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Nearby Cycles and Composition with a Nondegenerate Polynomial

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1 Introduction

Let X_j be smooth varieties over a field k of characteristic zero, for $1 \leq j \leq p$. Consider a family $\mathbf{f}$ of p functions $f_j : X_j \to \mathbf{A}^1_k$. We will denote also by f_j the function on the product $X = \prod_j X_j$ obtained by composition with the projection. We denote by $X_0(\mathbf{f})$ the set of common zeroes in X of the functions f_j. Let $P \in k[y_1, \ldots, y_p]$ be a polynomial, which we assume to be nondegenerate with respect to its Newton polyhedron. In the present paper, we will compute the motivic nearby cycles $\mathcal{S}_{P(\mathbf{f})}$ on $X_0(\mathbf{f})$ of the composed function $P(\mathbf{f})$ on X as a sum over the set of compact faces δ of the Newton polyhedron of P. For every such δ, we denote by P_δ the corresponding quasihomogeneous polynomial. We associate to such a quasihomogeneous polynomial a convolution operator Ψ_{P_δ}, which in the special case where P_δ is the polynomial $\Sigma = y_1 + y_2$ is nothing but the operator Ψ_Σ considered in [9]. For such a compact face δ, one may also define generalized nearby cycles $\mathcal{S}_{\mathbf{f}}^{\sigma(\delta)}$, constructed as the limit, as $T \mapsto \infty$, of certain truncated motivic zeta functions.

Our main result, Theorem 3.2, follows from additivity from the following statement, Theorem 3.3:

$$i^*\mathcal{S}_{P(\mathbf{f}),U} = \sum_{\delta \in \Gamma^{\varnothing}} \Psi_{P_\delta}\big(\mathcal{S}_{\mathbf{f}}^{\sigma(\delta)}\big). \tag{1.1}$$

Here U denotes the complement of the locus where at least one function f_j vanishes, $\Gamma^{\varnothing}$ denotes the set of compact faces of the Newton polyhedron of P not contained in any

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coordinate hyperplane, $\mathcal{S}_{P(f),U}$ refers to the extension of $\mathcal{S}_{P(f)}$ constructed in [1, 9], and i^* denotes restriction to $X_0(f)$.

When $p = 2$ and $P = \Sigma$, one recovers the motivic Thom-Sebastiani formula (cf. [5, 6, 10]) in the way stated in [9]. When f is the set of coordinate functions on the affine space $\mathbf{A}_k^p$, our result is equivalent to a result obtained by Guibert in [8].

This paper is a natural continuation of [9], from which part of the notation and several results are borrowed.

2 Preliminaries

2.1 Grothendieck rings

Throughout the paper, k will be a field of characteristic zero. By a variety over k, we mean a separated and reduced scheme of finite type over k. If a linear algebraic group G acts on a variety X, we say the action is good if every G-orbit is contained in an affine open subset of X. We denote by $\mathrm{Var}^{G,eq}$ the category of varieties with good G-action, morphisms being G-equivariant morphisms. If S is a variety with good G-action, we denote by $\mathrm{Var}_S^{G,eq}$ the category of objects over S, that is, the category whose objects are morphisms $Y \to S$ in $\mathrm{Var}^{G,eq}$, morphisms in $\mathrm{Var}^{G,eq}$ being defined in the standard way. Let Y be a variety over k and let $p : A \to Y$ be an affine bundle for the Zariski topology (the fibers of p are affine spaces and the transition morphisms between trivializing charts are affine). In particular, the fibers of p have the structure of affine spaces. Let G be a linear algebraic group. A good action of G on A is said to be affine if it is a lifting of a good action on Y and its restriction to all fibers is affine.

One defines $K_0(\mathrm{Var}_S^{G,eq})$ as the free abelian group on isomorphism classes of objects $Y \to S$ in $\mathrm{Var}_S^{G,eq}$, modulo the relations

$$[Y \longrightarrow S] = [Y' \longrightarrow S] + [Y \setminus Y' \longrightarrow S] \tag{2.1}$$

for Y' closed G-invariant in Y and, for $f : Y \to S$ in $\mathrm{Var}_S^{G,eq}$,

$$\left[Y \times \mathbf{A}_k^n \longrightarrow S, \sigma\right] = \left[Y \times \mathbf{A}_k^n \longrightarrow S, \sigma'\right] \tag{2.2}$$

if σ and σ' are two liftings of the same G-action on Y to an affine action, the morphism $Y \times \mathbf{A}_k^n \to S$ being composition of f with projection on the first factor. Fiber product over S induces a product in the category $\mathrm{Var}_S^{G,eq}$, which allows to endow $K_0(\mathrm{Var}_S^{G,eq})$ with a natural ring structure. Note that the unit 1_S for the product is the class of the identity morphism $S \to S$.

2.2 $\mathbf{G}_m^s$-actions

Let s denote a positive integer and let S be a k-variety. From now on, we will consider only $\mathbf{G}_m^s$-actions on $S \times \mathbf{G}_m^r$ which are trivial on the first factor.

We consider the category $\mathcal{C}$ whose objects are finite morphisms of group schemes $\varphi : \mathbf{G}_m^s \to \mathbf{G}_m^{s'}$, a morphism between $\varphi : \mathbf{G}_m^s \to \mathbf{G}_m^{s'}$ and $\varphi' : \mathbf{G}_m^s \to \mathbf{G}_m^{s''}$ being a finite morphism $\vartheta : \mathbf{G}_m^{s'} \to \mathbf{G}_m^{s''}$ such that $\vartheta \circ \varphi = \varphi'$.

We consider also the full subcategory $\mathcal{C}'$ of $\mathcal{C}$, the objects of which are finite morphisms $\varphi : \mathbf{G}_m^s \to \mathbf{G}_m^s$. The subcategory $\mathcal{C}'$ is final in $\mathcal{C}$ in the language of [11].

A morphism $\varphi : \mathbf{G}_m^s \to \mathbf{G}_m^{s'}$ induces a natural functor

$$\Phi : \mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^{s'},\mathrm{eq}} \longrightarrow \mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^{s},\mathrm{eq}}, \tag{2.3}$$

where an object $Y \to S \times \mathbf{G}_m^r$ with a good $\mathbf{G}_m^{s'}$-action is sent on the same underlying object of $\mathrm{Var}_{S\times\mathbf{G}_m^r}$ with the $\mathbf{G}_m^s$-action induced via φ.

The functor Φ induces a morphism

$$K_0(\varphi) : K_0\left(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^{s'},\mathrm{eq}}\right) \longrightarrow K_0\left(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^{s},\mathrm{eq}}\right). \tag{2.4}$$

We will denote by $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\varphi,\mathrm{eq}})$ the image of the morphism $K_0(\varphi)$.

For every morphism ϑ between φ and φ' in $\mathcal{C}$, we get a morphism

$$K_0(\vartheta) : K_0\left(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\varphi',\mathrm{eq}}\right) \longrightarrow K_0\left(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\varphi,\mathrm{eq}}\right), \tag{2.5}$$

where a class of a good $\mathbf{G}_m^s$-action induced by a $\mathbf{G}_m^{s''}$-action via φ' on an object of $\mathrm{Var}_{S\times\mathbf{G}_m^r}$ is sent on the class of the same $\mathbf{G}_m^s$-action as induced by a $\mathbf{G}_m^{s'}$-action via φ. As a particular case, taking $\varphi = \mathrm{Id}$, we get the natural inclusion of $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\varphi,\mathrm{eq}})$ into $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s,\mathrm{eq}})$.

We define the Grothendieck ring $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s})$ as the colimit along $\mathcal{C}$ (or along $\mathcal{C}'$, which amounts to the same) of the rings $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\varphi,\mathrm{eq}})$.

Note that we could have also defined the rings $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\varphi,\mathrm{eq}})$ and $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s})$ as suitable Grothendieck rings of the essential image $\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\varphi,\mathrm{eq}}$ of Φ and of the colimit $\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s}$ along $\mathcal{C}$ (or $\mathcal{C}'$) of the categories $\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\varphi,\mathrm{eq}}$, respectively.

There is a natural structure of $K_0(\mathrm{Var}_k)$-module on $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s})$. We denote by $\mathbf{L}_{S\times\mathbf{G}_m^r} = \mathbf{L}$ the element $\mathbf{L}\cdot 1_{S\times\mathbf{G}_m^r}$ in this module, and we set

$$\mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s} := K_0\left(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s}\right)[\mathbf{L}^{-1}]. \tag{2.6}$$

Note that when $s = r$ the above definitions of $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s})$ and $\mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s}$ coincide with that of [9] by [9, Section 2.7].

A morphism $\vartheta : \mathbf{G}_m^s \to \mathbf{G}_m^{s'}$ induces a morphism from $\mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^{s'}}$ to $\mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s}$. For example, the diagonal morphism $\mathbf{G}_m \to \mathbf{G}_m^r$ yields a canonical morphism

$$\Delta : \mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r} \longrightarrow \mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m}. \tag{2.7}$$

Through this morphism, the class of a $\mathbf{G}_m^r$-action α on an object of $\mathrm{Var}_{S\times\mathbf{G}_m^r}$ is sent on the class of $\mathbf{G}_m$-actions induced by α via a finite group morphism from $\mathbf{G}_m$ to $\mathbf{G}_m^r$.

If $f : S \to S'$ is a morphism of varieties, composition with f leads to a pushforward morphism $f_! : \mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s} \to \mathcal{M}_{S'\times\mathbf{G}_m^r}^{\mathbf{G}_m^s}$, while fiber product leads to a pullback morphism $f^* : \mathcal{M}_{S'\times\mathbf{G}_m^r}^{\mathbf{G}_m^s} \to \mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^s}$.

2.3 Limits of rational series

Let A be one of the rings $\mathbb{Z}[\mathbf{L}, \mathbf{L}^{-1}]$, $\mathbb{Z}[\mathbf{L}, \mathbf{L}^{-1}, (1/(1-\mathbf{L}^{-i}))_{i>0}]$, $\mathcal{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m}$, and so forth. We denote by $A[[T]]_{sr}$ the A-submodule of $A[[T]]$ generated by 1 and by finite sums of products of terms $p_{e,i}(T) = (\mathbf{L}^e T^i)/(1-\mathbf{L}^e T^i)$, with e in $\mathbb{Z}$ and i in $\mathbb{N}_{>0}$. There is a unique A-linear morphism

$$\lim_{T\mapsto\infty} : A[[T]]_{sr} \longrightarrow A \tag{2.8}$$

such that

$$\lim_{T\mapsto\infty}\left(\prod_{i\in I} p_{e_i,j_i}(T)\right) = (-1)^{|I|}, \tag{2.9}$$

for every family $((e_i, j_i))_{i\in I}$ in $\mathbb{Z}\times\mathbb{N}_{>0}$, with I finite, may be empty.

2.4 Motivic zeta functions

We denote as usual by $\mathcal{L}_n(X)$ the space of arcs of order n, also known as the nth jet space on X. It is a k-scheme whose set of K-points, for K a field containing k, is the set of morphisms $\varphi : \operatorname{Spec} K[t]/t^{n+1} \to X$. There are canonical morphisms $\mathcal{L}_{n+1}(X) \to \mathcal{L}(X)$ and the arc space $\mathcal{L}(X)$ is defined as the projective limit of this system. We denote by $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ the canonical morphism. There is a canonical $\mathbf{G}_m$-action on $\mathcal{L}_n(X)$ and on $\mathcal{L}(X)$ given by $a \cdot \varphi(t) = \varphi(at)$.

Let X be a smooth variety over k of pure dimension d and $g : X \to \mathbf{A}^1_k$. Set $X_0(g)$ for the zero locus of g, and define, for $n \geq 1$, the variety

$$\mathcal{X}_n(g) := \left\{\varphi \in \mathcal{L}_n(X) \mid \operatorname{ord}_t g(\varphi) = n\right\}. \tag{2.10}$$

Note that $\mathcal{X}_n(g)$ is invariant by the $\mathbf{G}_m$-action on $\mathcal{L}_n(X)$ and that furthermore g induces a morphism $g_n : \mathcal{X}_n(g) \to \mathbf{G}_m$, assigning to a point φ in $\mathcal{L}_n(X)$ the coefficient of t^n in $g(\varphi)$, which we will denote by $\operatorname{ac}(g)(\varphi)$. We have $g_n(a \cdot \varphi) = a^n g_n(\varphi)$, hence with the terminology of [9] g_n is diagonally monomial of weight n with respect to the $\mathbf{G}_m$-action on $\mathcal{X}_n(g)$. In particular, we may consider the class $[\mathcal{X}_n(g)]$ of $\mathcal{X}_n(g)$ in $\mathcal{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$ and the motivic zeta function

$$Z_g(T) := \sum_{n\geq 1} \left[\mathcal{X}_n(g)\right]\mathbf{L}^{-nd}T^n \tag{2.11}$$

in $\mathcal{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}[[T]]$.

Denef and Loeser showed in [3, 6], see also [9, 10], that $Z_g(T)$ is a rational series in $\mathcal{M}^{\mathbf{G}_m}_{S\times\mathbf{G}_m}[[T]]_{\mathrm{sr}}$ by giving a formula for $Z_g(T)$ in terms of a resolution of f we will recall in Section 2.5.

2.5 Resolutions

Let us introduce some notation and terminology. Let X be a smooth variety of pure dimension d and let F be a closed subset of X of codimension everywhere ≥ 1. By a log-resolution $h : Y \to X$ of (X, F), we mean a proper morphism $h : Y \to X$ with Y smooth such that the restriction of $h : Y \setminus h^{-1}(F) \to X \setminus F$ is an isomorphism, and $h^{-1}(F)$ is a divisor with simple normal crossings. We denote by E_i, i in A, the set of irreducible components of the divisor $h^{-1}(F)$. For $I \subset A$, we set

$$E_I := \bigcap_{i \in I} E_i, \qquad E_I^\circ := E_I \setminus \bigcup_{j \notin I} E_j. \tag{2.12}$$

We denote by ν_{E_i} the normal bundle of E_i in Y and by ν_{E_I} the fiber product of the restrictions to E_I of the bundles ν_{E_i}, i in I. We will denote by U_{E_i} the complement of the zero section in ν_{E_i} and by U_I the fiber product of the restrictions of the spaces U_{E_i}, i in I, to E_I°.

If $\mathcal{I}$ is an ideal sheaf defining a closed subscheme Z of X and $h^{-1}(\mathcal{I})\mathcal{O}_Y$ is locally principal, we define $N_i(\mathcal{I})$, the multiplicity of $\mathcal{I}$ along E_i, by the equality of divisors

$$h^{-1}(Z) = \sum_{i \in A} N_i(\mathcal{I}) E_i. \tag{2.13}$$

If $\mathcal{I}$ is principal generated by a function g we write $N_i(g)$ for $N_i(\mathcal{I})$. Similarly, we define integers ν_i by the equality of divisors

$$K_Y = h^* K_X + \sum_{i \in A} (\nu_i - 1) E_i. \tag{2.14}$$

2.6 The class $[U_I]$

Assume again g is a function on a smooth variety X of pure dimension d. Let F be a reduced divisor containing $X_0(g)$ and let $h : Y \to X$ be a log-resolution of (X, F). We explain how g induces a morphism $g_I : U_I \to \mathbf{G}_m$. Note that the function $g \circ h$ induces a function

$$\bigotimes_{i \in I} \nu_{E_i}^{\otimes N_i(g)}{}_{|E_I} \longrightarrow \mathbf{A}_k^1, \tag{2.15}$$

vanishing only on the zero section. We define $g_I : \nu_{E_I} \to \mathbf{A}_k^1$ as the composition of this last function with the natural morphism $\nu_{E_I} \to \bigotimes_{i \in I} \nu_{E_i}^{\otimes N_i(g)}{}_{|E_I}$, sending (u_i) to $\bigotimes u_i^{\otimes N_i(g)}$. We still denote by g_I the induced morphism from U_I to $\mathbf{G}_m$.

We view U_I as a variety over $X_0(g) \times \mathbf{G}_m$ via the morphism $(h \circ \pi_I, g_I)$. The group $\mathbf{G}_m$ has a natural action on each U_{E_i}, so the diagonal action induces a $\mathbf{G}_m$-action on U_I. Furthermore, the morphism g_I is monomial, in the terminology of [9], hence $U_I \to X_0(g) \times \mathbf{G}_m$ has a class in $\mathcal{M}_{X_0(g) \times \mathbf{G}_m}^{\mathbf{G}_m}$ which we will denote by $[U_I]$.

2.7 Motivic Milnor fiber

We now assume that $F = X_0(g)$, that is, $h : Y \to X$ is a log-resolution of $(X, X_0(g))$. In this case, h induces a bijection between $\mathcal{L}(Y) \setminus \mathcal{L}(|h^{-1}(X_0(g))|)$ and $\mathcal{L}(X) \setminus \mathcal{L}(X_0(g))$.

One deduces from [4, Lemma 3.4], in a way completely similar to [3, 6], the equality

$$Z_g(T) = \sum_{\varnothing \neq I \subset A} [U_I] \prod_{i \in I} \frac{1}{T^{-N_i(g)} \mathbf{L}^{\nu_i} - 1} \tag{2.16}$$

in $\mathcal{M}^{\mathbf{G}_m}_{X_0(g) \times \mathbf{G}_m}[[T]]$.

In particular, the function $Z_g(T)$ is rational and belongs to $\mathcal{M}^{\mathbf{G}_m}_{X_0(g) \times \mathbf{G}_m}[[T]]_{sr}$, with the notation of Section 2.3, hence we can consider $\lim_{T \mapsto \infty} Z_g(T)$ in $\mathcal{M}^{\mathbf{G}_m}_{X_0(g) \times \mathbf{G}_m}$ and set

$$\mathcal{S}_g := -\lim_{T \mapsto \infty} Z_g(T), \tag{2.17}$$

which by (2.16) may be expressed on a resolution h as

$$\mathcal{S}_g = -\sum_{\varnothing \neq I \subset A} (-1)^{|I|} [U_I] \tag{2.18}$$

in $\mathcal{M}^{\mathbf{G}_m}_{X_0(g) \times \mathbf{G}_m}$. The element $\mathcal{S}_g$ is called the motivic Milnor fiber or the motivic nearby fiber of f. It was first considered by Denef and Loeser (cf. [3, 6, 7]). For recent results concerning $\mathcal{S}_g$, we refer the reader to [1, 8, 9].

2.8 The zeta function $Z_{\mathbf{f}}^{C,\ell}(T)$

Consider a family $\mathbf{f}$ of p functions $f_j : X \to \mathbf{A}^1_k$, $1 \leq j \leq p$. We denote by $X_0(\mathbf{f})$ the set of common zeroes of the functions f_j, $1 \leq j \leq p$, and by F the product function $f_1 \cdots f_p$.

We fix a rational polyhedral convex cone C in $\mathbb{R}^p_{>0}$ and an integral linear form ℓ on $\mathbb{Z}^p$ which is positive on $\bar{C} \setminus \{0\}$, where $\bar{C}$ denotes the closure of C in $\mathbb{R}^p$.

We will consider the modified zeta function $Z_{\mathbf{f}}^{C,\ell}$ defined as follows: for a vector $\mathbf{n}$ in $\mathbb{N}^p_{>0}$, we denote by $s(\mathbf{n})$ the sum of its components and we consider, similarly as in (2.10), the variety

$$\mathcal{X}_{\mathbf{n}}(\mathbf{f}) := \left\{ \varphi \in \mathcal{L}_{s(\mathbf{n})}(X) \mid \operatorname{ord} f_j(\varphi) = n_j,\ 1 \leq j \leq p \right\}. \tag{2.19}$$

Note that $\mathcal{X}_{\mathbf{n}}(\mathbf{f})$ is stable under the $\mathbf{G}_m$-action on $\mathcal{L}_n(X)$ and that $\mathbf{f}$ induces a morphism

$$\mathbf{f}_{\mathbf{n}} : \mathcal{X}_{\mathbf{n}}(\mathbf{f}) \longrightarrow \mathbf{G}_m^p, \tag{2.20}$$

whose components are $\mathrm{ac}(f_j)$, $1 \leq j \leq p$, defined similarly as in Section 2.4. Since $\mathbf{f}_n(a \cdot \varphi) = a^n \mathbf{f}_n(\varphi)$, we may consider the class $[\mathcal{X}_n(\mathbf{f})]$ of $\mathcal{X}_n(\mathbf{f}) \to X_0(\mathbf{f}) \times \mathbf{G}_m^p$ in $\mathcal{M}^{\mathbf{G}_m}_{X_0(\mathbf{f}) \times \mathbf{G}_m^p}$. We set

$$Z_{\mathbf{f}}^{C,\ell}(T) := \sum_{\mathbf{n} \in C} \left[\mathcal{X}_{\mathbf{n}}(\mathbf{f})\right] \mathbf{L}^{-s(\mathbf{n})d}\, T^{\ell(\mathbf{n})} \tag{2.21}$$

in $\mathcal{M}^{\mathbf{G}_m}_{X_0(\mathbf{f}) \times \mathbf{G}_m^p}[[T]]$.

2.9 The class $\mathcal{S}_{\mathbf{f}}^{C,\ell}$

Let $h : Y \to X$ be a log-resolution of the set $X_0(F)$. We keep the notations of Section 2.5. In particular, we denote by A the set of irreducible components of $h^{-1}(X_0(F))$. For i in A, we will denote by N_i the integral vector of the orders $N_i(f_j)$ of the functions f_j, $1 \leq j \leq p$, along the divisor E_i. We denote by B the set of all subsets I of A such that $h(E_I^\circ)$ is contained in $X_0(\mathbf{f})$. For I in B, we denote by N_I the linear map

$$N_I : \begin{cases} \mathbb{R}^I_{>0} \longrightarrow \mathbb{R}^p_{>0}, \\ \mathbf{k} \longmapsto \sum_{i \in I} k_i N_i. \end{cases} \tag{2.22}$$

Similarly, the set of integers ν_i defines a linear integral form $\nu_I : \mathbf{k} \mapsto \sum_{i \in I} k_i \nu_i$ on $\mathbb{R}^I_{>0}$.

Using [4, Lemma 3.4] similarly as for the proof of (2.16) (see, e.g., [6, 10]), one gets the following formula for the zeta function $Z_{\mathbf{f}}^{C,\ell}(T)$ in terms of the resolution:

$$Z_{\mathbf{f}}^{C,\ell}(T) = \sum_{I \in B} [U_I] \sum_{\{\mathbf{k} \in \mathbb{N}^p_{>0} | N_I(\mathbf{k}) \in C\}} \prod_{i \in I} \left(T^{\ell(N_i)} \mathbf{L}^{-\nu_i}\right)^{k_i}. \tag{2.23}$$

Here, for I in B, $[U_I]$ stands for the class in $\mathcal{M}^{\mathbf{G}_m}_{X_0(\mathbf{f}) \times \mathbf{G}_m^p}$ of the morphism $(h, \mathbf{f}_I) : U_I \to X_0(\mathbf{f}) \times \mathbf{G}_m^p$.

It follows that $Z_{\mathbf{f}}^{C,\ell}(T)$ belongs to $\mathcal{M}^{\mathbf{G}_m}_{X_0(\mathbf{f}) \times \mathbf{G}_m^p}[[T]]_{\mathrm{sr}}$, hence we may set

$$\mathcal{S}_{\mathbf{f}}^{C,\ell} := \lim_{T \mapsto \infty} Z_{\mathbf{f}}^{C,\ell}(T) \tag{2.24}$$

in $\mathcal{M}^{\mathbf{G}_m}_{X_0(\mathbf{f}) \times \mathbf{G}_m^p}$. By [9, section 2.9], we have

$$\mathcal{S}_{\mathbf{f}}^{C,\ell} = \sum_{I \in B} \chi\left(N_I^{-1}(C)\right) [U_I], \tag{2.25}$$

where χ denotes Euler characteristic with compact supports. Note that this is independent of ℓ, so we may write $\mathcal{S}_{\mathbf{f}}^{C}$ instead of $\mathcal{S}_{\mathbf{f}}^{C,\ell}$.

3 Composition with a nondegenerate polynomial

3.1 The generalized convolution Ψ_P

Let P be a quasihomogeneous polynomial function on $\mathbf{G}_m^p$, that is, P is homogeneous for a $\mathbf{G}_m$-action α on $\mathbf{G}_m^p$ monomial of weight $w = (w_1, \dots, w_p)$.

Let X be a smooth variety. We will denote by pr_1 the projection of $X \times \mathbf{G}_m^p \times \mathbf{G}_m$ on $X \times \mathbf{G}_m$ (forgetting the $\mathbf{G}_m^p$ factor) and by i the inclusion of the complement of $X \times P^{-1}(0)$ into $X \times \mathbf{G}_m^p$.

For a variety A of dimension e in $\mathrm{Var}_{X \times \mathbf{G}_m^p}$, the function P induces by composition with the second projection a function on A we still denote by P:

$$P : A \longrightarrow \mathbf{A}_k^1. \tag{3.1}$$

We now define the (augmented) zeta function $Z_P^0(T)$ as

$$Z_P^0(T) = \sum_{n \geq 0} \left[\mathcal{X}_n(P)\right] \mathbf{L}^{-ne} T^n = \left[\mathcal{X}_0(P)\right] + Z_P(T), \tag{3.2}$$

where $\mathcal{X}_n(P)$ is

$$\mathcal{X}_n(P) := \left\{\varphi \in \mathcal{L}_n(A) \mid \mathrm{ord}_t\, P(\varphi) = n\right\}, \tag{3.3}$$

for $n \geq 0$. It belongs to $\mathcal{M}_{X \times \mathbf{G}_m^p \times \mathbf{G}_m}^{\mathbf{G}_m}[[T]]_{\mathrm{sr}}$. We define $\Psi_P^0(A)$ as the limit, as $T \mapsto \infty$, of the opposite $-Z_P^0(T)$. Thus, with the notations of [**9**], it is nothing but

$$-\lim_{T \to \infty} Z_P^0(T) = -\left[A \setminus P^{-1}(0)\right] + \mathcal{S}_P\left([A]\right). \tag{3.4}$$

It is an object in $\mathcal{M}_{X \times \mathbf{G}_m^p \times \mathbf{G}_m}^{\mathbf{G}_m}$, the $\mathbf{G}_m$-action and the morphism to $\mathbf{G}_m$ being the usual ones. On $A \setminus P^{-1}(0)$, the $\mathbf{G}_m$-action is trivial and the morphism to $\mathbf{G}_m$ is the restriction of P to $A \setminus P^{-1}(0)$. Taking the direct image by the projection pr_1, we get the following object in $\mathcal{M}_{X \times \mathbf{G}_m}^{\mathbf{G}_m}$:

$$\Psi_P^0(A) := \mathrm{pr}_{1!}\left(-\left[A \setminus P^{-1}(0)\right] + \mathcal{S}_P(A)\right). \tag{3.5}$$

One may then extend uniquely this construction to an $\mathcal{M}_k$-linear group morphism

$$\Psi_P^0 : \mathcal{M}_{X \times \mathbf{G}_m^p} \longrightarrow \mathcal{M}_{X \times \mathbf{G}_m}^{\mathbf{G}_m}. \tag{3.6}$$

If A is endowed with a $\mathbf{G}_m$-action α for which the morphism to $\mathbf{G}_m^p$ is monomial of weight w, $A \setminus P^{-1}(0)$ is endowed with an additional action which is homogeneous with respect to the composed morphism to $\mathbf{G}_m$. Hence, we may attach to $A \setminus P^{-1}(0)$ a class $[A \setminus P^{-1}(0)]$ in $\mathcal{M}^{\mathbf{G}_m^2}_{X\times\mathbf{G}_m^p\times\mathbf{G}_m}$. In [9, Section 3.10], we attached to such an A with the action α an element $\mathcal{S}_P(A)$ in $\mathcal{M}^{\mathbf{G}_m^2}_{X\times\mathbf{G}_m^p\times\mathbf{G}_m}$. Hence, we can consider $\mathrm{pr}_{1!}(-[A \setminus P^{-1}(0)] + \mathcal{S}_P(A))$ as an element of $\mathcal{M}^{\mathbf{G}_m^2}_{X\times\mathbf{G}_m}$. Composing with the canonical morphism $\mathcal{M}^{\mathbf{G}_m^2}_{X\times\mathbf{G}_m} \to \mathcal{M}^{\mathbf{G}_m}_{X\times\mathbf{G}_m}$ induced by the diagonal action, we get an element of $\mathcal{M}^{\mathbf{G}_m}_{X\times\mathbf{G}_m}$ we will denote by $\Psi_P(A)$. This construction extends uniquely to an $\mathcal{M}_k$-linear group morphism

$$\Psi_P : \mathcal{M}^{\mathbf{G}_m}_{X\times\mathbf{G}_m^p} \longrightarrow \mathcal{M}^{\mathbf{G}_m}_{X\times\mathbf{G}_m}. \tag{3.7}$$

Remark 3.1. When P is the sum of coordinates Σ on $\mathbf{G}_m^2$, then Ψ_Σ is nothing but the convolution product from [9]. More precisely, the convolution product Ψ_Σ defined in [9] is equal to the composition of the morphism Ψ_Σ defined in this paper with the morphism Δ defined in (2.7).

3.2 Composed maps

For $1 \leq j \leq p$, let $f_j : X_j \to \mathbf{A}^1_k$ be a function on a smooth k-variety X_j. By composition with the projection, f_j becomes a function on the product $X = \prod_j X_j$. We write d for the dimension of X. Define $\mathbf{f}$ as the family of the f_j on X, $1 \leq j \leq p$. The product of the log-resolutions of the $X_{j,0}(f_j)$ is a log-resolution $h : Y \to X$ of $X_0(F)$ (recall that $F = f_1 \cdots f_p$).

Let $P = \sum_{\alpha\in\mathbb{N}^p} a_\alpha y^\alpha$ be a polynomial in $k[y_1, \ldots, y_p]$. We denote by $\mathrm{supp}(P)$ the set of exponents α in $\mathbb{N}^p$ with $a_\alpha \neq 0$. The Newton polyhedron Γ of P is the convex hull of $\mathrm{supp}(P) + \mathbb{R}^p_+$. For a compact face δ of Γ, we denote by P_δ the sum of the monomials of P supported in δ:

$$P_\delta = \sum_{\alpha\in\delta} a_\alpha y^\alpha. \tag{3.8}$$

We say P is nondegenerate with respect to its Newton polyhedron Γ, if, for every compact face δ of Γ, the function P_δ is smooth on $\mathbf{G}_m^p$.

To the Newton polyhedron Γ one may associate a fan of rational polyhedral cones subdividing $\mathbb{R}^p_+$ as follows. We consider the function ℓ_Γ assigning to a vector a in $\mathbb{R}^p_+$ the value $\inf_{b\in\Gamma}\langle a, b\rangle$, with $\langle , \rangle$ the standard inner product. For any a in $\mathbb{R}^p_+$, we may consider the compact face

$$\delta_a = \left\{ b \in \Gamma' \mid \langle a, b\rangle = \ell_\Gamma(b) \right\}, \tag{3.9}$$

with Γ' the union of all compact faces of Γ.

For a compact face δ of the Newton polyhedron Γ, we denote by $\sigma(\delta)$ its dual cone $\{a \in \mathbb{R}_+^p \mid \delta_a = \delta\}$. The cones $\sigma(\delta)$, for δ running over the compact faces of Γ, form a fan partitioning $\mathbb{R}_+^p$ into rational polyhedral cones. The function ℓ_Γ is linear on each cone $\sigma(\delta)$.

We write Γ_c for the set of compact faces of Γ. For J a subset of $\{1, \ldots, p\}$, we denote by Γ^J the set of compact faces of Γ contained in the coordinate hyperplanes $x_i = 0$ for i in J, and in no other coordinate hyperplane, so that Γ_c is the disjoint union of the subsets Γ^J. Note that ℓ_Γ is positive on $\overline{\sigma(\delta)} \setminus \{0\}$ if and only if δ is in $\Gamma^\varnothing$. We denote by X_J the closed subset of X defined by the vanishing of the functions f_i, $i \in J$, and by $\mathbf{f}_J : X_J \to \mathbf{A}^{\{1,\ldots,p\}\setminus J}$ the morphism induced by the functions f_j, $j \notin J$.

For every variety Z containing $X_0(\mathbf{f})$, we denote by i^* the restriction morphisms

$$\begin{aligned} &\mathcal{M}^{\mathbf{G}_m}_{Z\times\mathbf{G}_m} \longrightarrow \mathcal{M}^{\mathbf{G}_m}_{X_0(\mathbf{f})\times\mathbf{G}_m}, \\ &\mathcal{M}^{\mathbf{G}_m}_{Z\times\mathbf{G}_m}[[T]] \longrightarrow \mathcal{M}^{\mathbf{G}_m}_{X_0(\mathbf{f})\times\mathbf{G}_m}[[T]]. \end{aligned} \tag{3.10}$$

Theorem 3.2. With the previous notations and hypotheses, the following formula holds for $i^*\mathcal{S}_{P(\mathbf{f})}$ in $\mathcal{M}^{\mathbf{G}_m}_{X_0(\mathbf{f})\times\mathbf{G}_m}$:

$$i^*\mathcal{S}_{P(\mathbf{f})} = \sum_{J\subset\{1,\ldots,p\}} \sum_{\delta\in\Gamma^J} \Psi_{P_\delta}\left(\mathcal{S}^{\sigma(\delta),\ell_\Gamma}_{\mathbf{f}_J}\right). \tag{3.11}$$

□

Proof. Following [9], for γ in $\mathbb{N}_{>0}$, we consider the constructible set

$$\mathcal{X}_n^{\gamma n} := \left\{\varphi \in \mathcal{L}_{\gamma n}(X) \mid \mathrm{ord}_t\, P(\mathbf{f})(\varphi) = n,\ \mathrm{ord}_t\, F(\varphi) \leq \gamma n\right\} \tag{3.12}$$

together with the morphism $\mathrm{ac}(P(\mathbf{f})) : \mathcal{X}_n^{\gamma n} \to \mathbf{G}_m$, giving rise to a class $[\mathcal{X}_n^{\gamma n}]$ in $\mathcal{M}^{\mathbf{G}_m}_{X_0(F)\times\mathbf{G}_m}$. By [9, Proposition 3.8], for $\gamma \gg 0$, the corresponding zeta function

$$Z^\gamma_{P(\mathbf{f}),X\setminus X_0(F)}(T) := \sum_{n>0} \left[\mathcal{X}_n^{\gamma n}\right] \mathbf{L}^{-\gamma n d} T^n \tag{3.13}$$

lies in $\mathcal{M}^{\mathbf{G}_m}_{X_0(F)\times\mathbf{G}_m}[[T]]_{sr}$ and its limit as $T \mapsto \infty$ is independent of γ, so we may set

$$\mathcal{S}_{P(\mathbf{f}),X\setminus X_0(F)} := -\lim_{T\mapsto\infty} Z^\gamma_{P(\mathbf{f}),X\setminus X_0(F)}(T). \tag{3.14}$$

Furthermore, by additivity of $\mathcal{S}_{P(\mathbf{f})}$ (cf. [9, Theorem 3.12]), we have

$$\mathcal{S}_{P(\mathbf{f})} = \sum_{J\subset\{1,\ldots,p\}} \mathcal{S}_{P(\mathbf{f})_{|X_J},X_J^\circ}, \tag{3.15}$$

with X_J° the largest open subset in X_J, where no f_j, $j \notin J$, vanishes. Theorem 3.2 now follows directly from Theorem 3.3. ■

Theorem 3.3. With the previous notation, the following holds:

$$i^*\mathcal{S}_{P(\mathbf{f}),X\setminus X_0(F)} = \sum_{\delta\in\Gamma^J} \Psi_{P_\delta}\big(\mathcal{S}_{\mathbf{f}}^{\sigma(\delta)}\big). \tag{3.16}$$

□

Proof. We fix a log-resolution $h : Y \to X$ of $X_0(F)$. We will keep the notations of Section 2.5.

Fix a subset of I of A and $\mathbf{k} = (k_i)_{i\in I}$ in $\mathbb{N}^I_{>0}$. For φ in $\mathcal{L}_{\gamma n}(Y)$ with $\varphi(0)$ in E_i, we set $\mathrm{ord}_{E_i}\,\varphi := \mathrm{ord}_t\, z_i(\varphi)$, for z_i any local equation of E_i at $\varphi(0)$. We denote by $\mathcal{X}_{n,\mathbf{k}}$ the set of arcs φ in $\mathcal{L}_{\gamma n}(Y)$ such that $\varphi(0)$ is in E_I° and $\mathrm{ord}_{E_i}\,\varphi = k_i$ for $i \in I$. We also consider the subset $\mathcal{Y}_{n,\mathbf{k}}$ of $\mathcal{X}_{n,\mathbf{k}}$ consisting of arcs φ such that $\mathrm{ord}_t(P(\mathbf{f}) \circ h)(\varphi) = n$. The variety $\mathcal{Y}_{n,\mathbf{k}}$ is stable by the usual $\mathbf{G}_m$-action on $\mathcal{L}_{\gamma n}(Y)$ and the morphism $\mathrm{ac}(P(\mathbf{f}) \circ h)$ defines a class $[\mathcal{Y}_{n,\mathbf{k}}]$ in $\mathcal{M}^{\mathbf{G}_m}_{X_0(F)\times\mathbf{G}_m}$. Note that $\mathcal{Y}_{n,\mathbf{k}} = \varnothing$ if $n < \ell_\Gamma(N_I(\mathbf{k}))$.

By a now standard calculation, using [4, Lemma 3.6], $Z^\gamma_{P(\mathbf{f}),X\setminus X_0(F)}$ may be expressed on the log-resolution Y as

$$Z^\gamma_{P(\mathbf{f}),X\setminus X_0(F)} = \sum_{\varnothing\neq I\subset A}\ \sum_{\substack{N_I(\mathbf{k})\in\sigma(\delta)\\ n=\ell_\Gamma(N_I(\mathbf{k}))\\ \langle N_I(\mathbf{k}),\mathbf{1}\rangle\leq\gamma n}} \big[\mathcal{Y}_{n,\mathbf{k}}\big]\mathbf{L}^{-\sum_{i\in I}(\nu_i-1)k_i}\mathbf{L}^{-\gamma n d}T^n. \tag{3.17}$$

As in Section 2.9, we denote by B the set of all subsets I of A such that $h(E_I^\circ)$ is contained in $X_0(\mathbf{f})$. We fix I in B and $\mathbf{k} = (k_i)_{i\in I}$ in $\mathbb{N}^I_{>0}$. Note that there is a unique compact face δ of Γ such that $N_I(\mathbf{k})$ lies in $\sigma(\delta)$.

To go further on, we will use the following variant of the classical deformation to the normal cone already considered in [9]. We consider the affine line $\mathbf{A}^1_k = \mathrm{Spec}\, k[u]$ and the subsheaf

$$\mathcal{A}_{\mathbf{k}} := \sum_{\mathbf{n}\in\mathbb{N}^I} \mathcal{O}_{Y\times\mathbf{A}^1_k}\Bigg(-\sum_{i\in I} n_i\Big(E_i \times \mathbf{A}^1_k\Big)\Bigg)u^{-\sum_{i\in I}k_i n_i} \tag{3.18}$$

of $\mathcal{O}_{Y\times\mathbf{A}^1_k}[u^{-1}]$. It is a sheaf of rings and we set

$$CY_{\mathbf{k}} := \mathrm{Spec}\,\mathcal{A}_{\mathbf{k}}. \tag{3.19}$$

The natural inclusion $\mathcal{O}_{Y\times\mathbf{A}^1_k} \to \mathcal{A}_{\mathbf{k}}$ induces a morphism $\pi : CY_{\mathbf{k}} \to Y \times \mathbf{A}^1_k$, hence a morphism $p : CY_{\mathbf{k}} \to \mathbf{A}^1_k$. Via the same inclusion, the functions $P(\mathbf{f}) \circ h$ and $F \circ h$ are, in $\mathcal{A}_{\mathbf{k}}$, divisible by $u^{\ell_\Gamma(N_I(\mathbf{k}))}$ and by $u^{\langle N_I(\mathbf{k}),\mathbf{1}\rangle}$, where $\mathbf{1}$ denotes the vector with all coordinates equal to 1, and we denote the corresponding quotients by $\widetilde{P}(\mathbf{f})_{\mathbf{k}}$ and $\widetilde{F}_{\mathbf{k}}$, respectively.

We denote by $\widetilde{E}_i$ the pullback of the divisor $E_i \times \mathbf{A}^1_k$ by π, by D the divisor globally defined on $CY_{\mathbf{k}}$ by $u = 0$, and by CE_i the divisors $\widetilde{E}_i - k_i D$, i in I (resp., $\widetilde{E}_i$, i not in I).

We denote by CY_k° the complement in CY_k of the union of the CE_i, i in A, and by Y° the complement in Y of the union of the E_i, i in A.

As proved in [9, Lemma 5.12], the scheme CY_k is smooth, the morphism π induces an isomorphism above $\mathbf{A}_k^1 \setminus \{0\}$, the morphism p is a smooth morphism and its fiber $p^{-1}(0)$ may be naturally identified with the bundle ν_{E_I}. Furthermore, when restricted to CY_k°, the fiber of p above 0 is naturally identified with U_I and π induces an isomorphism between $CY_k^\circ \setminus p^{-1}(0)$ and $Y^\circ \times \mathbf{A}_k^1 \setminus \{0\}$. The restrictions of $\widetilde{P}(\mathbf{f})_\mathbf{k}$ and $\widetilde{F}_\mathbf{k}$ to the fiber $U_I \subset p^{-1}(0)$ are, respectively, equal to $P_\delta(\mathbf{f}_I)$ and F_I.

The ring $\mathcal{A}_k$ being a graded subring of the ring $\mathcal{O}_Y[u, u^{-1}]$, we may consider the $\mathbf{G}_m$-action σ on CY_k, leaving sections of $\mathcal{O}_Y$ invariant and acting on u by $\sigma(\lambda) : u \mapsto \lambda^{-1}u$. It restricts on U_I to the diagonal action induced by the canonical $\mathbf{G}_m^I$-action on U_I via the finite morphism $\lambda \mapsto \lambda^k$. We have now two different $\mathbf{G}_m$-actions on $\mathcal{L}_n(CY_\mathbf{k}^\circ)$: the one induced by the standard $\mathbf{G}_m$-action on arc spaces and the one induced by σ. We denote by $\widetilde{\sigma}$ the action given by the composition of these two (commuting) actions.

We denote by $\widetilde{\mathcal{L}}_{\gamma n}(CY_\mathbf{k}^\circ)$ the set of arcs φ in $\mathcal{L}_{\gamma n}(CY_\mathbf{k}^\circ)$ such that $p(\varphi(t)) = t$ (in particular, $\varphi(0)$ is in U_I). For such an arc φ, composition with π sends φ to an arc in $\mathcal{L}_{\gamma n}(Y \times \mathbf{A}_k^1)$ which is the graph of an arc in $\mathcal{L}_{\gamma n}(Y)$ not contained in the union of the divisors E_i, i in I. Note that $\widetilde{\mathcal{L}}_n(CY_\mathbf{k}^\circ)$ is stable by $\widetilde{\sigma}$.

Lemma 3.4. Let I be in B and $\mathbf{k}$ in $\mathbb{N}_{>0}^I$. Assume $n \geq k_i$ for i in I. The morphism $\widetilde{\pi}$: $\widetilde{\mathcal{L}}_n(CY_\mathbf{k}^\circ) \to \mathcal{X}_{n,\mathbf{k}}$ induced by the projection $CY_\mathbf{k}^\circ \to Y$ is an affine bundle with fiber $\mathbf{A}_k^{\sum_I k_i}$. Furthermore, if $\widetilde{\mathcal{L}}_n(CY_\mathbf{k}^\circ)$ is endowed with the $\mathbf{G}_m$-action induced by $\widetilde{\sigma}$ and $\mathcal{X}_{n,\mathbf{k}}$ with the standard $\mathbf{G}_m$-action, $\widetilde{\pi}$ is $\mathbf{G}_m$-equivariant and the action of $\mathbf{G}_m$ on the affine bundle is affine. Furthermore, if $n \geq \ell_\Gamma(N_I(\mathbf{k}))$, then for every φ in $\widetilde{\mathcal{L}}_{\gamma n}(CY_\mathbf{k}^\circ)$

$$\mathrm{ac}\left(P(\mathbf{f}) \circ h\right)\left(\widetilde{\pi}(\varphi)\right) = \mathrm{ac}\left(\widetilde{P}(\mathbf{f})_\mathbf{k}(\varphi)\right). \tag{3.20}$$

When $P_\delta(\mathbf{f}_I)(\varphi(0)) \neq 0$, hence $(\mathrm{ord}_t(P(\mathbf{f})) \circ h)(\widetilde{\pi}(\varphi)) = \ell_\Gamma(N_I(\mathbf{k}))$, it holds that

$$\mathrm{ac}\left(P(\mathbf{f}) \circ h\right)\left(\widetilde{\pi}(\varphi)\right) = P_\delta\left(\mathbf{f}_I\right)\left(\varphi(0)\right). \tag{3.21}$$

□

Proof. The first part of the statement is contained in [9, Lemma 5.13] and the rest follows from its proof. ■

We then define $\widetilde{\mathcal{Y}}_{n,\mathbf{k}}$ as the inverse image of $\mathcal{Y}_{n,\mathbf{k}}$ by the fibration $\widetilde{\pi}$. It is the subset of arcs φ in $\mathcal{L}_{\gamma n}(CY_\mathbf{k}^\circ)$ such that $\mathrm{ord}_t\, \widetilde{P}(\mathbf{f})_\mathbf{k}(\varphi) = n - \ell_\Gamma(N_I(\mathbf{f}))$. We denote by $[\widetilde{\mathcal{Y}}_{n,\mathbf{k}}]$ the class of $\widetilde{\mathcal{Y}}_{n,\mathbf{k}}$ in $\mathcal{M}_{X_0(F)\times\mathbf{G}_m}^{\mathbf{G}_m}$, the morphism $\widetilde{\mathcal{Y}}_{n,\mathbf{k}} \to \mathbf{G}_m$ being $\mathrm{ac}(\widetilde{P}(\mathbf{f})_\mathbf{k})$ and the $\mathbf{G}_m$-action being induced by $\widetilde{\sigma}$. We denote by $[U_I \setminus (P_\delta(\mathbf{f}_I)^{-1}(0))]$ the class of $U_I \setminus (P_\delta(\mathbf{f}_I)^{-1}(0))$ in $\mathcal{M}_{X_0(F)\times\mathbf{G}_m}^{\mathbf{G}_m}$, the $\mathbf{G}_m$-action being the natural diagonal action of weight $\mathbf{k}$ on $U_I \setminus (P_\delta(\mathbf{f}_I)^{-1}(0))$ and the morphism to $\mathbf{G}_m$ being the restriction of $P_\delta(\mathbf{f}_I)$. We also consider the class $[\mathbf{G}_m \times F_I^{-1}(0)]$ of $\mathbf{G}_m \times P_\delta(\mathbf{f}_I)^{-1}(0)$ in $\mathcal{M}^{\mathbf{G}_m}_{X_0(F)\times\mathbf{G}_m}$, the $\mathbf{G}_m$-action on the second factor being the diagonal one and the morphism to $\mathbf{G}_m$ being the first projection.

Lemma 3.5. Let I be in B and $\mathbf{k}$ in $\mathbb{N}^I_{>0}$. The following equalities hold in $\mathcal{M}^{\mathbf{G}_m}_{X_0(F)\times\mathbf{G}_m}$:

(1) $[\widetilde{\mathcal{Y}}_{n,\mathbf{k}}] = \mathbf{L}^{\gamma n d}[U_I \setminus (P_\delta(\mathbf{f}_I)^{-1}(0))]$, if $n = \ell_\Gamma(N_I(\mathbf{k}))$,

(2) $[\widetilde{\mathcal{Y}}_{n,\mathbf{k}}] = \mathbf{L}^{\gamma n d - m}[\mathbf{G}_m \times P_\delta(\mathbf{f}_I)^{-1}(0)]$, if $n - \ell_\Gamma(N_I(\mathbf{k})) = m > 0$. □

Proof. As we assume P is nondegenerate with respect to its Newton polyhedron, P_δ is smooth on $\mathbf{G}^p_m$ and the composed map $P_\delta(\mathbf{f}_I)$ is smooth on U_I. It follows that the morphism $(\widetilde{P}(\mathbf{F})_\mathbf{k}, u) : CY^\circ_\mathbf{k} \to \mathbf{A}^2_\mathbf{k}$ is smooth on a neighborhood of U_I in $CY^\circ_\mathbf{k}$, so one can argue similarly as in the proof of [9, Lemma 5.14]. ■

Using Lemmas 3.4 and 3.5, we may rewrite (3.17) as

$$i^* Z^\gamma_{P(\mathbf{f}),X\setminus X_0(F)} = \sum_{\substack{\delta\in F(\Gamma)\\ I\in B}} Z_{\delta,I}(T), \tag{3.22}$$

with

$$Z_{\delta,I}(T) = \left[U_I \setminus \left(P_\delta\left(\mathbf{f}_I\right)^{-1}(0)\right)\right]\Phi_{\delta,I}(T) + \left[\mathbf{G}_m \times P_\delta\left(\mathbf{f}_I\right)^{-1}(0)\right]\Psi_{\delta,I}(T), \tag{3.23}$$

where

$$\begin{aligned}\Phi_{\delta,I}(T) &= \sum_{\substack{N_I(\mathbf{k})\in\sigma(\delta)\\ \langle N_I(\mathbf{k}),\mathbf{1}\rangle\le\gamma\ell_\Gamma(N_I(\mathbf{k}))}} T^{\ell_\Gamma(N_I(\mathbf{k}))}\,\mathbf{L}^{-\sum_i \nu_i k_i},\\ \Psi_{\delta,I}(T) &= \sum_{\substack{N_I(\mathbf{k})\in\sigma(\delta),\,n>0\\ \langle N_I(\mathbf{k}),\mathbf{1}\rangle\le\gamma\ell_\Gamma(N_I(\mathbf{k}))+\gamma n}} T^{\ell_\Gamma(N_I(\mathbf{k}))+n}\,\mathbf{L}^{-\sum_i \nu_i k_i}.\end{aligned} \tag{3.24}$$

If δ is not contained in a coordinate hyperplane, for γ large enough, the inequality

$$\left\langle N_I(\mathbf{k}), \mathbf{1}\right\rangle \le \gamma\ell_\Gamma\left(N_I(\mathbf{k})\right) + \gamma n \tag{3.25}$$

holds for every $N_I(\mathbf{k})$ in $\sigma(\delta)$ and every $n \ge 0$. It follows that

$$\lim_{T\mapsto\infty} \Phi_{\delta,I}(T) = \lim_{T\mapsto\infty} \Psi_{\delta,I}(T) = \chi\left(N_I^{-1}\left(\sigma(\delta)\right)\right). \tag{3.26}$$

If δ is contained in some coordinate hyperplane, it follows from [9, Lemma 2.10] that

$$\lim_{T\mapsto\infty} \Phi_{\delta,I}(T) = \lim_{T\mapsto\infty} \Psi_{\delta,I}(T) = 0. \tag{3.27}$$

The result follows now from the definition of Ψ_{P_δ} and (2.25). ■

Example 3.6. When $p = 2$ and $P = \Sigma$, one recovers the motivic Thom-Sebastiani formula (cf. [5, 6, 10]) in the way stated in [9]. When f is the family of coordinate functions on the affine space $\mathbf{A}^p_k$, formula (3.16) specializes to the one given by Guibert [8, Proposition 2.1.6].

Remark 3.7. Restricting to a given point x of $X_0(f)$ and applying the Hodge spectrum map Sp of [9, Section 6] to (3.11), one gets a formula for the Hodge-Steenbrink spectrum (cf. [12, 13, 15]) of $P(f)$ at x. It is not immediately clear whether this formula coincides with the one obtained by Terasoma (see [14, Theorem 3.6.1]).

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DISCRETE OSCILLATION THEORY

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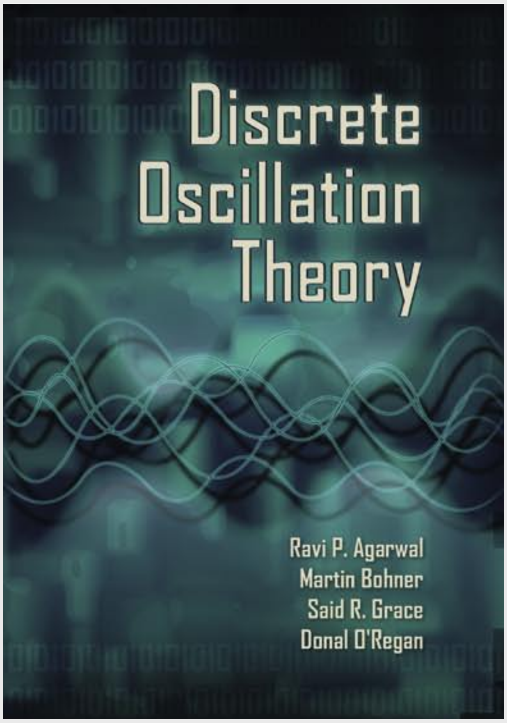

This book is devoted to a rapidly developing branch of the qualitative theory of difference equations with or without delays. It presents the theory of oscillation of difference equations, exhibiting classical as well as very recent results in that area. While there are several books on difference equations and also on oscillation theory for ordinary differential equations, there is until now no book devoted solely to oscillation theory for difference equations. This book is filling the gap, and it can easily be used as an encyclopedia and reference tool for discrete oscillation theory.

In nine chapters, the book covers a wide range of subjects, including oscillation theory for second-order linear difference equations, systems of difference equations, half-linear difference equations, nonlinear difference equations, neutral difference equations, delay difference equations, and differential equations with piecewise constant arguments. This book summarizes almost 300 recent research papers and hence covers all aspects of discrete oscillation theory that have been discussed in recent journal articles. The presented theory is illustrated with 121 examples throughout the book. Each chapter concludes with a section that is devoted to notes and bibliographical and historical remarks.

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