The geometry behind non-archimedean integrals

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Counting subgroups of finite index

Let G be a group and let $a_n(G)$ be the number of its subgroups of index n, which we assume to be finite (this is the case for finitely generated groups).

One considers the generating function

$$\xi_G(s) = \sum_{n=0}^{\infty} a_n(G) n^{-s}.$$

When G is nilpotent,

$$\xi_G(s) = \prod_{p \,\text{prime}} \xi_{p,G}(s),$$

with

$$\xi_{p,G}(s) = \sum_{i=0}^{\infty} a_{p^i}(G) p^{-is}.$$

THEOREM (GRUNEWALD-SEGAL-SMITH, 1988)

If G is a finitely generated group (torsion free) nilpotent group, $\xi_{p,G}(s)$ is a rational series in p^{-s}.

The main steps in the proof are

1. To express $\xi_{p,G}(s)$ as a p-adic integral.
2. To use a general result of Jan Denef on the rationality of such integrals.

Reminder: Let p be a prime. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the non-archimedean norm $|x|_p := p^{-v_p(x)}$.

The ring $\mathbb{Z}_p$ of p-adic integers is the subring of $\mathbb{Q}_p$ consisting of elements x with $|x|_p \leq 1$.

Elements of $\mathbb{Z}_p$ can be written as infinite series $\sum_{i\geq 0} a_i p^i$, with a_i in $\{0,\ldots,p-1\}$. They are added and multiplied by rounding up to the right.

Similarly, elements of $\mathbb{Q}_p$ can be written as infinite series $\sum_{i\geq -\alpha} a_i p^i$, with a_i in $\{0,\ldots,p-1\}$ and $\alpha \geq 0$.

The field $\mathbb{Q}_p$ endowed with the norm $||_p$ being locally compact, $\mathbb{Q}_p^n$ admits a canonical Haar measure μ_p, normalized by $\mu_p(\mathbb{Z}_p^n) = 1$.

In many cases, the p-adic volume of a subset $X \subset \mathbb{Z}_p^n$ may be computed as

$$\mu_p(X) = \lim_{r \mapsto \infty} (\operatorname{card} X_r)\, p^{-(r+1)n}$$

with X_r the image of X in $(\mathbb{Z}/p^{r+1}\mathbb{Z})^n$ (a finite set).

Let k be a field.

Semialgebraic subsets of k^n may be defined as the smallest collection of subsets of k^n (variable n)

1. containing the zero locus of polynomials $f \in k[x_1, \cdots, x_n]$
2. stable by boolean operations (complement, union, intersection)
3. stable by linear projection $k^{n+1} \to k^n$.

A function $g : k^n \to k^m$ is semialgebraic if its graph is.

These definitions correspond to the standard ones for $k = \mathbb{R}$ but also make sense for $k = \mathbb{Q}_p$.

Now we can state

THEOREM (DENEF, 1984)

Let V be a bounded semialgebraic subset of $\mathbb{Q}_p^n$ and let $g : \mathbb{Q}_p^n \to \mathbb{Q}_p$ be a semialgebraic function bounded on V. The integral

$$\int_V |g(x)|^s |dx|$$

is a rational function of p^{-s}.

To prove his theorem, Denef needs to have a firmer grasp on p-adic semialgebraic sets than the one given by the definition. This is provided by a theorem of Macintyre we now describe.

Over the reals, a famous result of A. Tarski (quantifier elimination) states that a subset of $\mathbb{R}^n$ is semialgebraic if and only if it is a finite boolean combination of subsets of the form $f(x) \geq 0$ with $f \in \mathbb{R}[x_1, \cdots, x_n]$.

Macintyre's theorem is the p-adic analogue of Tarski's.

Theorem (Macintyre, 1976)

A subset of $\mathbb{Q}_p^n$ is semialgebraic if and only if it is a finite boolean combination of subsets of the form "$f(x)$ is a d-th power", for some integer d and some $f \in \mathbb{Q}_p[x_1, \cdots, x_n]$.

Recently, E. Hrushovski and B. Martin proved rationality results for zeta functions obtained by counting isomorphism classes of irreducible representations of finitely generated nilpotent groups.

For such a result a good description of quotients of p-adic semialgebraic subsets by semialgebraic equivalence relations is needed. This is provided by more advanced model theoretical tools (elimination of imaginaries).

Algebraic varieties

Let k be a field. Let F be a family of polynomials $f_1, \ldots, f_r \in k[T_1, \ldots, T_N]$.

The set of k-points of the corresponding (affine) algebraic variety X_F is the set of points in k^N which are common zeroes of the polynomials f_i, that is,

$$X_F(k) = \{x = (x_1, \ldots, x_n) \in k^N : (\forall i)(f_i(x_1, \ldots, x_N) = 0)\}.$$

For any ring K containing k, we can also consider the set of K-points

$$X_F(K) = \{x = (x_1, \ldots, x_n) \in K^N : (\forall i)(f_i(x_1, \ldots, x_N) = 0)\}.$$

In particular, if $r = 0$, we get the affine space $\mathbb{A}^N$ with $\mathbb{A}^N(K) = K^N$ for every K containing k.

If $F' = F \cup_{i \in I} \{g_i\}$, we have

$$X_{F'}(K) \subset X_F(K)$$

for all K. We write $X_{F'} \subset X_F$ and we say $X_{F'}$ is a (closed) subvariety of X_F.

General algebraic varieties are defined by gluing affine varieties and the notion of (closed) subvariety can be extended to that setting.

If X' is a subvariety of X, there is a variety $X \setminus X'$ such that, for every K, $(X \setminus X')(K) = X(K) \setminus X'(K)$.
We have also a natural notion of products and natural notion of morphisms between algebraic varieties. Essentially morphisms are induced by "polynomial transformations".

In particular, there is a notion of isomorphism of algebraic varieties.

For instance, $T \mapsto (T^2, T^3, T^{-2})$ induces an isomorphism between $\mathbb{A}^1 \setminus \{0\}$ and the variety defined by

$$X_1^3 - X_2^2 = 0 \quad \text{and} \quad X_1 X_3 - 1 = 0.$$

Universal additive invariants

Let $K_0(\mathrm{Var}_k)$ denote the free abelian group on isomorphism classes $[S]$ of objects of Var_k mod out by the relations

$$[S] = [S'] + [S \setminus S']$$

for S' subvariety of S.

Setting

$$[S] \cdot [S'] = [S \times S']$$

endows $K_0(\mathrm{Var}_k)$ with a natural ring structure.

Denote by $\mathbb{L}$ the class of the affine line $\mathbb{A}^1_k$ in $K_0(\mathrm{Var}_k)$, and set

$$M_k := K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}],$$

that is M_k is the ring obtained by inverting $\mathbb{L}$ in $K_0(\mathrm{Var}_k)$.

Examples of additive invariants

One may view the mapping $X \mapsto [X]$ assigning to an algebraic variety X over k its class in M_k as the universal additive and multiplicative invariant (not vanishing on $\mathbb{A}^1_k$) on the category of algebraic varieties.

Two examples of additive invariants

1. if k is a subfield of $\mathbb{C}$ and X is a k-algebraic variety, $\mathrm{Eu}(X) := \mathrm{Eu}(X(\mathbb{C}))$, where Eu is the Euler characteristic with compact supports, is additive
2. counting points over finite fields is additive.

Recall that for every prime number p, and every $f \geq 1$, there exists a unique finite field $\mathbb{F}_q$ having $q = p^f$ elements.

Furthermore, for every $e \geq 1$, $\mathbb{F}_{q^e}$ is the unique field extension of degree e of $\mathbb{F}_q$.

If $k = \mathbb{F}_q$ and X is a k-algebraic variety, since $X(\mathbb{F}_{q^e})$ is finite, we may set

$$N_{q^e}(X) := |X(\mathbb{F}_{q^e})|.$$

$X \mapsto N_{q^e}(X)$ is an additive invariant.

When $k = \mathbb{Q}$, and X is a variety over k, we may at the same time

1. view $\mathbb{Q}$ as a subfield of $\mathbb{C}$ and consider $\mathrm{Eu}(X)$
2. reduce the equations of X mod p, for p not dividing the denominators of the equations of f, in order to get a variety X_p over $\mathbb{F}_p$. For such a p, we may consider, via counting, the number $N_{p^e}(X_p)$.

Is there any relation between $N_{p^e}(X_p)$ and $\mathrm{Eu}(X)$?

Yes!

The following is a consequence of results by A. Grothendieck going back to the 60's:

Theorem (Crude Form)

Given a X, for almost all p,

$$\lim_{e \longmapsto 0} N_{p^e}(X_p) = \mathrm{Eu}(X).$$

More precisely:

Theorem (Precise Form)

Given a X, for almost all p, there exists finite families of complex numbers α_i, $i \in I$, and β_j, $j \in J$, depending only on X and p, such that

$$N_{p^e}(X_p) = \sum_I \alpha_i^e - \sum_J \beta_j^e$$

and

$$\mathrm{Eu}(X) = |I| - |J|.$$

So, Euler characteristics may be computed by counting in finite fields!

We assume $k = \mathbb{C}$.

Let X and Y be two smooth and connected complex algebraic varieties (not necessarily affine). A morphism $h : Y \to X$ is called a modification or a birational morphism if

1. h is proper (i.e. h^{-1}(compact) = compact)
2. h is an isomorphism outside a subvariety F of Y, $F \neq Y$.

If, moreover, F is a union of smooth connected hypersurfaces (= of complex codimension 1) E_i, $i \in A$, of Y, which we also assume to be mutually transverse, we say h is a DNC modification.

To a DNC modification $h\colon Y \to X$ we assign the following combinatorics:

For $I \subset A$, we set

$$E_I^\circ := \cap_{i\in I} E_i \setminus \cup_{j\notin I} E_j.$$

Note that $E_\emptyset^\circ = Y \setminus F$ and Y is the disjoint union of all the E_I°'s.

For i in A, we set

$$n_i = 1 + (\text{order of vanishing of the jacobian of } h \text{ along } E_i)$$

and

$$n_I = \prod_{i\in I} n_i.$$

We can now state the following resulting, obtained in 1987 and published in 1992:

THEOREM (DENEF AND L.)

For any DNC *modification* $h : Y \to X$ *the relation*

$$\mathrm{Eu}(X) = \sum_{I \subset A} \frac{\mathrm{Eu}(E_I^\circ)}{n_I}$$

holds.

The proof was by no means direct. Main steps:

1. Reduce to data defined over $\mathbb{Q}$ (or a number field)
2. For general p, evaluate the p-adic volume of $X(\mathbb{Q}_p)$ as a p-adic integral on $Y(\mathbb{Q}_p)$ involving the order of jacobian of h via "change of variables formula"
3. Express these integrals as number of points on varieties over a finite field
4. Conclude by using Grothendieck's result relating Eu to number of points.

Challenging problem: Find a direct proof …

[Nowadays two other proofs: one by motivic integration, another by direct application of the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk.]

Inspired by mirror symmetry, physicists were led to conjecture the following statement:

“Birational Calabi-Yau have the same Betti numbers.”

In 1995, this was proved by V. Batyrev by using p-adic integrals in a way similar to the one just explained and the part of the Weil conjectures proved by Deligne (which allows for projective varieties to deduce not only Euler characteristics, but also Betti numbers, from counting in finite fields).

Shortly afterwards, M. Kontsevich found a direct approach to Batyrev’s Theorem, avoiding the use of p-adic integrals and involving arc spaces, which he explained in his seminal Orsay talk of December 7, 1995, entitled “String cohomology”.

Motivic integration was born …

The original construction

Motivic integration is a geometric analogue of p-adic integration with $\mathbb{Q}_p$ replaced by $k((t))$.

Here k is a field (say of characteristic zero) and $k((t))$ denotes the field of Laurent formal series with coefficients in k.

The most naive idea is to try to construct a real valued measure on a large class of subsets of $k((t))^n$ similarly as in the p-adic case.

Such an attempt is doomed to fail immediately since, as soon as k is infinite, $k((t))$ is not locally compact.

When Kontsevich invented Motivic Integration in 1995, his real breakthrough was to realize that a sensible measure on subsets of $k((t))^n$ could in fact be constructed once

the value group of the measure $\mathbb{R}$ is replaced by the ring M_k (or its completion) constructed in terms of geometric objects defined over k.

Let $X = X_F$ be a variety over the field k.

The arc space $\mathscr{L}(X)$ is defined by

$$\mathscr{L}(X)(K) := X(K[[t]])$$

for any field K containing k.

If X is defined by the vanishing of a family of polynomials f_i in the variables $x_1, \dots, x_N$, one gets equations for $\mathscr{L}(X)$ by writing $x_j = \sum_{\ell \geq 0} a_{j,\ell} t^\ell$, developing $f_i(x_1, \dots, x_N)$ into $\sum F_{i,\ell} t^\ell$, and asking all the polynomials $F_{i,\ell}$ in the variables $a_{j,\ell}$ to be zero.

Note that, in general $\mathscr{L}(X)$ is an infinite dimensional variety over k since it involves an infinite number of variables.

On the other hand, for $n \geq 0$, the space $\mathscr{L}_n(X)$ defined similarly as $\mathscr{L}(X)$ with $K[[t]]$ replaced by $K[[t]]/t^{n+1}$ is of finite type over k.

The original construction, outlined by Kontsevich in 1995, and developed by Denef-L. and Batyrev uses a limiting process similar to the one we saw in the p-adic case:

The basic idea is to use the truncation morphisms:

$$\pi_n : \mathscr{L}(X) \to \mathscr{L}_n(X).$$

For reasonable subsets A of $\mathscr{L}(X)$

$$\mu(A) := \lim_{n \mapsto \infty} [\pi_n(A)] \, \mathbb{L}^{-(n+1)d},$$

with d the dimension of X, in some completion $\widehat{M}_k$ of M_k (note the analogy with the p-adic case).

As we already mentioned, the very first application of motivic integration was made by Kontsevich, who used it to get a proof of Batyrev's Theorem without p-adic integration.

Similarly, one can avoid the use of p-adic integration in the proof of the Denef-L. Theorem.

What is the underlying idea?

If $h\colon Y \to X$ is a birational morphism, one can express the motivic volume of $\mathscr{L}(X)$ as a motivic integral on $\mathscr{L}(Y)$ involving the order of vanishing of the jacobian.

This is achieved by using an analogue of the "change of variables formula" in this setting.

Why does it work?

A modification $h\colon Y \to X$ induces an isomorphism outside a subset of finite positive codimension (usually one) $F \subset Y$.

At the level of arc spaces h induces a morphism between $\mathscr{L}(Y)$ and $\mathscr{L}(X)$ which restricts to a bijection between $\mathscr{L}(Y) \setminus \mathscr{L}(F)$ and $\mathscr{L}(X) \setminus \mathscr{L}(h(F))$, that is, between arcs in Y not completely contained in F and arcs in X not completely contained in $h(F)$.

But $\mathscr{L}(F)$ is of infinite codimension in $\mathscr{L}(Y)$, hence $\mathscr{L}(F)$ and $\mathscr{L}(h(F))$ have measure zero in $\mathscr{L}(Y)$ and $\mathscr{L}(X)$, respectively.

This is why measure theoretic tools seem to be so well adapted to birational geometry.

Finite group actions

Let G be a finite group. A linear action of G on a complex vector space V has a canonical decomposition $\oplus V_\alpha$ parametrized by characters.

Let now G act on a complex algebraic variety X.

There is no decomposition as above.

But there exists one at the level of arc spaces!

Indeed, let x be a point of X and denote by $G(x)$ the isotropy subgroup at x, consisting of those elements of G fixing x.
Denote by $\mathscr{L}(X)_x$ the space of arcs on X with origin at x.

There is a canonical decomposition

$$\mathscr{L}(X)_x = \bigsqcup_{\gamma \in \mathrm{Conj}\, G(x)} \mathscr{L}(X)_x^{\gamma} \sqcup B,$$

with $\mathrm{Conj}\, G(x)$ the set of conjugacy classes in $G(x)$ and B a subset of infinite codimension in $\mathscr{L}(X)_x$.

This explains the use of motivic integration in relation with the McKay correspondence between

1. certain resolutions of the quotient space X/G
2. group theoretical invariants of the action.

Work of Batyrev, Kontsevich, Denef-L., Yasuda, …

Let X be a smooth complex algebraic variety and $f : X \to \mathbb{C}$ a function (a morphism to the affine line).

Let x be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$.

Fix $0 < \eta \ll \varepsilon \ll 1$. The morphism f restricts to a fibration (the Milnor fibration)

$$B(x, \varepsilon) \cap f^{-1}(B(0, \eta) \setminus \{0\}) \to B(0, \eta) \setminus \{0\}.$$

Here $B(a, r)$ denotes the closed ball of center a and radius r.

The Milnor fiber at x,

$$F_x = f^{-1}(\eta) \cap B(x, \varepsilon)$$

has a diffeomorphism type that does not depend on η and ε and it is endowed with an automorphism, the monodromy M_x, induced by the characteristic mapping of the fibration.

In particular one can consider the n-th Lefschetz number

$$\Lambda^n(M_x) := \sum_j (-1)^j \operatorname{tr}(M_x^n; H^j(F_x)).$$

Denote by $\mathscr{X}_n$ the set of arcs φ in $\mathscr{L}_n(X)$ with $\varphi(0) = x$ such that

$$f(\varphi(t)) = t^n + (\text{higher order terms}).$$

THEOREM (DENEF-L.)

For $n \geq 1$,

$$\Lambda^n(M_x) = \operatorname{Eu}(\mathscr{X}_n).$$

Challenging Problem: Find a direct, geometric proof.

Let us note that this result has been restated and widely extended in the framework of rigid analytic geometry by Nicaise and Sebag.

In fact, the spaces $\mathscr{X}_n$ do contain much more information about the Milnor fiber and the monodromy:

Denef and L. proved that the series

$$Z(T) := \sum_{n\geq 1} [\mathscr{X}_n]\mathbb{L}^{-dn}T^n,$$

with d the dimension of X, has a limit as $T \mapsto \infty$, which is a motivic incarnation of the Milnor fiber together with the (semi-simplification of the) monodromy action on it.

One can show that this construction is compatible with a recent construction by J. Ayoub.

Work of Denef-L., Guibert, Bittner, Guibert-L.-Merle, etc.

Definable sets

A first order ring formula is a formula written with symbols 0, $+$, $-$, 1, $\times$, $=$, logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (negation), quantifiers $\exists$, $\forall$, and variables.

Now consider a formula φ with n free variables. If K is a field, we may set

$$X_\varphi(K) := \left\{(x_1,\ldots,x_n) \in K^n \,\middle|\, \varphi(x_1,\ldots,x_n)\,\text{holds}\right\}.$$

More generally one may extend this language to the valued ring language which admits symbols to express that the valuation is larger than something, or the initial coefficient of a series is equal to someting.

Objects of the form X_φ are called definable sets.

General motivic measure

With Raf Cluckers we recently constructed a very general theory of motivic integration based on cell decomposition.

In our theory, motivic integrals take place in a ring N_k which is obtained from the Grothendieck ring $K_0(\mathrm{Def}_k)$ of sets definable over k (in the ring language), by inverting $\mathbb{L}$ and $1-\mathbb{L}^i$ for $i \neq 0$ (here again $\mathbb{L}$ stands for the class of the affine line).

There is a natural morphism $N_k \to \widehat{M}_k$.

Our construction assigns to a bounded definable subset A of $k((t))^n$ in the valued field language a motivic volume $\mu(A)$ in N_k compatible with the previous construction.

Our construction relies on a cell decomposition theorem due to Denef and Pas (1989). Such cell decomposition results go back to the work of P. Cohen (1969).

Denef-Pas cell decomposition allows one to cut a definable subset A of $k((t))^{n+1} = k((t))^n \times k((t))$ into 0-dimensional cells (graphs of functions defined on a definable subset B of $k((t))^n$) and 1-dimensional cells (relative balls over B), maybe after adding some auxiliary parameters over the residue field and the value group.

This allows us to define the measure by induction on the valued field dimension.

The main difficulty is to prove that the measure is well defined (independent of the decomposition, in particular independent of the ordering of coordinates in the ambient affine space).

One of the main advantage of our construction is that it allows to consider integrals depending on parameters:

we introduce a natural class of constructible motivic functions and we prove stability of that class with respect to integration depending on parameters.

One can also extend the construction to allow motivic analogues of exponential functions.

Once developed, this new framework is as easy to use and flexible as standard Lebesgue integration, with Fubini theorems, change of variable theorems, Fourier inversion, distributions, etc.

AX-KOCHEN-ERŠOV

A (first order ring) sentence is a formula with no free variable.

For instance, for every $d > 0$, there is a sentence S_d expressing that any homogeneous polynomial of degree $d^2 + 1$ with coefficients in a field k has a non trivial zero in that field.

The following statement shows that for $p \gg 0$, $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ look very much the same:

THEOREM (AX-KOCHEN-ERŠOV)

Let φ be a first order sentence. For almost all prime number p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.

As a corollary one gets that S_d holds in $\mathbb{Q}_p$ for p large enough, since it holds in $\mathbb{F}_p((t))$ by Tsen and Lang.

Generalization to definable sets

How can one extend the Ax-Kochen-Eršov Theorem to formulas with free variables?

Theorem (Denef-L.)

Let φ be a formula in the valued ring language. Then, for almost all p, the sets $X_\varphi(\mathbb{Q}_p)$ and $X_\varphi(\mathbb{F}_p((t)))$ have the same volume.
Furthermore this volume is equal to the number of points in $\mathbb{F}_p$ of a motive M_φ canonically attached to φ.

There is a similar statement for integrals. This shows that p-adic integrals have a strongly uniform pattern as p varies: they are fully controlled by a single geometric object.

When φ has not free variable, one recovers the original form of the Ax-Kochen-Eršov Theorem.

An example of the transfer principle for parametrized integrals

We start by an example of an equality between two complicated integrals. Let E/F be a non ramified degree two extension of non archimedean local fields of residue characteristic $\neq 2$.

Let ψ be an additive character of F non trivial on $\mathcal{O}_F$ but trivial on the maximal ideal $\mathfrak{M}_F$.

Let N_n be the group of upper triangular matrices with 1's on the diagonal and consider the character $\theta : N_n(F) \to \mathbb{C}^\times$ given by

$$\theta(u) := \psi(\sum_i u_{i,i+1}).$$

For a the diagonal matrix $(a_1, \cdots, a_n)$ with a_i in $F^\times$, we consider the following complicated integral $I(a)$ defined in terms of F

$$I(a) := \int_{N_n(F)\times N_n(F)} \mathbf{1}_{M_n(\mathcal{O}_F)}({}^t u_1 a u_2)\,\theta(u_1 u_2)\,du_1\,du_2,$$

with the normalisation $\int_{N_n(\mathcal{O}_F)} du = 1$.

One may also consider a similar integral $J(a)$ defined in terms of E by replacing $N_n(F)\times N_n(F)$ by $N_n(E)$ and involving the non trivial element of the Galois group $x \mapsto \bar{x}$

$$J(a) := \int_{N_n(E)} \mathbf{1}_{M_n(\mathcal{O}_E)\cap H_n}({}^t \bar{u} a u)\,\theta(u\bar{u})\,du,$$

with H_n the set of Hermitian matrices.

The Jacquet-Ye Conjecture asserts that

$$I(a) = \gamma(a)\, J(a) \qquad (\Diamond)$$

with

$$\gamma(a) := \prod_{1\le i\le n-1} \eta(a_1 \cdots a_i),$$

and η the multiplicative character of order 2 on $F^\times$.

When $n = 2$, the Jacquet-Ye Conjecture essentially reduces to classical Gauss some identities, but already for $n = 3$ proof by direct computation is quite hard.

The full Jacquet-Ye Conjecture over finite field extensions of $\mathbb{F}_q((t))$ has been proved by Ngô in 1999 and in general by Jacquet in 2004.

Ngô's proof goes by reduction to a purely geometrical statement over algebraic varieties over $\mathbb{F}_q$ (which is not possible in the p-adic case), which he can prove by using fully the powerful machinery of ℓ-adic perverse sheaves over such varieties.

This is a typical instance of the general principle "complicated identities between character sums over finite fields are better proved by geometrical tools".

Hence it is natural to ask:

Assume we only know Jacquet-Ye holds over finite field extensions of $\mathbb{F}_q((t))$ is it possible to deduce it from general principles for p-adic fields?

Note that it makes no sense to compare the values of the integrals themselves, since a does not run over the same space in the characteristic 0 and p cases.

Answer: Yes!

The uniformity result of Denef and L. may be extended in the following way:

THEOREM (CLUCKERS-L.)

All p-adic integrals depending on parameters that are definable in a precise sense may be obtained by specialization of canonical motivic integrals of constructible functions for almost all p, and similarly for $\mathbb{Q}_p$ replaced by $\mathbb{F}_p((t))$.

Transfer Theorem (Cluckers-L.)

A given equality between definable integrals depending on parameters holds for $\mathbb{Q}_p$ if and only if it holds for $\mathbb{F}_p((t))$, when $p \gg 0$.

This applies in particular to the Jacquet-Ye Conjecture.

Cluckers, Hales and L. have recently proved that the transfer principle applies also to the so called Fundamental Lemma of Langlands theory.

The Fundamental Lemma was proved recently by Laumon and Ngô in the unitary case and by Ngô in the general case over finite extensions of $\mathbb{F}_p((t))$ by geometrical methods. Using specific group theoretical techniques, Waldspurger had already proved previously that one can then deduce it for p-adic fields.

It is natural to expect that relations between non archimedean integrals holding over all local fields of large residual characteristic already hold at the motivic level, as equalities between constructible motivic functions, but this seems to be presently out of reach.

Final comments

Time did not allow us to mention some other recent applications of Model Theory to Geometry over valued fields:

Hrushovski and Kazhdan developed a geometric integration theory for general complete valued fields (with residue characteristic zero) based on Robinson's quantifier elimination for algebraically closed valued fields.

Haskell, Hrushovski and Macpherson proved elimination of imaginaries for algebraically closed valued fields and introduced the notion of stably dominated type. Such more advanced model theoretic tools seem to have very promising applications to the study of the geometry of Berkovich spaces (work in progress by Hrushovski and L.).

Many thanks for your attention!