5ECM Amsterdam 2008

Geometry and non-archimedean integrals

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Abstract. Non-archimedean integrals are ubiquitous in various parts of mathematics. Motivic integration allows to understand them geometrically and to get strong uniformity statements. In these notes, intended for a general audience, we start by giving various examples of situations where one can get new geometric results by using p-adic or motivic integrals. We then present some more recent results in this area, in particular a Transfer Principle allowing to transfer identities involving functions defined by integrals from one class of local fields to another. Orbital integrals occurring in the Fundamental Lemma of Langlands Theory form a natural family of functions falling within the range of application of this Transfer Principle.

Mathematics Subject Classification (2010). Primary 11G25, 14B05, 14E18, 32S25, 32S45; Secondary 03C10, 03C65, 03C98, 14E16.

Keywords. Motivic integration, arcs, p-adic integration, birational geometry, Milnor fiber.

1. Introduction

This paper intends to provide a leisurely introduction to recent work related to the use of non-archimedean integrals in geometry. Our aim is to convey some flavour of the topic to a general audience, without entering too much into technicalities. Interested readers will find more detailed accounts in the recent surveys [46], [8], [37] and [24].

2. Using p-adic integrals and Denef's rationality theorem

2.1. An example: counting subgroups of finite index in nilpotent groups. As a motivating example, we shall start with an example of application of p-adic integration coming from group theory. Let G be a group and let $a_n(G)$ be the number of its subgroups of index n, which we assume to be finite. This is the case when G is a finitely generated group. To study the asymptotic behaviour of $a_n(G)$, it is natural to introduce the generating function

$$\zeta_G(s) = \sum_{n=0}^{\infty} a_n(G) n^{-s}.$$

When G is nilpotent the function ζ may be expressed as an Euler product

$$\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{p,G}(s)$$

of local factors

$$\zeta_{G,p}(s) = \sum_{i=0}^{\infty} a_{p^i}(G) p^{-is}.$$

For instance, if G is the subgroup of $\mathrm{GL}_3(\mathbb{Z})$ of matrices having zero entries under the diagonal and ones on the diagonal, one has

$$\zeta_{G,p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-3)\zeta_p(3s-3)^{-1},$$

with $\zeta_p(s) = (1-p^{-s})^{-1}$, so $\zeta_{G,p}(s)$ is rational in p^{-s}. This is a very special case of a general result proved by Grunewald, Segal and Smith in 1988:

2.1.1 Theorem (Grunewald–Segal–Smith [27], 1988). *If G is a finitely generated torsion free nilpotent group, then $\zeta_{G,p}(s)$ is a rational series in p^{-s}.*

How to prove such a result? The main idea is to express $\zeta_{G,p}(s)$ as a p-adic integral, and then to use a general result of Jan Denef on the rationality of such integrals we shall explain now.

2.2. p-adic integrals. Given a prime p, let us recall that the field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the non-archimedean norm $|x|_p := p^{-v_p(x)}$, with v_p the p-adic valuation. The ring $\mathbb{Z}_p$ of p-adic integers is the subring of $\mathbb{Q}_p$ consisting of elements x with $|x|_p \leq 1$. Elements of $\mathbb{Z}_p$ can be written as infinite series $\sum_{i \geq 0} a_i p^i$, with a_i in $\{0, \ldots, p-1\}$. They are added and multiplied by rounding up to the right. Similarly, elements of $\mathbb{Q}_p$ can be written as infinite series $\sum_{i \geq -\alpha} a_i p^i$, with a_i in $\{0, \ldots, p-1\}$ and $\alpha \geq 0$.

The field $\mathbb{Q}_p$ endowed with the norm $|\ |_p$ being locally compact, $\mathbb{Q}_p^n$ admits a canonical Haar measure μ_p, normalized by $\mu_p(\mathbb{Z}_p^n) = 1$.

In many cases, the p-adic volume of a subset $X \subset \mathbb{Z}_p^n$ may be computed as

$$\mu_p(X) = \lim_{r \to \infty} (\operatorname{card} X_r) p^{-(r+1)n}$$

with X_r the image of X in $(\mathbb{Z}/p^{r+1}\mathbb{Z})^n$ (a finite set).

Let k be a field. Let us denote by C_n, the smallest collection of subsets of k^n, $n \in \mathbb{N}$, such that:

(1) the zero locus of a polynomial $f \in k[x_1, \dots, x_n]$ is an element of C_n;

(2) C_n is stable by boolean operations (complement, union, intersection);

(3) if π denotes the linear projection $k^{n+1} \to k^n$ on the first n factors and A is in C_{n+1}, then $\pi(A)$ is in C_n.

Elements of C_n are called semi-algebraic subsets of k^n and a function $g\colon k^n \to k^m$ is said to be semi-algebraic if its graph is.

When k is the field of real numbers $\mathbb{R}$ one recovers the standard definitions of semi-algebraic sets and functions.

Now we can state the general result of Denef that Grunewald, Segal and Smith use in the proof of their theorem.

2.2.1 Theorem (Denef [12], 1984). *Let V be a bounded semi-algebraic subset of $\mathbb{Q}_p^n$ and let $g\colon \mathbb{Q}_p^n \to \mathbb{Q}_p$ be a semi-algebraic function bounded on V. Then the integral*

$$\int_V |g(x)|^s |dx|$$

is a rational function of p^{-s}.

To prove his theorem, Denef needs to have a firmer grasp on p-adic semi-algebraic sets than the one given by the above definition. Let us recall that over the reals, a classical result of A. Tarski (quantifier elimination) states that a subset of $\mathbb{R}^n$ is semi-algebraic if and only if it is a finite boolean combination of subsets of the form $f(x) \geq 0$ with $f \in \mathbb{R}[x_1, \dots, x_n]$. In 1976, A. Macintyre proved the following analogue of Tarski's theorem (note that over the reals the condition $f(x) \geq 0$ may be restated as $f(x)$ being a square):

2.2.2 Theorem (Macintyre [39], 1976). *A subset of $\mathbb{Q}_p^n$ is semi-algebraic if and only if it is a finite boolean combination of subsets of the form "$f(x)$ is a d-th power", for some integer d and some $f \in \mathbb{Q}_p[x_1, \dots, x_n]$.*

Recently, E. Hrushovski and B. Martin [32] proved rationality results for zeta functions counting isomorphism classes of irreducible representations of finitely generated nilpotent groups. They use a good description of quotients of p-adic semi-algebraic subsets by semi-algebraic equivalence relations which is provided by recent work of Haskell, Hrushovski and Macpherson [28], who proved elimination of imaginaries for algebraically closed valued fields.

3. Additive invariants

3.1. Algebraic varieties. Let k be a field and let F be a family of polynomials $f_1, \ldots, f_r \in k[T_1, \ldots, T_N]$. The set of k-points of the corresponding (affine) algebraic variety X_F is the set of points in k^N which are common zeroes of the polynomials f_i, that is,

$$X_F(k) = \{(x_1, \ldots, x_n) \in k^N : f_i(x_1, \ldots, x_N) = 0 \text{ for all } i\}.$$

For any ring K containing k, we can also consider the set of K-points

$$X_F(K) = \{(x_1, \ldots, x_n) \in K^N : f_i(x_1, \ldots, x_N) = 0 \text{ for all } i\}.$$

In particular, if $r = 0$, we get the affine space $\mathbb{A}^N$ with $\mathbb{A}^N(K) = K^N$ for every K containing k. If $F' = F \bigcup_{i \in I} \{g_i\}$, we have

$$X_{F'}(K) \subset X_F(K)$$

for all K. We write $X_{F'} \subset X_F$ and we say $X_{F'}$ is a (closed) subvariety of X_F.

General algebraic varieties are defined by gluing affine varieties and the notion of (closed) subvariety can be extended to that setting. If X' is a subvariety of X, there is a variety $X \setminus X'$ such that, for every K, $(X \setminus X')(K) = X(K) \setminus X'(K)$. There is also a natural notion of products and a natural notion of morphisms between algebraic varieties. Basically, morphisms are induced by "polynomial transformations". In particular, there is a notion of isomorphism of algebraic varieties. For instance, $T \mapsto (T^2, T^3, T^{-2})$ induces an isomorphism between $\mathbb{A}^1 \setminus \{0\}$ and the variety defined by

$$X_1^3 - X_2^2 = 0 \quad \text{and} \quad X_1 X_3 - 1 = 0.$$

3.2. Universal additive invariants. Let $K_0(\mathrm{Var}_k)$ denote the free abelian group on isomorphism classes $[S]$ of objects of Var_k mod out by the subgroup generated by the relations of the form

$$[S] = [S'] + [S \setminus S']$$

for S' a (closed) subvariety of S. Setting

$$[S] \cdot [S'] = [S \times S']$$

endows $K_0(\mathrm{Var}_k)$ with a natural ring structure. Denote by $\mathbb{L}$ the class of the affine line $\mathbb{A}_k^1$ in $K_0(\mathrm{Var}_k)$, and set

$$M_k := K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}],$$

that is, M_k is the ring obtained by inverting $\mathbb{L}$ in $K_0(\mathrm{Var}_k)$.

One may view the mapping $X \mapsto [X]$ assigning to an algebraic variety X over k its class in M_k as the universal additive and multiplicative invariant (not vanishing on $\mathbb{A}_k^1$) on the category of algebraic varieties.

3.3. Euler characteristics versus counting. Amongst all additive invariants, the most fundamental ones may well be given by the Euler characteristic with compact supports and by counting points over finite fields.

If k is a subfield of $\mathbb{C}$ and X is a k-algebraic variety, one sets $\mathrm{Eu}(X) := \mathrm{Eu}(X(\mathbb{C}))$, where Eu is the Euler characteristic with compact supports. This an additive invariant that factors through a morphism $\mathrm{Eu}\colon \mathcal{M}_k \to \mathbb{Z}$.

Also counting points over finite fields is additive. Recall that for every prime number p, and every $f \geq 1$, there exists a unique finite field $\mathbb{F}_q$ having $q = p^f$ elements. Furthermore, for every $e \geq 1$, $\mathbb{F}_{q^e}$ is the unique field extension of degree e of $\mathbb{F}_q$. If $k = \mathbb{F}_q$ and X is a k-algebraic variety, since $X(\mathbb{F}_{q^e})$ is finite, we may set

$$N_{q^e}(X) := |X(\mathbb{F}_{q^e})|.$$

Clearly, $X \mapsto N_{q^e}(X)$ is an additive invariant and factors through a morphism $N_{q^e}\colon \mathcal{M}_k \to \mathbb{Z}[p^{-1}]$.

When $k = \mathbb{Q}$, and X is a variety over k, we may at the same time view $\mathbb{Q}$ as a subfield of $\mathbb{C}$ and consider $\mathrm{Eu}(X)$, and reduce the equations of X mod p, for p not dividing the denominators of the equations of f, in order to get a variety X_p over $\mathbb{F}_p$. For such a p, we may consider, via counting, the number $N_{p^e}(X_p)$, for any $e \geq 1$.

It is a very striking fact, that these two invariants – apparently of a very different nature – are related. Indeed, it follows from important results by A. Grothendieck going back to the 60s that given an X, for almost all p,

$$\lim_{e \to 0} N_{p^e}(X_p) = \mathrm{Eu}(X),$$

so, Euler characteristics may be computed by counting in finite fields! Of course there is literally no meaning to taking the limit as $e \to 0$ of $N_{p^e}(X_p)$. Here is the technically correct statement:

3.3.1 Theorem (Grothendieck). *Given an X, for almost all p, there exists finite families of complex numbers α_i, $i \in I$, and β_j, $j \in J$, depending only on X and p, such that*

$$N_{p^e}(X_p) = \sum_I \alpha_i^e - \sum_J \beta_j^e$$

and

$$\mathrm{Eu}(X) = |I| - |J|.$$

It is difficult to give precise references for this result since it is merely a potpourri of various results scattered in the literature. The main ingredients are the rationality of zeta functions of varieties over finite fields due to B. Dwork [21], the cohomological interpretation of these zeta functions and general comparison results for étale cohomology both due to A. Grothendieck, for which we refer to [23] and [40].

4. Using p-adic integrals in birational geometry

In this section we outline some applications of p-adic integration to birational geometry.

4.1. Birational geometry. We assume $k = \mathbb{C}$. Let X and Y be two smooth and connected complex algebraic varieties (not necessarily affine). A morphism $h\colon Y \to X$ is called a modification or a birational morphism if h is proper (i.e., h^{-1}(compact) $=$ compact) and h is an isomorphism outside a subvariety F of Y, $F \neq Y$. If, moreover, F is a union of smooth connected hypersurfaces E_i, $i \in A$, of Y, which we also assume to be mutually transverse, we say h is a DNC modification (here DNC stands for "divisor with normal crossings"). To a DNC modification $h\colon Y \to X$ we assign the following combinatorics:

For $I \subset A$, we set

$$E_I^\circ := \bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j.$$

Note that $E_\emptyset^\circ = Y \setminus F$ and Y is the disjoint union of all the E_I°'s.

For i in A, we set

$$n_i = 1 + (\text{order of vanishing of the jacobian of } h \text{ along } E_i)$$

and, for $I \subset A$, we set

$$n_I = \prod_{i \in I} n_i.$$

4.2. Euler characteristics. We can now state the following resulting, obtained in 1987 and published in 1992:

4.2.1 Theorem (Denef–Loeser [13], 1992). *For any* DNC *modification* $h\colon Y \to X$ *the relation*

$$\mathrm{Eu}(X) = \sum_{I \subset A} \frac{\mathrm{Eu}(E_I^\circ)}{n_I}$$

holds.

The proof was by no means direct. The main steps were:

(1) To reduce to data defined over a ring of finite type over $\mathbb{Z}$. For simplicity of exposition, we shall assume everything is defined over a localization of $\mathbb{Z}$.

(2) For general p, to evaluate the p-adic volume of $X(\mathbb{Q}_p)$ as a p-adic integral on $Y(\mathbb{Q}_p)$ involving the order of jacobian of h via "change of variables formula" for p-adic integrals.

(3) To express these integrals as number of points on varieties over a finite field.

(4) To conclude by using Grothendieck's result relating Eu to number of points.

Nowadays, two other proofs are available: one by motivic integration that we shall outline in 5.2, another one by direct application of the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [1]. It is still a challenging problem to find a direct, geometric, proof.

4.3. Betti numbers of birational Calabi–Yau varieties. Inspired by mirror symmetry, physicists were led to conjecture the following statement: "Birational Calabi–Yau have the same Betti numbers". This was proved by V. Batyrev in [3] by using p-adic integrals in a way similar to the one just explained, with as additional ingredient the part of the Weil conjectures proved by Deligne, which allows for smooth projective varieties to recover not only Euler characteristics, but also Betti numbers, from counting in finite field. Shortly afterwards, M. Kontsevich found a direct approach to Batyrev's Theorem, avoiding the use of p-adic integrals and involving arc spaces. This was explained in his famous Orsay talk of December 7, 1995, entitled "String cohomology", which marked the official birth of motivic integration.

5. Motivic integration

5.1. The original construction. Motivic integration is a geometric analogue of p-adic integration with $\mathbb{Q}_p$ replaced by $k((t))$. Here k is a field (say of characteristic zero) and $k((t))$ denotes the field of formal Laurent series with coefficients in k. The most naive idea is to try to construct a real valued measure on a large class of subsets of $k((t))^n$ similarly as in the p-adic case. Such an attempt is doomed to fail immediately since, as soon as k is infinite, $k((t))$ is not locally compact.

When Kontsevich invented Motivic Integration in 1995, a real breakthrough was to realize that a sensible measure on subsets of $k((t))^n$ could in fact be constructed once the value group of the measure $\mathbb{R}$ is replaced by the ring M_k (or some of its completions) constructed in terms of geometric objects defined over k.

Let X be a variety over the field k. The arc space $\mathcal{L}(X)$ is defined by

$$\mathcal{L}(X)(K) := X(K[[t]])$$

for any field K containing k. If X is affine and defined by the vanishing of a family of polynomials f_i in the variables $x_1, \ldots, x_N$, one gets equations for $\mathcal{L}(X)$ by writing $x_j = \sum_{\ell \geq 0} a_{j,\ell} t^\ell$, developing $f_i(x_1, \ldots, x_N)$ into $\sum F_{i,\ell} t^\ell$, with $F_{i,\ell}$ polynomials in the variables $a_{j,\ell}$, and asking all the polynomials $F_{i,\ell}$ to be zero.

Note that, in general, $\mathcal{L}(X)$ is an infinite-dimensional variety over k since it involves an infinite number of variables. On the other hand, for $n \geq 0$, the space $\mathcal{L}_n(X)$ defined similarly as $\mathcal{L}(X)$ with $K[[t]]$ replaced by $K[[t]]/t^{n+1}$ is of finite type over k. The original construction, outlined by Kontsevich in 1995, and developed by Denef–Loeser [15] and Batyrev [4] uses a limiting process similar to the one we saw in the p-adic case:

The basic idea is to use truncation morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X).$$

For reasonable subsets A of $\mathcal{L}(X)$,

$$\mu(A) := \lim_{n\to\infty} [\pi_n(A)]\, \mathbb{L}^{-(n+1)d}$$

with d the dimension of X, in some completion $\widehat{M}_k$ of M_k, in complete analogy with the p-adic case.

5.2. First application: birational geometry. As we already mentioned, the very first application of motivic integration was made by Kontsevich, who used it to get a proof of Batyrev's Theorem mentioned in 4.3 without p-adic integration. Similarly, one can avoid the use of p-adic integration in the proof of the Denef–Loeser Theorem 4.2.1.

Let us explain the underlying idea. If $h : Y \to X$ is a birational morphism, one can express the motivic volume of $\mathcal{L}(X)$ as a motivic integral on $\mathcal{L}(Y)$ involving the order of vanishing of the jacobian. This is achieved by using an analogue of the "change of variables formula" in this setting. This may work for the following reason: a modification $h : Y \to X$ induces an isomorphism outside a subset $F \subset Y$ of finite positive codimension (usually one), but at the level of arc spaces h induces a morphism between $\mathcal{L}(Y)$ and $\mathcal{L}(X)$ which restricts to a bijection between $\mathcal{L}(Y) \setminus \mathcal{L}(F)$ and $\mathcal{L}(X) \setminus \mathcal{L}(h(F))$, that is, between arcs in Y not completely contained in F and arcs in X not completely contained in $h(F)$. The key fact, making measure theoretic tools so well adapted to birational geometry, is that $\mathcal{L}(F)$ is of infinite codimension in $\mathcal{L}(Y)$, hence $\mathcal{L}(F)$ and $\mathcal{L}(h(F))$ have measure zero in $\mathcal{L}(Y)$ and $\mathcal{L}(X)$, respectively.

5.3. Second application: finite group actions. Let G be a finite group. A linear action of G on a complex vector space V has a canonical decomposition $\bigoplus V_\alpha$ parametrized by characters. If G acts on a complex algebraic variety X, there is of course no decomposition as above. But there exists one at the level of arc spaces!

Indeed, let x be a point of X and denote by $G(x)$ the isotropy subgroup at x, consisting of those elements of G fixing x. Denote by $\mathcal{L}(X)_x$ the space of arcs on X with origin at x. It was proved in [19] that there is a canonical decomposition

$$\mathcal{L}(X)_x = \bigsqcup_{\gamma \in \operatorname{Conj} G(x)} \mathcal{L}(X)_x^\gamma \sqcup B,$$

with Conj $G(x)$ the set of conjugacy classes in $G(x)$ and B a subset of infinite codimension in $\mathcal{L}(X)_x$ (hence of motivic measure zero).

This explains the use of motivic integration in relation with the McKay correspondence to get results relating certain resolutions of the quotient space X/G with group theoretical invariants of the action, cf. [5], [19], [48], [49].

5.4. Third application: motivic Milnor fiber. Let X be a smooth complex algebraic variety and $f : X \to \mathbb{C}$ a function (a morphism to the affine line). Let x be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$. Fix $0 < \eta \ll \varepsilon \ll 1$. The morphism f restricts to a fibration – called the Milnor fibration –

$$B(x, \varepsilon) \cap f^{-1}(B(0, \eta) \setminus \{0\}) \longrightarrow B(0, \eta) \setminus \{0\}.$$

Here $B(a, r)$ denotes the closed ball of center a and radius r. The Milnor fiber at x,

$$F_x = f^{-1}(\eta) \cap B(x, \varepsilon),$$

has a diffeomorphism type that does not depend on η and ε, and it is endowed with an automorphism, the monodromy M_x, induced by the characteristic mapping of the fibration.

In particular, the trace of the action of the n-th iterate of the monodromy M_x on the cohomology of the Milnor fiber F_x,

$$\Lambda^n(M_x) := \sum_j (-1)^j \ \mathrm{tr}\,(M_x^n; H^j(F_x)),$$

is an invariant of the singularity. Quite surprisingly, this invariant may be expressed in purely algebraic terms using arcs. Consider the set $\mathcal{X}_n$ consisting of (truncated) arcs $\varphi(t)$ in $\mathcal{L}_n(X)$ such that $\varphi(0) = x$ and $f(\varphi(t)) = t^n +$ (higher order terms).

5.4.1 Theorem (Denef–Loeser [18], 2002). *For $n \geq 1$, we have*

$$\Lambda^n(M_x) = \mathrm{Eu}(\mathcal{X}_n).$$

The proof of this result is not very enlightening: one computes both sides of the equality on a resolution of singularities of $f = 0$ using the change of variables formula and checks that they are equal. Finding a direct, fully geometric proof, not using resolution of singularities, still represents a quite challenging problem.

Nicaise and Sebag ([44], [43]) have shown that this result may be naturally reformulated and generalized within the framework of rigid analytic geometry as a trace formula connecting the Euler characteristic of a motivic Serre invariant (cf. [38]) with the trace of the monodromy on the analytic Milnor fiber.

In fact, the spaces $\mathcal{X}_n$ do contain much more information about the Milnor fiber and the monodromy. Denef and Loeser (cf. [20]) proved that the series

$$Z(T) := \sum_{n\geq 1} [\mathcal{X}_n]\mathbb{L}^{-dn}T^n,$$

with d the dimension of X, is rational in T and has a limit $-\mathcal{S}_f$ as $T \to \infty$ in $\mathcal{M}_k$. In fact, by considering an equivariant version of $Z(T)$, one can define $\mathcal{S}_f$ as an element of an equivariant version of $\mathcal{M}_k$, cf. [20]. It is called the motivic Milnor fiber of f at x, and can be viewed as a motivic incarnation of the Milnor fiber together with the (semi-simplification of the) monodromy action on it. We refer to papers by Denef–Loeser [14], Bittner [6] and Guibert–Loeser–Merle [25] and [26] for more on this topic. Let us mention that the motivic Milnor fiber is used in an essential way by Kontsevich and Soibelman in their recent work [35] on motivic Donaldson–Thomas invariants and cluster transformations.

6. Motivic measure for definable sets and uniformity of p-adic integrals

6.1. Definable sets. A first order formula in the language of rings is a formula written with symbols $0, +, -, 1, \times, =$, logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (negation), quantifiers $\exists$, $\forall$, and variables. If k is a ring, we may extend the language by adding constants for every element of k and consider formulas in this extended language, which we call ring formulas over k. Now consider a ring formula $\varphi(x_1, \ldots, x_n)$ over k with all its free variables belonging to $\{x_1, \ldots, x_n\}$. If K is a field containing an homomorphic image of k, we may consider the set

$$X_\varphi(K) := \big\{(x_1, \ldots, x_n) \in K^n \mid \varphi(x_1, \ldots, x_n) \text{ holds}\big\}.$$

Objects of the form X_φ are called definable sets over k.

More generally one can consider natural extensions of the ring language to valued ring languages admitting symbols to express that the valuation is larger than something, or that the initial coefficient of a series is equal to something. This leads also to a notion of definable sets in the corresponding language.

6.2. General motivic measure. With Raf Cluckers we constructed in [7] a general theory of motivic integration based on cell decomposition. In our theory, motivic integrals take place in a ring N_k which is obtained from the Grothendieck ring $K_0(\mathrm{Def}_k)$ of definable sets over k (in the ring language) by inverting $\mathbb{L}$ and $1 - \mathbb{L}^i$ for $i \neq 0$ (here again $\mathbb{L}$ stands for the class of the affine line). There is a natural morphism $N_k \to \widehat{\mathcal{M}}_k$.

Our construction assigns to a bounded definable subset A of $k((t))^n$ in the valued field language a motivic volume $\mu(A)$ in N_k compatible with the construction in 5.1.

It relies on a cell decomposition theorem due to Denef and Pas [45] (such cell decomposition results trace back to the work of P. Cohen [11]). The result by Denef and Pas tells us that one can cut a definable subset A of $k((t))^{n+1} = k((t))^n \times k((t))$ into 0-dimensional cells (graphs of functions defined on a definable subset B of $k((t))^n$) and 1-dimensional cells (relative balls over B), maybe after adding some auxiliary parameters over the residue field and the value group. This allows us to define the measure by induction on the valued field dimension. One of the main difficulties is to prove that the measure is well defined, that is, independent of the cell decomposition. In particular, one has to prove the non obvious fact that it is independent of the ordering of coordinates in the ambient affine space, which is a form of a Fubini theorem.

6.3. Constructible motivic functions. In fact, from the start our construction is relative: we define a natural class of constructible motivic functions and we prove stability of that class with respect to integration depending on parameters. Once the new framework is developed, integration of constructible motivic functions behaves very similar to the more classical Lebesgue integration with Fubini theorems, change of variable theorems, distributions, etc. Also one can extend the construction to allow motivic analogues of exponential functions and Fourier inversion [9].

6.4. Ax–Kochen–Eršov. Though they are basically different, for instance they have different characteristics, $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ look asymptotically when $p \gg 0$ very much the same:

6.4.1 Theorem (Ax–Kochen–Eršov [2], [22]). *Let φ be a first order sentence (that is, a formula with no free variables) in the language of rings. For almost all prime numbers p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.*

For instance, for every $d > 0$, there is a sentence S_d in the language of rings expressing that any homogeneous polynomial of degree $d^2 + 1$ with coefficients in a field k has a non trivial zero in that field. Since, by work of Tsen and Lang, S_d holds in $\mathbb{F}_p((t))$, it follows from Theorem 6.4.1 that S_d holds in $\mathbb{Q}_p$ for p large enough.

6.5. Generalization to definable sets. How can one extend the Ax–Kochen–Eršov Theorem to formulas with free variables?

6.5.1 Theorem (Denef–Loeser [17]). *Let φ be a formula in the valued ring language. Then, for almost all p, the sets $X_\varphi(\mathbb{Q}_p)$ and $X_\varphi(\mathbb{F}_p((t)))$ have the same volume. Furthermore this volume is equal to the number of points in $\mathbb{F}_p$ of a motive M_φ canonically attached to φ.*

When φ has no free variables, one recovers the original form of the Ax–Kochen–Eršov Theorem. There is a similar statement for integrals. This shows that p-adic integrals have a strongly uniform pattern as p varies: they are fully controlled by a single geometric object. On the other hand, it is a priori unclear what an Ax–Kochen–Eršov Theorem for integrals depending on parameters could be, since there seems to be no way to compare functions defined over different spaces. Before going more into that direction, let us look at an example.

6.6. An example. Let E/F be a non ramified degree two extension of non-archimedean local fields of residue characteristic different from 2. Let ψ be an additive character of F which is non trivial on $\mathcal{O}_F$ but trivial on the maximal ideal $\mathfrak{M}_F$. Let N_n be the group of upper triangular matrices with 1's on the diagonal and consider the character $\theta\colon N_n(F) \to \mathbb{C}^\times$ given by

$$\theta(u) := \psi\Big(\sum_i u_{i,i+1}\Big).$$

For a the diagonal matrix $(a_1, \dots, a_n)$ with a_i in $F^\times$, Jacquet and Ye considered the following complicated integral $I(a)$ defined in terms of F:

$$I(a) := \int_{N_n(F)\times N_n(F)} \mathbf{1}_{M_n(\mathcal{O}_F)}({}^t u_1 a u_2)\, \theta(u_1 u_2)\, du_1 du_2,$$

with the normalisation $\int_{N_n(\mathcal{O}_F)} du = 1$. They also considered a similar integral $J(a)$ defined in terms of E by replacing $N_n(F) \times N_n(F)$ by $N_n(E)$ and involving the non trivial element of the Galois group $x \mapsto \bar{x}$:

$$J(a) := \int_{N_n(E)} \mathbf{1}_{M_n(\mathcal{O}_E)\cap H_n}({}^t \bar{u} a u)\, \theta(u\bar{u})\, du,$$

with H_n the set of Hermitian matrices.

The Jacquet–Ye Conjecture asserts that

$$I(a) = \gamma(a)\, J(a) \tag{6.6.1}$$

with

$$\gamma(a) := \prod_{1\le i\le n-1} \eta(a_1 \dots a_i),$$

and η the multiplicative character of order 2 on $F^\times$.

When $n = 2$, the Jacquet–Ye Conjecture essentially reduces to classical Gauss sum identities, but already for $n = 3$ a proof by direct computation is quite hard. The full Jacquet–Ye Conjecture over finite field extensions of $\mathbb{F}_q((t))$ has been proved by Ngô in 1999 [41] and over any non-archimedean local field by Jacquet in 2004

[33]. Ngô's proof goes by reduction to a purely geometrical statement over algebraic varieties over $\mathbb{F}_q$ (which is not possible in the p-adic case), which he can prove by fully using the powerful machinery of ℓ-adic perverse sheaves over such varieties. This is a typical instance of the general principle "complicated identities between character sums over finite fields are better proved by geometrical tools".

Hence it is natural to ask if assuming we only know (6.6.1) holds over finite field extensions of $\mathbb{F}_q((t))$ whether it is possible to deduce it from general principles for p-adic fields. Note that it makes no sense to compare the values of the integrals themselves, since a does not run over the same space in the characteristic 0 and p cases. The answer is yes as we shall see now.

6.7. The transfer principle. The uniformity result given by Theorem 6.5.1 may be extended in the following way:

6.7.1 Theorem (Cluckers–Loeser [9]). *All p-adic integrals depending on parameters that are definable in a precise sense may be obtained by specialization of canonical motivic integrals of constructible functions for almost all p, and similarly for $\mathbb{Q}_p$ replaced by $\mathbb{F}_p((t))$.*

6.7.2 Transfer Principle (Cluckers–Loeser [9]). *A given equality between definable integrals depending on parameters holds for $\mathbb{Q}_p$ if and only if it holds for $\mathbb{F}_p((t))$, when $p \gg 0$.*

With Cluckers and Hales [10] we have recently proved that the range of application of the transfer principle contains in particular the so called Fundamental Lemma of Langlands theory. One can also check that it applies to the Jacquet–Ye Conjecture. Recall that the Fundamental Lemma was proved recently by Laumon and Ngô [36] in the unitary case and by Ngô in the general case over finite extensions of $\mathbb{F}_p((t))$ [41] by geometrical methods. Using specific techniques, Waldspurger [47] had already previously proved that one can then deduce it for p-adic fields. It is natural to expect that relations between non-archimedean integrals holding over all local fields of large residual characteristic already hold at the motivic level, as equalities between constructible motivic functions, but this seems to be presently out of reach.

6.8. Recent developments. Let us close this brief survey by mentioning some other recent applications of Model Theory to Geometry over valued fields. Hrushovski and Kazhdan [30] developed a geometric integration theory for general complete valued fields (with residue characteristic zero) based on Robinson's quantifier elimination for algebraically closed valued fields.

On the other hand Haskell, Hrushovski and Macpherson [29] recently introduced the notion of stably dominated type and studied it in great detail for algebraically closed valued fields. Such more advanced model theoretic tools seem to have very promising applications to the study of the geometry of Berkovich spaces, cf. [31].

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