset K(f(y))^{\mathrm{h}}$, so $K(y)^{\mathrm{h}} = K(f(y))^{\mathrm{h}}$. ☐

4.8. Definition. — Let $f\colon Y \to X$ be a morphism between algebraic varieties over some valued field K, let V be a K-definable subset of Y and let U be a K-definable subset of X. We shall say that f induces a (K-definable) *Nash isomorphism* between V and U if V is contained in the étale locus of f and f induces a K-definable isomorphism between V and U inducing in turn a *homeomorphism* between $\widehat{V}$ and $\widehat{U}$.

4.9. Example. — Assume that f is a map from Y to a Zariski-open subset Ω of X containing U, and that V is contained in the étale locus of f. Moreover we assume to be given a K-regular function φ on Ω and K-regular function ψ on X such that the following hold:

◇ $|\varphi| < 1$ on Ω ;

◇ for every $z \in U$ there exists a unique element $\sigma(z)$ in the fiber $f^{-1}(z)$ with $|\psi(\sigma(z))| = |\varphi(z)|$;

◇ for every $z \in U$ and every $t \neq \sigma(z)$ in in $f^{-1}(z)$ one has $|\psi(t)| = 1$;

◇ $\sigma(U) = V$.

By replacing Y with the étale locus of f we can assume that f is étale. Then f induces a Nash isomorphism between V and U. Indeed, by replacing Y with the étale locus of f (which contains V) we can assume that f is étale.

By construction, f induces a definable bijection $V \simeq U$ (with inverse bijection σ), and in turn a continuous bijection $\widehat{V} \to \widehat{U}$.

By proposition 4.1, f induces an open map from $\widehat{Y'}$ to $\widehat{X}$; by base-change, $f^{-1}(\widehat{U}) \to \widehat{V}$ is open. As V is by definition the subset of $f^{-1}(U)$ defined by the condition $|\psi| < 1$, its stable completion $\widehat{V}$ is open in $f^{-1}(\widehat{U})$, so f induces an open map from $\widehat{V}$ to $\widehat{U}$, hence a homeomorphism $\widehat{V} \simeq \widehat{U}$.

4.10. Lemma. — *Let $f\colon Y \to X$ a morphism between K-algebraic varieties, and let U and V be K-definable subsets of X and Y respectively, such that f induces a Nash isomorphism between V and U.*

(1) *The set U admits a nice K-definable exhaustion if and only if so does V.*

(2) *The set $\widehat{V}$ is open and closed in $f^{-1}(\widehat{U})$.*

(3) *The set $\widehat{U}$ is open in $\widehat{X}$ if and only if $\widehat{V}$ is open in $\widehat{Y}$.*

Proof. — By definition f induces a homeomorphism between $\widehat{V}$ and $\widehat{U}$. So if $(U_t)_t$ is a nice K-definable exhaustion of U then $(f^{-1}(U_t) \cap V)_t$ is a nice K-deifnable exhaustion of $\widehat{V}$, and if $(V_t)_t$ is a nice K-definable exhaustion of V then $(f(V_t))_t$ is a nice K-definable exhaustion of U, whence (1).

Let $\sigma\colon U \to V$ be the converse bijection of $f|_V\colon V \to U$. As our varieties are separated by definition, their stable completions are Hausdorff, so the continuous section σ of $f^{-1}(\widehat{U}) \to \widehat{U}$ has closed image; this means that $\widehat{V}$ is closed in $f^{-1}(\widehat{U})$.

It remains to prove that $\widehat{V}$ is open in $f^{-1}(\widehat{U})$ as well as (3). For both properties we can replace Y by the étale locus of f (which contains V) and assume that f is étale. Then by the local structure of an étale map there exists a finite cover (Y_i) of Y by K-Zariski open affine subsets and, for each i, an affine K-Zariski open subset X_i of X such that f induces an elementary étale map $Y_i \to X_i$; i.e., $X_i = \operatorname{Spec} A_i$ and Y_i is the open subset $D(P_i(T))$ of $\operatorname{Spec} A_i[T]/P_i$ for some monic polynomial $P_i \in A_i[T]$.

Set $U_i = U \cap X_i \cap \sigma^{-1}(Y_i)$ and $V_i = \sigma(U_i) = V \cap Y_i$. It suffices to prove that $\widehat{V_i}$ is open in $f^{-1}(\widehat{U_i}) \cap \widehat{Y_i}$ for all i, and that each $\widehat{V_i}$ is open in $\widehat{Y_i}$ if and only if $\widehat{U_i}$ is open in $\widehat{X_i}$. So we can assume that $X = \operatorname{Spec} A$ and Y is the open subset $D(P'(T))$ of $\overline{Y} := \operatorname{Spec} A[T]/P$ for some monic polynomial $P = T^n + \sum_{1 \leqslant i \leqslant n} a_i T^{n-i} \in A[T]$; we still denote by f the structure map from $\overline{Y}$ to X (that extends the original f).

Let $\Phi\colon X \to \Gamma$ the K-definable map $\max|a_i|^{1/i}$ from X to Γ, and let Ψ be the map $|T - T \circ \sigma \circ f|$ from $\overline{Y}$ to Γ. Denote by Ω the K-definable subset of Y defined by the inequality $\Psi < |P'(T)| \cdot \Phi^{1-n}$. By a standard computation, for all models M of ACVF containing K and all $x \in U(M)$ such that $\sigma(x) \in \Omega$, there are no roots in M of the polynomial $T^n + \sum_i a_i(x)T^{n-i}$ different from $T(\sigma(x))$ and whose distance to the latter is strictly bounded by $P'(T(\sigma(x))(\max_i |a_i(x)|^{1/i})^{1-n}$. Otherwise said $\sigma(x)$ is the unique point of $f^{-1}(x)$ inside Ω. As a consequence one has $\Omega \cap f^{-1}(U) = V$, so $\widehat{\Omega} \cap f^{-1}(\widehat{U}) = \widehat{V}$, hence $\widehat{V}$ is open in $f^{-1}(\widehat{U})$ inside $\widehat{\overline{Y}}$, and a fortiori inside $\widehat{Y}$. It clearly follows that if $\widehat{U}$ is open in $\widehat{X}$ then $\widehat{V}$ is open in $\widehat{Y}$.

Conversely if $\widehat{V}$ is open in $\widehat{Y}$ then $\widehat{U} = f(\widehat{V})$ is open in $\widehat{X}$ since étale maps are flat and thus open (Proposition 4.1). □

4.11. — Let K be a valued field, let X be a K-algebraic variety and let U be a K-definable subset of X.

We will use the notation $\mathscr{Z}_K(U)$ (where X is implicitly understood) for the ordered and filtered set of all K-Zariski open subsets of X that contain U, and $\mathscr{E}_K(U)$ for the ordered collection of triples (Y, V, f) where Y is a K-variety, V is a K-definable subset of Y and $f : Y \to X$ is a map inducing a Nash isomorphism $V \simeq U$. We observe that $\mathscr{E}_K(U)$ is filtered: if (Y, V, f) and (Z, W, g) belong to $\mathscr{E}_K(U)$ then $(Y \times_X Z, W \times_U V, h)$, where $h : Y \times_X Z \to X$ is the structure map, belongs to $\mathscr{E}_K(U)$ and refine both (Y, V, f) and (Z, W, g). We also remark that one can see $\mathscr{Z}_K(U)$ as contained in $\mathscr{E}_K(U)$.

We then set

$$K[U] = \operatorname{colim}_{Y \in \mathscr{Z}_K(U)} K[Y] \text{ and } K\langle U\rangle = \operatorname{colim}_{(Y,V,f)\in\mathscr{E}_K(U)} K[Y].$$

By construction $K\langle U\rangle$ is a K-algebra, and by flatness of étale maps $K[U]$ is a subalgebra of $K\langle U\rangle$. It follows from the construction that every $\varphi \in K\langle U\rangle$ induces K-definable maps $\varphi : U \to \mathbf{A}^1$ and $|\varphi| : U \to \Gamma$.

Let $(Y, V, f) \in \mathscr{E}_K(U)$. By Lemma 4.7 $K(f(v))^{\mathrm{h}} = K(v)^{\mathrm{h}}$ for every v in V; it follows that $\varphi(u) \in K(u)^{\mathrm{h}}$ for all $u \in U$ and all $\varphi \in K\langle U\rangle$; of course if $\varphi \in K[U]$ then $\varphi(u) \in K(u)$.

Note that for every $(Y, V, f) \in \mathscr{E}_K(U)$ there is a natural map from $K\langle V\rangle$ to $K\langle U\rangle$ which is an isomorphism; it is compatible with the identifications between $K(f(v))^{\mathrm{h}}$ and $K(v)^{\mathrm{h}}$ for every $v \in V$.

4.11.1. — Assume that U admits a nice L-definable exhaustion for some valued extension L of K (this amounts to requiring that it admits such an exhaustion over K except in the trivially valued case, see Remark 4.4). If (Y, V, f) belongs to $\mathscr{E}_U(K)$, it follows from Lemma 4.10 that V also admits a nice L-definable exhaustion, so $\widehat{V}$ has compact closure in $\widehat{Y}$ in view of Lemma 4.6. Hence $\varphi|_{\widehat{V}}$ is bounded for every $\varphi \in K[Y]$; as a consequence, the function $|\varphi| : U \to \Gamma$ is bounded for every $\varphi \in K\langle U\rangle$.

We then set

$$\mathsf{RV}(K)[U] = \coprod_{r\in|K^\times|^{\mathbf{Q}}} \{\varphi \in K[U],\ \|\varphi\|_{\infty,U} \leqslant r\}/\{\varphi \in K[U],\ \|\varphi\|_{\infty,U} < r\};$$

and

$$\mathsf{RV}(K)\langle U\rangle = \coprod_{r\in|K^\times|^{\mathbf{Q}}} \{\varphi \in K\langle U\rangle,\ \|\varphi\|_{\infty,U} \leqslant r\}/\{\varphi \in K\langle U\rangle,\ \|\varphi\|_{\infty,U} < r\}.$$

These are generalized reduced $\mathsf{RV}(K)$-algebras, $\mathsf{RV}(K)[U]$ embeds naturally in $\mathsf{RV}(K)\langle U\rangle$, and $\mathsf{RV}(K)\langle U\rangle$ is the colimit of the $\mathsf{RV}(K)[V]$, taken on all triples $(Y, V, f) \in \mathscr{E}_K(U)$. If $(Y, V, f) \in \mathscr{E}_K(U)$ then $\mathsf{RV}(K)[U]$ embeds into $\mathsf{RV}(K)[V]$, and $\mathsf{RV}(K)\langle U\rangle \simeq \mathsf{RV}(K)\langle V\rangle$.

If $\varphi \in K\langle U\rangle$ and $\|\varphi\|_{U,\infty} \neq 0$, we shall often denote by $\mathsf{rv}_U(\varphi)$ the image of φ in the $\|\varphi\|_{U,\infty}$-summand of $\Lambda_K(U)$.

4.11.2. — Assume that $U(M)$ is open in $X(M)$ for the valuative topology for some model M of ACVF containing K (this is then the case for all such models), and that U is contained in the reduced locus on X (but we do not assume anymore that U admits a nice exhaustion over some extension of K). We observe then that if (Y, V, f) belongs to $\mathscr{E}_K(U)$ then V is contained in the reduced locus of Y (by étaleness of f) and that $V(M)$ is open in $Y(M)$ for any model M of ACVF containing K (as $V(M)$ is open in $f^{-1}(U(M))$, for $\widehat{V}$ is open in $f^{-1}(\widehat{U})$ by Lemma 4.10).

Then $K\langle U\rangle$ embeds into the set of K-definable functions on U. Indeed, let $\varphi \in K\langle U\rangle$ inducing the zero map on U. We want to prove that $\varphi = 0$. For proving this we can replace (X, U) by (Y, V) for any (Y, V, f) in $\mathscr{E}_K(U)$; so we can assume that φ comes from some function in $K[X]$, which we still denote by φ; and we can also assume by shrinking it if needed that X is reduced. As U is open for the valuative topology, its intersection with every irreducible component Z of X contains some point z whose K-Zariski-closure is Z [**Duc12**, Lemme 1.1], It follows that the pointwise vanishing locus of φ, which contains U by assumption, contains a K-Zariski neighborhood X' of U in X; as X is reduced $\varphi|_{X'} = 0$; a fortiori $\varphi = 0$ when viewed in $K\langle U\rangle$.

In this setting we may think of $K\langle U\rangle$ as a model-theoretic substitute for the ring of analytic functions on U, which is an ACVF analog of the ring of Nash functions in RCF. We will therefore call the elements of $K\langle U\rangle$ *Nash functions* on U.

Be aware that we do not pretend to have defined a *sheaf* (for whatever Grothendieck topology) of Nash functions; since we will morally only deal with "Stein" spaces, global rings defined as above will be sufficient for our purposes.

4.12. Lemma. — *Let K be a henselian valued field, let X be a K-algebraic variety and let U be a K-definable subset of X. Let L be a finite extension of K.*

(1) *Zariski-open subsets defined over K are cofinal in $\mathscr{Z}_L(U)$.*

(2) *Triples defined over K are cofinal in $\mathscr{E}_L(U)$.*

(3) *The natural maps*

$$K[U] \otimes_K L \to L[U] \quad \textit{and} \quad K\langle U\rangle \otimes_K L \to L\langle U\rangle$$

are isomorphisms.

Proof. — Let M be a model of ACVF containing L and let E be the set of K-embeddings from L into M; we denote by ι the inclusion $L \subset M$.

We start with (1). Let $Y \in \mathscr{Z}_L(U)$. For every $\sigma \in E$, let Y^σ be the M-Zariski open subset of X deduced from Y through σ. Then $\bigcap_{\sigma\in E} Y^\sigma$ is a K-definable Zariski-open subset of X, containing U and contained in Y, whence (1).

Now let us prove (2); the idea is basically the same, except that intersections has to be replaced by fiber products. So let $(Y, V, f) \in \mathscr{E}_L(U)$. Our purpose is to exhibit a triple (Z, W, g) in $\mathscr{E}_K(U)$ which refines (Y, V, f). We first note that as f is étale at every point of V we can shrink Y so that f is étale on the whole of Y. For every $\sigma \in E$, let $(Y^\sigma, V^\sigma, f^\sigma)$ be the element of $\mathscr{E}_M(U)$ deduced from (Y, V, f) through σ, and set $Z = \prod_X Y^\sigma$; note that $(Y^\iota, V^\iota, f^\iota) = (Y, V, f)$. Then $Z \to X$ is an étale map. As Y is in fact defined over the separable closure of K in L by topological invariance of the étale site, étale descent shows that the X-étale scheme Z is defined over K; let g denote the structure map from Z to X. Let W be the K-definable subset $\prod_X V^\sigma$ of Z. We are going to show that (Z, W, g) fulfills our requirements. One has by definition $g(W) \subset f(V) = U$.

First we note that the projection $p\colon Z \to Y$ (corresponding to $\sigma = \iota$) is an X-morphism, so is étale; moreover $f \circ p = g$, and p is defined over L. Hence it suffices to prove that g induces an L-definable Nash isomorphism $W \simeq U$.

Let $u \in U(M)$. As f induces an L-definable Nash isomorphism $V \to U$, the map f^σ induces for every σ an M-definable Nash isomorphism $V^\sigma \to U$; for every σ, let v^σ be the unique pre-image of u in $V^\sigma(M)$. Then $(v_\sigma)_\sigma$ is clearly the unique pre-image of u in W, so g induces a K-definable isomorphism $W \simeq U$.

The map $\widehat{W} \to \widehat{U}$ induced by g is then a continuous bijection. Let us prove that this map is closed – it will then be a homeomorphism. Let (U_t) be a K-definable nice exhaustion of U; for each t, set $V_t = f^{-1}(U_t)$. Using the definition of (definably) closedness and compactness through limits of definable types, we see that it suffices to prove that $g^{-1}(\widehat{U_t})$ is compact for every t. But $g^{-1}(\widehat{U_t})$ is equal to the pre-image of $\prod_{\widehat{X}} (f^\sigma)^{-1}(U_t)$ under the canonical map $\widehat{Z} \to \prod_{\widehat{X}} \widehat{Y^\sigma}$, which is proper (one can check it using the characterization of compactness through limits of definable types). As each of the $(f^\sigma)^{-1}(U_t)$ is compact, $g^{-1}(\widehat{U_t})$ is compact and (2) is proved.

Now (3) just follows from (1) and (2) by a limit argument, in view of the fact that $L[Z] = K[Z] \otimes_K L$ for every K-variety Z. □

4.13. Reminders on tame extensions. — Let K be a henselian valued field and let K^{sep} be a separable closure of K; set $G = \text{Gal}(K^{\text{sep}}/K)$.

4.13.1. — Let us first remind the classical formulation of the theory.

4.13.1.1. — One defines the *inertia* subgroup of G as the kernel I of the natural action of G on the residuel field $\mathsf{k}(K^{\text{sep}})$, and an algebraic extension L of K is called *unramified* if I acts trivially on the corresponding G-module $\text{Hom}_K(L, K^{\text{sep}})$. If $[L : K] < \infty$ this amounts to requiring that $\mathsf{k}(L)$ be separable and of degree $[L : K]$ over $\mathsf{k}(K)$, and also that the valuation ring of L be étale over that of K.

4.13.1.2. — One defines the *ramification* subgroup W of G as follows: it is trivial if $\mathsf{k}(K)$ is of characteristic 0; if $\mathsf{k}(K)$ is of characteristic $p > 0$ then W is the unique pro-p-Sylow of I (proving the uniqueness is part of the theory). An algebraic extension L of K is called *tamely ramified* if W acts trivially on the corresponding G-module $\text{Hom}_K(L, K^{\text{sep}})$. If $[L : K] < \infty$ this amounts to requiring that $\mathsf{k}(L)$ be separable over $\mathsf{k}(K)$, that $(|L^\times| : |K^\times|)$ be prime to the residue characteristic exponent, and that

$$|\mathsf{k}(L) : \mathsf{k}(K)] \cdot (|L^\times| : |K^\times|) = [L : K].$$

If K is of residue characteristic zero every algebraic extension of K is tame. In general, every finite extension of K whose degree is prime to the residue characteristic exponent of K is tame.

4.13.2. — There is an alternative handling of tame ramification, carried out by the first author in the first section of [**Duc13**], which uses RV and the theory of generalized ring extensions, and make the theory fully analogous to that of unramified extensions.

In this approach the group W is directly defined as the kernel of the natural action of G on the generalized residue field $\mathsf{RV}(K^{\mathrm{sep}})$, and one proves that a finite extension L of K is tamely ramified if and only if $\mathsf{RV}(L)$ is separable and of degree $[L:K]$ over $\mathsf{RV}(K)$.

4.14. Corollary. — *Let K be a henselian valued field, let X be a K-algebraic variety and let U be a K-definable subset of X admitting a nice K-definable exhaustion. Let L be a tamely ramified finite extension of K. The natural maps*

$$\mathsf{RV}(K)[U]\otimes_{\mathsf{RV}(K)}\mathsf{RV}(L)\to\mathsf{RV}(L)[U] \quad \textit{and} \quad \mathsf{RV}(K)\langle U\rangle\otimes_{\mathsf{RV}(K)}\mathsf{RV}(L)\to\mathsf{RV}(L)\langle U\rangle$$

are isomorphisms.

Proof. — The generalized ring $\mathsf{RV}(K)\langle U\rangle$ is the colimit of the $\mathsf{RV}(K)[V]$ for (Y,V,f) running through $\mathscr{E}_K(U)$, and by 4.12 (2) $\mathsf{RV}(L)\langle U\rangle$ is the colimit of the $\mathsf{RV}(L)[V]$ for (Y,V,f) also running through $\mathscr{E}_K(U)$, so it suffices to prove that the canonical map $\mathsf{RV}(K)[U]\otimes_{\mathsf{RV}(K)}\mathsf{RV}(L)\to\mathsf{RV}(L)[U]$ is an isomorphism.

Let $a_1,\ldots,a_r$ be elements of $L^\times$ such that $(\mathsf{rv}(a_i))_i$ is a basis of $\mathsf{RV}(L)$ over $\mathsf{RV}(K)$. As L is tamely ramified over K it follows from [**Duc13**, 2.21 and Lemme 2.3] that (a_i) is a basis of L over K and that for every henselian valued extension F of K and every element $\lambda=\sum\lambda_i\otimes a_i$ of the tensor product $L\otimes_K F$ one has $\|\lambda\|_\infty=\max_i|\lambda_i|\cdot|a_i|$. Here $\|\cdot\|_\infty$ denotes the spectral norm of $L\otimes_K F$; the latter ring is a product $\prod F_j$ of finitely many finite separable extensions of F, and $\|\lambda\|_\infty$ is then equal to the maximum of the absolute values of the components of λ for the decomposition $L\otimes_k F=\prod F_j$.

By Lemma4.12 (3) one has $L[U]=L\otimes_K K[U]$, so $L[U]$ is a free module over $K[U]$ with basis (a_i). Let f be an element of $L[U]$. Let us write $f=\sum a_i f_i$ with each f_i in $K[U]$. Let p be a quantifier-free type over K lying on U and let $\{p_j\}$ be the set of quantifier-free types over L lying above p; there is a natural isomorphism $L\otimes_K K(p)^{\mathrm{h}}\simeq\prod L(p_j)^{\mathrm{h}}$, which maps $\sum_i a_i\otimes f_i(p)$ to $(f(p_j))_j$. As a consequence,

$$\max_j|f(p_j)|=\max_i|a_i|\cdot|f_i(p)|.$$

It follows that

$$\|f\|_{U,\infty}=\max_i|a_i|\cdot\|f_i\|_{U,\infty}.$$

But this immediately implies that $(\mathsf{rv}(a_i))_i$ is a basis of $\mathsf{RV}(L)[U]$ over $\mathsf{RV}(K)[U]$, which ends the proof. □

5. Abstract open polydiscs and their Nash functions

5.1. — We fix from now on a henselian valued field K, an algebraic variety X over K, a K-definable subset U of X, and an algebraic map $f\colon X\to\mathbf{A}^n_K$ inducing a Nash isomorphism between U and a K-definable open polydisc D centered at the origin; since D is K-definable, its polyradius $(r_1, \ldots, r_n)$ has components in $|K^\times|^{\mathbf{Q}}$. The variety $\mathbf{A}^n$ is reduced, $\widehat{D}$ is open in $\widehat{\mathbf{A}^n}$ and D admits a nice definable exhaustion over any non-trivially valued extension of K.

We can thus apply 4.11.1 and 4.11.2. It follows that elements of $K\langle U\rangle$ are actual functions on U (which are called Nash functions), that they are bounded, and that $K[U]$ and $K\langle U\rangle$ give rise to two generalized rings $\mathsf{RV}(K)[U]$ and $\mathsf{RV}(K)\langle U\rangle$. The map f induces isomorphisms $K\langle U\rangle \simeq K\langle D\rangle$ and $\mathsf{RV}(K)\langle U\rangle \simeq \mathsf{RV}(K)\langle D\rangle$.

We denote by u the unique pre-image of 0 on X. The point u is K-definable, and $K(u)$ is finite and separable over K as f is étale at u; as we assume here that K is henselian, this implies that u is a K-point.

5.2. Conventions. — For simplicity we shall often write elements of $\mathscr{E}_K(U)$ as pairs (Y, V), the map $g\colon Y \to X$ being implicit; and the image of f in $K[Y]$, which is rigorously equal to $f \circ g$, will still be denoted by f; note that with this convention f induces a Nash isomorphism between V and D.

For studying functions of $K\langle U\rangle$ it will often be useful to replace (X, U) by some suitable (Y, V) in $\mathscr{E}_K(U)$ fulfiling some property in which case we will say "up to refining (X, U) we can assume that this or that property holds...".

Typically, we will often refine (X, U) so that:

- ◇ f is étale, hence X is smooth – this can be achieved by replacing X by the étale locus of f;
- ◇ X is connected – as $\widehat{U}$ is connected, being homeomorphic to the stable completion of an open polydisc, this can be achieved by replacing X by its unique connected component containing U, which is K-definable;
- ◇ a given element φ of $K\langle U\rangle$ comes from $K[X]$; note that under the assumption that X is smooth and connected $K[X]$ embeds into $K[U] \subset K\langle U\rangle$, for it embeds in $K[X']$ for every K-Zariski neighborhood X' of X, so in this situation we will consider $K[X]$ as a subring of $K\langle U\rangle$ and simply say that φ belongs to $K[X]$;

If $n \geqslant 1$ and if f is assumed to be étale and X connected, then $X' := \{f_n = 0\}$ is a smooth hypersurface of X, equipped with an étale map $(f_1, \ldots, f_{n-1})$ to $\mathbf{A}^{n-1}$, and it will also be possible to also assume that X' is connected: indeed, if U' denotes the intersection $X' \cap U$ then $(f_1, \ldots, f_{n-1})$ induces a homeomorphism between $\widehat{U'}$ and the stable completion of an $(n-1)$-dimensional open polydisc, so $\widehat{U'}$ is connected, hence U intersects only one irreducible component of X', so one can remove the other components from X.

When we perform such a replacement of (X, U) by (Y, V), we of course replace u by its unique pre-image v on V. This operation does not preserve the K-algebraic local ring $\mathscr{O}_{X,u}$ in general, but it preserves its henselization, and a fortiori its (formal) completion: indeed, by étaleness at v the map $Y \to X$ induces an isomorphism $\mathscr{O}^{\mathrm{h}}_{X,u} \simeq \mathscr{O}^{\mathrm{h}}_{Y,v}$; it therefore also preserves its (formal) completion.

Of course we will also proceed to such replacements for studying $K[U]$, in which case they are much simpler: one only has to replace X with a K-Zariski open neighborhood of U; for instance, the connected component of the étale locus of f that contains U.

We shall say that a n-uple $g = (g_1, \ldots, g_n)$ of elements of $K\langle U\rangle$ induces a Nash isomorphism between U and some K-definable subset Δ of $\mathbf{A}^n$ (in practice, Δ will be an open polydisc) if there is some $(Y, V) \in \mathscr{E}_K(U)$ such that g_i is induced for every i by some element of $K[Y]$ (still denoted by g_i) and g induces a K-definable Nash isomorphism between V and Δ. This will then be the case for every $(Y, V) \in \mathscr{E}_K(U)$ such that $g_i \in K[Y]$ for all i (by étale descent of étaleness). If g induces a Nash isomorphism between U and Δ it induces a homeomorphism between $\widehat{U}$ and $\widehat{\Delta}$.

5.2.1. — Let $\varphi \in K\langle U\rangle$. We will associate to it a "Taylor expansion at u" as follows. First, by refining (X, U) we can assume that $\varphi \in K[U]$. Now as f induces an isomorphism $\widehat{\mathscr{O}_{\mathbf{A}^n,0}} \simeq \widehat{\mathscr{O}_{X,u}}$ the image of φ in $\widehat{\mathscr{O}_{X,u}}$ can be naturally written as a formal power series $\sum a_I f^I$.

Let φ be an element of $K\langle U\rangle$ whose Taylor expansion at u is zero, and let us prove that $\varphi = 0$. By refining (X, U) we can assume that f is étale, X is connected and $\varphi \in K[X]$. As X is smooth and connected, taking the Taylor expansion at u defines an embedding $K[X] \hookrightarrow K[\![f]\!]$, so $\varphi = 0$.

As a consequence, taking the Taylor expansion at u defines an embedding $K\langle U\rangle \hookrightarrow K[\![f]\!]$; in particular, $K\langle U\rangle$ is a domain.

5.3. Lemma. — *Let $\varphi = \sum a_I f^I$ be a function in $K\langle U\rangle$ and let be its Taylor expansion. Assume $n \geqslant 1$ and write formally the power series φ as $\sum_i A_i f_1^i$, where each A_i belongs to $K[\![f_1, \ldots, f_{n-1}]\!]$. Let X' be the zero-locus of f_n on X and set $U' = X' \cap U$.*

(1) *Each A_i is the Taylor expansion at u of some element φ_i of $K\langle U'\rangle$, and one has $\varphi_0 = \varphi|_{U'}$; if moreover f is étale, X and X' are connected and $\varphi \in K[X]$, then each φ_i belongs to $K[X']$.*

(2) *The following are equivalent:*

   (i) $\varphi \in f_n K\langle U\rangle$;

   (ii) $A_0 = 0$;

   (iii) $\varphi|_{U'} = 0$.

   If these conditions are fulfilled the Taylor expansion of the function φ/f_n of $K\langle U\rangle$ at u is equal to $\sum_{i\geqslant 1} A_i f_n^{i-1}$. If we assume moreover that f is étale, X and X' are connected and $\varphi \in K[X]$, then $\varphi \in f_n K[X]$.

5.4. Comments. — By construction $(f_1, \ldots, f_{n-1})$ induces a Nash isomorphism between U' and the open polydisc of polyradius $(r_1, \ldots, r_{n-1})$, hence $K\langle U'\rangle$ embeds into $K[\![f_1, \ldots, f_{n-1}]\!]$ trough Taylor expansion at u and the lemma's statement makes sense.

Proof of Lemma 5.3. — By refining (X, U) we can assume that f is étale, X is connected and $\varphi \in K[X]$; then X is smooth and as the pre-image of a coordinate hyperplane under an étale map, X' is smooth as well. The formal completion of X along its closed subscheme X' can be identified to $X' \times_K \operatorname{Spf} K[\![f_n]\!]$. Let $\sum \varphi_i f_n^i$ be the image of φ in $\mathscr{O}(X' \times_K \operatorname{Spf} K[\![f_n]\!])$, where each φ_i is a regular function on Y. One can see each φ_i as an element of $K[X']$, and by construction its expansion in $\widehat{\mathscr{O}_{X',u}} \simeq K[\![f_1, \ldots, f_{n-1}]\!]$ is precisely A_i ; moreover, $\varphi_0 = \varphi|_{X'}$. This ends the proof of (1).

Let us prove (2). As the Taylor expansion map $K\langle U'\rangle \to K[\![f_1, \ldots, f_{n-1}]\!]$ is injective, (ii) $\iff$ (iii). It is clear that (i)$\Rightarrow$(iii).

Now assume that (iii) holds. Up to refining (X, U) we can assume that X' is connected. Then $K[X']$ embeds into $\widehat{\mathscr{O}_{X',u}}$, so the equality $A_0 = 0$ implies that $\varphi_0 = 0$, hence that φ belongs to $f_n K[X]$ and that the Taylor expansion of φ/f_n at u is equal to $\sum_{i \geqslant 1} A_i f_n^{i-1}$. □

5.5. — Assume that $n \geqslant 1$. As in the proof of the Lemma above, let us denote by X' the closed subscheme $\{f_n = 0\}$, of X, and by U' the intersection $X' \cap U$. Set $f' = (f_1, \ldots, f_{n-1})$. The map $f' \colon X' \to \mathbf{A}^{n-1}$ induces a Nash isomorphism between U' and the open polydisc D' of polyradius $r' := (r_1, \ldots, r_{n-1})$. Let $\sigma \colon D' \to U'$ be the converse bijection. Then $\rho := \sigma \circ f' \colon U \to Z$ is a K-definable retraction of the inclusion $Z \hookrightarrow U$, which induces a continuous retraction of $\widehat{Z} \hookrightarrow \widehat{U}$. Moreover, since $f'|_{U'}$ preserves henselizations by Lemma 4.7, so does σ, whence a natural embedding of valued fields $K(\rho(x))^{\mathrm{h}} \hookrightarrow K(x)^{\mathrm{h}}$ for every $x \in U$.

Let ψ be an element of $K\langle U'\rangle$. For every $x \in U$ we have a well-defined element $\psi(\rho(x))$ of $K(\rho(x))^{\mathrm{h}} \subset K(x)^{\mathrm{h}}$. We note that $(\psi \circ \rho)|_Z = \psi$.

5.6. Proposition. — *We keep the notation above.*

(1) *For every $\psi \in K\langle U'\rangle$ the function $\psi \circ \rho$ belongs to $K\langle U\rangle$.*

(2) *Let $\varphi = \sum a_I f^I$ be an element of $K\langle U\rangle$. The following are equivalent:*

   (i) *there exists $\psi \in K\langle U'\rangle$ such that $\varphi = \psi \circ \rho$;*

   (ii) *the function f_n does not actually appear in the expansion $\sum_I a_I f^I$ (otherwise said $a_I = 0$ as soon as the n-th component of I is non-zero).*

   Moreover if these conditions are satisfied then ψ is necessarily equal to $\varphi|_{U'}$ and its Taylor expansion at u is also $\sum_I a_I f^I$, viewed as a power series in $(f_1, \ldots, f_{n-1})$.

Proof. — By refining (X, U) we can assume that f is étale (then X is smooth).

5.6.1. *A general construction.* — Let (Y', V', g') be an element of $\mathscr{E}_K(U')$, and suppose that g' is étale. Set $Z = X \times_{\mathbf{A}^{n-1}} Y'$ where the fiber product is understood along $f' \colon X \to \mathbf{A}^{n-1}$ and the étale map $f' \circ g' \colon Y' \to \mathbf{A}^{n-1}$. The K-variety Z is equipped with two natural morphisms $h \colon Z \to X$ (which is étale) and $p \colon Z \to Y'$. The composition $Y' \to X' \hookrightarrow X$ furnishes a section τ of p; let Z' denote its image.

The algebraic map $\tau \circ p$ is then a retraction of the closed embedding $Z' \hookrightarrow Z$. Let V be the K-definable subset $U \times_{\mathbf{A}^{n-1}} V'$ of $Z = X \times_{\mathbf{A}^{n-1}} Y'$.

A point of Z is a pair (x, y') with $x \in X$ and $y \in Y'$ such that $f'(x) = f'(g'(y))$. A point (x, y') of Z belongs to Z' if and only if and only if $x = g'(y')$, and the algebraic retraction $\tau \circ \rho$ maps an arbirary point (x, y') of Z to $(g'(y'), y')$. The natural isomorphism between Y' and Z' is $y' \mapsto (g'(y'), y')$; hence modulo this isomorphism the map h corresponds to g'.

A point (x, y') of Z belongs to V if and only if $x \in U$ and $y' \in V'$; a point of V can therefore be described as a pair (u, v') with $u \in U, v' \in V'$ such that $f'(g'(v)) = f'(u)$; this exactly means that $g'(v') = \rho(u)$. A point (u, v') of V belongs to Z' if and only if $u = g'(v')$, which amounts to saying that $u = \rho(u)$, or also that $u \in U'$. As a consequence $Z' \cap V$ is exactly the pre-image of U' under the first projection, which is also the zero-locus of $f_n \circ h$ on V.

The algebraic retraction $\tau \circ p$ maps an arbitrary point (u, v') of V to the pair $(g'(v'), v') = (\rho(u), v')$.

By assumption g' induces a Nash isomorphism between V' and U'; let σ' be the converse bijection. The map h then induces a K-definable bijection $V \simeq U$ with converse bijection $\beta \colon u \mapsto (u, \sigma'(\rho(u))$. Both components of β induce a contiuous map at the level of stable completions, so β itself induces a continuous map $\widehat{U} \to \widehat{V}$ (because the natural map $\widehat{Z} \to \widehat{X} \times_{\widehat{\mathbf{A}^{n-1}}} \widehat{Y'}$ is topologically proper, as one can check by using the definition of definable compactness through definable types); therefore h induces a Nash isomorphism between V and U.

The Zariski-closed subsets Z' and $\{f_n \circ h = 0\}$ of Z have the same intersection with V. This intersection identifies through h with U', which itself identifies through f' with an $(n-1)$-dimensional open polydisc; it thus admits a point v (in some model) such that $K(v)$ has transcendence degree $n-1$ over K. Hence the smooth K-varieties Z' and $\{h_n \circ f = 0\}$ have at least a common irreducible (and connected) component Z'' which meets V. As the stable completion of $Z' \cap V = \{f_n \circ h = 0\} \cap V$ is connected (for it is definably homeomorphic to the stable completion of an open polydisc), no other connected connected component of Z' or $\{f_n \circ h = 0\}$ can meet V. Hence if we denote by Ω the connected component of

$$(Z \backslash (\{f_n \circ h = 0\} \cup Z')) \cup Z''$$

that contains Z'', the following holds: (Ω, V, h) is an objet of $\mathscr{E}_K(U)$, the map h is étale, Ω is connected, and the natural retraction from V to $\{f_n \circ h = 0\} \cap V$ is induced by an algebraic retraction from Ω to its Zariski-closed subset $\Omega' : \{f_n \circ h = 0\}$. By construction Ω' can be identified with an open subset of Y' containing V', and $h|_{\Omega'}$ can be identified with g'.

5.6.2. *Proof of (1).* — One can prove (1) after refining (X, U). So starting from some (Y', V', g') in $\mathscr{E}_K(U')$ such that g' is étale and $\psi \in K[Y']$, and using the construction of (Ω, V, h) carried out in 5.6.1, we can then assume that X is connected, that the retraction ρ is induced by a K-algebraic retraction from $X \to X'$ (so X' is connected) and that $\psi \in K[X']$. But then $\psi \circ \rho \in K[X]$ and we are done.

5.6.3. *Proof of (2).* — By refining suitably (X, U) with the help of 5.6.1, we can assume for proving both implications that X is connected, that $\varphi \in K[X]$, and that the retraction ρ is induced by a K-algebraic retraction from $X \to X'$ (so X' is connected), and for (i)⇒(ii) we can moreover assume that $\psi \in K[X']$. But with these new assumptions both implications are obvious, as well as the final statement. □

5.7. Lemma. — *Let $\varphi \in K\langle U\rangle$ and let $d \geqslant 0$. Assume that the Taylor expansion of φ at u belongs to $(f_1, \ldots, f_n)^d K[\![f_1, \ldots, f_n]\!]$. Let $\mathscr{I}_d$ be the set of multi-indices I with $|I| = d$.*

(1) *The function φ can be written $\sum_{I \in \mathscr{I}_d} f^I \varphi_I$ where each φ_I belongs to $K\langle U\rangle$ and where the Taylor expansions of the functions $f^I \varphi_I$ at u have pairwise disjoint supports.*

(2) *Assume that $n = 1$, so $\mathscr{I}_d = \{d\}$, and that $\varphi \in K[U]$. Then φ_d belongs to $K[U]$.*

5.8. Comments. — In assertion (2) the equality $\varphi = f^d \varphi_d$ determines uniquely φ_d since $K\langle U\rangle$ is a domain, so there is no ambiguity.

Proof of Lemma 5.7. — We argue by induction on n. The case $n = 0$ is trivial. Let us prove the case $n = 1$, and write f instead of f_1. By refining (X, U) we can assume that f is étale, X is connected and $\varphi \in K[X]$. Then X is smooth and f is a local parameter at u that does not vanish elsewhere on U. As a consequence the rational function φ/f^d is regular in a K-Zariski neighborhood of U, so it belongs to $K[U]$. This proves (2) as well as (1) when $n = 1$. Now we assume that $n > 1$ and that the Lemma holds in dimension $n - 1$. We insere here a second induction here, on the degree d. If $d = 0$ there is nothing to prove (the required decomposition is $\varphi = \varphi$). So we know assume that $d > 0$ and that the Lemma holds in dimension $n - 1$ for any degree, and in dimension n for $d - 1$.

Let $\sum a_I f^I$ be the Taylor expansion of φ at u.

Let $\mathscr{J}$ be the set of multi-indices $(i_1, \ldots, i_n)$ with $i_n = 0$; one then has formally

$$\sum a_I f^I = \sum_{I \in \mathscr{J}} a_I f^I + f_n \sum_{I \notin \mathscr{J}} a_I f^{I-(0,\ldots,0,1)}.$$

Set $\psi = \varphi|_{U'}$. Then $\psi \in K\langle U'\rangle$ and its Taylor expansion at u is $\sum_{I \in J} a_I f^I$. As ψ belongs to $(f_1, \ldots, f_{n-1})^d K[\![f_1, \ldots, f_{n-1}]\!]$ the induction hypothesis ensures that ψ can be written $\sum_{I \in \mathscr{J}_d} f^I \psi_I$ where $\mathscr{J}_d = \mathscr{J} \cap \mathscr{I}_d$, where the ψ_I belong to $K\langle U'\rangle$ and where the Tayor expansion of the $f^I \psi_I$ have pairwise disjoint support.

For every $I \in \mathscr{J}_d$ set $\varphi_I = \psi_I \circ \rho$. By proposition 5.6 each φ_I belongs to $K\langle U\rangle$ and has the same Taylor expansion at u than ψ_I, so $\sum_{I \in \mathscr{J}_d} f^I \varphi_I$ is an element of $K\langle U\rangle$ whose Taylor expansion at u is $\sum_{I \in \mathscr{J}_d} a_I f^I$, and the Taylor expansions of the $f^I \varphi_I$ for $I \in \mathscr{J}_d$ are pairwise disjoint. Set $\chi = \varphi - \sum_{I \in \mathscr{J}_d} f^I \varphi_I$. By construction, χ's Taylor expansion is a multiple of f_n, which implies that f_n divides χ in $K\langle U\rangle$ (Lemma 5.3 (2)).

Let $\varphi_n \in K\langle U\rangle$ such that $\chi = f_n \varphi_n$. The Taylor development of φ_n at u belongs to $(f_1, \ldots, f_n)_{d-1} K[\![f_1, \ldots, f_n]\!]$, so by induction we can write $\chi = \sum_{I \in \mathscr{I}^{d-1}} f^I \chi_I$ where the χ_I belong to $K\langle U\rangle$ and where the Taylor expansions of the $f^I\chi_I$ have pairwise disjoint supports. Then we have

$$\varphi = \sum_{I\in\mathscr{J}_d} f^I\varphi_I + \sum_{I\in\mathscr{I}_{d-1}} f^{I+(0,\ldots,0,1)}\chi_I,$$

and this writing fulfills the requirement of the Lemma. □

5.9. Remark. — If $n \geqslant 2$ and $\varphi \in k[U]$ it seems unlikely that it is possible in general to achieve (1) with the $\varphi_I \in K[U]$; at least the functions φ_I provided by our proof definitely do not belong to $K[U]$ in general.

Indeed, let us consider the following example. We assume that the residue characteristic of K is not 2, and we take for X the quadratic cover

$$\operatorname{Spec} K[T_1, T_2][S]/(S^2 - 1 - T_1 - T_2)$$

of $\mathbf{A}^2$, and we still use the notation T_1 and T_2 for their pull-back on X (so they play the role of f_1 and f_2 above). Let U be the K-definable subset $\{|S-1| < 1\}$ of X. Then (T_1, T_2) induces a Nash isomorphism between U and the open unit bi-disc; the unique pre-image u of the origin is characterized by the equalites $T_1(u) = T_2(u) = 0$ and $S(u) = 1$.

The function $S-1$ vanishes at u, so the Lemma above furnishes a decomposition $S-1 = T_1\varphi_1 + T_2\varphi_2$ with φ_1 and φ_2 in $K\langle U\rangle$ (and where the Taylor expansions of the $T_i\varphi_i$ are disjoint). Looking at its proof, we see that the function φ_2 it actually exhibits is such that the Taylor expansion of $T_2\varphi_2$ at u is equal to $\sqrt{1+T_2} - 1$. Therefore φ_2 does not belong to $K[U]$: if one wants to realize it as a regular function, one has to go to the double cover of X obtained by adjoining a square root of $1 + T_2$ (such a square root does not exist in $K(X)$, as $(1+T_2)/(1+T_1+T_2)$ is not a square in $K(T_1, T_2)$).

If $s = (s_1, \ldots, s_n)$ is a polyradius $< r$ consisting of elements of $|M^\times|$ for some model M of ACVF containing K we shall denote by η_s the point of $\widehat{D}(M)$ given by the valuation $\sum a_I T^I \mapsto \max|a_i| \cdot s^I$ on $M[T_1, \ldots, T_n]$.

5.10. Theorem. — *Let s be as above. Let U_s be the pre-image on U of the closed polydisc with polyradius s, and let u_s be the unique pre-image of η_s in $\widehat{U}$. Let $\varphi = \sum a_I f^I$ be an element of $K\langle U\rangle$.*

One has then

$$\|\varphi\|_{U_s,\infty} = |\varphi(u_s)| = \max_I |a_I| \cdot s^I,$$

and the maximum is achieved for only finitely many coefficients a_I.

Proof. — The statement is insensitive to ground field extension, so we can assume that K is a model of ACVF and that each s_i belongs to $|K^\times|$; then η_s and u_s belogn respectively to $\widehat{D}(K)$ and $\widehat{U}(K)$. We argue by induction on n; there is nothing to prove if $n = 0$. The key point for the induction step will be the case $n = 1$ which we handle now.

5.10.1. *The case* $n = 1$. — We assume that $n = 1$, and we write T, f, r, s instead of T_1, f_1, r_1, s_1. If $a_i = 0$ for every i then φ is zero in the connected component of the smooth locus of X that contains U; in particular $\varphi|_U = 0$ and the result is obvious. So we assume from now on that $\{i, a_i \neq 0\}$ is non-empty.

5.10.1.1. — Let us fix i with $a_i \neq 0$ and first prove that $|\varphi(u_s)| = \|\varphi\|_{U_s,\infty} \geqslant |a_i| \cdot s^i$. If there is some index $j < i$ such that $|a_j| \cdot s^j \geqslant |a_i| \cdot s^i$ it suffices to prove that $|\varphi(u_s)| = \|\varphi\|_{U_s,\infty} \geqslant |a_j| \cdot s^j$, so we can replace i by j; hence we reduce to the case where $|a_j| \cdot s^j < |a_i| s^i$ for all $j < i$. Then $|\sum_{j<i} a_j f^j|$ is everywhere $< |a_i| \cdot s^i$ on $\widehat{U}_s$, so it suffices to prove that $\|\varphi - \sum_{j<i} a_j f^j\|_{U_s,\infty}$ is achieved at u_s and that it is greater or equal than $|a_i| \cdot s^i$. Otherwise said, we can assume that $a_i f^i$ is the first term of φ's expansion. Then by Lemma 5.3 (2) the quotient φ/f^i is well-defined in $K\langle U\rangle$.

By construction U is contained in the smooth locus of X, the function $f|_U$ vanishes only at u, and generates the maximal ideal of the discrete valuation ring $\mathscr{O}_{X,u}$; therefore the rational function (g/f^i) is defined on a K-Zariski neighborhood of U, and can thus be seen as an element of $K\langle U\rangle$. As $|f^i(u_s)| = s^i$ and as $s^i = \|f\|_{U_s,\infty}$, it suffices to show that $\|(\varphi/f^i)\|_{U_s,\infty} = |(\varphi/f^i)(u_s)| \geqslant |a_i|$. Hence we can replace φ by φ/f^i; that is, we can assume that $i = 0$. By refining (X, U), we can assume that X is smooth and connected and that $\varphi \in K[X]$.

If we prove that $\|\varphi\|_{U_s,\infty}$ is achieved at u_s we shall be done because of the obvious minoration $\|\varphi\|_{U_s,\infty} \geqslant |\varphi(u)| = |a_0|$. We argue by contradiction, so we assume that $|\varphi(u_s)| < \|\varphi\|_{U_s,\infty}$. As $\widehat{U_s}$ is definably homeomorphic to the stable completion of the closed polydisc D_s of radius s, it is compact, so $|\varphi|$ achieves its maximum $\|\varphi\|_{U_s,\infty}$ on $\widehat{U_s}$. The set of points of $\widehat{U_s}$ at which $|\varphi| = \|\varphi\|_{s,\infty}$ is closed. We are going to prove that it is also open; as $\widehat{U}_s$ is connected (it is indeed homeomorphic to $\widehat{D_s}$), this will force $|\varphi|$ to be constant of $\widehat{U_s}$, and imply in particular that $|\varphi(u_s)| = |\varphi(u)| = |a_0|$, contradiction.

So let v be a point of $\widehat{U_s}$ such that $|\varphi(v)| = \|\varphi\|_{U_s,\infty}$. By our assumption $v \neq u_s$. As η_s is the topological boundary of $\widehat{D_s}$ in $\widehat{D}$, the point u_s is the topological boundary of $\widehat{U_s}$ in $\widehat{U}$, so $\widehat{U_s} \backslash \{u_s\}$ is open in $\widehat{U}$, hence in $\widehat{X}$. As a consequence $|\varphi|$ has a local maximum at v in $\widehat{X}$ (which is non-zero, as $a_0 = \varphi(u) \neq 0$, so $\|\varphi\|_{U_s,\infty} \neq 0$). If v is a simple point, then $|\varphi|$ is constant around v as $\varphi(v) \neq 0$. If v is not simple, then for every germ of interval I emanating from v on $\widehat{X}$, the function $|\varphi|$ is non-increasing along I. Then the generalized rational function $\mathsf{rv}(\varphi(v))$ on C_v has no poles, so is constant, which means that it is equal to $\mathsf{rv}(\mu)$ for some $\mu \neq 0$ in K. But φ is then equal to $\mu + h$ where h is such that $|h(v)| < |\mu|$. As a consequence $|\varphi| = |\mu|$ around v and we are done.

5.10.1.2. — The map f induces a homeomorphism between $\widehat{U}$ and $\widehat{D}$. Moreover it follows from Lemma 4.7 (applied to any realization of v) that f induces an isomorphism between the residue curve at u_s and that at η_s, which is isomorphic to the projective line over $\mathsf{k}(K)$. A more natural way to see these curves is as generalized varieties over the generalized residue field $\mathsf{RV}(K)$, so that C_{η_s} is the generalized projective line over $\mathsf{RV}(K)$ with coordinate $\mathsf{rv}(T(\eta_s))$; under this identification a germ of interval emanating from η_s lies in $\widehat{D_s}$ if and only if the corresponding closed point of C_{η_s} is not the point at infinity. It follows that C_{u_s} is the generalized projective line over $\mathsf{RV}(K)$ with coordinate $\mathsf{rv}(f(v_s))$, and that a germ of interval emanating from v_s lies in $\widehat{U_s}$ if and only if the corresponding closed point of C_{u_s} is not the point at infinity.

We have seen that $|\varphi(v_s)| = \|\varphi\|_{U_s,\infty}$. This implies that on every germ of interval emanating from u_s and drawn on $\widehat{U_s}$ the function $|\varphi|$ is less than or equal to $|\varphi(u_s)|$. As a consequence, $\mathsf{rv}(\varphi(u_s))$ cannot have any pole on C_{u_s} except at the point at infinity; it follows that $\mathsf{rv}(\varphi(u_s))$ is a polynomial in $f(u_s)$. This means that there exists a finite set of integers J and a finite non-empty family $(b_j)_{j\in J}$ of elements of K such that $|b_j|s^j = \|\varphi\|_{U_s,\infty}$ for all j and $\mathsf{rv}(\varphi(u_s)) = \sum \mathsf{rv}(b_j)\mathsf{rv}(f(u_s))^j$; one can rephrase the latter equality by saying that $|\varphi(u_s) - \sum_j b_j f(u_s)^j| < \|\varphi\|_{U_s,\infty}$. The Taylor expansion at u of $\varphi - \sum_j b_j f^j$ is $\sum_i c_i f^i$ with $c_i = a_i$ for $i \notin J$ and $c_j = a_j - b_j$ for every $j \in J$. In view of 5.10.1.1 the inequality $|\varphi(u_s) - \sum_j b_j f(u_s)^j| < \|\varphi\|_{U_s,\infty}$ implies that $|c_i|s^i < \|\varphi\|_{U_s,\infty}$ for all i. Otherwise said, $|a_i|s^i < \|\varphi\|_{U_s,\infty}$ for all $i \notin J$ and $|a_j|s^j = \|\varphi\|_{U_s,\infty}$ for all $j \in J$, which ends the proof when $n = 1$.

5.10.2. — Now we handle the general case, assuming that the result holds in dimension $n-1$. We write formally $\varphi = \sum_i A_i f_n^i$, so that the coefficient A_i is (again, formally) equal to $\sum_{J\in\mathbf{Z}^{n-1}} a_{(J,i)}(f')^J$ where $f' = (f_1, \ldots, f_{n-1})$.

We denote by $D_{s'}$ the closed polydisc of polyradius $s' := (s_1, \ldots, s_{n-1})$, by X' the closed subscheme of X defined by the equation $f_n = 0$, and by U' be the intersection $X' \cap U$. Then f' induces a K-definable Nash isomorphism between U' and D', and by Lemma 5.3 (1) there is for each i some φ_i in $K\langle U'\rangle$ whose Taylor expansion at u is equal to A_i, and we can moreover assume, up to refining (X, U), that f is étale, X is connected and every φ_i belongs to $K[X]$.

Let v be any point of U' and let Z_v be the closed K-subscheme of X defined by the equations $f_j = f_j(v)$ for $1 \leqslant j \leqslant n-1$. Then f induces an étale map from Z_v to the subscheme $\{T_j = f_j(v)\}_{1\leqslant j\leqslant n-1}$ of $\mathbf{A}^n_K$, which we identify to the affine line with coordinate function T_n. The map f_n then induces a definable bijection $Z_v \cap U \simeq \Delta$, where Δ is the one-dimensional open disc with radius r_n. Let $w \in Z \times U$ be the unique pre-image of the origin of Δ (otherwise said, the unique point of $Z \cap U$ at which f_n vanishes, or the unique point of $Z \cap U'$); the expansion of $\varphi|_Z$ in $\widehat{\mathscr{O}_{Z,w}} \simeq K[\![f_n]\!]$ is then by construction equal to $\sum_i \varphi_i(v) f_n^i$.

5.10.2.1. — Let M be a model of ACVF containing K and let $v \in U'(M)$. Let us denote by $\rho(v)$ the maximum of $|\varphi|$ on Z_v, and let t_v be any point of Z_v such that the type of $f_n(t_v)$ over M is equal to η_{s_n}. By the above and by 5.10.1 one has

$$\rho(v) = |\varphi(t_v)| = \max_i |\varphi_i(v)| s_n^i,$$

and this maximum is achieved for only finitely many indices i.

Now let ω be a point of U' whose image under $(f_1, \ldots, f_{n-1})$ induces the type $\eta_{s_1,\ldots,s_{n-1}}$ on M. The following facts follow from the induction hypothesis:

- for all i the maximum θ_i of the function $|\varphi_i|$ on U' is achieved at ω;

- ⋄ for all i we have $\theta_i = \max_J |a_{J,i}|(s')^J$ where $s' = (s_1, \ldots, s_{n-1})$, and this maximum is achieved for only finitely many multi-indices J.

As a consequence, $\|\varphi\|_{U_s,\infty}$ is achieved at t_ω, it is equal to $\max_I |a_I| \cdot s^I$, and this maximum is achieved for only finitely many multi-indices I. Since the type of $f(t_\omega)$ over M is equal to η_s by construction, this ends the proof. □

5.11. Corollary. — *Let $\varphi = \sum a_I f^I$ be an element of $K\langle U\rangle$. Then $\max_I |a_i| \cdot r^I$ exists and $\|g\|_{U,\infty} = \max_I |a_I| \cdot r^I$.*

5.12. Comments. — One cannot in general expect the maximum of the $|a_I| \cdot r^I$ to be achieved for only finitely many indices. For instance, on the one-dimensional open unit disc, $(1-T)^{-1}$ has Taylor expansion $\sum_i T^i$.

Proof of Corollary 5.11. — Let M be a model of ACVF containing K and some λ with $|\lambda| < 1$ and $(\lambda, 1) \cap |K^\times|^{\mathbf{Q}} = \varnothing$.

Set $s_i = r_i|\lambda|$ for all i. It follows from Theorem 5.10 that $\max_I |a_I| \cdot s^I$ exists (and is achieved for only finitely many indices, but we shall not need it here). Let J such that $|a_J| \cdot s^J$ realizes this maximum. Then for all I one has $|a_I| \cdot s^I \leqslant |a_J| \cdot s^J$, and since non-strict inequalities are preserved when one mods out by a convex subgroup, this implies that $|a_I| \cdot r^I \leqslant |a_J| \cdot r^J$ for all I, and we are done.

It follows then again from Theorem 5.10, by letting s tend to r in any model of ACVF containing K, that $\|\varphi\|_{U,\infty} = \max_I |a_I| \cdot r^I$. □

5.13. Corollary. — *Let φ be an invertible element in $K\langle U\rangle$. Then $|\varphi|$ is constant on U.*

Proof. — For all $s < r$ it follows from Theorem 5.10 that both $|\varphi|$ and $|\varphi^{-1}|$ achieve their maximum on $\widehat{U_s}$ at the same point, namely u_s. This implies that $|\varphi|$ is constant on U_s for all s, so it is constant on the whole of U. □

6. Tame descent of abstract open polydiscs

6.1. Theorem. — *Let X be an algebraic variety over a henselian valued field K, and let U be a K-definable subset of X. Let $f = (f_1, \ldots, f_n)$ be an algebraic map from X to $\mathbf{A}^n_K$ inducing a Nash isomorphism between U and an open polydisc D.*

1. *The elements $\mathsf{rv}_U(f_1), \ldots, \mathsf{rv}_U(f_n)$ of $\mathsf{RV}(K)[U]$ are algebraically independent over $\mathsf{RV}(K)$.*
2. *Let A be the completion of $\mathsf{RV}(K)[\mathsf{rv}_U(f_1), \ldots, \mathsf{rv}_U(f_n)]$ along its ideal generated by the $\mathsf{rv}(f_i)$, and let $\mathfrak{n}$ be its maximal ideal. The generalized ring $\mathsf{RV}(K)\langle U\rangle$ embbedds into A. It is local, its maximal ideal $\mathfrak{m}$ is generated by $(\mathsf{rv}_U(f_1), \ldots, \mathsf{rv}_U(f_n))$, a system of generators of $\mathfrak{m}$ is minimal if and only if it has cardinality n, and $\mathfrak{m}^d = \mathfrak{n}^d \cap \mathsf{RV}(K)\langle U\rangle$ for all d.*
3. *The generalized ring $\mathsf{RV}(K)[U]$ is local as well, and the inclusion of $\mathsf{RV}(K)[U]$ in $\mathsf{RV}(K)\langle U\rangle$ is local. If $n = 1$ the maximal ideal $\mathfrak{p}$ of $\mathsf{RV}(K)[U]$ is generated by $\mathsf{rv}_U(f_1)$, a system of generators of $\mathfrak{p}$ is minimal if and only if it has cardinality 1, and $\mathfrak{p}^d = \mathfrak{n}^d \cap \mathsf{RV}(K)[U]$ for all d.*

(4) *Let* $g = (g_j)_{1 \leqslant j \leqslant n}$ *be a family of non-zero elements of* $K\langle U\rangle$*, with cardinality* n*. The following are equivalent:*

(i) *the family* (g_j) *induces a Nash isomorphism between* U *and an open polydisc;*

(ii) *the family* $(\mathsf{rv}_U(g_j))$ *is a system of generators of* $\mathfrak{m}$.

Proof. — Let $(r_1, \ldots, r_n)$ be the polyradius of D; otherwise said, $r_i = \|f_i\|_{U,\infty}$; let u be the unique pre-image of the origin under f on D.

6.1.1. *Proof of (1).* — For every polynomial $\sum a_I T^I$ in $K[T]$ one has

$$\left\|\sum a_I f^I\right\|_{U,\infty} = \left\|\sum a_I T^I\right\|_{D,\infty} = \max|a_I| \cdot r^I$$

(as one deals with polynomials here, one does not have to use Corollary 5.11); this shows that the functions $\mathsf{rv}_U(f_i)$ are algebraically independent over $\mathsf{RV}(K)$.

6.1.2. *Proof of (2) and (3).* — Let $\varphi = \sum a_I f^I$ be a non-zero element of $K\langle U\rangle$ at u. By Corollary 5.11, one has $\|\varphi\|_{U,\infty} = \max_I |a_I| \cdot r^I$; let us denote by $N(\varphi)$ the set of those I such that $|a_I| \cdot r^I$. If $\psi = \sum b_I f^I$ is another non-zero element of $K\langle U\rangle$ it is immediate that $\mathsf{rv}_U(\varphi) = \mathsf{rv}_U(\psi)$ if and only if

$$\sum_{I \in N(\varphi)} \mathsf{rv}(a_I)\mathsf{rv}_U(f)^I = \sum_{I \in N(\psi)} \mathsf{rv}(b_I)\mathsf{rv}_U(f)^I,$$

so $\varphi = \sum_I a_I f^I \mapsto \sum_{I \in N(\varphi)} \mathsf{rv}(a_I)\mathsf{rv}_U(f)^I$ induces an embedding from $\mathsf{RV}(K)\langle U\rangle$ into A which we shall use from now on to see $\mathsf{RV}(K)\langle U\rangle$ as a subring of A; with this identification one has $\mathsf{rv}_U(\varphi) = \sum_{I \in N(\varphi)} \mathsf{rv}(a_I)\mathsf{rv}_U(f)^I$.

6.1.2.1. — Let us prove that an element of the generalized ring $\mathsf{RV}(K)\langle U\rangle$, resp. $\mathsf{RV}(K)[U]$, is invertible if and only if it is invertible in A, i.e., if and only if its constant coefficient is non-zero. The "only if" part is obvious. Let us now consider a non-zero $\varphi = \sum a_I f^I$ in $K\langle U\rangle$, resp. $K[U]$, such that the constant term of $\mathsf{rv}_U(\varphi)$ is non-zero, which exactly means that $0 \in N(\varphi)$, that is, that $|a_0| = \|\varphi\|_{U,\infty}$. One then has $|a_I| r^I \leqslant |a_0|$ for all $I \neq 0$, and thus $|a_I| s^I < |a_0|$ for all $I \neq 0$ and every $s = (s_1, \ldots, s_n)$ with $|s_i| < r_i$ for all i. Set $\psi = \varphi - a_0$; then $\psi = \sum_{I \neq 0} a_I f^I$, so by the above and Theorem 5.10 one has for every $s = (s_1, \ldots, s_n)$ with $|s_i| < r_i$ for all i the majoration $\|\psi\|_{V_s,\infty} < |a_0|$, where $V_s = \{|f| \leqslant s\}$. It follows that $\varphi = a_0 + \psi$ does not vanish on any V_s as above, so φ does not vanish on U. Then φ^{-1} is an element of $K\langle U\rangle$, resp. $K[U]$: indeed in both cases we can refine (X, U) so that f is étale, X is connected and $\varphi \in K[X]$. Then $U \subset D(\varphi)$ so $\varphi^{-1} \in K[U] \subset K\langle U\rangle$. Then $\mathsf{rv}_U(\varphi^{-1})$ is an inverse of $\mathsf{rv}_U(\varphi)$ in $\mathsf{RV}(K)[U]$ and in $\mathsf{RV}(K)\langle U\rangle$.

It follows from the above that $\mathsf{RV}(K)\langle U\rangle$ and $\mathsf{RV}(K)[U]$ are local, and that in both cases, the maximal ideal is that of series with zero constant term; one has therefore $\mathfrak{m} = \mathfrak{n} \cap \mathsf{RV}(K)\langle U\rangle$, and the inclusion map $\mathsf{RV}(K)[U] \hookrightarrow \mathsf{RV}(K)\langle U\rangle$ is local.

6.1.2.2. — Let us fix an integer d, and denote by $\mathscr{I}_d$ the set of multi-indices I with $|I| = d$. It is obvious that $\mathfrak{m}^d$ contains each of the $\mathsf{rv}_U(f)^I$ for $I \in \mathscr{I}_d$, and that $\mathfrak{m}^d$ is contained in $\mathfrak{n}^d \cap K\langle U\rangle$. We shall prove that $\mathfrak{n}^d \cap \mathsf{RV}(K)\langle U\rangle$ is contained in the ideal $(\mathsf{rv}_U(f)^I))_{I\in\mathscr{I}_d}$ of $\mathsf{RV}(K)\langle U\rangle$, which will prove that $\mathfrak{m}^d=\mathfrak{n}^d\cap K\langle U\rangle$ and that it is generated by the $\mathsf{rv}_U(f)^I$; this will imply (by taking $d=1$) that $\mathfrak{m}$ is generated by the $\mathsf{rv}_U(f_i)$.

Let $\varphi=\sum a_I f^I$ be a non-zero element of $K\langle U\rangle$ whose reduction $\mathsf{rv}_U(\varphi)$ belongs to $\mathfrak{n}^d\cap K\langle U\rangle$. This means that $N(\varphi)$ does nt contain any multi-index I with $|I|<d$. But one has then $\mathsf{rv}_U(\varphi)=\mathsf{rv}_U(\varphi-\sum_{|I|<d}a_I f^I)$, which allows to assume that $a_I=0$ for all I with $|I|<d$.

Then by Lemma 5.7 one can write $\varphi=\sum_{I\in\mathscr{I}_d}f^I\varphi_I$ where each φ_I belongs to $K\langle U\rangle$ and where the Taylor expansions of the $f^I\varphi_I$ have pairwise disjoint support. But one then has in view of Corollary 5.11 $\|\varphi|\|_{U,\infty}=\max_{I\in\mathscr{I}_d}\|f^I\varphi_I\|_{U,\infty}$ and

$$\mathsf{rv}_U(\varphi)=\sum_{I\in\mathscr{I}'}\mathsf{rv}_U(f)^I\mathsf{rv}_U(\varphi_I),$$

where $\mathscr{I}'$ is the set of $I\in\mathscr{I}_d$ such that $\|f^I\varphi_I\|_{U,\infty}=\|\varphi\|_{U,\infty}$; this the kind of writing we were looking for.

6.1.2.3. — As $(\mathsf{rv}_U(f_i))_i$ is a minimal generating system of the maximal ideal of A, it is a fortiori minimal as a generating system of $\mathfrak{m}$. And by the generalized Nakayama Lemma, all minimal generating systems of $\mathfrak{m}$ have the same cardinality, namely the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over $\mathsf{RV}(K)$; hence they all have cardinality n.

6.1.2.4. — Let us assume in this paragraph that $n=1$, write f instead of f_1, and let $\mathfrak{p}$ be the maximal ideal of $\mathsf{RV}(K)[U]$. It is clear that $\mathsf{rv}_U(f)$ belongs to $\mathfrak{p}$. We shall prove that $\mathfrak{n}^d\cap\mathsf{RV}(K)[U]$ is contained in the ideal $(\mathsf{rv}_U(f))$ of $\mathsf{RV}(K)[U]$, which will end the proof of (4) (reasoning like in 6.1.2.2).

Let $\varphi=\sum a_i f^i$ be a non-zero element of $K[U]$ whose reduction $\mathsf{rv}_U(\varphi)$ belongs to $\mathfrak{n}^d\cap\mathsf{RV}(K)[U]$. This means that $N(\varphi)$ does not contained any integer $<d$ or, otherwise said, that $|a_i|<\|\varphi\|_{U,\infty}$ for all $i<d$. But one has then $\mathsf{rv}_U(\varphi)=\mathsf{rv}_U(\varphi-\sum_{i<d}a_i f^i)$, which allows to assume that $a_i=0$ for $i<d$.

Then by Lemma 5.7 one can write $\varphi=f^d\psi$ for some $\psi\in K[U]$. Thus $\mathsf{rv}_U(\varphi)$ is equal to $\mathsf{rv}_U(f)^d\mathsf{rv}_U(\psi)$, so $\mathsf{rv}_U(\varphi)$ belongs to the ideal $(\mathsf{rv}_U(f)^d)$.

6.1.3. *Proof of (4).* — It follows from (2) that (i)⇒(ii). So let us now assume that (ii) holds and prove (i). For every index j write $g_j=\sum_I a_{I,j}f^I$; one has then $\mathsf{rv}_U(g_j)=\sum_I\alpha_{I,j}\mathsf{rv}_U(f)^I$ where $\alpha_{I,j}=0$ if $I\notin N(g_j)$ and $\alpha_{I,j}=\mathsf{rv}(a_{I,j})$ otherwise. Set $\gamma_j=\|g_j\|_{U,\infty}$. In view of (2), sssumption (ii) amounts to saying that $\alpha_{0,j}=0$ for all j and (by Nakayama) that the matrix $(\alpha_{i,j})_{i,j}$ is invertible. Let M, resp. N, denote the K-vector space K^n equipped with the norm $(\lambda_1,\ldots,\lambda_n)\mapsto\max|\lambda_i|\cdot\gamma_i$, resp. $\max|\lambda_i|r_i$.

By construction $|a_{ij}|\leqslant\gamma_j r_i^{-1}$ for all i; hence the matrix (a_{ij}) induces a linear map from M to N that does not increase norms. This map in turn induces a $\mathsf{RV}(K)$-linar map between the graded reductions $\mathsf{RV}(M)$ and $\mathsf{RV}(N)$, precisely given by the matrix (α_{ij}) in the canonical bases of both $\mathsf{RV}(K)$-vector spaces. The fact that (α_{ij}) is bijective can then be rephrased by saying that (a_{ij}) induces an isometry ι between M and N. The isometries ι and ι^{-1}, viewed as K-endomorphisms of $\mathbf{A}^n_K$, map open balls to open balls; hence for proving (i) one can replace g by $\iota^{-1}\circ g$, which amounts to assume that (α_{ij}) is the identity matrix (in fact we have achieved more, since (a_{ij}) itself is the identity matrix; but we shall need to modify the f_j later in the proof, and (a_{ij}) might be modified through these changes, while (α_{ij}) will remain unchanged). In other words, one has $\mathsf{rv}_U(g_j) = \mathsf{rv}_U(f_j)$ modulo $\mathfrak{m}^2$ for all j (this implies that $\gamma_j = r_j$ for all j).

6.1.3.1. — Our first aim is now to prove that g induces a Nash isomorphism between U and D. The ground field does not matter for this purpose, so we can assume that K is a model of ACVF. The assumption that $\mathsf{rv}(g_j) = \mathsf{rv}(f_j)$ modulo $\mathfrak{m}^2$ can be rephrased by saying that the Taylor expansion $\sum a_{I,j} f^I$ of g_j at u fulfills the following conditions, if we denote by δ_i the n-uple whose i-th component is 1 and all of whose other components are zero.

- $|a_{I,j}| \cdot r^I \leqslant r_j$ for all I;
- $a_{\delta_j,j}$ if of the form $1 + \varepsilon_j$ with $|\varepsilon_j| < 1$;
- $|a_{0,j}| < r_j$ and $|a_{\delta_i,j}| \cdot r_i < r_j$ for all $i \neq j$.

It follows from the above that for every $s < r$ and every j one has $\max_I |a_{I,j}| s^I < r_j$, which implies in view of Theroem 5.10 that $|g_j|$ is strictly bounded by r_j on $\{|f| \leqslant s\}$; as this holds for all $s < r$, we see that the supremum norm $\|g_j\|_{U,\infty} = r_j$ is not achieved on U. As a consequence, $g(U) \subset D$.

Let m be the number of indices j such that $g_j = f_j$. We argue by descending induction on m. There is nothing to prove if $m = n$, so we assume $m < n$ and that the result holds for $m + 1$. Since $m < n$ there exists j such that $g_j \neq f_j$. Up to renumbering the g_j and the f_j we can assume that $g_1 \neq f_1$. We are then going to prove that $(g_1, f_2, \ldots, f_n)$ induces a Nash isomorphism between U and D. Setting $h_1 = g_1$ and $h_j = f_j$ if $j > 1$ we will have $\mathsf{rv}(g_j) = \mathsf{rv}(h_j)$ for all j, and there will be $m + 1$ indices j with $g_j = h_j$; so the induction hypothesis (applied with h instead of f) will then ensure that $(g_1, g_2, \ldots, g_n)$ induces a K-definable Nash isomorphism between U and D.

Let Y be the zero-locus of $(f_2, \ldots, f_n)$ and set $V = Y \cap U$. Then $f_1|_Y : Y \to \mathbf{A}^1$ induces a K-definable Nash isomorphism between V and the open one-dimensional disc Δ of radius r_1. The point u belongs to V, and the Taylor expansion in f_1 of $g_1|_Y$ at u is the pure f_1-part of that of g_1 at u; i.e., this is $\sum_i b_i f_1^i$ where $b_i = a_{(i,0,\ldots,0),1}$. One has then $|b_0| < r_1$, $b_1 = 1 + \varepsilon_1$ (with $|\varepsilon_1| < 1$), and $|b_i| r_1^i \leqslant r_1$ for all i. This implies that $\mathsf{rv}_V(g_1) = \mathsf{rv}_V(f_1)$ (in particular $\|g_1\|_{V,\infty} = r_1$).

6.1.3.2. *Let us prove that g_1 has a unique zero on V.* — By Corollary 5.11 one has $\|g_1\|_{V,\infty} = \max |b_i| r_1^i$, and the latter is equal to r_1 by the above. As $|g_1(u)| = |b_0| < r_1$ we see that $|g_1|$ is not constant on V, so it is not invertible on V by Corollary 5.13; hence g_1 vanishes at some point of V. Let us prove that this point is unique. Let v be a zero of g_1 and set $h = f_1 - f_1(v)$. Then h induces a Nash isomorphism between V and Δ, and since $|f_1(v)| < r_1$ one has $\mathsf{rv}_V(h) = \mathsf{rv}_V(f_1)$, so $\mathsf{rv}_V(g_1) = \mathsf{rv}_V(h)$. This means that the Taylor expansion of g_1 at v in h is of the form $\sum c_i h^i$ with $c_0 = 0$ (since g_1 vanishes at v), with $c_1 = 1 + \varepsilon'$ with $|\varepsilon'| < 1$, and with $|c_i| \cdot r_1^i \leqslant r_1$ for all i. Let $s \in [0, 1)$ and let $E(v, s)$ be the pre-image under $h = f_1 - v$ of the closed disc of radius s. Then for all $i \geqslant 2$ one has

$$|c_i| \cdot s^i = (s/r_1)^i |c_i| \cdot r_1^i \leqslant (s/r_1)^i r_1 < s,$$

so by Theorem 5.10 one has

$$\|g_1 - h\|_{E(v,s),\infty} = \|g_1 - f_1 + f_1(v)\|_{E(V,s),\infty} = \left\| \sum_{i \geqslant 2} c_i f_1^i \right\|_{E(v,s),\infty} < s.$$

Now let w be another zero of g_1 and set $s = |f_1(v) - f_1(w)|$. Then $E(v,s) = E(w,s)$ and by the above one has

$$\|g_1 - f_1 + f_1(v)\|_{E(V,s),\infty} < s \text{ and } \|g_1 - f_1 + f_1(w)\| < s,$$

whence the inequality $|f_1(v) - f_1(w)| < s$, contradiction.

6.1.3.3. *The map $(g_1, f_2, \ldots, f_n)$ induces a bijection $U \simeq D$.* — We have already seen (6.1.3.1) that $g(U) \subset D$. By the paragraph above, the origin has a unique pre-image in U under $(g_1, f_2, \ldots, f_n)$.

Now let $(d_1, \ldots, d_n)$ be any point of $D(K)$. Then $(f_j - d_j)_j$ induces a Nash isomorphism between U and D, and $\mathsf{rv}(g_1 - d_1) = \mathsf{rv}(g_1) = \mathsf{rv}(f_1)\mathsf{rv}(f_1 - d_1)$. So by the above applied to $(f_j - d_j)_j$ instead of $(f_j)_j$ we see that there is a unique K-point of U at which $(g_1 - d_1, f_2 - d_2, \ldots, f_n - d_n)$ vanishes; this proves that $(d_1, \ldots, d_n)$ has a unique pre-image on U under $(g_1, f_2, \ldots, f_n)$.

6.1.3.4. *The map $(g_1, f_2, \ldots, f_n)$ induces a homeomorphism $\widehat{U} \simeq \widehat{D}$.* — We already know that it induces a continuous bijection $\widehat{U} \simeq \widehat{D}$; it remains to show that the latter is topologically proper. So fix $s < r$ and let us prove that the pre-image V under $(g_1, f_2, \ldots, f_n)$ of the the closed polydisc of polyradius s has a compact stable completion. It suffices to prove that $\widehat{V}$ is contained in some compact subset of $\widehat{U}$ or, equivalently, that it is contained in $\{|f| \leqslant t\}$ for some $t < r$; and this can be checked at the level of the definable sets themselves, without refereeing to their stable completions.

So let $v \in V$. Our purpose it to bound for every j the element $|f_j(v)|$ of Γ by some quantity $< r_j$ independent of v. As $|f_j(v)| \leqslant s_j$ as soon as $j \geqslant 2$, it remains to bound $|f_1(v)|$.

One has $g_1 = \sum_I a_{I,1} f^I$. We can split formally this sum and write

$$g_1 = \sum_{I \in \mathscr{J}} a_{I,1} f^I + \sum_{I \notin \mathscr{J}} a_{I,1} f^I,$$

where $\mathscr{J}$ denotes the set of indices I whose first component is non-zero. It now follows from Lemma 5.3 and Proposition 5.6 that this formal splitting actually underlies a genuine decomposition $g_1 = f_1 \varphi + \psi$ where φ and ψ belong to $K\langle U \rangle$ and have

$$\sum_{I \in \mathscr{J}} a_{I,1} f^{I - \delta_1} \text{ and } \sum_{I \notin \mathscr{J}} a_{I,1} f^I$$

as respective Taylor expansions.

We then have $f_1\varphi = g_1 - \psi$. As the Taylor expansion of ψ does not involve f_1, it follows from Theorem 5.10 that

$$\|\psi\|_{V,\infty} = \max_{I\notin \mathscr{J}} |a_I| \cdot s^I < r_1,$$

where the latter inequality comes from the fact that $r_1 = \|g_1\|_{U,\infty} = \max_I |a_{I,1}| f^I$, again by Theorem 5.10. As $\|g_1\|_{V,\infty} = s_1 < r_1$, it now suffices to show that $|\varphi|$ is identically equal to 1 on V.

The Taylor expansion of φ is equal to $a_{\delta_1,1} + \sum_{I\in\mathscr{J},|I|\geqslant 2} a_{I,1} f^{I-\delta_1}$. One has $a_{\delta_1,1} = 1 + \varepsilon_1$ with $|\varepsilon|_1 < 1$, so $|a_{\delta_1,1}| = 1$. Set $\chi = \varphi - a_{\delta_1,1}$. For all $I \in \mathscr{J}$ with $|I| \geqslant 2$ one has $|a_{I,1}| \cdot r^I \leqslant r_1$, so $|a_{I,1}| \cdot r^{I-\delta_1} \leqslant 1$. Let t_1 be any element of Γ with $t < r_1$, and set $t_j = s_j$ if $j \geqslant 2$. Then as $|I| \geqslant 2$ one has $|a_{I,1}| \cdot t^{I-\delta_1} < 1$. By Theorem 5.10 it follows that the maximum of χ on $\{|f| \leqslant t\}$ is < 1. Since t_1 was chosen arbitrary, $|\chi| < 1$ at every point of $\{|f_2| \leqslant s_2, \ldots, |f_n| \leqslant s_n\}$, and in particular at every point of V; as a consequence, $|\varphi| = |\chi + a_{\delta_1,1}|$ is identically equal to 1 on V, which we wanted to show.

6.1.3.5. — We have shown that g induces a K-definable bijection $U \simeq D$ and even a K-definable homeomorphism $\widehat{U} \simeq \widehat{D}$. It remains to ensure that g is étale at every point of U. So let $v = (v_1, \ldots, v_n)$ be a point of U. For all j one has $|g_j(v)| < r_j$ so that we have $\mathsf{rv}_U(g_j - g_j(v)) = \mathsf{rv}_U(g_j) = \mathsf{rv}_U(f_j)$; as the translation by $(g(v_1), \ldots, g(v_n))$ is an automorphism of $\mathbf{A}^n$, it suffices to show that $(g - g(v_1), \ldots, g - g(v_n))$ is étale at v. So we can assume that $g(v) = 0$.

Now $(f - f(v_1), \ldots, f - f(v_n))$ also induces a Nash isomorphism between U and D, and $\mathsf{rv}_U(f_j - f_j(v)) = \mathsf{rv}_U(f_j)$ for all j. So we can replace f by $f - f(v)$, which amounts to assuming that $f(v) = 0$, that is, that $v = u$.

As $g(u) = 0$ the Taylor expansion of every g_j at u has no constant term. And as $|a_{\delta_j,j}| = 1$ and $|a_{\delta_j,i}| < 1$ for all j and all $i \neq j$, $|\det((a_{\delta_j,i})_{i,j})| = 1$, and in particular the matrix $(a_{\delta_j,i})_{i,j})$ is invertible. By Nakayama's Lemma this implies that (g_j) is a regular system of parameters of X at u, so g is étale at u. □

***6.2. Theorem* (tame descent for abstract open polydiscs)**
Let X be a K-algebraic variety and let U be a K-definable subset of X. Assume that there exist a tamely ramified finite extension L of K and a tuple $f = (f_1, \ldots, f_n)$ of functions in $L\langle U\rangle$ inducing a Nash isomorphism between U and an open polydisc. Then there exists a tuple $g = (g_1, \ldots, g_n)$ of functions in $K\langle U\rangle$ inducing a Nash isomorphism between U and an open polydisc. Moreover, if $n = 1$ and $f = f_1 \in L[U]$ then $g = g_1$ can be chosen in $K[U]$.

Proof. — As L is a tamely ramified extension of K, it follows from Corollary 4.14 that

$$\mathsf{RV}(L)\langle U\rangle = \mathsf{RV}(K)\langle U\rangle \otimes_{\mathsf{RV}(K)} \mathsf{RV}(L) \quad \text{and} \quad \mathsf{RV}(L)[U] = \mathsf{RV}(K)[U] \otimes_{\mathsf{RV}(K)} \mathsf{RV}(L).$$

By Theorem 6.1 (2), the generalized ring $\mathsf{RV}(L)\langle U\rangle$ is local, hence $\mathsf{RV}(K)\langle U\rangle$ is local as well by the generalized going-up lemma. Let (α_i) be a family of generators of the maximal ideal of $\mathsf{RV}(K)\langle U\rangle$. The $\mathsf{RV}(K)\langle U\rangle/(\alpha_i)$-algebra $\mathsf{RV}(L)\langle U\rangle/(\alpha_i)$ is (again by the going-up lemma) local artinian, and also étale since $\mathsf{RV}(L)$ is separable over $\mathsf{RV}(K)$ by one of the possible definitions of a tamely ramified extension. Hence $\mathsf{RV}(L)\langle U\rangle/(\alpha_i)$ is a field, so (α_i) generates the maximal ideal of $\mathsf{RV}(L)\langle U\rangle$. By Theorem 6.1 (2) there is a sub-family of (α_i) of cardinality n that already generates the maximal ideal of $\mathsf{RV}(L)\langle U\rangle$. If g denotes any lifting of this subfamily in $K\langle U\rangle$, Theorem 6.1 (4) ensures that g induces a Nash isomorphism between U and an open polydisc. This ends the proof of the first statement.

Assume now that $n = 1$ and that $f \in L[U]$. Reasoning as above and using assertion (3) (instead of (2)) of Theorem 6.1, we ge the existence of $g \in K[U]$ such that $\mathsf{rv}(g)$ generates the maximal ideal of $\mathsf{RV}(K)[U]$. But by using assertion (2) and (3) of loc. cit., we see that this amounts to saying first that $\mathsf{rv}(g)$ generates the maximal ideal of $\mathsf{RV}(K)[\![f]\!]$, and then that it generates the maximal ideal of $K\langle U\rangle$; and we conclude once again by using Theorem 6.1 (4).

□

7. Abhyankar points lie on atomic abstract open polydiscs

7.1. Lemma. — *Let K be a valued field and let $f\colon Y \to X$ be a morphism between K-varieties. Let $x \in X$ and let y be a pre-image of x. Assume that f is quasi-finite at y, that the valued field $K(x)$ is defectless and that $K(y)$ is an immediate extension of $K(x)$. Then there exists a K-subvariety X' of X containing x, a K-subvariety Y' of $Y \times_X X'$ containing y which is finite and étale over X', a K-regular function ψ on Y', a K-regular function φ on X' with $|\psi(y)| = |\varphi(x)| < 1$ and a K-definable subset D of X' containing x such that the following hold for every $z \in D$:*

- ◇ *$|\varphi(z)| < 1$ and there exists one and only one element $\sigma(z)$ of Y' such that $f(\sigma(z)) = z$ and $|\psi(\sigma(z))| = |\varphi(z)|$;*
- ◇ *$|\psi(t)| = 1$ for every $t \neq \sigma(z)$ in the fiber $f^{-1}(z)$;*
- ◇ *$K(\sigma(z))^{\mathrm{h}} = K(z)^{\mathrm{h}}$.*

Proof. — Up to replacing X by the reduced K-Zariski closure of x and Y by the reduced K-Zariski-closure of y we can assume that X and Y are integral and x and y are K-Zariski dense. And then up to replacing Y by the quasi-finite locus of f we can assume that f is quasi-finite.

The field $K(y)$ is a finite extension of $K(x)$. Thus $K(y)^{\mathrm{h}}$ is a finite extension of $K(x)^{\mathrm{h}}$. By assumption this finite extension is immediate and since $K(x)$ is defectless, so is $K(x)^{\mathrm{h}}$; we therefore have $K(y)^{\mathrm{h}} = K(x)^{\mathrm{h}}$. As a by-product $K(y) \subset K(x)^{\mathrm{h}}$, so $K(y)$ is separable over $K(x)$. As Y is reduced and x is K-Zariski-generic, this implies that f is étale at y; so by replacing X and Y by suitable dense K-Zariski open subsets we can assume that f is finite étale.

By the above, y appears as a $K(x)^{\mathrm{h}}$-rational point of the $K(x)$-variety $f^{-1}(x)$. As $K(x)^{\mathrm{h}}$ is contained in the definable closure of $K(x)$, the point y is definable over $K(x)$, so it is equal to $\sigma(x)$ for some suitable K-definable map. Then every point t of $f^{-1}(x)$ having the same type than y over K satisfies the equation $\sigma((f(t)) = t$, so that $t = y$. Therefore y is, as a point of the fiber $f^{-1}(x)$, entirely determined by its type over K; otherwise said, it is entirely determined by the induced valuation v on $K(Y)$. As

$Y \to X$ is finite, the set $\mathscr{V}$ of valuations on $K(Y)$ extending the valuation u on $K(X)$ induced by x is finite, and is a system of independent valuations. Therefore there exists $\psi \in K(Y)$ such that $v(\psi) < 1$ and $w(\psi) = 1$ for every valuation $w \in \mathscr{V}\backslash\{v\}$. Up to replacing ψ by ψ^N for N large enough we can assume that $v(\psi) \in u(K(X))^\times$, so we can find a rational function φ on X with $u(\varphi)) < 1$ and $v(\psi) = u(\varphi)$. Then $|\varphi(x)| < 1$, the point y is the only point of $f^{-1}(x)$ at which $|\psi| = |\varphi|$, and $|\psi(t)| \geqslant 1$ for all $t \neq y$ in $f^{-1}(x)$. Let us shrink X and Y so that φ is a regular function on X and ψ is a regular function on Y. Then we can take for D the set of those points $z \in X$ such that $|\varphi(z)| < 1$, such that there exists one and only one point $\sigma(z)$ in the fiber $f^{-1}(z)$ at which $|\psi| = |\varphi|$ and such that $|\psi(t)| = 1$ for all $t \neq \sigma(z)$ in $f^{-1}(z)$.

And since f is étale at every point of Y, Lemma 4.7 above then ensures that $K(\sigma(z))^{\mathrm{h}} = K(z)^{\mathrm{h}}$ for all $z \in D$. □

7.2. ***Corollary.*** — *Let K be a defectless valued field and let X be an n-dimensional K-variety. Let $f_1, \dots, f_n$ be n functions belonging to $K[X]$; let f denote the K-morphism $(f_1, \dots, f_n)\colon X \to \mathbf{A}^n_K$. Let $x \in X$ be such that the f_i are invertible at x, the $\mathsf{rv}(f_i(x))$ are algebraically independent over $\mathsf{RV}(K)$, and $\mathsf{RV}(K(x)) = \mathsf{RV}(K)(\mathsf{rv}(f_1(x)), \dots, \mathsf{rv}(f_n(x)))$. Then there exist a K-Zariski open subset X' of X containing x, a K-Zariski open subset U of $\mathbf{A}^n_K$ such that f induces a finite étale map from X' to U, a K-regular function φ on U and a K-regular function ψ on X' with $|\psi(x)| = |\varphi(f(x))| < 1$, and a K-definable subset D of U containing $f(x)$ satisfying the following:*

- *for every $z \in D$ one has $|\varphi(z)| < 1$ and there exists one and only one element $\sigma(z)$ of X' such that $f(\sigma(z)) = z$ and $|\psi(\sigma(z))| = |\varphi(z)|$;*
- *for every $z \in D$ and for every $t \neq \sigma(z)$ in $f^{-1}(z)$ one has $|\psi(t)| = 1$;*
- *for all $z \in D$ one has $K(\sigma(z))^{\mathrm{h}} = K(z)^{\mathrm{h}}$.*

Proof. — As $\dim X = n$, the transcendence degree of $K(x)$ over K is bounded by n. But since $\mathrm{trdeg}(\mathsf{RV}(K(x))/\mathsf{RV}(K)) = n$ by our assumptions, $\mathrm{trdeg}(K(x)/K)$ is actually equal to n, the valued field $K(x)$ is an Abhyankar extension of K, and $(f_1(x), \dots, f_n(x))$ is an Abhyankar basis of $K(x)$ over K. The point $f(x)$ is therefore K-Zariski generic on $\mathbf{A}^n_K$, which implies that f is quasi-finite at x. The field $K(f(x))$ is the rational function field generated by the algebraically independent elements $f_1(x), \dots, f_n(x)$ and its valuation is the Gauss valuation with parameters $|f_1(x)|, \dots, |f_n(x)|$. Therefore $\mathsf{RV}(f(x))$ is the rational function field over $\mathsf{RV}(K)$ generated by the algebraically independent elements $(\mathsf{rv}(f_i(x)))_i$, so $\mathsf{RV}(K(f(x))$ is equal to $\mathsf{RV}(K(x))$; otherwise said, $K(x)$ is an immediate extension of $K(f(x))$. And as K is defectless, so is its Abhyankar extension $K(f(x))$. The corollary then follows directly from the lemma above. □

7.3. ***Theorem.*** — *Let K be an algebraically closed valued field and let X be a K-variety. Let x be a point of X such that $K(x)$ is an Abhyankar extension of K, let n be the transcendence degree of $K(x)$ over K and e be the rational rank of $|K(x)^\times|$ over $|K^\times|$; let $\gamma = (\gamma_1, \dots, \gamma_e)$ in $(|K(x)^\times|)^e$ be a basis of the free abelian group $|K(x)^\times|/|K^\times|$, and let $f_1, \dots f_m$ be regular invertible functions on a K-Zariski* *neighborhood of x such that the* $\mathsf{rv}(f_i(x))$ *generate* $\mathsf{RV}(K(x))$ *over* $\mathsf{RV}(K)$*. Set* $d = \mathsf{rv}(f(x)) \in \mathsf{RV}(K(x))^m$ *and* $\lambda = \mathsf{rv}(f(\cdot))\colon X \to \mathsf{RV}^m$*. There exist:*

- ◇ *an n-dimensional integral K-subvariety X' of X containing x on which all f_i are defined;*
- ◇ *a finite étale K-morphism of schemes $\pi\colon X' \to \Omega$ where Ω is a dense affine K-Zariski open subset of $\mathbf{A}^n$;*
- ◇ *a K-regular function φ on Ω and a K-regular function ψ on X' satisfying the condition $|\psi(x)| = |\varphi(\pi(x))| < 1$;*
- ◇ *a Kd-definable and* $\mathsf{RV}(K)$*-Zariski dense point c of $\mathbf{A}^{\gamma}_{\mathsf{RV}} \times \mathbf{A}^{n-e}_{\mathsf{k}}$ such that the open Kd-definable ball $D := \mathsf{rv}^{-1}(c) \subset \Omega$ contains $\pi(x)$ and is Kd-atomic (see 1.10 for the meaning of $\mathbf{A}^{\gamma}_{\mathsf{RV}}$);*
- ◇ *a finite Kc-definable subset A of* RV^m *containing d,*

such that

- $\pi^{-1}(D) = \lambda^{-1}(A)$*;*
- *for all $z \in D$ one has $|\varphi(z)| < 1$, there exists one and only one element $\sigma(z)$ of $\pi^{-1}(z) \cap \lambda^{-1}(d)$ at which $|\psi| = |\varphi|$ and for every $t \neq \sigma(z)$ in $\pi^{-1}(z) \cap \lambda^{-1}(d)$ one has $|\psi(t)| = 1$.*

Comments. — As c is Zariski-generic, the Abhyankar inequality ensures that $\mathsf{rv}^{-1}(c)$ consists itself of K-Zariski dense points of $\mathbf{A}^n_K$. This is the reason why $\mathsf{rv}^{-1}(c)$ is automatically contained in Ω without having to enforce this explicitly.

Also note that one has necessarily $\sigma(\pi(x)) = x$.

Proof. — For proving the Theorem, we can replace X by any of its K-subschemes containing x. So we can assume that X is integral of dimension n and that x is K-Zariski generic. Note that by the Abhyankar inequality every point of $\lambda^{-1}(d)$ is K-Zariski generic on X, so shrinking the K-scheme X when needed will be harmless.

7.3.1. — By the general theory of finitely generated abelian goupes (applied to the free group $|K(X)^\times|/|K^\times|$) there exists a matrix M in $\mathrm{GL}_m(\mathbf{Z})$ such that if one denotes by f'_i (for $1 \leqslant i \leqslant m$) the i-th component of $M(f_1, \ldots, f_m)$ for M acting monomially, then $|f'_i(x)| = \gamma_i ;\bmod |K^\times|$ for $1 \leqslant i \leqslant e$ and $|f'_i(x)| \in |K^\times|$ for $e+1 \leqslant i \leqslant m$. Choose elements $\ell_1, \ldots, \ell_m$ of $K^\times$ such that $|f'_i(x)/\ell_i| = \gamma_i$ for all $1 \leqslant i \leqslant e$ and $|f'_i(x)/\ell_i| = 1$ for some $e+1 \leqslant i \leqslant m$. Set $g_i = f'_i/\ell_i$ and $\mu = \mathsf{rv}(g(\cdot))\colon X \to \mathsf{RV}^m$. It follows from our constructions than (g_i) can be obtained from (f_i) by a bijective monomial transformation with coefficients in $|K^\times|$ and integral exponents whose inverse is of the same form. As a consequence the $\mathsf{rv}(g_i(x))$ also generate $\mathsf{RV}(K(x))$ over $\mathsf{RV}(K)$, the element $\mu(x)$ of RV^m is inter-definable with d, and $\lambda^{-1}(d) = \mu^{-1}(\mu(x))$. We thus can replace the f_i by the g_i, which amounts to assuming that $|f_i(x)| = \gamma_i$ for $i \leqslant e$ and that $|f_i(x)| = 1$ for $i > e$. We then have $d = (\delta, z)$ for some $\delta = (\delta_i)_{1 \leqslant i \leqslant e}$ with each δ_i non-zero of degree γ_i and with $z \in \mathsf{k}^{m-e}$, and the $\mathsf{RV}(K)$-Zariski- closure of d is equal to $\mathbf{A}^{\gamma}_{\mathsf{RV}} \times Z$ for some irreducible $\mathsf{k}(K)$-algebraic subvariety Z of $\mathbf{A}^{n-e}_{\mathsf{k}}$. As $\mathsf{RV}(K)(d) = \mathsf{RV}(K(x))$ and as $K(x)$ is Abhyankar over K, the dimension of $\mathbf{A}^{\gamma}_{\mathsf{RV}} \times Z$ is equal to n, so $\dim(Z) = n - e$.

7.3.2. — By generic smoothness there exists a family $h = (h_{e+1}, \ldots, h_n)$ of polynomials in $\mathsf{k}(K)[T_{e+1}, \ldots, T_m]$ that induces a generically étale map from Z to $\mathbf{A}^{n-e}_{\mathsf{k}}$. Choose $H_{e+1}, \ldots, H_n \in K[T_{r+1}, \ldots, T_m]$ with integral coefficients that lift $h_{r+1}, \ldots, h_n$, and set

$$\pi = (f_1, \ldots, f_r, H_{e+1}(f_{e+1}, \ldots, f_m), \ldots, H_n(f_{e+1}, \ldots, f_m));$$

this is a map from X to $\mathbf{A}^n$. Set $c = (\mathrm{Id}, h)(d) = (\delta, h(z))$; this is a Kd-definable element of $\mathbf{A}^{\gamma}_{\mathsf{RV}} \times \mathbf{A}^{n-e}_{\mathsf{k}}$, and $\mathsf{rv}(\pi(x)) = c$. As the map $h|_Z \colon Z \to \mathbf{A}^{n-e}_{\mathsf{k}}$ is generically étale, $h(z)$ is $\mathsf{k}(K)$-Zariski dense in $\mathbf{A}^{n-e}_{\mathsf{k}}$, so c is $\mathsf{RV}(K)$-Zariski dense in $\mathbf{A}^{\gamma}_{\mathsf{RV}} \times \mathbf{A}^{n-e}_{\mathsf{k}}$. Therefore by the Abhyankar inequality $\pi(x)$ is Zariski-generic, and π is generically finite; by construction, it induces the valuation $\eta_{K,(\gamma,1)}$ on $K(T_1, \ldots, T_n)$ (here the 1 in the index stands for $n - e$ terms all equal to 1, and we keep track of the ground field K in the notation since we will also have to consider this Gauss valuation with parameter $(\gamma, 1)$ on a rational function field over some valued extension of K), thus $\mathsf{RV}(K(\pi(x)) = \mathsf{RV}(K)(\mathsf{rv}(\pi(x))) = \mathsf{RV}(K)(c)$.

Since $K(\pi(x))$ is an Abhyankar extension of K (which is itself algebraically closed) $K(\pi(x))$ is defectless, so that the degree of $K(x)^{\mathrm{h}}$ over $K(\pi(x))^{\mathrm{h}}$ is equal to the residue degree, namely that of $\mathsf{RV}(K)(d)$ over $\mathsf{RV}(K)(c)$. As $\mathsf{RV}(K)(d)$ is by construction separable over $\mathsf{RV}(K)(c)$, it follows that $K(x)^{\mathrm{h}}$ is separable (and even tamely ramified) over $K(\pi(x))^{\mathrm{h}}$, so $K(x)$ is separable over $K(\pi(x))$. Moreover it follows from the fact that X is integral and $\pi(x)$ is K-Zariski generic that the scheme-theoretic fiber $\pi^{-1}(\pi(x))$ is reduced. Hence π is étale at x and up to shrinking X we can thus assume that π is finite étale over some dense K-Zariski affine open subset Ω of $\mathbf{A}^n_K$.

7.3.3. — Set $A = (\mathrm{Id}, h)^{-1}(c)$. By the Abhyankar inequality, the $\mathsf{RV}(K)$-definable subset $\lambda(X)$ is of dimension $\leqslant n$, so its fiber over c is necessarily finite by $\mathsf{RV}(K)$-genericity of c; therefore A is finite.

Set $D = \mathsf{rv}^{-1}(c)$. Then D is a Kc-definable n-dimensional open polydisc in $\mathbf{A}^n$, which is contained in Ω since each of its point is K-Zariski-generic by Abhyankar inequality, and $\pi^{-1}(D) = \lambda^{-1}(A)$. Moreover the polydisc D is $\mathsf{acl}(Kc)$-atomic: using the fact that c does not belong to $\mathsf{acl}(K)$ (unless $n = 0$ in which case c is empty, D is a point and the statement is obvious!), one sees it through a straightforward adaptation of the proof of [**HK06**, Lemma 3.8] (be aware that in loc. cit., the word *transitive* is used instead of *atomic*). As $d \in \mathsf{acl}(Kc)$ the polydisc D is particular Kd-atomic.

Set $F = \lambda^{-1}(d)$. The set F is Kd-definable, and $\pi(F) \subset D$ by definition. By Kd-atomicity $\pi(F)$ is either empty, either equal to the whole of D. But as $F \neq \varnothing$ (it contains x), we have $\pi(F) = D$. If G is any Kd-definable subset of F, it follows once again from atomicity that all fibers of $\pi|_G$ have the same cardinality $\nu(G)$; let us choose such a G with $x \in G$ and with $\nu(G)$ minimal; note that $\nu(G) > 0$ since G is non-empty (it contains x). If G' is any Kd-definable subset of G then either G' or its complement G'' contains x, so that $\nu(G') = \nu(G)$ or $\nu(G'') = \nu(G)$ by minimality of $\nu(G)$; it follows that $G' = G$ or $G'' = G$, so G is Kd-atomic. Therefore the type over Kd of any point $t \in G$ does not depend on t; a fortiori, its type over K does not depend on t and we denote it by p. We denote by v the corresponding valuation on $K(X)$. It is an extension of the valuation $\eta_{K,(\gamma,1)}$ along the finite extension $K(T_1, \ldots, T_n) \hookrightarrow K(X)$.

Since the extensions of $\eta_{K,(\gamma,1)}$ to $K(X)$ form a system of independent valuations there exist $\psi \in K(X)$ such that $v(\psi) < 1$ and $w(\psi) = 1$ for every extension $w \neq v$ of $\eta_{K,(\gamma,1)}$ to $K(X)$. Up to replacing ψ by ψ^N for N large enough, we may assume that there exists $\varphi \in K(T_1, \ldots, T_n)$ such that $v(\psi) = \eta_{K,(\gamma,1)}(\varphi)$. Up to shrinking Ω (and X) we can assume that φ is a K-regular function on Ω and ψ is a K-regular function on X. By construction $|\varphi| < 1$ at every point of D and $|\psi| = |\varphi|$ identically on G. As $\nu(G) > 0$ every point z of D has a pre-image s on G, and $|\psi(s)|$ is then equal to $|\varphi(z)|$.

Our purpose is now to show that for every $z \in D$ there is a unique element s of $\pi^{-1}(z)$ with $\lambda(s) = d$ and $|\psi(s)| = |\varphi(z)|$; by Kd-atomicity of D it suffices to exhibit one model N of ACVF containing Kd and one point $z \in D(N)$ for which this property holds.

7.3.4. — Let M be an algebraically closed valued field containing K such that $|M^\times| = |K^\times|$ and $z \in \mathsf{k}(M)$. Let N be a model of ACVF containing M and an M-Zariski generic n-uple $\xi = (\xi_1, \ldots, \xi_n)$ such that the valuation of $M(T_1, \ldots, T_n)$ induced by ξ is the composition $u \circ \eta_{M,(\gamma,1)}$ where u is the lexicographic valuation centered at the $\mathsf{k}(M)$-rational point $h(z)$ of $\mathbf{A}^{n-r}_{\mathsf{k}(M)}$, with respect to the regular system of parameters $(T_i - h_i(z))_{r+1 \leqslant i \leqslant n}$. The point ξ belongs to $D(N)$ and we are going to show that that there is a unique $\omega \in \pi^{-1}(\xi)$ with $\lambda(\omega) = d$ and $|\psi(\omega)| = |\varphi(\xi)|$. We know by the above that there is at least one such ω, so we fix one.

7.3.5. — The point ω is M-Zariski generic on X, and the induced valuation on $M(X)$ is of the form $\theta \circ w$, where w is an extension of $\eta_{\gamma,1}$ to $M(X)$ and θ is some valuation on $\mathsf{k}((M(X), w))$ lying above u. Its restriction to $K(T_1, \ldots, T_n)$ is thus equal to $u' \circ \eta_{K,(\gamma,1)}$, where u' is the restriction of u to $\mathsf{k}(K)(T_{r+1}, \ldots, T_n)$. But by construction the center of u on $\mathbf{A}^{n-r}_{\mathsf{k}(M)}$ lies above the generic point of $\mathbf{A}^{n-r}_{\mathsf{k}(K)}$, so the valuation u' is trivial, and the valuation on $K(T_1, \ldots, T_n)$ induced by ω is finally equal to $\eta_{K,(\gamma,1)}$. Hence the valuation induced by ω on $K(X)$ is an extension of $\eta_{K,(\gamma,1)}$, and the equality $|\psi(\omega)| = |\pi(\xi)|$ ensures (by the choice of ψ) that this extension is precisely v. Otherwise said, the type of ω over K is equal to p.

Now as v is a valuation extending the Gauss valuation $\eta_{K,(\gamma,1)}$ to $K(X)$, and w is an valuation extending the Gauss valuation $\eta_{M,(\gamma,1)}$ to $M(X)$, it follows from [**DHLY24**, §A.1.3] that w is the unique extension of $\eta_{M,(\gamma,1)}$ to $M(X)$ restricting to v, and that $\mathsf{RV}(M(X), w)$ is the (generalized) fraction field of the tensor product $\mathsf{RV}(K(X), v) \otimes_{\mathsf{RV}(K)} \mathsf{RV}(M)$. As $|M^\times| = |K^\times|$ this means that $w(M(X)^\times)$ is equal to $v(K(X)^\times)$ and that $\mathsf{k}(M(X), w)$ is the usual fraction field of $\mathsf{k}(K(X), v) \otimes_{\mathsf{k}(K)} \mathsf{k}(M)$.

But v is the valuation on $K(X)$ induced by x, for the latter lies on G. So one has $\mathsf{k}(K(X), v) = \mathsf{k}(K)(Z')$, $\mathsf{k}(M(X), w) = \mathsf{k}(M)(Z')$ and

$$w(M(X)^\times) = v(K(X)^\times) = |K^\times| \gamma^{\mathbf{Z}}.$$

The valuation induced on $M(X)$ by ω is equal to $\theta \circ w$. As $\lambda(\omega) = d = (\gamma, z)$, it follows that $\mathsf{rv}(f_i(\omega)) = z_i$ for all $i \geqslant r + 1$, so that the valuation θ of $\mathsf{k}(M)(Z')$ is necessarily centered at the $\mathsf{k}(M)$-rational point z of the scheme $Z'_{\mathsf{k}(M)}$.

7.3.6. — The map $h\colon Z'_{\mathsf{k}(M)} \to \mathbf{A}^{n-r}_{\mathsf{k}(M)}$ is étale. The valuation θ is therefore the only valuation on $\mathsf{k}(M)(Z')$ centered at z and lying above u, and one has moreover the equality $(\mathsf{k}(M)(Z'),\theta)^{\mathrm{h}} = (\mathsf{k}(M)(T_{r+1},\dots,T_n),u)^{\mathrm{h}}$.

Indeed, let S denote the spectrum of the ring of integers of $(\mathsf{k}(M)(T_{r+1},\dots,T_n),u)^{\mathrm{h}}$, and let s be its closed point. The special fiber of the S-scheme $Z'_{\mathsf{k}(M)} \times_{\mathbf{A}^n_{\mathsf{k}(M)}} S$ has only one pre-image t over z, and t and s have the same residue field since z and $h(z)$ have the same residue field $\mathsf{k}(M)$. As S is henselian the connected component S' of $Z'_{\mathsf{k}(M)} \times_{\mathbf{A}^n_{\mathsf{k}(M)}} S$ that contains t is the spectrum of a finite local S-scheme; since $Z'_{\mathsf{k}(M)} \times_{\mathbf{A}^n_{\mathsf{k}(M)}} S$ is étale over S and since s and t have the same residue field, $S' \simeq S$. Now is ϖ is any valuation on $\mathsf{k}(M)(Z')$ centered at z and lying above u and if S'' denotes the spectrum of the ring of integers of $(\mathsf{k}(M)(Z'),\varpi)^{\mathrm{h}}$, then S'' dominates S', so is equal to S' as the latter is already the spectrum of a henselian valuation ring (because $S' \simeq S$). This proves at the same time the uniqueness of ϖ and the fact that its henselization coincides with that of u.

7.3.7. — By the above, the valuations w and θ are uniquely determined; i.e., they do not depend on the choice of ω; therefore the type of ω over M (which is entirely encoded in the valuation $\theta \circ w$) also does not depend on ω. And since θ and u have the same henselization, the former is an immediate extension of the latter. We have also seen at the end of 7.3.5 that $w(M(X))^{\times}) = |K^{\times}|\gamma^{\mathbf{Z}}$, which implies that $w(M(X))^{\times}) = \eta_{M,(\gamma,1)}(M(T_1,\dots,T_n)^{\times})$; thus $\theta \circ w$ is an immediate extension of $u \circ \eta_{M,(\gamma,1)}$.

7.3.8. — Now consider all pairwise distinct pre-images $\omega = \omega_1,\dots,\omega_\ell$ of ξ on $\Delta(N)$ at which $|\psi| = 1$. Fix j. It follows from 7.3.7 that $M(\omega)$ is an immediate extension of $M(\xi)$. As the latter is Abhyankar over M, which is itself algebraically closed, it is defectless. Lemma 7.1 then ensures that $M(\omega)^{\mathrm{h}} = M(\xi)^{\mathrm{h}}$. As a consequence ω is definable over $M(\xi)$, so there exists an M-definable map τ from $\mathbf{A}^n$ to X such that $\tau(\xi) = \omega$, so $\tau(\pi(\omega)) = \omega$. But by 7.3.7, all ω_j have the same type over M, so $\tau(\pi(\omega_j)) = \omega_j$ for all j, which means that $\omega_j = \omega$.

7.3.9. *Conclusion.* — We have thus proved that for every $z \in D$ there is a unique element s of $\pi^{-1}(z)$ such that $\lambda(s) = d$ and $|\psi(s)| = |\varphi(z)|$, which we denote by $\sigma(z)$. Now let t be an element of $\pi^{-1}(z) \cap \lambda^{-1}(d)$ different from $\sigma(z)$. The valuation w induced by t on $K(X)$ is an extension of $\eta_{K,(\gamma,1)}$, and it is not equal to v since $|\psi(t)| \neq 1$. Then $w(\psi) = 1$, which exactly means that $|\psi(t)| = 1$, and we are done (there is no explicit X' because we have shrunken X several times during the proof). □

8. Nash structure of curves: covering by nice charts

Our purpose is now to elucidate the structure of the stable completion of a curve beyond its topology. The genuine structure theorem we have in mind will in fact be stated and proved only in the next section, but it will rest on a first structure theorem (with a less tractable formulation) which is the object of the current section.

We will use freely the results of sections 2 and 3 on the topology of curves, and especially the notion of an admissible skeleton (Definition 2.3) and the associated maps $\rho, \widehat{\rho}, \mathfrak{r}$ and $\widehat{\mathfrak{r}}$ (2.7, 3.8).

8.1. Definition. — Let K be a valued field and let X be an algebraic K-curve. A *nice K-chart* on X is a pair (U, f) where U is a K-definable subset of X and where $f\colon X' \to \Omega$ is an étale map defined over K whose source X' is a K-Zariski open subset of X containing U and whose target Ω is a dense K-Zariski open subset of $\mathbf{A}^1_K$; moreover we require that there exist a K-regular map ψ on X', a K-regular map φ on Ω, and that one of the following hold:

- ◇ (Charts of type 1) The set Ω contains an open disc D centered at the origin and:
  - for all $z \in D$ one has $|\varphi(z)| < 1$ and there exists a unique element $\sigma(z)$ in $f^{-1}(z)$ such that $|\psi(\sigma(z)| = |\varphi(z)|$;
  - for every $z \in D$ and every $t \neq \sigma(z)$ in $f^{-1}(z)$ one has $|\psi(t)| = 1$;
  - $\sigma(D) = U$.
- ◇ (Charts of type 2) The map f is finite, the chart (U, f) comes implicitly with an extra datum consisting of a K-definable point s of $\widehat{U}$ called the *vertex* of the chart such that $\{s\}$ is an admissible skeleton of the stable completion $\widehat{U}$, the image $f(s)$ is equal to $\eta_{0,|f(s)|}$ and s is the unique pre-image of $\eta_{0,|f(s)|}$ on $\widehat{U}$, and there exist a family $h = (h_1, \ldots, h_m)$ of regular functions on X', and for every $\zeta \in C_{s,\widehat{U}}$ an open disc D_ζ of radius $|f(s)|$, a finite subset A_ζ of k^m and an element a_ζ of A, satisfying the following, where λ denotes the map $\mathsf{rv} \circ h\colon X' \to \mathsf{RV}^m$:
  - D_ζ, A_ζ and a_ζ depend K-definably on ζ;
  - for every ζ one has $f^{-1}(D_\zeta) = \coprod_{a \in A_\zeta} \lambda^{-1}(a)$;
  - for every $\zeta \in C_{s,\widehat{U}}$ and every $z \in D_\zeta$ one has $|\varphi(z)| < 1$;
  - for every $\zeta \in C_{s,\widehat{U}}$ and every $z \in D_\zeta$ there is a unique point $\sigma(z)$ in $f^{-1}(z) \cap \lambda^{-1}(a_\zeta)$ such that $|\psi(\sigma(z))| = |\varphi(z)|$;
  - for every $\zeta \in C_{s,\widehat{U}}$, every $z \in D_\zeta$ and every $t \neq \sigma(z)$ in $f^{-1}(z) \cap \lambda^{-1}(a_\zeta)$ one has $|\psi(t)| = 1$;
  - for every $\zeta \in C_{s,\widehat{U}}$ one has $\sigma(D_\zeta) = \mathfrak{r}^{-1}_{U,\{s\}}(\zeta)$.
- ◇ (Charts of type 3) The map f is finite, the set Ω contains an open annulus $A = \{R_1 < |T| < R_2\}$ of $\mathbf{A}^1$ with R_1 and R_2 in $|K| \cup \{\infty\}$ such that:
  - $|\varphi(z)| < 1$ for all $z \in A$;
  - for all $z \in A$ there exists a unique element $\sigma(z)$ in $f^{-1}(z)$ such that $|\psi((\sigma(z))| = |\varphi(z)|$;
  - for every $z \in A$ and every $t \neq \sigma(z)$ in $f^{-1}(z)$ one has $|\psi(t)| = 1$;
  - $\sigma(A) = U$.

8.2. — Let K be a valued field and let (U, f) be a nice K-chart. In what follows, we will distinguish three cases according to the type of the chart. For each type we will use the notation of the corresponding definition above.

8.2.1. *The case where (U, f) is of type 1.* — It follows then from Example 4.9 that f induces a Nash isomorphism between U and D, and from Lemma 4.10 that $\widehat{U}$ is open in $\widehat{X'}$, hence in $\widehat{X}$, and that it is closed in $f^{-1}(\widehat{D})$.

8.2.2. *The case where (U, f) is of type 2.* — Let $\zeta \in C_{s,\widehat{U}}$. The pre-image $f^{-1}(\widehat{D_\zeta})$ is an open $K\zeta$-definable subset of $\widehat{X'}$ (so it is open in $\widehat{X}$ too). By assumption one has $f^{-1}(\widehat{D_\zeta}) = \coprod_{a \in A_\zeta} \widehat{\lambda^{-1}(a)}$; each of the $\widehat{\lambda^{-1}(a)}$ is open in $\widehat{X'}$ (it is described by strict inequalities), so in particular $\widehat{\lambda^{-1}(a_\zeta)}$ is open and closed in $f^{-1}(\widehat{D_\zeta})$.

As $\widehat{\mathfrak{r}}^{-1}(\zeta)$ is by definition of a nice K-chart of type 2 the intersection of $\widehat{\lambda^{-1}(a_\zeta)}$ with $\{|\psi| < 1\}$, it is open in $f^{-1}(\widehat{D_\zeta})$. As a consequence, and since the étale map $X' \to \Omega$ induces an open map at the level of stable completions (Proposition 4.1), the definable bijection $f|_{\mathfrak{r}_{U,s}^{-1}(\zeta)} \colon \mathfrak{r}^{-1}(\zeta) \to D_\zeta$ induces a definable homeomorphism $\widehat{\mathfrak{r}}_{U,s}^{-1}(\zeta) \simeq \widehat{D_\zeta}$, so f induces a Nash ismorphism $\mathfrak{r}_{U,s}^{-1}(\zeta) \simeq D_\zeta$. In view of Lemma 4.10, $\widehat{\mathfrak{r}}_{U,s}^{-1}(\zeta)$ is open in $\widehat{X'}$, hence in $\widehat{X}$, and it is closed in $f^{-1}(\widehat{D}_\zeta)$.

8.2.3. *The case where (U, f) is of type 3.* — It follows then from Example 4.9 that f induces a Nash isomorphism between U and A, and from Lemma 4.10 that $\widehat{U}$ is open in $\widehat{X'}$, hence in $\widehat{X}$, and that it is closed in $f^{-1}(\widehat{A})$.

8.3. Remark. — Let (U, f) be a nice K-chart of type (1) or (3). It follows from 4.7 that for every model M of ACVF containing K and every M-point of x of U the valued fields $M(u)$ and $M(f(u))$ have the same henselization; in particular they have the same value group and the same residue field. It follows that if x is a non-simple point of $\widehat{U}$, the map f induces an isomorphism between the residue curves C_x and $C_{f(x)}$.

8.4. Remark. — If $a \in K^\times$ and (U, f) is a nice chart of type 1, resp. 2, resp. 3, then so is $(U, f/a)$ (with the same vertex in the case of type 2). Therefore if $|K^\times|$ is divisible, so that every K-definable element of Γ belongs to $|K^\times|$, every nice chart of type 1 (resp. 2) can be rescaled so that D is of radius 1 (resp. so that the vertex lies above $\eta_{0,1}$).

8.5. Theorem. — *Let K be an algebraically closed valued field, let X be a smooth algebraic K-curve and let U be a K-definable subset of X without any isolated simple point. There exist finitely many nice K-charts (U_i, f_i) on X such that $\widehat{U} = \bigcup \widehat{U_i}$.*

Proof. — It is quite long, with several steps. Its core relies on Theorem 7.3 that exhibits atomic balls.

8.5.1. *Preliminaries.* — We can assume that K is maximally complete. Indeed, if K is trivially valued there is nothing to do. If not, K is a model of ACVF and as a nice K-chart can be described using finitely many parameters in K, and as being a nice K-chart is a first-order formula in those parameters, we can prove the theorem on any model larger than K, which allows one to assume that K is maximally complete.

Let $x \in \widehat{U}(M)$ for some model M of ACVF containing K. By compactness it suffices to exhibit a nice K-chart V such that $x \in \widehat{V}$. Let $\alpha \in U(N)$ be a realization of x (which is a type over M) for some model $N \supset M$. We shall distinguish three cases according to the type of α *over* K.

8.5.2. *The "immediate extension" case.* — Assume that $K(\alpha)$ is an immediate extension of K. Since K is maximally complete this amounts to saying that $\alpha \in X(K)$. Our purpose is to build a nice K-chart (V, f) of type 1 such that $\alpha \in V$, which will ensure that $x \in \widehat{V}$.

As the curve X is smooth, there exists some K-Zariski neighborhood X' of α in X and a K-definable étale map f from X' to $\mathbf{A}^1$ mapping α to the origin. In view of the local structure of étale maps we can assume that there exists an affine K-Zariski open neighborhood $\Omega = \operatorname{Spec} R$ of 0 in $\mathbf{A}^1_K$ and a monic polynomial $P \in R[\tau]$ such that X' can be identified with the invertible locus of P' in $\operatorname{Spec} R[\tau]/P$. Now $\beta = \tau(\alpha)$ is a simple root of P_0, where P_0 denotes the polynomial in $K[\tau]$ obtained by evaluating each coefficient of P at 0. By suitably translating τ we can assume that $\beta = 0$, so $P = \sum a_i \tau_i$ with $a_0(0) = 0$ and $a_1(0) \neq 0$ (because $\beta = 0$ is a simple root of P_0).

We handle first the case where K is trivially valued. Let F be any model of ACVF containing K and let z be an element of F with $|z| < 1$. As all a_i are rational functions with coefficients in K and no poles at the origin, and as $a_1(0) \neq 0$ does not vanish at the origin, we have $|a_0(z)| < 1$, $|a_1(z)| = 1$ and $|a_i(z)| \leqslant 1$ for all $i \geqslant 2$. Then the basic theory of Newton polygons show that the monic polynomial P_z has a unique root $\sigma(z)$ with $|\sigma(z)| < 1$ and that $|\sigma(z)| = |a_0(z)|$, and that $|\gamma| = 1$ for every root $\gamma \neq \sigma(z)$ of P; it is also clear that $\sigma(z) \in X'$. Let D be the open unit disc centered at the origin (note that $D \subset \Omega$ since K is trivially valued) and set $V = \sigma(D)$; note that $\sigma(0)$ is necessarily equal to α. Then (V, f) is a nice K-chart of type 1 that contains α (it fulfills the definition with $\varphi = a_0$ and $\psi = 1$).

Assume now that the valuation of K is non-trivial, so K is a model of ACVF. Again, the theory of Newton polygons ensures that there exists an open K-definable disc D centered at the origin and contained in Ω such that:

- ◇ a_1 does not vanish on D;
- ◇ for every point z of the disc D, the polynomial R_z has a unique root $\sigma(z)$ such that $|\sigma(z)| = |a_0(z)/a_1(z)|$, and $\sigma(z)$ lies in X';
- ◇ there exist some $\lambda \in |K^\times|$ such that for every $z \in D$ and every root $\gamma \neq \sigma(z)$ of P_z one has $|\gamma| > |\lambda| > |a_0(z)/a_1(z)|$.

Now after shrinking if needed Ω (and X') so that $a_1 \in \mathscr{O}(\Omega)^\times$, we set $\varphi = \frac{1}{\lambda} \cdot \frac{a_0}{a_1}$; this is a K-regular function on Ω. And after shrinking once again if needed Ω and X' so that for all $z \in \Omega$ the polynomial P_z does not vanish at λ, we set $\psi = \frac{\tau}{\tau - \lambda}$; this is a K-regular function on X'. By construction for every $z \in D$ and every $t \in f^{-1}(z)$ either $t = \sigma(z)$ and $|\psi(t)| = |\varphi(z)| < 1$, or $t \neq \sigma(z)$ and $|\psi(t)| = 1$. Set $V = \sigma(D)$. Then (V, f) is a nice K-chart of type 1 that contains α.

8.5.3. *The "group extension" case.* — Assume now that $|K(\alpha)^\times| \supsetneq |K^\times|$. Using Abhyankar inequality and the fact that $|K^\times|$ is divisible, we see that $|K(\alpha)^\times|$ is equal to $|K^\times| \oplus r^{\mathbf{Z}}$ for some $r \notin |K^\times|$, and that $\mathsf{k}(K(\alpha)) = \mathsf{k}(K)$. Choose a K-Zariski open subset X' in X such that $\alpha \in X'$ and a K-regular function f on X' such that $|f(\alpha)| = r$. Then $f(\alpha)$ does not belong to K, so f is quasi-finite at α; and $K(\alpha)$ is by construction an immediate extension of $K(f(\alpha))$, which is Abhyankar over K and thus defectless. By Lemma 7.1 there exists a K-Zariski open neighborhood X' of α in X such that f induces a finite étale map from X' to some K-Zariski dense open Ω of $\mathbf{A}^1_K$, a K-regular function ψ on X' such that $|\psi(\alpha)| < 1$, a K-regular function φ on Ω and a K-definable subset D of Ω containing $f(\alpha)$ and satisfying the following property: for every $z \in D$ one has $|\varphi(z)| < 1$, there is a unique element $\sigma(z)$ of $f^{-1}(z)$ such that $|\psi(\sigma(z))| = |\varphi(z)|$, and $|\psi(t)| = 1$ for every $t \neq \sigma(z)$ in $f^{-1}(z)$.

Set $\beta = f(\alpha)$. As $|\beta|$ is not torsion modulo the divisible group $|K^\times|$ one has $|\sum \lambda_i \beta^i| = \max|\lambda_i| \cdot |\beta|^i$ for every polynomial $\sum \lambda_i T^i$ in $K[T]$; thus if $P \in K[T]$ is such that $|P(\beta)| \leqslant 1$ (resp. < 1) then there exists a K-definable open annulus containing β on which $|P| \leqslant 1$ (resp. < 1). Consequently, the K-definable subset $\sigma^{-1}(U)$ of D, which contains β, contains a K-definable open annulus A with $\beta \in A$. Set $V = \sigma(A)$. Then by construction (V, f) is a nice K-chart of type 3 contained in U; since $\alpha \in V$, the stable completion of V contains x.

8.5.4. *The "residue extension" case.* — We finally assume that $\mathsf{k}(K(\alpha)) \supsetneq \mathsf{k}(K)$, so by Abhyankar inequality and by divisibility of $|K^\times|$ the group $|K(\alpha)^\times|$ is equal to $|K^\times|$ and the field $\mathsf{k}(K(\alpha))$ is a function field of transcendence degree 1 over $\mathsf{k}(K)$. Up to shrinking X (i.e., replacing it by a K-Zariski open subset containing α) we can choose a family $\lambda = (f_1, \ldots, f_m)$ of K-regular invertible functions on X such that $|f_i(\alpha)| = 1$ for all i and such that the $\mathsf{rv}(\lambda_i(\alpha))$ generate $\mathsf{k}(K(\alpha))$ over $\mathsf{k}(K)$. We set $\lambda = (\mathsf{rv} \circ f_i)_i \colon X \to \mathsf{RV}^m$ and $d = \lambda(\alpha)$.

8.5.4.1. — In view of Theorem 7.3 we can shrink X once again to ensure that there exist:

- ◇ a K-definable finite étale map $f \colon X \to \Omega$ where Ω is a K-Zariski dense affine open subset of $\mathbf{A}^1_K$, a K-regular function φ on Ω and a K-regular function ψ on X;
- ◇ a K-transcendental d-definable element c of k, such that the open Kd-definable ball $D := \mathsf{rv}^{-1}(c) \subset \Omega$ contains $f(\alpha)$ and is Kd-atomic;
- ◇ a finite c-definable subset A of $\mathsf{k}(K)^m$ containing d,

such that:

- $f^{-1}(D) = \coprod_{a \in A} \lambda^{-1}(a)$;
- for every $z \in D$ one has $|\varphi(z)| < 1$ and there exists a unique element $\sigma(z)$ of $f^{-1}(z) \cap \lambda^{-1}(d)$ such that $|\psi(\sigma(z))| = |\varphi(z)|$;
- for every $z \in D$ one has $|\varphi(z)| > 1$ and and for every $t \neq \sigma(z)$ in $f^{-1}(z) \cap \lambda^{-1}(d)$ one has $|\psi(t)| = 1$.

Set $E = \sigma(D)$. This is a Kd-definable subset of X definably isomorphic to D via f, so it is in particular Kd-atomic. We also have $E = f^{-1}(D) \cap \lambda^{-1}(d) \cap \{|\psi| = |\varphi|\}$

as well as $E = f^{-1}(D) \cap \lambda^{-1}(d) \cap \{|\psi| < 1\}$; applying stable completion to both sides of the latter equality shows that $\widehat{E}$ is open in $\widehat{X}$. By openness of étale (in fact, even flat) maps at the level of stable completions (Proposition 4.1), the map f induces a Nash isomorphism between E and D.

The type of α over K is stably dominated, so it defines a point s of $\widehat{X}(K)$, which lies on $\widehat{U}(K)$. As $\mathsf{rv}(f(\alpha)) = c$, which is K-transcendental, the image $f(s)$ is the Gauss point $\eta_{0,1}$ of $\widehat{\mathbf{A}^1_K}$. Note that $\eta_{0,1} \notin \widehat{D}$, so $s \notin \widehat{E}$. The fiber $f^{-1}(\eta_{0,1})$ in $\widehat{U}$ is finite and K-definable. In view of the explicit description of the topology of $\widehat{X}$ there exists a K-definable subset U' of U, defined by a conjunction of strict inequalities, such that s is the unique pre-image of $\eta_{0,1}$ on $\widehat{U'}$. Then as $s \in \widehat{U'}$ the point α belongs to U', so $U' \cap E$ is a non-empty Kd-definable subset of E, hence $U' \cap E = E$ by Kd-atomicity, and $E \subset U'$. Set $U'' = U' \cap \{|\psi| = |\varphi|\}$. Then $E \subset U''$ and s is the only pre-image of $\eta_{0,1}$ under f in $\widehat{U''}$.

Let us choose a K-definable admissible skeleton Σ of $\widehat{U''}$ containing s and set $W = \mathfrak{r}_{U'',\Sigma}^{-1}(s) \cap U''$. Then W is a K-definable subset of U'', and $\widehat{W} = \widehat{\mathfrak{r}}_{U'',\Sigma}^{-1}(s)$, so $\{s\}$ is a K-definable admissible skeleton of $\widehat{W}$. As $s \in \widehat{W}$ the simple point α belongs to W. The intersection $W \cap E$ is then a non-empty Kd-definable subset of E, so it is equal to the whole of E; hence $E \subset W$.

8.5.4.2. — Set $\xi = \mathfrak{r}_{W,\{s\}}(\alpha)$. Our purpose is now to show that ξ is inter-definable with d over K and that $\widehat{E} = \widehat{\mathfrak{r}}_{W,s}^{-1}(\xi)$ (and so that $E = \mathfrak{r}_{W,s}^{-1}(\xi)$).

Let o be a point of $\widehat{E}$ and let I be the unique segment on $\widehat{W}$ with endpoints o and s; let $t \in (o, s]$ be the "smallest" point of I outside $\widehat{E}$. Since f induces a definable homemorphism from $\widehat{E}$ to $\widehat{D}$, the image $f([o, t))$ is an interval on $\widehat{D}$ that is closed in $\widehat{D}$; since moreover $f([o, t])$ is compact, one has necessarily $f(t) = \eta_{0,1}$. As s is the only pre-image of $\eta_{0,1}$ on $\widehat{W}$, we see that $t = s$, so $[o, s) \subset \widehat{E}$ and $\widehat{E} \cup \{s\}$ identifies to the canonical compactification of $\widehat{E}$ (which is a tree with one branch at infinity, since it is homeomorphic to $\widehat{D}$). If o' is another point of E, and if I' denotes the unique segment on $\widehat{W}$ joining o' to s, then $I' \cap I = [o, s] \cap [o', s]$ and it is of the form $[\tau, s]$ for some $\tau \in \widehat{E}$; hence the germs of I and I' at s are the same, so $\widehat{\mathfrak{r}}_{W,s}(o) = \widehat{\mathfrak{r}}_{W,s}(o')$. Thus $\widehat{\mathfrak{r}}_{W,s}$ is constant on $\widehat{E}$ with value ξ, for $\widehat{\mathfrak{r}}_{W,s}(\alpha) = \xi$. The set E being Kd-definable, this already shows that ξ is Kd-definable.

Now let w be a point of $\widehat{W}$ with $\widehat{\mathfrak{r}}_{W,s}(w) = \xi$. Then the unique interval $[w, s]$ on $\widehat{W}$ has the same germ at s than $[\alpha, s]$, so in particular there exists $\tau \in \widehat{E}$ with $\tau \in [w, s]$. The intersection $\widehat{E} \cap [w, s)$ is then a non-empty definable open subset of $[w, s)$ (we use the fact that $\widehat{E}$ is open in $\widehat{X}$). If it were not equal to the whole of $[w, s)$ the connected component of τ in this intersection would be of the form (τ', s); then τ' would belong to $\partial\widehat{E} = \{s\}$, whence a contradiction. As a consequence $w \in \widehat{E}$ and $\widehat{E} = \widehat{\mathfrak{r}}_{W,s}^{-1}(\xi)$. As λ takes the constant value d on E we see that d is $K\xi$-definable, as announced.

8.5.4.3. *Conclusion.* — Let Θ be the subset of $C_{s,\widehat{W}}$ consisting of points ζ such that there exist:

— a $K\zeta$-definable element c_ζ of $\mathsf{k}(K)$;

— a finite $K\zeta$-definable subset A_ζ of $\mathsf{k}(K)^m$;

— a $K\zeta$-definable element a_ζ of A_ζ

such that the following hold, where D_ζ denotes the open unit ball $\mathsf{rv}^{-1}(c_\zeta)$:

- $f^{-1}(D_\zeta) = \coprod_{a \in A_\zeta} \lambda^{-1}(a)$;
- $|\varphi| < 1$ everywhere on D_ζ;
- $\mathfrak{r}_{W,s}^{-1}(\zeta) = f^{-1}(D_\zeta) \cap \lambda^{-1}(a_\zeta) \cap \{|\psi| = |\varphi|\}$;
- $\mathfrak{r}_{W,s}^{-1}(\zeta) = f^{-1}(D_\zeta) \cap \lambda^{-1}(a_\zeta) \cap \{|\psi| < 1i\}$;
- the map $\mathfrak{r}_{W,s}^{-1}(\zeta) \to D_\zeta$ induced by f is bijective.

The set Θ is ind-K-definable, and it contains ξ which is $\mathsf{k}(K)$)-Zariski generic (indeed c is $K\zeta$-definable and is transcendental over $\mathsf{k}(K)$). Thus Θ contains a $\mathsf{k}(K)$-Zariski-dense open subset C' of $C_{s,\widehat{W}}$ and we can assume that c_ζ, A_ζ and a_ζ depend K-definably and uniformly on the element ζ of C'. Set $V = \mathfrak{r}^{-1}(C')$. It follows from our construction than (V, f) is a nice K-chart of type 2; and as $\xi \in C'$ by $\mathsf{k}(K)$-Zariski genericity, the point a belongs to V, so $x \in \widehat{V}$. □

9. Nash structure of curves: triangulations

We fix for the whole section a valued field K and an algebraic closure $\overline{K}$ of K, equipped with an extension of the valuation of K.

What will actually matter to us here is K-definability rather than K-rationality, so it would be essentially harmless to replace K by its definable closure, i.e., to assume that K is perfect Henselian; but this would not make the proofs nor the statements simpler. We denote by G the automorphism group of $\overline{K}$ over K (as valued fields; so this is the decomposition group of the chosen valuation on $\overline{K}$).

9.1. Split and non-split definable intervals. — Let I be a K-definable topological space which is $\overline{K}$-definably homeomorphic to a (generalized) open interval J. As G acts trivially on Γ_0, it acts trivially on J and so acts through a finite quotient on I (namely through $\mathrm{Aut}(L/K)$ where L is any finite Galois extension of K inside $\overline{K}$ over which I is homeomorphic to J). So if some $\sigma \in G$ preserves the two orientations of I (or said otherwise, preserves its two branches at infinity), then σ acts on I as a monotonic torsion homeomorphism of an interval, so it acts trivially. As a consequence either G acts trivially on I, or it acts through a finite quotient of order two whose non-trivial element acts as an order-reversing involution on I; such an involution has a unique fixed point, which is a K-definable point of I (and even the unique K-definable point of I) and will be called its *center*.

9.2. Definition. — Let Σ be a Γ-internal one-dimensional K-definable topological space. A K-definable *vertex set* of Σ is a finite K-definable subset $\mathscr{V}$ of Σ such that each connected component of $\Sigma\backslash\mathscr{V}$ is $\overline{K}$-definably homeomorphic to a (generalized) open interval.

9.3. — Let Σ be a Γ-internal one-dimensional K-definable topological space and let $\mathscr{V}$ be a K-definable vertex set of Σ.

9.3.1. — If $\mathscr{W}$ is a K-definable subset of Σ containing $\mathscr{V}$, it is immediate that $\mathscr{W}$ is also a K-definable vertex set of Σ.

9.3.2. — Let $I \in \pi_0(\Sigma\backslash\mathscr{V})$. It follows from 9.1 that we are in one (and only one) of these two cases:

- the stabilizer of I in G acts trivially on I; this happens if and only if I is L-definably homeomorphic to an open interval for every $K \subset L \subset \overline{K}$ such that I is L-definable, or also if and only if I is inter-definable with at least one of its two branches at infinity (it will then be inter-definable with both);
- the stabilizer of I in G acts on I through a quotient of order two, whose non-trivial element acts on I as an orientation-reversing involution having exactly one fixed point v_I, inter-definable with I, which we call the center of I.

9.4. Lemma-Definition. — *Let Σ be a Γ-internal one-dimensional K-definable topological space and let $\mathscr{V}$ be a K-definable vertex set of* Σ.

(1) *The following are equivalent:*

  (i) *there exists a K-definable family $(o_I)_{I\in\pi_0(\Sigma\backslash\mathscr{V})}$ where each o_I is an orientation of I;*

  (ii) *for every $I \in \pi_0(\Sigma\backslash\mathscr{V})$ the two orientations of I are (individually) inter-definable with I.*

  When they are fulfilled we say that $(\Sigma,\mathscr{V})$ is K-definably orientable*.*

(2) *If $(\Sigma,\mathscr{V})$ is K-definably orientable, so is $(\Sigma,\mathscr{W})$ for any finite K-definable set $\mathscr{W}$ containing $\mathscr{V}$.*

(3) *In general, there always exists some finite K-definable set $\mathscr{W} \supset \mathscr{V}$ such that $(\Sigma,\mathscr{W})$ is K-definably orientable.*

Proof. — Let us prove (1). If (i) holds then for each $\sigma \in G$ one has $\sigma(o_I) = o_{\sigma(I)}$, so if $\sigma(I) = I$ then σ preserves o_I, whence (ii). Let us now assume that (ii) holds. Pick a system of reprentatives $\mathscr{J}$ of the orbits of $\pi_0(\Sigma\backslash\mathscr{V})$ under G and choose an arbitrary orientation o_J on J for every $J \in \mathscr{J}$. Now if $I \in \pi_0(\Sigma\backslash\mathscr{V})$ there exist some (unique) $J \in \mathscr{J}$ and some $\sigma \in \mathrm{Aut}(\overline{K}/K)$ and by (ii) the orientation $\sigma(o_J)$ only depends on I; we denote it by o_I, and the family (o_I) is K-definable, whence (i).

Now let us prove (2). Assume that $(\Sigma,\mathscr{V})$ is K-definably orientable. If J is a connected component of $\Sigma\backslash\mathscr{W}$ it is contained in some connected component $c(J)$ of $\Sigma\backslash\mathscr{V}$; then if $(o_I)_I$ is as in (i), the family $(o_{c(J)}|_J)_{J\in\pi_0(\Sigma\backslash\mathscr{W})}$ will be a K-definable system of orientations, so $(\Sigma,\mathscr{W})$ is K-definably orientable.

Let us finally prove (3). Let $\mathscr{I}$ be the set of connected components of $\Sigma\backslash\mathscr{V}$ which are not inter-definable with their orientations. Let $I \in \mathscr{I}$; for such an I, denote its center by v_I. Then $I\backslash\{v_I\}$ has two connected components, and each of them is inter-definable with its two orientations (individually), for one of these orientations is the one pointing toward v_I. Set $\mathscr{W} = \mathscr{V} \cup \{v_I\}_{I\in\mathscr{I}}$; then $\mathscr{W}$ is K-definable and $(\Sigma, \mathscr{W})$ is K-definably orientable. □

9.5. Definition. — Let K be a valued field and let $\overline{K}$ be an algebraic closure of K equipped with an extension of the valuation of K. Let X be an algebraic K-curve and let U be a K-definable subset of X. A K-skeleton Σ of $\widehat{U}$ is called *Nash-admissible* if it is admissible and if there exists a K-definable vertex set $\mathscr{V}$ of Σ containing all simple points of Σ and fulfilling the following properties:

(1) the pair $(\Sigma, \mathscr{V})$ is K-definably orientable;

(2) for every connected component I of $\Sigma\backslash V$ there exists a $\overline{K}$-regular function f_I defined on a $\overline{K}$-Zariski neighborhood of $\rho_{U,\Sigma}^{-1}(I) \cap U$ such that $(\rho_{U,\Sigma}^{-1}(I) \cap U, f_I)$ is a nice $\overline{K}$-chart of type 3;

(3) one has $\rho_{U,\Sigma}^{-1}(v) = \{v\}$ for every $v \in \mathscr{V} \cap U$ (as $\mathscr{V}$ contains all simple points of Σ this means that Σ contains all U-branches at is simple points, so $\mathfrak{r}_{U,\Sigma}$ and $\widehat{\mathfrak{r}}_{U,\Sigma}$ make sense);

(4) for every vertex $v \in \mathscr{V}\backslash U$:

  (4a) there exist a finite $\overline{K}$-definable subset F_v of $C_{v,\widehat{U}\backslash\Sigma}$, a $\overline{K}$-regular function f_v defined on a $\overline{K}$-Zariski neighborhood of $\mathfrak{r}_{U,\Sigma}^{-1}(C_{v,\widehat{U}\backslash\Sigma}\backslash F_v) \cap U$ such that $(\mathfrak{r}_{U,\Sigma}^{-1}(C_{v,\widehat{U}\backslash\Sigma}\backslash F_v) \cap U, f_v)$ is a nice $\overline{K}$-chart of type 2 with vertex v;

  (4b) for every $\zeta \in F_v$, there exists a $\overline{K}$-regular function f_ζ defined on a $\overline{K}$-Zariski neighborhood of $\mathfrak{r}_{U,\Sigma}^{-1}(\zeta) \cap U$ such that $(\mathfrak{r}_{U,\Sigma}^{-1}(\zeta) \cap U, f_\zeta)$ is a nice $\overline{K}$-chart of type 1.

9.6. Comments. — Let us insist on the rationality assumptions in the above definition. The skeleton Σ is assumed to be K-definable, as well as the set $\mathscr{V}$, and $(\Sigma, \mathscr{V})$ is assumed to be K-definably orientable.

All individual elements of $\mathscr{V}$, all nice charts involved and all connected components of $\Sigma\backslash\mathscr{V}$ are a priori merely $\overline{K}$-definable.

If L is a valued extension of K, the property for a K-skeleton of being Nash-admissible as an L-skeleton is a priori weaker than being Nash-admissible as an K-skeleton; but it turns out that both are in fact equivalent, see Theorem 9.15(2) below.

9.7. — Let K, X and U be as in the definition above. Let Σ be a Nash-admissible K-skeleton of $\widehat{U}$ and let $\mathscr{V}$ be a K-definable vertex set of Σ witnessing its Nash-admissibility. Let Ω be a connected component of $\widehat{U}\backslash\mathscr{V}$. Taking into account the fact that $\rho_{U,\Sigma}^{-1}(v) = \{v\}$ for every $v \in \mathscr{V}\backslash U$, we see in view of 3.16.3 that Ω can be of one of the two following kinds:

(a) $\Omega = \widehat{\rho}_{U,\Sigma}^{-1}(I)$ for some connected component I of $\Sigma\backslash\mathscr{V}$;

(b) $\Omega = \widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ for some $v \in \mathscr{V} \backslash U$ and $\zeta \in C_{v, \widehat{U} \backslash \Sigma}$.

In both cases Ω is open in $\widehat{X}$, and of the form $\widehat{V}$ for V a definable subset of U (definable over $\overline{K}$ in case (a) and over $\overline{K}\zeta$ in case (b)).

Moreover in case (a) the function f_I induces a $\overline{K}$-definable Nash isomorphism between V and A for some open annulus A in $\mathbf{A}^1$; in particular f identifies I with a closed subset of $\widehat{A}$ homeomorphic to an open interval, which is necessarily the only interval containing the two branches at infinity of $\widehat{A}$; therefore Ω is a tree with exactly two branches at infinity and I is the unique interval joining them, which we call the canonical skeleton of Ω.

In case (b) either the function f_v or the function f_ζ (if $\zeta \in F_v$) induces a definable Nash isomorphism between V and D for some open $\overline{K}\zeta$-definable disc D in $\mathbf{A}^1$. In particular, Ω is a tree with exactly one branch at infinity.

It follows that Σ can be entirely reconstructed from $\mathscr{V}$: it is the union of $\mathscr{V}$ and of the canonical skeletons of the finitely many connected components of $\widehat{U} \backslash \mathscr{V}$ having two branches at infinity.

9.8. Definition. — Let K be a valued field. A *K-triangulation* of $\widehat{U}$ is a finite K-definable vertex set of a Nash-admissible K-skeleton Σ of $\widehat{U}$ that witnesses the Nash-admissibility of Σ. By the above Σ is then entirely determined by $\mathscr{V}$ and is called the skeleton of the triangulation $\mathscr{V}$.

Any family of data $((f_I)_{I \in \pi_0(\Sigma \backslash \mathscr{V})}, (f_v, F_v, (f_\zeta)_{\zeta \in F_v})_{v \in \mathscr{V}'})$ as in Definition 9.5 (where $\mathscr{V}'$ denotes the set of non-simple points of $\mathscr{V}$) will be called a *$\mathscr{V}$-atlas* on $\widehat{U}$.

9.9. Remark. — Let $\mathscr{V}$ be a finite K-definable subset of $\widehat{U}$ and let Σ be a K-skeleton of $\widehat{U}$ containing $\mathscr{V}$. It follows from the definitions (see also Comments 9.6) that $\mathscr{V}$ is a K-triangulation with skeleton Σ if and only if $\mathscr{V}$ is a $\overline{K}$-triangulation with skeleton Σ and $(\Sigma, \mathscr{V})$ is K-definably orientable.

9.10. Inherited triangulation. — Let $\mathscr{V}$ be a triangulation of $\widehat{U}$, let Σ be its skeleton, and let $((f_I)_I, (f_v, F_v, (f_\zeta)_\zeta)_v)$ be a $\mathscr{V}$-atlas on $\widehat{U}$. Let E be a K-definable subset of $\pi_0(\Sigma \backslash \mathscr{V})$ and let $\mathscr{W}$ be a K-definable subset of $\mathscr{V}$. Let T be the K-definable subset $\mathscr{W} \cup \bigcup_{I \in E} I$ of Σ, and set $V = \rho_{U,\Sigma}^{-1}(\mathrm{T})$; we will implicitly use the properties of V and T stated in 3.15. It follows from the definitions that $\pi_0(\mathrm{T} \backslash \mathscr{W}) = E$, that $\mathscr{W}$ is a triangulation of $\widehat{V}$ with skeleton T and that $((f_I)_{I \in E}, (f_v, F_v, (f_\zeta)_{\zeta \in F_v})_{v \in \mathscr{W}'})$ is a $\mathscr{W}$-atlas of $\widehat{V}$.

9.11. First examples: around the projective line. — Let Σ be the K-skeleton $[0, \infty] = \{\eta_{0,r}\}_{0 \leqslant r \leqslant \infty}$ of $\widehat{\mathbf{P}^1}$. It is K-definably homeomorphic to an interval, is admissible and contains all $\mathbf{P}^1$-branches at its simple points. The map $\rho_{\mathbf{P}^1,\Sigma}$ maps 0 to 0, ∞ to ∞, and any non-zero a to $\eta_{a,|a|}$. As a consequence, for every definable interval I contained in $[0, \infty]$ we have $\rho_{\mathbf{P}^1,\Sigma}^{-1}(I) = \{|T| \in I\}$.

9.11.1. — Let E be a finite (possibly empty) subset of $|\overline{K}^{\times}|$. It follows from Example 3.13 that $\mathscr{V} := \{\eta_{0,r}\}_{r\in E} \cup \{0,\infty\}$ is a K-triangulation of $\widehat{\mathbf{P}}^1$ with skeleton Σ, and that $((T)_{I\in\pi_0(\Sigma\backslash\mathscr{V})},(T,\varnothing)_{r\in E})$ is a $\mathscr{V}$-atlas of $\widehat{\mathbf{P}}^1$.

9.11.2. — Let R_1 and R_2 be two elements of $|\overline{K}| \cup \{\infty\}$ with $R_1 \leqslant R_2$ and let T be an interval of Σ with endpoints η_{0,R_1} and η_{0,R_2} (each of them may belong to T or not). Let E be a finite subset of $|\overline{K}| \cup \{\infty\}$ such that $\eta_{0,r} \in \mathrm{T}$ for all $r \in E$ and R_1 (resp. R_2) belongs to E if η_{0,R_1} (resp. η_{0,R_2}) belongs to T. Set $\mathscr{W} = \{\eta_{0,r}\}_{r\in E}$, and let $\mathscr{V}$ be the set $\{0,\infty\} \cup \{\eta_{0,R_1},\eta_{0,R_2}\} \cup \mathscr{W}$. Then $\mathscr{V}$ is a K-triangulation of $\widehat{\mathbf{P}}^1$ with skeleton Σ, and T is the union of $\mathscr{W}$ and of finitely many connected components of $\Sigma\backslash\mathscr{W}$. So in view of 9.11.1 and of 9.10, $\mathscr{W}$ is a K-triangulation of $\widehat{\rho}^{-1}_{\mathbf{P}^1,\Sigma}(\mathrm{T})$ with skeleton T; and $((T)_{I\in\pi_0(\mathrm{T}\backslash\mathscr{W})},(T,\varnothing)_{r\in E\cap|\overline{K}^{\times}|})$ is a $\mathscr{W}$-atlas of $\widehat{\rho}^{-1}_{\mathbf{P}^1,\Sigma}(J)$.

9.11.3. — We keep the notation of 9.11.2 but we assume that $R_1 = 0$, that $\eta_{0,0} = 0$ belongs to T and that E contains at least one element of $|\overline{K}^{\times}|$; we denote by e the smallest element of $E \cap |\overline{K}^{\times}|$, and by V the pre-image $\rho^{-1}_{\mathbf{P}^1,\Sigma}(\mathrm{T})$. Not that by our assumptions V is either an open or closed disc with radius in $|\overline{K}^{\times}|$ or the affine line or the projective line. Let T' be the intersection of T and of $\{\eta_{0,r}\}_{r\geqslant e}$. Then T' is a sub-interval of T and there is a K-definable strong deformation retraction from T to T', so T' is a K-admissible skeleton of $\widehat{V}$. Moreover for every $v \in \widehat{V}$ we are by construction in one of the following two cases:

- ◇ if $\widehat{\rho}_{V,\mathrm{T}}(v) \in \mathrm{T}'$ then $\widehat{\rho}_{V,\mathrm{T}'}(v) = \widehat{\rho}_{V,\mathrm{T}}(v)$;
- ◇ if $\widehat{\rho}_{V,\mathrm{T}}(v) \notin \mathrm{T}'$ then $\widehat{\rho}_{V,\mathrm{T}'}(v) = \eta_{0,e}$ and $[v,\eta_{0,e}] = [v,\widehat{\rho}_{V,T}(v)] \cup [\widehat{\rho}_{V,T}(v),\eta_{0,e}]$.

We thus see that $\rho^{-1}_{V,\mathrm{T}'}(\eta_{0,e})$ is the closed disc of radius e.

It follows that $\mathscr{W}\backslash\{0\}$ is still a triangulation of $\widehat{V}$, with skeleton T', and that $((T)_{I\in\pi_0(\mathrm{T}'\backslash\mathscr{W})},(T,\varnothing)_{r\in E\cap|\overline{K}^{\times}|})$ is a $(\mathscr{W}\backslash\{0\})$-atlas of $\widehat{V}$.

9.12. Triangulations of charts. — Let (U,f) be a nice K-chart on some K-algebraic curve. We are going to exhibit K-triangulations on U. The simplest case will be that of type (2), where the construction will directly follow from the definition; the cases of type (1) and (3) will be dealt with using the Nash isomorphism f to pull-back the triangulations on discs and annuli built above.

9.12.1. *The case where (U,f) is of type 2.* — Let v be the vertex of the chart. It follows directly from the definition that $\{v\}$ is a K-triangulation of $\widehat{U}$ with skeleton $\{v\}$ and that $(f,\varnothing)$ is an $\{v\}$-atlas of $\widehat{U}$.

9.12.2. *The case where (U,f) is a of type 1.* — Set $D = f(U)$. Let r be any element of $|\overline{K}|$ strictly smaller than the radius ρ of D and let x be the unique pre-image of $\eta_{0,r}$ in $\widehat{U}$ under f. It follows from 9.11.3, which we apply with $R_1 = 0, R_2 = 1, \eta_{0,\rho} \notin \mathrm{T}$ and $E = \{r\}$, that $\{\eta_{0,r}\}$ is a K-triangulation of $\widehat{D}$ with skeleton $[\eta_{0,r},\eta_{0,1})$, and that $\{T\}$ (resp. $(T,(T,\varnothing))$) is an $\{\eta_{0,r}\}$-atlas of $\widehat{D}$ if $r = 0$ (resp. if $r \neq 0$). So $\{x\}$ is a K-triangulation of $\widehat{U}$ with skeleton the pre-image of $[\eta_{0,r},\eta_{0,1})$, which is the semi-open interval of $\widehat{U}$ starting at x and defining the unique branch at infinity of $\widehat{U}$; if $r = 0$ (that is, if x is a simple point) then $\{f\}$ is an $\{x\}$-atlas of $\widehat{U}$; if not, then $(f, (f, \varnothing))$ is an $\{x\}$-atlas of $\widehat{U}$.

9.12.3. *The case where (U, f) is of type 3.* — Let E be any finite subset (possibly empty) of elements of $|\overline{K}^{\times}| \cap (R_1, R_2)$, where R_1 and R_2 are the two radii of the annulus $f(A)$. It follows from 9.11.2 (applied assuming that neither η_{0,R_1} nor η_{0,R_2} lies on T) that $\mathscr{V} := \{\eta_{0,r}\}_{r \in E}$ is a K-triangulation of $\widehat{A}$ with skeleton the canonical skeleton Σ of $\widehat{A}$, and that $((T)_{I \in \pi_0(\Sigma \setminus \mathscr{V})}, (T, \varnothing)_{r \in E})$ is a $\mathscr{V}$-atlas of $\widehat{A}$. So $\mathscr{W} := f^{-1}(\mathscr{V})$ is a K-triangulation of $\widehat{U}$ with skeleton the canonical skeleton Σ' of $\widehat{U}$, and $((f)_{I \in \pi_0(\Sigma' \setminus \mathscr{W})}, (f, \varnothing)_{w \in \mathscr{W}})$ is a $\mathscr{W}$-atlas of $\widehat{U}$.

9.13. Theorem. — *Let U be a K-definable subset of a K-algebraic curve X, and let $\mathscr{V}$ be a K-triangulation of $\widehat{U}$ with skeleton Σ. Let $((f_I)_I, (f_v, F_v, (f_\zeta)_{\zeta \in F_v})_v)$ be a $\mathscr{V}$-atlas on $\widehat{U}$ and let E be the finite set consisting of all f_I, f_v, f_ζ.*

1. *The set $\Sigma_{\mathsf{RV},U}$ is RV-internal. More precisely there exist a $\overline{K}$-definable topological embedding $\Sigma \hookrightarrow [0, \infty]^m$ and a $\overline{K}$-definable embedding $\Sigma_{\mathsf{RV},U} \hookrightarrow \mathsf{RV}^m$ such that the diagram*

$$\begin{array}{ccc} \Sigma_{\mathsf{RV},U} & \hookrightarrow & \mathsf{RV}^m \\ \sigma\downarrow & & \downarrow \\ \Sigma & \hookrightarrow & [0,\infty]^m \end{array}$$

   commutes.

2. *Let $\zeta \in \Sigma_{\mathsf{RV}}$. There exists $f \in E$ defined on a $\overline{K}$-Zariski neighborhood of the connected component $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ of $\widehat{U} \setminus \Sigma$ that induces a Nash isomorphism between $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ and a $\overline{K}\zeta$-definable open disc of $\mathbf{A}^1$.*
3. *The map $\rho_{U,\Sigma}$ is topologically proper; otherwise said, Σ contains all branches at infinity of $\widehat{U}$.*
4. *Any finite K-definable subset of Σ containing $\mathscr{V}$ is still a K-triangulation of $\widehat{U}$ with skeleton Σ.*
5. *Let T be an admissible skeleton containing Σ and let $\mathscr{W}$ be the union of $\mathscr{V}$ and of the set of topological nodes of T, i.e., those points that do not have a neighborhood in T homeomorphic to an open interval. Then T is Nash-admissible and $\mathscr{W}$ is a K-triangulation of T.*

Proof. — We prove each statement separately.

9.13.1. *Proof of (1).* — Let I be a connected component of $\Sigma \setminus \mathscr{V}$ and let R_1 and R_2 be the infimum and the supremum of $|f_I|$ on I. The map f_I induces a Nash isomorphism between $\rho_{U,\Sigma}^{-1}(I)$ and the open annulus $R_1 < |T| < R_2$; it follows then from Example 3.13 that we have a commutative diagram

$$\begin{array}{ccc} \sigma^{-1}(I) & \xrightarrow{\sim} & \{\mathsf{rv}(\lambda)\}_{R_1<|\lambda|<R_2} \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ I & \xrightarrow[|f_I|]{\sim} & (R_1, R_2) \end{array}$$

with $\overline{K}$-definable arrows, the bottom map being a homeomorphism. Let v be an element of $\mathscr{V}$ belonging to ∂I; so at least one branch of I has limit v. If v is not simple then f_I is defined at v and $|f_I(v)| = R_i$ for a necessarily unique $i \in \{1, 2\}$ (which shows that only one branch at infinity of I has limit v: $\overline{I}$ is not a circle). If v is simple then the limit of $|f_I|$ along any branch at infinity of I having limit v necessarily belongs to $\{0, \infty\}$ (if not, $|f_I|$ would be constant along J, for it would be defined and invertible at the pre-image v' of v on the normalization of X defined by the branch J). If follows that $|f_I|$ induces a homeomorphism between $\overline{I}$ and either some interval stuck between (R_1, R_2) and $[R_1, R_2]$, or the quotient $[0, \infty]/\sim$ where $\sim$ is the identification between 0 and ∞; the last case occurs if and only if $R_1 = 0, R_2 = \infty$ and $\overline{I}$ is a circle, the two branches at infinity of I having the same limit $v \in \mathscr{V}$, which is necessarily a singular point of X, belonging to the indeterminacy locus of the rational function f_I. We remark that the mapping t to (t, t) if $t \leqslant 1$ and to (t^{-1}, t^{-2}) otherwise defines a $\varnothing$-definable embedding from $[0, \infty]/\sim$ into $[0, \infty]^2$. So (by lifiting if necessary the preceding embedding to the RV-sort) we eventually get a commutative diagram

$$\begin{array}{ccc} \sigma^{-1}(I) & \hookrightarrow & \mathsf{RV}^m \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \overline{I} & \hookrightarrow & [0, \infty]^m \end{array}$$

for $m \in \{1, 2\}$, where all maps are $\overline{K}$-definable, and the bottom map is a topological embedding. Since $|f_I(v)| = |f_J(v)|$ modulo $|\overline{K}^\times|$ for all non-simple $v \in \mathscr{V}$ and all pairs (I, J) of connected components of $\Sigma\backslash\mathscr{V}$ having v as a limit point (because v is $\overline{K}$-definable) one can glue the above diagrams for $I \in \pi_0(\Sigma\backslash\mathscr{V})$ by making each of them live in some $\overline{K}$-definable "affine coordinate subspace" and get a commutative diagram

$$\begin{array}{ccc} \Sigma_{\mathsf{RV},U}\backslash\sigma^{-1}(\mathscr{V}) & \hookrightarrow & \mathsf{RV}^m \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \Sigma & \hookrightarrow & [0, \infty]^m \end{array}$$

where all maps are $\overline{K}$ definable, where the bottom map is a topological embedding, and where m is an integer. The existence of a commutative diagram like required by statement (1) then follows from the fact that for each $v \in \mathscr{V}$ the pre-image $\sigma^{-1}(v)$ is the RV-internal (it is even k-internal) curve $C_{v,\widehat{U}\backslash\Sigma}$ (one may have to slightly enlarge the integer m).

9.13.2. *Proof of (2) and (3).* — Set $v = \sigma(\zeta)$; this is also the constant value of $\widehat{\rho}_{U,\Sigma}$ on the component $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$. If v does not belong to $\mathscr{V}$ then it lies on some connected component I of $\Sigma\backslash\mathscr{V}$, and $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ is a connected component of $\widehat{\rho}_{U,\Sigma}^{-1}(I)\backslash I$; it follows then from Example 3.13 that f_I induces a Nash isomorphism between $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ and a $\overline{K}\zeta$-definable open disc. Assume that v belong to $\mathscr{V}$. Then $\zeta \in C_{v,\widehat{U}\backslash\Sigma}$. If $\zeta \in F_v$, resp. if $\zeta \notin F_v$, the map f_ζ, resp. f_v, induces a $\overline{K}$-definable Nash isomorphism between $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ and an $\overline{K}\zeta$-definable open disc, ending the proof of (2).

In particular, each connected component of $\widehat{U}\backslash\Sigma$ is a tree with exactly one branch at infinity, which implies (3) (see the discussionin 3.16.3).

9.13.3. *Proof of (4).* — Let $\mathscr{W}$ be a finite K-definable subset of Σ containing $\mathscr{V}$. Then $\mathscr{W}$ is a K-definable vertex set of Σ, and $(\Sigma, \mathscr{W})$ is K-definably orientable. For showing that $\mathscr{W}$ is a K-triangulation of $\widehat{U}$, it therefore suffices to show that it is a $\overline{K}$-triangulation of $\widehat{U}$, so we can assume $K = \overline{K}$. Then we can argue by induction on the cardinality of $\mathscr{W}\backslash\mathscr{V}$, reducing to the case where $\mathscr{W} = \mathscr{V} \cup \{w\}$ for some $w \in \Sigma\backslash\mathscr{V}$. Let J be the connected component of $\Sigma\backslash\mathscr{V}$ that contains w. It then follows from 9.12.3 that $\mathscr{W}$ is a K-triangulation of $\widehat{U}$ with skeleton Σ, which admits an atlas consisting of the family $((f_I)_{I\neq J}, (f_v, F_v, (f_\zeta)_{\zeta\in F_v})_v)$ together with one copy of f_J for each connected component of $J\backslash\{w\}$ and for the vertex w, and with the empty set of points of the curve $C_{w,\widehat{U}\backslash\Sigma}$.

9.13.4. *Proof of (5).* — The set $\mathscr{W}$ is K-definable. Let $(o_I)_{I\in\pi_0(\Sigma\backslash\mathscr{V})}$ be a K-definable system of orientations.

If Υ is a connected component of $\mathrm{T}\backslash\Sigma$ then $\partial\Upsilon$ consists of one point $w \in \Sigma$ and $\overline{\Upsilon}$ is a (compact) tree; note that if $w \notin \mathscr{V}$ then w is necessarily a topological node of T, so $w \in \mathscr{W}$ in any case. Then every connected component J of $\overline{\Upsilon}\backslash\mathscr{W}$ is $\overline{K}$-definably homeomorphic to an open interval, and for such a J we denote by o_J its orientation pointing toward w in the tree $\overline{\Upsilon}$.

If J is a connected component of $\mathrm{T}\backslash\mathscr{W}$ then either J is a connected component of $\Upsilon\backslash\mathscr{W}$ for some connected component Υ of $\mathrm{T}\backslash\Sigma$, either it is contained in some connected component I of $\Sigma\backslash\mathscr{V}$, in which case we set $o_J = o_I|_J$. By construction the system of orientations $(o_J)_{J\in\pi_0(\mathrm{T}\backslash\mathscr{W})}$ is K-definable, so $\mathscr{W}$ is a K-definable vertex set of T and $(\mathrm{T}, \mathscr{W})$ is K-definably orientable. It thus suffices to prove that $\mathscr{W}$ is a $\overline{K}$-definable triangulation of T, so we can now assume that $K = \overline{K}$.

As T is K-admissible, it can be built from Σ by adding successively K-definable topological edges (the assumption that K is algebraically closed is used to ensure their K-definability). Arguing by induction on the number of edges we can assume that T is equal to $\Sigma \cup [x, w]$ where x is some K-definable point of $\widehat{U}\backslash\Sigma$ and w is a K-definable point of Σ. By (4) $\mathscr{V} \cup \{w\}$ is still a K-triangulation of $\widehat{U}$ with skeleton Σ, so we can assume that $w \in \mathscr{V}$. Then $\mathscr{W} = \mathscr{V} \cup \{x\}$.

Set $\xi = \mathfrak{r}_{U,\Sigma}(x)$; we can see it as a point of $C_{w,\widehat{U}\backslash\Sigma}$. Set $V = \mathfrak{r}_{U,\Sigma}^{-1}(\xi)$; this is a K-definable subset of U, and $\widehat{V} = \widehat{\rho}_{U,\mathrm{T}}^{-1}([x, w))$ in view of Lemma 3.18.

Set $g = f_w$ if $\xi \notin F_w$ and $g = f_\xi$ otherwise. Then g induces a Nash isomorphism between V and a K-definable open disc of $\widehat{\mathbf{A}}^1$, say of radius r. The point $g(x)$ is equal to $\eta_{a,s}$ for some $a \in K$ and some $s \in |K| \cap [0, r)$. Set $h = g - a$; by construction (V, h) is a nice K-chart of type 1 and $h(x) = \eta_{0,s}$ It follows then from 9.12.2 that $\{x\}$ is a K-triangulation of $\widehat{V}$ with skeleton $[x, v)$, that $\{h\}$ is an $\{x\}$-atlas of $\widehat{V}$ if x is simple, and that $(h, (h, \varnothing))$ is an $\{x\}$-atlas of $\widehat{V}$ if x is not simple.

We deduce from the above that $\mathscr{W} = \{x\} \cup \mathscr{V}$ is a K-triangulation of $\widehat{U}$ with skeleton Γ, and that a $\mathscr{W}$-atlas of $\widehat{U}$ can be built as follows: one concatenates the family $((f_I)_{I \in \pi_0(\Sigma \backslash \mathscr{V})}, (f_v, F_v, (f_\zeta)_{\zeta \in F_v})_{v \in \mathscr{V} \backslash \{w\}})$, with $(f_w, F_w \backslash \{\xi\}, (f_\zeta)_{\zeta \in F_w \backslash \{\xi\}})$ for the vertex w, the function h for the connected component (x, w) of $\Gamma \backslash \mathscr{W}$ and

- ◇ if x is simple (i.e., $s = 0$), nothing else;
- ◇ if x is not simple (i.e., $s \neq 0$), the function h for the vertex x, and the empty set of points of the residue curve $C_{x, \widehat{V} \backslash \Gamma}$. □

9.14. Lemma. — *Let U be a K-definable subset of an algebraic K-curve X. Let Σ be an admissible K-skeleton of $\widehat{U}$, and let (Σ_i) be a finite family of K-definable subsets of Σ covering Σ. For each i, set $U_i = \rho_{U,\Sigma}^{-1}(\Sigma_i)$, and assume that Σ_i is a Nash-admissible K-skeleton of $\widehat{U}_i$. Choose for every i a K-triangulation $\mathscr{V}_i$ of $\widehat{U}_i$ with skeleton Σ_i. Let $\mathscr{V}$ be the union of the $\mathscr{V}_i$ and of the $\partial\widehat{U}_i$ (note that $\partial\widehat{U}_i$ coincides with $\partial\Sigma_i \subset \Sigma$ by Corollary 2.11). Then Σ is Nash-admissible, and $\mathscr{V}$ is a K-triangulation of $\widehat{U}$ with skeleton Σ.*

Proof. — By Theorem 9.13 (4) the union $\mathscr{V}_i \cup (\mathscr{V} \cap \Sigma_i)$ is for every i still a K-triangulation with skeleton Σ_i, so we can assume that $\mathscr{V}_i = \mathscr{V} \cap \Sigma_i$ for all i. Choose for every i a $\mathscr{V}_i$-atlas $\mathscr{A}_i$ of $\widehat{U}_i$.

Let I be a connected component of $\Sigma \backslash \mathscr{V}$. As $\mathscr{V}$ contains $\partial\Sigma_i$ for all i there exists a (non necessarily unique) i with $I \subset \Sigma_i$, so I is a connected component of $\Sigma_i \backslash \mathscr{V}_i$; as a consequence I is $\overline{K}$-definably homeomorphic to an open interval, and it is interdefinable with each of its orientations. Let f_I be the function associated to I by the atlas $\mathscr{A}_i$. Then as $\rho_{U,\Sigma}^{-1}(I) = \rho_{U_i,\Sigma_i}^{-1}(I)$, the pair $(\rho_{U,\Sigma}^{-1}(I), f_I)$ is a nice $\overline{K}$-chart of type 3.

Let v be a simple point of $\mathscr{V}$. Then v belongs to some $\mathscr{V}_i$, and we then have $\rho_{U,\Sigma}^{-1}(v) = \rho_{U_i,\Sigma_i}^{-1}(v) = \{v\}$.

Now let v be a non-simple point of $\mathscr{V}$. Choose i so that v belongs to $\widehat{U}_i$. Let $(f_v, F_v, (f_\zeta)_{\zeta \in F_v})$ be the data associated with v provided by the atlas $\mathscr{A}_i$. Let F'_v be the union of F_v and of the finite set of all $\zeta \in C_{v, \widehat{U} \backslash \Sigma}$ that do not belong to $C_{v, \widehat{U}_i \backslash \Sigma_i}$. Then $\mathfrak{r}_{U,\Sigma}^{-1}(C_{v, \widehat{U} \backslash \Sigma} \backslash F'_v) = \mathfrak{r}_{U_i,\Sigma_i}^{-1}(C_{v, \widehat{U}_i \backslash \Sigma} \backslash F_v)$, so $(\mathfrak{r}_{U,\Sigma}^{-1}(C_{v, \widehat{U} \backslash \Sigma} \backslash F'_v), f_v)$ is a nice $\overline{K}$-chart of type 2. Now let $\zeta \in F'_v$. Then $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ does not meet the union of the $\partial\widehat{U}_j$, so there exists j such that $\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta)$ is entirely contained in $\widehat{U}_j$, so it coincides with $\widehat{\mathfrak{r}}_{U_j,\Sigma_j}^{-1}(\zeta)$. Now if $i = j$ then $\zeta \in F_v$ so $(\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta), f_\zeta))$ is a nice $\overline{K}$-chart of type 1. Assume now that $j \neq i$ and let $(g_v, G_v, (g_\xi)_{\xi \in G_v})$ be the data associated with v provided by the atlas $\mathscr{A}_j$. If $\zeta \in G_v$ then $(\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta), g_\zeta))$ is a nice $\overline{K}$-chart of type 1.

Assume now that $\zeta \notin G_v$. As F'_v is finite $\mathfrak{r}_{U,\Sigma}^{-1}(\zeta)$ is $\overline{K}$-definable, so it contains a $\overline{K}$-point u (this is obvious if K is non-tivially valued; if K is trivially valued then as v is $\overline{K}$-definable the image $g_v(v)$ is necessarily the unique $\overline{K}$-definable, non-simple point of $\mathbf{A}^1$, namely $\eta_{0,1}$, and $g_v(\mathfrak{r}_{U,\Sigma}^{-1}(\zeta))$ is then an open disc of radius 1 centered at $\overline{K}$-point, whence the claim). And now $(\widehat{\mathfrak{r}}_{U,\Sigma}^{-1}(\zeta), g_v - g_v(u))$ is a nice $\overline{K}$-chart of type 1, and we are done. □

***9.15. Theorem* (Triangulations of algebraic curves)**

Let K be a valued field and let X be a generically smooth algebraic K-curve. Let U be a K-definable subset of X.

(1) *The stable completion $\widehat{U}$ possesses a Nash-admissible K-skeleton (or, equivalently, a K-triangulation); moreover this Nash-admissible K-skeleton can be taken to contain any prescribed K-skeleton of $\widehat{U}$.*

(2) *Let L be a valued extension of K. A K-skeleton Σ of $\widehat{U}$ is Nash-admissible as a K-skeleton if and only if it is Nash-admissible as an L-skeleton.*

Proof. — We proceed in several steps.

9.15.1. — We first assume that $K = \overline{K}$ and we are going to prove that there exists a Nash-admissible K-skeleton on $\widehat{U}$.

9.15.1.1. *Modification of the problem.* — Let X' be the normalization of X, let S be the singular locus of X (which consists of finitely many points by our generic smoothness assumption) and let U' and S' be the pre-images of U and S on X'. Let U'' be the set obtained from U' by removing its finite set E of isolated points. Assume that we have proven the existence of Nash-admissible K-skeleton Σ'' of $\widehat{U''}$ containing $U'' \cap S'$. Then $\Sigma' := \Sigma'' \cup E$ is a Nash-admissible K-skeleton of $\widehat{U'}$ containing $U' \cap S'$. Since Σ' is Nash-admissible, any admissible K-homotopy homotopy of $\widehat{U'}$ with target Σ' fixes all simple points of Σ', and in particular all points of $\Sigma' \cap S'$. It follows that this homotopy descends to an admissible K-homotopy of $\widehat{U}$ with target the image Σ of Σ'. So Σ is an admissible K-skeleton of $\widehat{U}$ containing S and as $X' \to X$ is an isomorphism outside S the Nash-admissibility of Σ' implies that of Σ.

We can thus assume from now on that X is smooth and that U has no isolated points, and we want to prove the existence of a Nash-admissible K-skeleton Σ of $\widehat{U}$ containing a prescribed finite set S of simple points.

9.15.1.2. — By Theorem 8.5 there exists a finite family (U_i, f_i) of nice K-charts of $\widehat{U}$ such that $\widehat{U} = \bigcup \widehat{U_i}$. By 9.12 et sq., there exists for every i a Nash-admissible K-skeleton Σ_i of $\widehat{U}_i$. Now by 2.8 there exists an admissible K-skeleton T of $\widehat{U}$ which is adapted to every U_i, contains S and contains each Σ_i. Fix an index i. As T is adapted to U_i, $\mathrm{T} \cap \widehat{U}_i$ is an admissible K-skeleton of $\widehat{U}_i$; as it contains Σ_i, it is Nash-admissible by Theorem 9.13 (5). It now follows from Lemma 9.14 that T is a Nash-admissible K-skeleton of $\widehat{U}$.

9.15.2. *Proof of (2) for $L = \overline{K}$.* — If Σ is Nash admissible as a K-skeleton, it is obviously Nash-admissible as a $\overline{K}$-skeleton. Let us assume conversely that Σ is Nash-admissible as a $\overline{K}$-skeleton. Let $\mathscr{V}$ be a $\overline{K}$-triangulation of $\widehat{U}$ with skeleton Σ. Let $\mathscr{V}'$ denotes the union of all conjugates of $\mathscr{V}$ under G. Then $\mathscr{V}'$ is a finite K-definable subset of Σ; as it contains $\mathscr{V}$, this is still a vertex set of Σ. By Lemma-Definition 9.4 there exists a finite K-definable subset $\mathscr{V}''$ of Σ containing $\mathscr{V}'$ such that $(\Sigma, \mathscr{V}'')$ is K-definably orientable. As it contains $\mathscr{V}$, the set $\mathscr{V}''$ is in view of Theorem 9.13 (3) a $\overline{K}$-triangulation of $\widehat{U}$ with skeleton Σ; together with the fact that $(\Sigma, \mathscr{V}'')$ is K-definably orientable, this implies that $\mathscr{V}''$ is a K-triangulation of $\widehat{U}$ with skeleton Σ.

9.15.3. *Proof of (1).* — Let Σ be a K-skeleton of $\widehat{U}$. By the particular case of (1) handled in 9.15.1, there exists a Nash-admissible $\overline{K}$-skeleton T on $\widehat{U}$. Let T' be the union of all conjugates of T under G. Then T' is a K-definable skeleton of $\widehat{U}$, and so is $\mathrm{T}' \cup \Sigma$. The latter is therefore contained in some admissible K-skeleton Υ. As Υ is admissible and contains T, it is Nash-admissible as a $\overline{K}$-skeleton by Theorem 9.13 (5), hence it is Nash-admissible as a K-skeleton by statement (2) already proved for $L = \overline{K}$.

9.15.4. *Proof of (2) in general.* — As (2) holds for $L = \overline{K}$, it suffices to prove that Σ is Nash-admissible as a $\overline{K}$-skeleton if and only if it is Nash-admissible an $\overline{L}$-skeleton (where $\overline{L}$ is a valued algebraic closure of L containing $\overline{K}$), so we can assume that both K and L are algebraically closed.

9.15.4.1. — If the valuation of K is non-trivial, K and L are models of ACVF, and the result follows by model-completeness, since the set of data that define a triangulation is clearly ind-definable (one has to choose the set $\mathscr{V}$ and then the collection of nice charts).

9.15.4.2. — It remains to handle the case where K is trivially valued and we will only sketch it; details are left to the reader. First of all, let us assume that X is projective and smooth. Then $\widehat{X}(K) = X(K) \cup \{\eta\}$ where η is its Gauss type – corresponding to the trivial valuation on $K(X)$. Using several K-definable finite maps $X \to \mathbf{P}^1$ (sufficiently many so that each K-points of X lies in the étale locus of one of them) one shows essentially by the same kind of arguments as in 8.5.2 that $\{\eta\}$ is a K-triangulation of $\widehat{X}$ with skeleton $\{\eta\}$ and that each K-definable connected component of $\widehat{X}\backslash\{\eta\}$ contains exactly one K-point of X. It easily follows that any closed K-skeleton of $\widehat{X}$ either is a finite union of K-points or contains η. It also follows that if U is not a finite subset of X, then $\widehat{U}$ contains η and all connected components of $\widehat{X}\backslash\eta$ except possibly finitely many of them $\Omega_1, \ldots, \Omega_r$ which are K-definable; and for each i the intersection $\widehat{U} \cap \Omega_i$ can only be one of those three sets: $\varnothing$, $\{\omega_i\}$ or $\Omega_i\backslash\{\omega_i\}$, where ω_i is the unique K-point of X lying in Ω_i.

Now if we do not assume anymore that X is projective nor smooth we can use the above, together with standard arguments involving a compactification of X and its normalization, that a K-skeleton Σ of $\widehat{U}$ is Nash-admissible as an L-skeleton if and only if the following hold: the intersection of Σ with every connected component of

$\widehat{U}$ is non-empty and connected; the skeleton Σ contains all singular points of X lying on U as well as all branches at infinity of $\widehat{U}$; the skeleton Σ contains all U-branches at its simple points (if these conditons are fulfilled, $\Sigma \cap \widehat{X}(K)$ is an L-triangulation of $\widehat{U}$ with skeleton Σ). But these conditions do not involve the field L, which ends the proof. □

10. Tame henselian rationality and other applications

10.1. Theorem. — *Let K be any valued field. Let U be a K-definable subset of an algebraic K-curve X.*

(1) *The pro-K-definable sets $\breve{U}$ and $\widetilde{U}$ are K-definable.*

(2) *For every closed K-skeleton Σ of $\widehat{U}$, the K-definable set of connected components of $\widehat{U}\backslash\Sigma$ (Proposition 3.17 (2)) is* RV*-internal.*

(3) *The set of non-simple points of $\widehat{U}$ whose residue curve has positive genus is finite and K-definable.*

Proof. — We prove each statement separately.

10.1.1. *Proof of (1).* — Let Z be the normal projective completion of the normalization of X. There is a natural pro-K-definable isomorphism between Z and $\breve{Z}\backslash\widetilde{Z}$, wich maps a point z to the type induced by the discrete valuation on Z centered at z. Now $\breve{U}\backslash\widetilde{U}$ is cut out from $\breve{Z}\backslash\widetilde{Z}$ by finitely many definable conditions (the ones that define U), so it is K-definable. Hence it suffices to prove that $\widetilde{U}$ is K-definable. As it is already pro-K-definable, it suffices to prove that $\widetilde{U}$ is M-definable for some model M of ACVF containing K. Therefore we can assume that K is a model of ACVF.

First remind that $\widetilde{\mathbf{P}^1}$ is K-definable by 2.2. By Theorem 9.15 we know that $\widehat{U}$ admits a Nash-admissible K-skeleton Σ. Let $\mathscr{V}$ be a K-triangulation with skeleton Σ. We then have

$$U = \left(\coprod_{I \in \pi_0(\Sigma\backslash\mathscr{V})} \rho_{U,\Sigma}^{-1}(I)\right) \coprod \left(\coprod_{v \in \mathscr{V}} \rho_{U,\Sigma}^{-1}(v)\right).$$

For proving that $\widetilde{U}$ is K-definable it suffices to know that for every summand V of the above decomposition, $\widetilde{V}$ is K-definable. Assume first that $V = \rho_{U,\Sigma}^{-1}(I)$ for some connected component I of $\Sigma\backslash\mathscr{V}$. Then V is K-definably Nash-isomorphic to an open annulus $A = \{r < |T| < R\} \subset \mathbf{P}^1$ with r and R in $|K|$. So $\widetilde{A}$ is relatively K-definable in the K-definable set $\widetilde{\mathbf{P}^1}$, so it is K-definable and so is $\widetilde{V}$. Assume now that $V = \rho_{U,\Sigma}^{-1}(v)$ for some vertex $v \in \mathscr{V}$. If $v \in U(K)$ then $V = \{v\} = \widetilde{V}$ and the result is obvious. Assume that v is not a simple point and use the notation F_v, f_v, f_ζ of Definition 9.5 (4a) and (4b). Set $r = |f(v)|$ and $C'_v = C_{v,\widehat{U}\backslash\Sigma}\backslash F_v$, and let $C'_{\eta_{0,r}}$ be the image of C'_v in $C_{\eta_{0,r}}$. Set $W = \mathfrak{r}_{U,\Sigma}^{-1}(C'_v)$. Then V is the disjoint union of W and of the $V_\zeta := \mathfrak{r}_{U,\Sigma}^{-1}(\zeta)$ for $\zeta \in F_v$; each V_ζ is K-definably isomorphic to an open disc through f_ζ, so each $\widetilde{V_\zeta}$ is K-pro-definably isomorphic to a relatively K-definable subset of $\widetilde{\mathbf{P}^1}$, so it is K-definable. It remains to show that $\widetilde{W}$ is K-definable. Set $\Delta = \mathfrak{r}^{-1}_{\mathbf{A}^1,\eta_{0,r}}(C'_v)$. It follows once again from the K-definability of $\widetilde{\mathbf{P}^1}$ that $\widetilde{\Delta}$ is K-definable. But $\widetilde{W}$ can be now identified through $(f_v, \widetilde{\mathfrak{r}}_{W,v})$ to $\widetilde{\Delta} \times_{\widehat{C'_{\eta_{0,r}}}} \widehat{C'_v}$, so it is K-definable.

10.1.2. *Proof of (2).* — We know by Theorem 9.15 that there exists a Nash-admissible K-skeleton T of $\widehat{U}$ containing Σ. It follows from Theorem 9.13 that $\mathrm{T}_{\mathsf{RV},U}$ is RV-internal. Now Proposition 3.17 (1) show that $\pi_0(\widehat{U}\backslash\Sigma)$ is the union of a K-definable subset of $\mathrm{T}_{\mathsf{RV},U}$ and of a finite set, whence (2).

10.1.3. *Proof of (3).* — By Theorem 9.15 there exists a K-triangulation of $\widehat{U}$; let Σ be its skeleton. Let $\mathscr{V}'$ be the subset of $\mathscr{V}$ consisting of non-simple points whose residue curve has positive genus. Then $\mathscr{V}'$ is Galois-invariant, so it is K-definable. Let us show that $\mathscr{V}'$ is precisely the set of non-simple points of points of $\widehat{U}$ whose residue curve has positive genus, which will end the proof. Let x be a non-simple point of $\widehat{U}$ that does not belong to $\mathscr{V}'$; let is prove that C_x has genus zero. We distinguish three cases.

10.1.3.1. *The case where $x \in \mathscr{V}$.* — The genus of C_x is then zero by the very definition of $\mathscr{V}'$.

10.1.3.2. *The case where $x \in \Sigma\backslash\mathscr{V}$.* — Then x lies on some connected component I of $\Sigma\backslash\mathscr{V}$. By definition of a triangulation there exists some étale map from a $\overline{K}$-Zariski neighborhood of $\rho^{-1}_{U,\Sigma}(I)$ to $\mathbf{A}^1$ inducing a Nash isomorphism between $\rho^{-1}_{U,\Sigma}(I)$ and an open annulus. The point $f(x)$ is then a non-simple point of $\mathbf{A}^1$,xo it is fo the form $\eta_{a,r}$. The map f induces an isomorphism between C_x and $C_{\eta_{a,r}}$, and the latter is isomorphic to the projective line, so C_x is of genus zero.

10.1.3.3. *The case where $x \notin \Sigma$.* — The connected component of $\widehat{U}\backslash\Sigma$ that contains x is equal to $\widehat{V}$ for some definable subset V of U, and by definition of a triangulation there exists some étale map from a Zariski neighborhood of V to $\mathbf{A}^1$ inducing a Nash isomorphism between V and an open disc. The point $f(x)$ is then a non-simple point of $\mathbf{A}^1$, so it is fo the form $\eta_{a,r}$. The map f induces an isomorphism between C_x and $C_{\eta_{a,r}}$, and the latter is isomorphic to the projective line, so C_x is of genus zero. □

10.2. Remark. — The finiteness statement in (3) can also be deduced from the genus inequality proved in [**GMP89**, Theorem 3.1].

Now we will provide yet another proof of the semi-stable reduction theorem. It might seem in some sense unsatisfactory, since we get it only in height 1, and have to leave at some point the world of stable completions and use the genuine Berkovich formalism instead, and then rely on a lot of technical results in formal and rigid geometry.

But in a subsequent work we will use the formalism and the results of this current paper for investigating models of varieties over valuation rings of arbitrary height, and we plan to build directly from our triangulations semi-stable models for curves, remaining in the stable completion setting. We also intend to explore what can be done in higher dimensions using the methods developed in the present paper.

10.3. Theorem. — *Let K be a henselian valued field of height 1 and let X be a smooth projective curve over K. There exists a finite extension L of K such that X_L admits a semi-stable model over the ring of integers of L.*

Proof. — Let us first assume that K is complete, non-trivially valued and algebraically closed. Let Σ be a Nash-admissible K-skeleton of $\widehat{X}$. By Chapter 14 of [**HL16**], this skeleton Σ gives rise to a skeleton Υ on the Berkovich analytification X^{an} of the curve X, and the K-definable deformation retraction of $\widehat{X}$ toward Σ induces a deformation retraction from X^{an} toward Υ. Let U be a connected component of $X^{\mathrm{an}}\backslash\Upsilon$. By the analytic Nullstellensatz U has a K-point x, and it follows from the constructions in [**HL16**] that U is the Berkovich analytification V^{an} where V is the K-definable subset $\mathfrak{r}_{X,\Sigma}^{-1}(\mathfrak{r}_{X,\Sigma}(x)) \cap X$. By definition of a Nash-admissible K-skeleton, there exists a K-regular étale map f from a dense K-Zariski open subset X' of X to $\mathbf{A}^1$ that induces a K-definable bijection between V and an open disc $D \subset \mathbf{A}^1$. Then f induces an étale map $V^{\mathrm{an}} \to D^{\mathrm{an}}$ and moreover by Lemma 4.7 one has $K(f(y))^{\mathrm{h}} = K(y)^{\mathrm{h}}$ for every $y \in V$. It follows that the completion of $K(y)$ is equal to that of $K(f(y))$ when the valuation of $K(y)$ is of height 1, so f induces an isomorphism $\mathscr{H}(f(z)) \simeq \mathscr{H}(z)$ for every Berkovich point $z \in V^{\mathrm{an}}$. Being étale, f is then a local isomorphism around every point of V^{an}. Moreover the bijectivity of $f\colon V \to D$ implies the bijectivity of $V^{\mathrm{an}} \to D^{\mathrm{an}}$ in view of the preservation of completed residue fields and of the fact that fibers are finite (so that any pre-image on V of a point of D with height 1 valuation still has height 1 valuation). Then $f\colon V^{\mathrm{an}} \to D^{\mathrm{an}}$ is a bijective local isomorphism, so it is an isomorphism.

Therefore every connected component of $X^{\mathrm{an}}\backslash\Upsilon$ is isomorphic to an open disc. The same kind of transfer argument also ensures that there is a finite family $\mathscr{V}$ of vertices in Υ such that every connected component I of $\Upsilon\backslash\mathscr{V}$ is an open interval, whose pre-image under the retraction $X^{\mathrm{an}} \to \Sigma$ is isomorphic to an open annulus.

These facts are classically equivalent to the semi-stable reduction for X over K, see for instance Chapter 6 of [**Duc24**] (this relies on highly non-trivial results like Grauert's finiteness, Bosch's study of formal fibers, the algebraization of one-dimensional proper formal schemes, ...). Therefore X has a semi-stable model.

Now let us handle the general case. There is nothing to do if K is trivially valued. If it is not, let A be the ring of integers of K and let $\mathbb{K}$ be the completion of an algebraic closure of K. By the above, $X_{\mathbb{K}}$ admits a semi-stable model over the ring of integers B of $\mathbb{K}$. By approximation there exists a finitely presented integral A-scheme S having a B-point and a proper semi-stable S-scheme $\mathscr{X}$ equipped with an isomorphism $\mathscr{X}_{S_K} \simeq X \times_K S_K$. By quantifier elimination in ACVF, S also has a B'-point where B' is the ring of integers of the algebraic closure of K in $\mathbb{K}$ (the reader who would like at this stage to see a purely scheme-theoretic proof of the non-emptyness of $\mathscr{X}(B')$ may also refer to Theorem 7.4.8 of [**Duc18**]). So X has a semi-stable model over the ring of integers of an algebraic closure of K, and then over the ring of integers of a finite extension of K by approximation. □

Finally we are going to answer positively the original question by Franziska Jahnke and Franz-Viktor Kuhlmann, by providing a geometric and model theoretic proof of Kuhlmann's henselian rationality theorem for tame henselian fields from [**Kuh19**] (see more comments on this in the Introduction).

We recall that a henselian valued field K is called *tame* if every finite extension of K is tamely ramified. This amounts to requiring that K be defectless and $\mathsf{RV}(K)$ be perfect (this implies that K itself is perfect); and it is easily seen that $\mathsf{RV}(K)$ is perfect if and only if $\mathsf{k}(K)$ is perfect and $|K^\times|$ is p-divisible, where p is the residue characteristic exponent. If the residue characteristic of K is zero, K is tame.

***10.4. Theorem* (Kuhlmann).** — *Let K be a tame henselian valued field and let L be a finitely generated extension of transcendence degree 1 over K, equipped with an immediate extension v of the valuation of K. There exists an element t of L, transcendental over K, such that the inclusion $(K(t), v|_{K(t)}) \hookrightarrow (L, v)$ induces an isomorphism*

$$(K(t), v|_{K(t)})^{\mathrm{h}} \simeq (L, v)^{\mathrm{h}}.$$

Proof. — We first need to investigate the behavior of L under finite ground field extension.

10.4.1. — Let F be a finite extension of K. Write $L^{\mathrm{h}} \otimes_K F = \prod L_i$, where the L_i are finite separable extensions of L^{h}.

Since K is tame, F is tamely ramified over K. It follows then from [**Duc13**, 2.21] that the natural map $\mathsf{RV}(L) \otimes_{\mathsf{RV}(K)} \mathsf{RV}(F) \to \prod \mathsf{RV}(L_i)$ is an isomorphism. But as L is an immediate extension of K one has $\mathsf{RV}(L) = \mathsf{RV}(K)$, so $\prod \mathsf{RV}(L_i) = \mathsf{RV}(F)$. This means that $L^{\mathrm{h}} \otimes_K F$ is a field, and is an immediate extension of F. Otherwise said $L \otimes_K F$ is a field, and the valuation on L has a unique extension to $L \otimes_K F$, which makes it an immediate extension of F.

Let $\overline{K}$ be an algebraic closure of K. By the above and a colimit argument, $L \otimes_K \overline{K}$ is a field, to which v has a unique extension w and which makes $L \otimes_K \overline{K}$ an immediate extension of $\overline{K}$.

10.4.2. — Let X be an integral K-curve with function field L. The valuation w defines a quantifier-free type on X over K; let x be a realization of this type in some model. As w is the unique extension of v to $L \otimes_K \overline{K}$, the point x is Galois invariant so it is K-definable.

By Theorem 9.15, the stable completion of X has a Nash-admissible K-skeleton Σ. The image $y := \rho_{X,\Sigma}(x)$ is a K-definable point of Σ; as x is not simple, y is not simple. If F is a finite extension of K then F is tamely ramified over K, so by the properties recalled in 10.4.1 above $\mathsf{RV}(F(y)) = \mathsf{RV}(K(y)) \otimes_{\mathsf{RV}(K)} \mathsf{RV}(F)$. By a colimit argument this also holds for $F = \overline{K}$, which implies that the residue curve C_y is defined over K.

The image $\mathfrak{r}_{X,\Sigma}(x)$ is therefore an $\mathsf{RV}(K(x))$-definable point of the K-definable curve C_y. As (L, v) is an immediate extension of the valuation of K, we have $\mathsf{RV}(K(x)) = \mathsf{RV}(K)$, so the point $\mathfrak{r}_{X,\Sigma}(x)$ of C_y is $\mathsf{RV}(K)$-definable (by tameness of K the generalized field $\mathsf{RV}(K)$ is perfect, so $\mathfrak{r}_{X,\Sigma}(x)$ is even a $\mathsf{RV}(K)$-point of C_y).

Set $U := \mathfrak{r}_{X,\Sigma}^{-1}(\mathfrak{r}_{X,\Sigma}(x))$; this is a K-definable subset of X. By definition of a Nash-admissible K-skeleton there exist a $\overline{K}$-Zariski open subset X' of X containing U and a $\overline{K}$-regular étale map $f: X' \to \mathbf{A}^1$ that induces a Nash isomorphism between U and an open disc D. Let F be a finite extension of K over which f and X' are defined. Then f is an element of $F[U]$ inducing a Nash isomorphism between U and D. As K is tame, F is a tamely ramified extension of K. By tame descent for open (poly)discs (Theorem 6.2) there exists $g \in K[U]$ that induces a Nash isomorphism between U and an open disc Δ.

This implies that $K(g(x))^{\mathrm{h}} = K(x)^{\mathrm{h}}$. As $K(x)$ is isomorphic to L as a field and g is étale, $g(x)$ is transcendental over K. Identifying $K(x)$ with L and setting $t = g(x)$ we get the required statement. □

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