
ON SOME RATIONAL GENERATING SERIES OCCURRING IN ARITHMETIC GEOMETRY

by

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To Bernie's memory, with gratitude

Introduction

The main purpose of the present paper is to illustrate the following motto: “rational generating series occurring in arithmetic geometry are motivic in nature”. More precisely, consider a series $F = \sum_{n \in \mathbf{N}} a_n T^n$ with coefficients in \mathbf{Z} . We shall say F is motivic in nature if there exists a series $F_{\text{mot}} = \sum_{n \in \mathbf{N}} A_n T^n$, with coefficients A_n in some Grothendieck ring of varieties, or some Grothendieck ring of motives, such that a_n is the number of rational points of A_n in some fixed finite field, for all $n \geq 0$. Furthermore, we require F_{mot} to be canonically attached to F . Of course, such a definition is somewhat incomplete, since one can always take for A_n the disjoint union of a_n points. In the present paper, which is an update of a talk by the second author at the Conference “Geometric Aspects of Dwork’s theory” that took place in Bressanone in July 2001, we consider the issue of being motivic in nature for the following three types of generating series: Hasse-Weil series, Igusa series and Serre series. In section 4, we consider the easiest case, that of Igusa type series, for which being motivic in nature follows quite easily from Kontsevich’s theory of motivic integration as developed in [6] [7]. The Serre case is more subtle. After a false start in section 5, we explain in section 6 how to deal with it by using the work in [8] on arithmetic motivic integration. Finally, in section 7, we consider the case of Hasse-Weil series, which still remains very much open. Here there is a conjecture, which is due to M. Kapranov [20] and can be traced back to insights of Grothendieck cf. p. 184 of [16]. Since a “counterexample” to the conjecture recently appeared [21], we spend some time to explain the dramatic effects of inverting the class of the affine line in the Grothendieck group of varieties. This gives us the opportunity of reviewing some interesting recent work of Poonen [26], Bittner [3] and Larsen and Lunts [21] and allows us to propose a precised form of Kapranov’s conjecture that escapes Larsen and Lunts’ counterexample.

It was one of Bernie's insights that most, if not all, functions occurring in Number Theory should be of geometric origin. So we hope the present contribution will not be too inadequate as an homage to his memory.

1. Conventions and preliminaries

1.1. In this paper, by a variety over a ring R , we mean a reduced and separated scheme of finite type over $\text{Spec } R$.

1.2. Let A be a commutative ring. The ring of rational formal series with coefficients in A is the smallest subring of $A[[T]]$ containing $A[T]$ and stable under taking inverses (when they exist in $A[[T]]$).

2. Some classical generating series

2.1. Let X be a variety over \mathbf{F}_q . We set $N_n := |X(\mathbf{F}_{q^n})|$, for $n \geq 1$.

2.1.1. Theorem (Dwork [10]). — *The Hasse-Weil series*

$$Z(T) := \exp\left(\sum_{n \geq 1} \frac{N_n}{n} T^n\right)$$

is rational.

2.2. Let K be a finite extension of \mathbf{Q}_p with ring of integers \mathcal{O}_K and uniformizing parameter π . Let X be a variety over \mathcal{O}_K . We set $\tilde{N}_n := |X(\mathcal{O}_K/\pi^{n+1})|$, for $n \geq 0$.

2.2.1. Theorem (Igusa[19]). — *The series*

$$Q(T) := \sum_{n \geq 0} \tilde{N}_n T^n$$

is rational.

Strictly speaking this result is due to Igusa [19] in the hypersurface case and to Meuser in general [24]. However, as mentioned in the review MR 83g:12015 of [24], a trick by Serre allows to deduce the general case from the hypersurface case.

2.3. Let K be a finite extension of \mathbf{Q}_p with ring integers \mathcal{O}_K and uniformizing parameter π . We keep the notations of 2.2. For $n \geq 0$ we denote by \overline{N}_n the cardinality of the image of $X(\mathcal{O}_K)$ in $X(\mathcal{O}_K/\pi^{n+1})$. In other words, \overline{N}_n is the number of points in $X(\mathcal{O}_K/\pi^{n+1})$ (approximate solutions modulo π^{n+1}) that may be lifted to points in $X(\mathcal{O}_K)$ (actual solutions in \mathcal{O}_K). Clearly, \overline{N}_n is finite. Furthermore, when X is smooth, then $\tilde{N}_n = \overline{N}_n$ for every n .

2.3.1. Theorem (Denef [5]). — *The series*

$$P(T) := \sum_{n \geq 0} \overline{N}_n T^n$$

is rational.

2.3.2. Remark. — The problem of proving the analogue of Theorems 2.2.1 and 2.3.1 when K is a finite extension of $\mathbf{F}_q[[t]]$ still remains very much an open issue, but the level of difficulty seems quite different for $Q(T)$ or $P(T)$. While rationality of $Q(T)$ for function fields would follow using Igusa's proof once Hironaka's strong form of resolution of singularities is known in characteristic p , proving rationality of $P(T)$ for function fields would require completely new ideas, since no general quantifier elimination Theorem is known, or even conjectured, in positive characteristic.

3. Additive invariants of algebraic varieties

3.1. Let R be a ring. We denote by Var_R the category of algebraic varieties over R . An additive invariant

$$\lambda : \text{Var}_R \longrightarrow S,$$

with S a ring, assigns to any X in Var_R an element $\lambda(X)$ of S such that

$$\lambda(X) = \lambda(X')$$

for $X \simeq X'$,

$$\lambda(X) = \lambda(X') + \lambda(X \setminus X'),$$

for X' closed in X , and

$$\lambda(X \times X') = \lambda(X)\lambda(X')$$

for every X and X' .

Let us remark that additive invariants λ naturally extend to take their values on constructible subsets of algebraic varieties.

3.2. Examples. —

3.2.1. Euler characteristic. — Here $R = k$ is a field. When k is a subfield of \mathbf{C} , the Euler characteristic $\text{Eu}(X) := \sum_i (-1)^i \text{rk} H_c^i(X(\mathbf{C}), \mathbf{C})$ give rise to an additive invariant $\text{Eu} : \text{Var}_k \rightarrow \mathbf{Z}$. For general k , replacing Betti cohomology with compact support by ℓ -adic cohomology with compact support, $\ell \neq \text{char} k$, one gets an additive invariant $\text{Eu}_\ell : \text{Var}_k \rightarrow \mathbf{Z}$, which does not depend on ℓ .

3.2.2. Hodge polynomial. — Let us assume $R = k$ is a field of characteristic zero. Then it follows from Deligne's Mixed Hodge Theory that there is a unique additive invariant $H : \text{Var}_k \rightarrow \mathbf{Z}[u, v]$, which assigns to a smooth projective variety X over k its usual Hodge polynomial

$$H(u, v) := \sum_{p, q} (-1)^{p+q} h^{p, q}(X) u^p v^q,$$

with $h^{p, q}(X) = \dim H^q(X, \Omega_X^p)$ the (p, q) -Hodge number of X .

3.2.3. Virtual motives. — More generally, when $R = k$ is a field of characteristic zero, there exists by Gillet and Soulé [13], Guillen and Navarro-Aznar [17], a unique additive invariant $\chi_c : \text{Var}_k \rightarrow K_0(\text{CHMot}_k)$, which assigns to a smooth projective variety X over k the class of its Chow motive, where $K_0(\text{CHMot}_k)$ denotes the Grothendieck ring of the category of Chow motives over k (with rational coefficients).

3.2.4. Counting points. — Counting points also yields additive invariants. Assume $k = \mathbf{F}_q$, then $N_n : X \mapsto |X(\mathbf{F}_{q^n})|$ gives rise to an additive invariant $N_n : \text{Var}_k \rightarrow \mathbf{Z}$. Similarly, if R is (essentially) of finite type over \mathbf{Z} , for every maximal ideal \mathfrak{P} of R with finite residue field $k(\mathfrak{P})$, we have an additive invariant $N_{\mathfrak{P}} : \text{Var}_R \rightarrow \mathbf{Z}$, which assigns to X the cardinality of $(X \otimes k(\mathfrak{P}))(k(\mathfrak{P}))$.

3.3. There exists a universal additive invariant $[-] : \text{Var}_R \rightarrow K_0(\text{Var}_R)$ in the sense that composition with $[-]$ gives a bijection between ring morphisms $K_0(\text{Var}_R) \rightarrow S$ and additive invariants $\text{Var}_R \rightarrow S$. The construction of $K_0(\text{Var}_R)$ is quite easy: take the free abelian group on isomorphism classes $[X]$ of objects of Var_R and mod out by the relation $[X] = [X'] + [X \setminus X']$ for X' closed in X . The product is now defined by $[X][X'] = [X \times X']$.

We shall denote by \mathbf{L} the class of the affine line \mathbf{A}_R^1 in $K_0(\text{Var}_R)$. An important role will be played by the ring $\mathcal{M}_R := K_0(\text{Var}_R)[\mathbf{L}^{-1}]$ obtained by localization with respect to the multiplicative set generated by \mathbf{L} . This construction is analogue to the construction of the category of Chow motives from the category of effective Chow motives by localization with respect to the Lefschetz motive. (Remark that the morphism χ_c of 3.2.3 sends \mathbf{L} to the class of the Lefschetz motive.)

One should stress that very little is known about the structure of the rings $K_0(\text{Var}_R)$ and \mathcal{M}_R even when R is a field. Let us just quote a result by Poonen [26] saying that when k is a field of characteristic zero the ring $K_0(\text{Var}_k)$ is not a domain (we shall explain this result with more details in §7.3). For instance, even for a field k , it is not known whether the localization morphism $(\text{Var}_k) \rightarrow \mathcal{M}_k$ is injective or not (although the whole point of §7.5 relies on the guess it should not).

3.3.1. Remark. — In fact, the ring $K_0(\text{Var}_k)$ as well as the canonical morphism $\chi_c : K_0(\text{Var}_k) \rightarrow K_0(\text{CHMot}_k)$, were already considered by Grothendieck in a letter to Serre dated August 16, 1964, cf. p. 174 of [16].

4. Geometrization of $Q(T)$

4.1. Let k be a field. For every variety X over k , we denote by $\mathcal{L}(X)$ the corresponding space of arcs. It is a k -scheme, which satisfies

$$\mathcal{L}(X)(K) = X(K[[t]])$$

for every field K containing k . More precisely $\mathcal{L}(X)$ is defined as the inverse limit $\mathcal{L}(X) := \varprojlim \mathcal{L}_n(X)$, where $\mathcal{L}_n(X)$ represents the functor from k -algebras to sets sending a k -algebra R to $X(R[[t]]/t^{n+1}R[[t]])$. We shall always consider $\mathcal{L}(X)$ as

endowed with its reduced structure. We shall denote by π_n the canonical morphism $\mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$.

4.2. We consider the generating series

$$Q_{\text{geom}}(T) := \sum_{n \geq 0} [\mathcal{L}_n(X)] T^n$$

in $\mathcal{M}_k[[T]]$.

4.2.1. Theorem (Denef-Loeser). — Assume $\text{char } k = 0$.

1) The series $Q_{\text{geom}}(T)$ in $\mathcal{M}_k[[T]]$ is rational of the form

$$\frac{R(T)}{\prod (1 - \mathbf{L}^a T^b)},$$

with $R(T)$ in $\mathcal{M}_k[T]$, a in \mathbf{Z} and b in $\mathbf{N} \setminus \{0\}$.

2) If X is defined over some number field K , then, for almost all finite places \mathfrak{P} ,

$$N_{\mathfrak{P}}(Q_{\text{geom}}(T)) = Q_{X \otimes \mathcal{O}_{K_{\mathfrak{P}}}}(T).$$

Here we should explain what we mean by $N_{\mathfrak{P}}(Q_{\text{geom}}(T))$. For X a variety over K , $N_{\mathfrak{P}}(X)$ makes sense for almost all finite places \mathfrak{P} , by taking some model over $\mathcal{O}_{K_{\mathfrak{P}}}$. Now we apply this termwise to the series $Q_{\text{geom}}(T)$. This is possible since the series is rational by 4.2.1 1). The Theorem is proved for hypersurfaces in [6], and the general case is similar and may also be deduced from general results in [7] and [8].

Oversimplified sketch of proof the rationality. — Let us first recall Igusa's proof of Theorem 2.2.1 when X is an hypersurface defined by $f = 0$ in $\mathbf{A}_{\mathcal{O}_K}^m$. The basic idea is to express the series $Q(T)$ as the integral

$$I(s) := \int_{\mathcal{O}_K^m} |f|^s |dx|,$$

up to trivial factors, with $T = q^{-s}$, q the cardinality of the residue field. Then one may use Hironaka's resolution of singularities to reduce the computation of $I(s)$ to the case where $f = 0$ is locally given by monomials for which direct calculation is easy.

Our proof of the rationality of $Q_{\text{geom}}(T)$ follows similar lines. One express first our series as an integral, but here p -adic integration is replaced by **motivic integration**. If Y is a variety over k , motivic integration assigns to certain subsets A of the arc space $\mathcal{L}(Y)$ a motivic measure $\mu(A)$ in \mathcal{M}_k (or sometimes, but this will not be considered here, a measure in a certain completion of \mathcal{M}_k). Then, to be able to use Hironaka's resolution of singularities to reduce to the locally monomial case as in Igusa's proof, we have to use the fundamental **change of variable formula** established in §3 of [7]. \square

5. Geometrization of $P(T)$: I

5.1. In view of the previous section, it is natural to consider now the image $\pi_n(\mathcal{L}(X))$ of $\mathcal{L}(X)$ in $\mathcal{L}_n(X)$. Thanks to Greenberg's Theorem on solutions of polynomial systems in Henselian rings, we know that $\pi_n(\mathcal{L}(X))$ is a constructible subset $\mathcal{L}_n(X)$, hence we may consider its class $[\pi_n(\mathcal{L}(X))]$ in \mathcal{M}_k . We consider the generating series

$$P_{\text{geom}}(T) := \sum_{n \geq 0} [\pi_n(\mathcal{L}(X))] T^n$$

in $\mathcal{M}_k[[T]]$.

5.1.1. Theorem (Denef-Loeser [7]). — Assume $\text{char} k = 0$. The series $P_{\text{geom}}(T)$ in $\mathcal{M}_k[[T]]$ is rational of the form

$$\frac{R(T)}{\prod (1 - \mathbf{L}^a T^b)},$$

with $R(T)$ in $\mathcal{M}_k[T]$, a in \mathbf{Z} and b in $\mathbf{N} \setminus \{0\}$.

Oversimplified sketch of proof. — Let us first recall the strategy of the proof [5] of Theorem 2.3.1 in the p -adic case. One reduces to the case where X is a closed subvariety of $\mathbf{A}_{\mathcal{O}_K}^m$. Then one expresses the series $P(T)$ as the integral

$$J(s) := \int_{\mathcal{O}_K^m} d(x, X)^s |dx|,$$

up to trivial factors, with $T = q^{-s}$, similarly as in Igusa's case, where $d(x, X)$ is the function “distance to X ”. Here an essential new feature appears, the function $d(x, X)$ being in general not a polynomial function, but only a definable or semi-algebraic function. Then one is able to use Macintyre's quantifier elimination Theorem [23], a p -adic analogue of Tarski-Seidenberg's theorem, to prove rationality.

In the present setting our proof follows a similar pattern, replacing p -adic integration by motivic integration and the theory of p -adic semi-algebraic sets by a theory of $k[[t]]$ -semi-algebraic sets built off from a quantifier elimination Theorem due to Pas [25]. \square

5.2. When X is defined over a number field K , a quite natural guess would be, by analogy with what we have seen so far, that, for almost all finite places \mathfrak{P} , $N_{\mathfrak{P}}(P_{\text{geom}}(T)) = P_{X \otimes \mathcal{O}_{K_{\mathfrak{P}}}}(T)$. But such a statement cannot hold true. This is due to the fact that, in the very definition of $P(T)$, one is concerned in not considering extensions of the residue field, while in the definition of $P_{\text{geom}}(T)$ extensions of the residue field k are allowed. To remedy this, one needs to be more careful about rationality issues concerning the residue field, and for that purpose it is convenient to introduce **definable subassignments** as we do in the next section.

6. Geometrization of $P(T)$: II

6.1. Subassignments. — Fix a ring R . We denote by Field_R the category of R -algebras that are fields. For an R -scheme X , we denote by h_X the functor which to a field K in Field_R assigns the set $h_X(K) := X(K)$. By a **subassignment** $h \subset h_X$ of h_X we mean the datum, for every field K in Field_R , of a subset $h(K)$ of $h_X(K)$. We stress that, contrarily to subfunctors, no compatibility is required between the various sets $h(K)$.

All set theoretic constructions generalize in an obvious way to the case of subassignments. For instance if h and h' are subassignments of h_X , then we denote by $h \cap h'$ the subassignment $K \mapsto h(K) \cap h'(K)$, etc.

Also, if $\pi : X \rightarrow Y$ is a morphism of R -schemes and h is a subassignment of h_X , we define the subassignment $\pi(h)$ of h_Y by $\pi(h)(K) := \pi(h(K)) \subset h_Y(K)$.

6.2. Definable subassignments. — Let R be a ring. By a ring formula φ over R , we mean a first order formula in the language of rings with coefficients in R and free variables x_1, \dots, x_n . In other words φ is built out from boolean combinations (“and”, “or”, “not”) of polynomial equations over R and existential and universal quantifiers.

For example

$$(\exists x)(x^2 + x + y = 0 \text{ and } 4y \neq 1)$$

is a ring formula over \mathbf{Z} with free variable y .

To a ring formula φ over R with free variables x_1, \dots, x_n one assigns the subassignment h_φ of $h_{\mathbf{A}_R^n}$ defined by

$$(6.2.1) \quad h_\varphi(K) := \left\{ (a_1, \dots, a_n) \in K^n \mid \varphi(a_1, \dots, a_n) \text{ holds in } K \right\} \subset K^n = h_{\mathbf{A}_R^n}(K).$$

Such a subassignment of $h_{\mathbf{A}_R^n}$ is called a **definable subassignment**. More generally, using affine coverings, cf. [8], one defines definable subassignments of h_X for X a variety over R .

It is quite easy to show that if $\pi : X \rightarrow Y$ is an R -morphism of finite presentation, $\pi(h)$ is a definable subassignment of h_Y if h is a definable subassignment of h_X .

In our situation, we are concerned with the subassignment $\pi(h_{\mathcal{L}(X)}) \subset h_{\mathcal{L}_n(X)}$. Remark that $\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$ is not of finite type.

Nevertheless, we have the following:

6.2.1. Proposition ([8]). — $\pi(h_{\mathcal{L}(X)})$ is a definable subassignment of $h_{\mathcal{L}_n(X)}$.

6.3. Formulas and motives. — Let k be a field of characteristic zero. It follows from 3.2.3 that we have a canonical morphism $\chi_c : K_0(\text{Var}_k) \rightarrow K_0(\text{CHMot}_k)$. We shall denote by $K_0^{\text{mot}}(\text{Var}_k)$ the image of $K_0(\text{Var}_k)$ in $K_0(\text{CHMot}_k)$ under this morphism. Remark that the image of \mathbf{L} in $K_0^{\text{mot}}(\text{Var}_k)$ is not a zero divisor since it is invertible in $K_0(\text{CHMot}_k)$.

Let us explain now how to assign in a canonical way to a ring formula φ over k an element $\chi_c([\varphi])$ of $K_0^{\text{mot}}(\text{Var}_k) \otimes \mathbf{Q}$.

6.4. Let φ be a formula over a number field K . For almost all finite places \mathfrak{P} with residue field $k(\mathfrak{P})$, one may extend the definition in (6.2.1) to give a meaning to $h_\varphi(k(\mathfrak{P}))$. If φ and φ' are formulas over K , we set $\varphi \equiv \varphi'$ if $h_\varphi(k(\mathfrak{P})) = h_{\varphi'}(k(\mathfrak{P}))$ for almost all finite places \mathfrak{P} .

It follows from a fundamental result of J. Ax [2] that $\varphi \equiv \varphi'$ if and only if $h_\varphi(L) = h_{\varphi'}(L)$ for every pseudo-finite field L containing K . Let us recall that a pseudo-finite field is an infinite perfect field that has exactly one field extension of any given finite degree, and over which every geometrically irreducible variety has a rational point. Historically, the above result of Ax was one of the main motivation for introducing that notion. One way of constructing pseudo-finite fields is by taking infinite ultraproducts of finite fields.

Let us now introduce the Grothendieck ring of formulas over R , $K_0(\text{Field}_R)$, and $K_0(\text{PFF}_R)$ the Grothendieck ring of the theory of pseudo-finite fields over R . The ring $K_0(\text{Field}_R)$ (resp. $K_0(\text{PFF}_R)$) is the group generated by symbols $[\varphi]$, where φ is any ring formula over R , subject to the relations $[\varphi_1 \text{ or } \varphi_2] = [\varphi_1] + [\varphi_2] - [\varphi_1 \text{ and } \varphi_2]$, whenever φ_1 and φ_2 have the same free variables, and the relations $[\varphi_1] = [\varphi_2]$, whenever there exists a ring formula ψ over k that, when interpreted in any field (resp. any pseudo-finite field) K in Field_R , yields the graph of a bijection between the tuples of elements of K satisfying φ_1 and those satisfying φ_2 . The ring multiplication is induced by the conjunction of formulas in disjoint sets of variables.

There is a canonical morphism

$$K_0(\text{Field}_R) \longrightarrow K_0(\text{PFF}_R).$$

We can now state the following:

6.4.1. Theorem (Denef-Loeser [8],[9]). — *Let k be a field of characteristic zero. There exists a unique ring morphism*

$$\chi_c : K_0(\text{PFF}_k) \longrightarrow K_0^{\text{mot}}(\text{Var}_k) \otimes \mathbf{Q}$$

satisfying the following two properties:

- (i) *For any formula φ which is a conjunction of polynomial equations over k , the element $\chi_c([\varphi])$ equals the class in $K_0^{\text{mot}}(\text{Var}_k) \otimes \mathbf{Q}$ of the variety defined by φ .*
- (ii) *Let X be a normal affine irreducible variety over k , Y an unramified Galois cover⁽¹⁾ of X , and C a cyclic subgroup of the Galois group G of Y over X . For such data we denote by $\varphi_{Y,X,C}$ a ring formula, whose interpretation in any field K containing k , is the set of K -rational points on X that lift to a geometric point on Y with decomposition group C (i.e. the set of points on X that lift to*

⁽¹⁾Meaning that Y is an integral étale scheme over X with $Y/G \cong X$, where G is the group of all endomorphisms of Y over X .

a K -rational point of Y/C , but not to any K -rational point of Y/C' with C' a proper subgroup of C). Then

$$\chi_c([\varphi_{Y,X,C}]) = \frac{|C|}{|N_G(C)|} \chi_c([\varphi_{Y,Y/C,C}]),$$

where $N_G(C)$ is the normalizer of C in G .

Moreover, when k is a number field, for almost all finite places \mathfrak{P} , $N_{\mathfrak{P}}(\chi_c([\varphi]))$ is equal to the cardinality of $h_{\varphi}(k(\mathfrak{P}))$.

The above theorem is a variant of results in §3.4 of [8]. A sketch of proof is given in [9].

Some ingredients in the proof. — Uniqueness uses quantifier elimination for pseudo-finite fields (in terms of Galois stratifications, cf. the work of Fried and Sacerdote [12][11, §26]), from which it follows that $K_0(\text{PFF}_k)$ is generated as a group by classes of formulas of the form $\varphi_{Y,X,C}$. Thus by (ii) we only have to determine $\chi_c([\varphi_{Y,Y/C,C}])$, with C a cyclic group. But this follows directly from the following recursion formula:

$$(6.4.1) \quad |C| [Y/C] = \sum_{A \text{ subgroup of } C} |A| \chi_c([\varphi_{Y,Y/A,A}]).$$

This recursion formula is a direct consequence of (i), (ii), and the fact that the formulas $\varphi_{Y,Y/C,A}$ yield a partition of Y/C .

The proof of existence is based on work of del Baño Rollin and Navarro Aznar [4] who associate to any representation over \mathbf{Q} of a finite group G acting freely on an affine variety Y over k , an element in the Grothendieck group of Chow motives over k . By linearity, we can hence associate to any \mathbf{Q} -central function α on G (i.e. a \mathbf{Q} -linear combination of characters of representations of G over \mathbf{Q}), an element $\chi_c(Y, \alpha)$ of that Grothendieck group tensored with \mathbf{Q} . Using Emil Artin's Theorem, that any \mathbf{Q} -central function α on G is a \mathbf{Q} -linear combination of characters induced by trivial representations of cyclic subgroups, one shows that $\chi_c(Y, \alpha) \in K_0^{\text{mot}}(\text{Var}_k) \otimes \mathbf{Q}$. For $X := Y/G$ and C any cyclic subgroup of G , we define $\chi_c([\varphi_{Y,X,C}]) := \chi_c(Y, \theta)$, where θ sends $g \in G$ to 1 if the subgroup generated by g is conjugate to C , and else to 0. With some more work we prove that the above definition of $\chi_c([\varphi_{Y,X,C}])$ extends by additivity to a well-defined map $\chi_c : K_0(\text{PFF}_k) \longrightarrow K_0^{\text{mot}}(\text{Var}_k) \otimes \mathbf{Q}$. \square

Clearly $\chi_c(\varphi)$ depends only on h_{φ} and the construction easily extends by additivity to definable subassignments of h_X , for any variety X over k . So, to any such definable subassignment h , we may associate $\chi_c(h)$ in $K_0^{\text{mot}}(\text{Var}_k) \otimes \mathbf{Q}$.

6.4.2. Proposition (Denef-Loeser). — *Let k be a field of characteristic zero. For any definable subassignment h , $\text{Eu}(\chi_c(h))$ belongs to \mathbf{Z} .*

Proof. — It is enough to show that $\text{Eu}(\chi_c(\varphi_{Y,X,C}))$ belongs to \mathbf{Z} for every Y , X and C . Consider first the case C is the trivial subgroup e of G . We have

$$\chi_c(\varphi_{Y,X,e}) = \frac{1}{|G|} \chi_c(\varphi_{Y,Y,e}) = \frac{1}{|G|} [Y].$$

It follows that

$$\mathrm{Eu}(\chi_c(\varphi_{Y,X,e})) = \frac{1}{|G|} \mathrm{Eu}(Y) = \mathrm{Eu}(X) \in \mathbf{Z}.$$

When C is a non trivial cyclic subgroup of G , by induction on $|C|$, it follows from the recursion formula (6.4.1) that $\mathrm{Eu}(\chi_c(\varphi_{Y,X,C})) = 0$. \square

6.4.3. Example. — Let n be a integer ≥ 1 and assume k contains all n -roots of unity. Consider the formula $\varphi_n : (\exists y)(x = y^n \text{ and } x \neq 0)$; then $\chi_c(\varphi_n) = \frac{\mathbf{L}-1}{n}$. In particular $\mathrm{Eu}(\chi_c(\varphi_n)) = 0$ and $H(\chi_c(\varphi_n)) = \frac{uv-1}{n}$. This example contradicts the example on page 430 line -2 of [8] (page 3 line 4 in the preprint) which is unfortunately incorrect.

6.4.4. Remark. — It is the place to correct the following errors in the published version of [9]. On line 18 of the third page, after the word “motives” one has to insert “, and by killing all \mathbf{L} -torsion”. Once this correction is made, it is easily checked that $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)$ becomes the same in the present paper and in [9]. On line 6 of the eighth page, one has to delete the last sentence.

6.5. The series P_{ar} . — We now consider the series

$$P_{\mathrm{ar}}(T) := \sum_{n \geq 0} \chi_c(\pi_n(h_{\mathcal{L}(X)})) T^n$$

in $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$.

6.5.1. Theorem (Denef-Loeser [8]). — Assume $\mathrm{char} k = 0$.

1) The series $P_{\mathrm{ar}}(T)$ in $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ is rational of the form

$$\frac{R(T)}{\prod (1 - \mathbf{L}^a T^b)},$$

with $R(T)$ in $(\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q})[T]$, a in \mathbf{Z} and b in $\mathbf{N} \setminus \{0\}$.

2) If X is defined over some number field K , then, for almost all finite places \mathfrak{P} ,

$$N_{\mathfrak{P}}(P_{\mathrm{ar}}(T)) = P_{X \otimes_{\mathcal{O}_{K_{\mathfrak{P}}}}} (T).$$

In the proof of Theorem 6.5.1, one uses in an essential way **arithmetic motivic integration** a variant of motivic integration developed in [8]. The specialization statement 2) in Theorem 6.5.1 is a special case of the following results, which states that “natural p -adic integrals are motivic”.

6.5.2. Theorem (Denef-Loeser [8]). — Let K be a number field. Let φ be a first order formula in the language of **valued** rings with coefficients in K and free variables x_1, \dots, x_n . Let f be a polynomial in $K[x_1, \dots, x_n]$. For \mathfrak{P} a finite place of K , denote by $K_{\mathfrak{P}}$ the completion of K at \mathfrak{P} . Then there exists a **canonical** motivic integral which specializes to

$$\int_{h_{\varphi}(K_{\mathfrak{P}})} |f|_{\mathfrak{P}}^s |dx|_{\mathfrak{P}}$$

for almost all finite places \mathfrak{P} .

The formulation here is somewhat unprecise and we refer to [8] for details. Let us just say that formulas in the language of valued rings include expressions like $\text{ord}a \leq \text{ord}b$ or $\text{ord}c \equiv b \pmod{e}$.

6.6. Theorem 6.5.2 applies in particular to integrals occurring in p -adic harmonic analysis, like orbital integrals. This has led recently Tom Hales to speculate that many of the basic objects in representation theory are motivic in nature and to develop a beautiful conjectural program aiming to the determination of the (still conjectural) virtual Chow motives that control the behavior of orbital integrals and leading to a motivic fundamental lemma [18][14].

7. Geometrization of $Z(T)$

7.1. Let k be a field and let X be a variety over k . For $n \geq 0$, we denote by $X^{(n)}$ the n -fold symmetric product of X , i.e. the quotient of the cartesian product X^n by the symmetric group of n elements. Note that $X^{(0)}$ is isomorphic to $\text{Spec} k$.

Following Kapranov [20], we define the motivic zeta function of X as the power series

$$Z_{\text{mot}}(T) := \sum_{n=0}^{\infty} [X^{(n)}] T^n$$

in $K_0(\text{Var}_k)[[T]]$.

Also, when $\alpha : K_0(\text{Var}_k) \rightarrow A$ is a morphism of rings, we denote by $Z_{\text{mot},\alpha}(T)$ the power series $\sum_{n=0}^{\infty} \alpha([X^{(n)}]) T^n$ in $A[[T]]$. We shall write \mathbf{L} for $\alpha(\mathbf{L})$.

7.1.1. Proposition. — *If $k = \mathbf{F}_q$, and we write $N(S) = N_1(S) = |S(k)|$ for S a variety over k , then $Z_{\text{mot},N}(T)$ is equal to the Hasse-Weil zeta function considered in 2.1.*

Proof. — Rational points of $X^{(n)}$ over k correspond to degree n effective zero cycles of X , hence the result follows from the usual inversion formula between the number of effective zero cycles of given degree on X and the number of rational points of X over finite extensions of k . \square

In his paper [20], Kapranov proves the following result:

7.1.2. Theorem (Kapranov [20]). — *Let X be a smooth projective irreducible curve of genus g . Let $\alpha : K_0(\text{Var}_k) \rightarrow A$ be a morphism of rings with A a field, such that \mathbf{L} is non zero in A . Assume also there exists a degree 1 line bundle on X . Then:*

- 1) *The series $Z_{\text{mot},\alpha}(T)$ is rational. It is the quotient of a polynomial of degree $2g$ by $(1 - T)(1 - \mathbf{L}T)$.*
- 2) *The function $Z_{\text{mot},\alpha}(T)$ satisfies the functional equation*

$$Z_{\text{mot},\alpha}(\mathbf{L}^{-1}T^{-1}) = \mathbf{L}^{1-g}T^{2-2g} Z_{\text{mot},\alpha}(T).$$

The proof follows the lines of F. K. Schmidt's classical proof [27] of rationality and functional equation for the Hasse-Weil zeta function of a smooth projective curve.

In the same paper, Kapranov states “it is natural to expect that the motivic zeta functions are rational and satisfy similar functional equations?”.

7.1.3. Remark. — A generating series similar to Z_{geom} and the question of its rationality were already considered by Grothendieck in a letter to Serre dated September 24, 1964, cf. p. 184 of [16].

7.2. Stable birational invariants. — We now give a new presentation by generators and relations of $K_0(\text{Var}_k)$ due to F. Bittner [3]. We denote by $K_0^{\text{bl}}(\text{Var}_k)$ the quotient of the free abelian group on isomorphism classes of smooth proper varieties over k by the relation

$$[\text{Bl}_Y X] - [E] = [X] - [Y],$$

for Y and X smooth proper irreducible over k , Y closed in X , $\text{Bl}_Y X$ the blowup of X with center Y and E the exceptional divisor in $\text{Bl}_Y X$. As for $K_0(\text{Var}_k)$, cartesian product induces a product on $K_0^{\text{bl}}(\text{Var}_k)$ which endows it with a ring structure. There is a canonical ring morphism $K_0^{\text{bl}}(\text{Var}_k) \rightarrow K_0(\text{Var}_k)$, which sends $[X]$ to $[X]$.

7.2.1. Theorem (Bittner [3]). — *Assume k is of characteristic zero. The canonical ring morphism*

$$K_0^{\text{bl}}(\text{Var}_k) \rightarrow K_0(\text{Var}_k)$$

is an isomorphism.

The proof is based on Hironaka resolution of singularities and the weak factorization Theorem of Abramovich, Karu, Matsuki and Włodarczyk [1].

One deduces easily the following result, first proved by Larsen and Lunts [21].

7.2.2. Corollary (Larsen and Lunts [21]). — *Let us assume k is algebraically closed of characteristic zero. Let A be the monoid of isomorphism classes of smooth proper irreducible varieties over k and let $\Psi : A \rightarrow G$ be a morphism of commutative monoids such that*

- 1) *If X and Y are birationally equivalent smooth proper irreducible varieties over k , then $\Psi([X]) = \Psi([Y])$.*
- 2) *$\Psi([\mathbf{P}_k^n]) = 1$.*

Then there exists a unique morphism a rings

$$\Phi : K_0(\text{Var}_k) \longrightarrow \mathbf{Z}[G]$$

such that $\Phi([X]) = \Psi([X])$ when X is smooth proper irreducible.

We assume from now on that k is algebraically closed of characteristic zero. We denote by SB the monoid of equivalence classes of smooth proper irreducible varieties

over k under stably birational equivalence ⁽²⁾. It follows from Corollary 7.2.2 that there exists a universal stable birational invariant

$$\Phi_{\text{SB}} : K_0(\text{Var}_k) \longrightarrow \mathbf{Z}[\text{SB}].$$

7.2.3. Proposition (Larsen and Lunts [21]). — *The kernel of the morphism $\Phi_{\text{SB}} : K_0(\text{Var}_k) \rightarrow \mathbf{Z}[\text{SB}]$ is the principal ideal generated by $\mathbf{L} = [\mathbf{A}_k^1]$.*

Sketch of proof. — It is clear that \mathbf{L} lies in the kernel of Φ_{SB} . Conversely, take $\alpha = \sum_{1 \leq i \leq r} [X_i] - \sum_{1 \leq j \leq s} [Y_j]$ in the kernel of Φ_{SB} , with X_i and Y_j smooth, proper and irreducible. Since $\sum_{1 \leq i \leq r} [X_i] = \sum_{1 \leq j \leq s} [Y_j]$ in $\mathbf{Z}[\text{SB}]$, $r = s$ and, after renumbering the X_i 's, we may assume X_i is stably birational to Y_i for every i . Hence it is enough to show that if X and Y are smooth, proper and irreducible stably birationally equivalent, then $[X] - [Y]$ belongs to $\mathbf{L}K_0(\text{Var}_k)$. Since $[X] - [\mathbf{P}_k^r \times X]$ belongs to $\mathbf{L}K_0(\text{Var}_k)$, we can even assume X and Y are birationally equivalent and then the result follows easily from the weak factorization Theorem. \square

7.3. Back to Poonen's result. — As promised, we shall now give some explanations concerning the proof of Poonen's Theorem 7.3.3.

7.3.1. Key-Lemma (Poonen [26]). — *Let k be a field of characteristic zero. There exists abelian varieties A and B over k such that $A \times A$ is isomorphic to $B \times B$ but $A_{\bar{k}} \not\cong B_{\bar{k}}$.*

The proof relies on the following lemma:

7.3.2. Lemma (Poonen [26]). — *Let k be a field of characteristic zero. There exists an abelian variety A over k such that $\text{End}_k(A) = \text{End}_{\bar{k}}(A) \cong \mathcal{O}$, with \mathcal{O} the ring of integers of a number field of class number 2.*

When $k = \mathbf{C}$, one may take A an elliptic curve with complex multiplication by $\mathbf{Z}[\sqrt{-5}]$. The general case is much more involved and necessitates the use of modular forms and Eichler-Shimura Theory as well as some table checking, see [26].

Let us now explain how Poonen deduces from the Key-Lemma the following:

7.3.3. Theorem. — *The ring $K_0(\text{Var}_k)$ is not a domain, for k a field of characteristic zero.*

Proof. — Take A and B as in the Key-Lemma. We have $([A] + [B])([A] - [B]) = 0$ in $K_0(\text{Var}_k)$. To check that $[A] + [B]$ and $[A] - [B]$ are nonzero in $K_0(\text{Var}_k)$, it is enough to check that they have a nonzero image under the composition

$$K_0(\text{Var}_k) \rightarrow K_0(\text{Var}_{\bar{k}}) \rightarrow \mathbf{Z}[\text{SB}_{\bar{k}}] \rightarrow \mathbf{Z}[\text{AV}_{\bar{k}}],$$

where $\text{AV}_{\bar{k}}$ is the monoid of isomorphism classes of abelian varieties over \bar{k} and the last morphism is induced by the Albanese functor assigning to a smooth irreducible

⁽²⁾ X and Y are called stably birational if $X \times \mathbf{P}_k^r$ is birational to $Y \times \mathbf{P}_k^s$ for some $r, s \geq 0$.

variety its Albanese variety (which is indeed a stable birational invariant). To conclude we just have to remark that the Albanese variety of an abelian variety is equal to itself. \square

7.3.4. Remark. — Poonen’s proof does not tell us anything about zero divisors in \mathcal{M}_k . Indeed, it relies on the use of stable birational invariants, and after inverting L no (non trivial) such invariant is left.

7.4. Non rationality results. — Larsen and Lunts proved [21] the following non rationality Theorem:

7.4.1. Theorem (Larsen-Lunts). — *Let X be a smooth proper complex irreducible surface with geometric genus $p_g(X) \geq 2$. Then there exists a morphism $\alpha : K_0(\text{Var}_k) \rightarrow F$, with F a field, such that the zeta function $Z_{\text{mot},\alpha}$ attached to X is not rational.*

Some ideas from the proof. — Larsen and Lunts consider, for X a smooth proper complex irreducible variety of dimension d , the polynomial $\Psi_h(X) := \sum_{1 \leq i \leq d} h^{i,0} t^i$. Remark $p_g(X)$ is the leading coefficient of $\Psi_h(X)$. It is well known Ψ_h is a stable birational invariant, hence by Corollary 7.2.2 it gives rise to a ring morphism

$$\Phi_h : K_0(\text{Var}_k) \longrightarrow \mathbf{Z}[C],$$

with C the multiplicative monoid of polynomials in $\mathbf{Z}[t]$ with positive leading coefficient. Larsen and Lunts show that $\mathbf{Z}[C]$ is a domain and take for α the composition of Φ_h with the localization morphism from $\mathbf{Z}[C]$ to its fraction field F . A key ingredient in the proof is the fact that, for smooth projective surfaces with geometrical genus $r \geq 2$, $p_g(X^{(n)}) = \binom{r+n-1}{r-1}$. Here $p_g(X^{(n)})$ is by definition the geometric genus of any smooth proper variety birational to $X^{(n)}$. The Hilbert scheme $X^{[n]}$ parametrizing closed zero-dimensional subschemes of length n of X is such a variety and it follows from results of Göttsche and Soergel [15] that $p_g(X^{[n]}) = \binom{r+n-1}{r-1}$. Some more work is needed in order to deduce non rationality for the series $Z_{\text{mot},\alpha}$. \square

7.4.2. Remark. — Since the first version of the present paper was written, Larsen and Lunts wrote a very interesting sequel [22] of [21].

7.5. Rationality conjectures. — In view of Theorem 7.4.1, one cannot hope for the series Z_{mot} to be in general rational in $K_0(\text{Var}_k)[[T]]$.

Nevertheless, one can still believe in

7.5.1. Rationality conjecture (strong form). — *Let X be a variety over a field k . Then the series Z_{mot} attached to X is rational in $\mathcal{M}_k[[T]]$.*

7.5.2. Rationality conjecture (weak form). — *Let X be a variety over a field k . Then, for every morphism $\alpha : \mathcal{M}_k \rightarrow F$, with F a field, the series $Z_{\text{mot},\alpha}$ attached to X is rational in $F[[T]]$.*

7.5.3. Remarks. —

- 1) A posteriori it is not so surprising that we have to invert \mathbf{L} in order for rationality to conjecturally hold. Indeed, the guess that the motivic series should be rational comes from analogy with Dwork's Theorem 2.1.1. But counting rational points is certainly not a birational invariant!
- 2) When X is smooth and proper, one can conjecture strong and weak forms of functional equations.

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