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Regular elements and monodromy of discriminants of finite reflection groups

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1. INTRODUCTION

1.1. Let V be a vectorspace over $\mathbb{C}$ of finite dimension n, and let G be a *finite reflection group in* V, i.e. a finite subgroup of $GL(V)$ which is generated by reflections, see e.g. [17], [21], [8]. For each reflection hyperplane H of G we choose a linear form $\ell_H : V \to \mathbb{C}$ defining H, and we denote by $e(H)$ the order of the group of elements of G which fix H pointwise. Put

$$\delta = \prod_H \ell_H^{e(H)},$$

where the product is over all reflection hyperplanes of G. Let $\Delta : V/G \to \mathbb{C}$ be the map induced by δ, thus Δ is the *discriminant* of G. A subgroup of G is called *parabolic* if it is generated by all reflections of G fixing elementwise a given subspace of V. The *degrees* of G are denoted by $d_1, d_2, \ldots, d_n$. We call a degree of G *primitive* if it is bigger than the degrees of all proper parabolic subgroups of G. When $V = \mathbb{C}^n$ and $G \subset GL_n(\mathbb{R})$ we call G a finite *Coxeter group*.

1.2. We denote by F_0 the *Milnor fiber* of Δ at 0, and by $Z(T, G)$ the zeta function of local monodromy of Δ at 0, i.e.

$$Z(T,G) = \prod_i \det(1 - TM, H^i(F_0, \mathbb{C}))^{(-1)^{i+1}},$$

where M denotes the monodromy automorphism (see e.g. [2], [15]). Thus knowing $Z(T, G)$ is the same as knowing the alternating product of the characteristic polynomials of the monodromy acting on $H(F_0, \mathbb{C})$. In [7] we calculated $Z(T, G)$ for Coxeter groups by using the following recursion:

Theorem 1.3. *If G is a finite Coxeter group then*

$$\prod_{\mathcal{E}} Z(-T, G(\mathcal{E}))^{(-1)^{\sharp\mathcal{E}}} = \prod_{i=1}^{n} \frac{1 - T^{d_i}}{1 - T},$$

where the product at the left runs over all connected subgraphs $\mathcal{E}$ of the Coxeter diagram of G, $G(\mathcal{E})$ denotes the Coxeter group with Coxeter diagram $\mathcal{E}$ and $\sharp\mathcal{E}$ the number of vertices of $\mathcal{E}$. (Each edge of a subgraph has to have the same weight as in the original graph.)

1.4. We gave a case-free proof of this theorem in [7], using Macdonald's formula [13] (proved by Opdam [16])

$$(1.4.1) \quad \int_{\mathbb{R}^n} \delta(x)^s e^{-\|x\|^2} dx = \pi^{n/2} \prod_{i=1}^{n} \frac{\Gamma(d_i s + 1)}{\Gamma(s+1)} \quad (\text{assuming } \|\ell_i\| = 2),$$

and work of Anderson [1] and Loeser–Sabbah [12]. Indeed we showed in [7] that the precise form of the Γ factors in (1.4.1) is actually equivalent with Theorem 1.3.

1.5. In the present paper we calculate $Z(T, G)$ case by case for all irreducible finite reflection groups G. When G is not irreducible but essential $Z(T, G)$ equals 1, see Corollary 3.3 below. Write

$$(1.5.1) \quad Z(-T, G)^{(-1)^n} = \prod_i (1 - T^{|m_i|})^{\text{sign}(m_i)},$$

with i running over a finite index set and $m_i \in \mathbb{Z} \setminus \{0\}$, $m_i + m_j \neq 0$, $\text{sign}(m_i) = m_i/|m_i|$. The m_i are tabulated in 4.1 and 4.2. This yields the following experimental

Theorem 1.6. *Let G be an irreducible finite reflection group in $\mathbb{C}^n$ which can be generated by n reflections of order 2 (for example a finite Coxeter group). Then the m_i in 1.5.1 are given by $d, -(\deg(\delta))/d$ where d runs over all primitive degrees of G which divide* $\deg(\delta)$.

In the Coxeter case any primitive degree divides $\deg(\delta)$ (see [7]), but this is not true in general (e.g. $d = 12$ in G_{27}). For the Coxeter groups of type A_n Theorem 1.6 follows also from [9].

1.7. The method in the present paper is based on some new properties of Springer's regular elements [21] in finite reflection groups. In § 2 we show that the coexponents of the centralizer of a regular element $g \in G$ of order d are the coexponents of G which are congruent to 1 mod d (Theorem 2.8). We also prove that the intersection of a reflection arrangement with a regular eigenspace is again a reflection arrangement (Theorem 2.5). In § 3 we give an explicit formula (Theorem 3.2) for the Lefschetz numbers of the local monodromy of Δ, and express the orders of the regular elements in terms of the m_i (Corollary 3.4). We determine $Z(T, G)$ for all G and verify Theorem 1.6 in § 4, by simple case by case calculations. Finally in the second part of § 4 we determine the zeros of the Bernstein polynomial $b(s)$ of Δ. In the Coxeter group case a formula for $b(s)$ was conjectured by Yano and Sekiguchi [25] and proved by Opdam [16].

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2. THE COEXPONENTS OF THE CENTRALIZER OF A REGULAR ELEMENT

2.1. Assume the notation of (1.1), in particular V is an n-dimensional vector space over $\mathbb{C}$ and $G \subset GL(V)$ is a finite reflection group with degrees $d_1, \ldots, d_n$. A vector $v \in V$ is called *regular* if it is not contained in a reflection hyperplane of G. An element $g \in G$ is called *regular* if it has a regular eigenvector. Let $g \in G$ be regular, with order d. Choose any eigenspace V_g of g which contains a regular vector and let ξ be the corresponding eigenvalue. With these notations we have:

Theorem 2.2 (Springer [21])

(i) *The root of unity ξ has order d.*

(ii) $\dim V_g = \sharp\{i \mid d \text{ divides } d_i\}$,

(iii) *the centralizer C_g of g in G is a reflection group in V_g whose degrees are the d_i divisible by d and whose order is $\prod_{d \mid d_i} d_i$,*

(iv) *the conjugacy class of g consists of all elements of G having* $\dim V_g$ *eigenvalues equal to ξ,*

(v) *the eigenvalues of g are $\xi^{1-d_1}, \ldots, \xi^{1-d_n}$.*

2.3. The orders > 1 of the regular elements of G are called the *regular numbers* of G. Note that any divisor > 1 of a regular number is again a regular number. The reflection arrangement $\mathcal{A}_G$ of G is the union of the reflection hyperplanes of G. It is the set of elements of V which are fixed by a nonidentity element of G. Our next goal is to prove Theorem 2.5 that $\mathcal{A}_{C_g} = V_g \cap \mathcal{A}_G$.

2.4. For any $a \in V$ we denote by D_a the derivation with

$$(D_a h)(v) = \lim_{t \to 0} (h(v + ta) - h(v))/t$$

for any polynomial function h on V and $v \in V$. We learned the following lemma from Springer. It streamlined our original proof of Theorem 2.5 below.

Lemma 2.4. *Let f be a homogeneous G-invariant polynomial function on V of degree e. Let $\sigma \in G$ and $a, b \in V$ eigenvectors of σ with eigenvalues α, β. If $\alpha\beta^{e-1} \neq 1$ then $(D_a f)(b) = 0$.*

Proof. One verifies for any polynomial f and any $a \in V$ that

$$D_{\sigma a}(\sigma f) = \sigma(D_a f).$$

Hence

$$\alpha(D_a f)(b) = (D_{\sigma a}\sigma f)(b) = (\sigma(D_a f))(b) = (D_a f)(\beta^{-1}b)$$
$$= \beta^{-(e-1)}(D_a f)(b),$$

which yields the lemma. □

Theorem 2.5. *Let g be a regular element of the finite reflection group G and let V_g be an eigenspace of g containing a regular vector. If H is a reflection hyperplane of G then $H \cap V_g$ is a reflection hyperplane of the centralizer C_g of g considered as a reflection group in V_g. Hence the reflection arrangement of C_g equals $V_g \cap \mathcal{A}_G$.*

Proof. Let d be the order of g. Let $f_1, \ldots, f_n$, with $n = \dim V$, be homogeneous generators for the $\mathbb{C}$-algebra of G-invariant polynomial functions on V, with degrees $d_1, \ldots, d_n$. Suppose

(1) $\qquad d \mid d_1, d_2, \ldots, d_r \quad \text{and} \quad d \nmid d_{r+1}, \ldots, d_n.$

Note that $r = \dim V_g$, by Theorem 2.2. Choose an Hermitian scalar product on V which is preserved by G, and an orthonormal basis for V consisting of eigenvectors of g. Let $x = (x_1, \ldots, x_n)$ be the corresponding coordinate system with V_g the locus of $x_{r+1} = \cdots = x_n = 0$. Let $a = (a_1, \ldots, a_n) \in V \backslash \{0\}$ be orthogonal to the hyperplane H. Since V_g contains a regular vector, $V_g \not\subset H$ and not all $a_1, \ldots, a_r$ are zero. From Lemma 2.4, with σ a reflection with respect to H, we obtain that

(2) $\qquad D_a(f_i) = \sum_{j=1}^{n} a_j \dfrac{\partial f_i}{\partial x_j}$ is zero on H, for $i = 1, \ldots, n.$

Moreover again from Lemma 2.4, with σ replaced by g, we deduce that

(3) $\qquad \dfrac{\partial f_i}{\partial x_j}$ is zero on V_g, for $i = 1, \ldots, r$ and $j = r+1, \ldots, n,$

because of (1) and 2.2(i). Now (2) and (3) yield

$$\sum_{j=1}^{r} a_j \frac{\partial f_i}{\partial x_j} \text{ is zero on } V_g \cap H, \quad \text{for } i = 1, \ldots, r.$$

Since not all $a_1, \ldots, a_r$ are zero, the determinant

$$\det\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{i=1,\ldots,r\\ j=1,\ldots,r}}$$

Theorem 2.5 was recently obtained independently by G.I. Lehrer, using a different method. See corollary 5.8 in his paper 'Poincaré polynomials for unitary reflection groups', to appear in Inventiones Math.

is zero on $V_g \cap H$. But the restrictions of $f_1, \ldots, f_r$ to V_g are algebraically independent generators for the $\mathbb{C}$-algebra of C_g-invariant polynomial functions on V_g, see [21]. Thus the locus in V_g of the above determinant is the union of the reflection hyperplanes of C_g in V_g. Thus $V_g \cap H$ is such a reflection hyperplane. $\square$

2.6. We denote the $\mathbb{C}$-algebra of polynomial functions on V by $\mathbb{C}[V]$. The group $G \subset GL(V)$ acts on $\mathbb{C}[V] \otimes V$ by $\sigma(h \otimes v) = (h \circ \sigma^{-1}) \otimes \sigma(v)$, for $\sigma \in G$, $h \in \mathbb{C}[V]$, $v \in V$. Moreover $\mathbb{C}[V] \otimes V$ is graded by $\deg(h \otimes v) = \deg h$. Note that the $\mathbb{C}[G]$-algebra $\mathbb{C}[V] \otimes V$ is canonically isomorphic to the algebra of polynomial vector fields on V, by sending $h \otimes v$ to the derivation hD_v. The *coexponents* $c_1, c_2, \ldots, c_n$ of the finite reflection group G are the degrees of any homogeneous basis of the module $(\mathbb{C}[V] \otimes V)^G$ (or the module of G-invariant polynomial vector fields on V) over the ring $\mathbb{C}[V]^G$ of G-invariant polynomial functions on V. See [19, def. 6.50]; these are the numbers $n_1, n_2, \ldots$ in [17].

Theorem 2.6.1 (Orlik and Solomon [17]). *With the above notation we have*

$$\sum_i \dim H^i(V \setminus \mathcal{A}_G, \mathbb{C}) t^i = (1 + c_1 t) \ldots (1 + c_n t),$$

where $\mathcal{A}_G$ is the reflection arrangement of G and t a variable.

In particular the coexponents are completely determined by $\mathcal{A}_G$. The numbers $d_i - 1$ for $i = 1, \ldots, n$, are called the exponents of G. It is well known that if G is a finite Coxeter group then the coexponents are equal to the exponents.

2.7. As above $c_1, \ldots, c_n$ denote the coexponents of G. Let $g \in G$ be a regular element of order $d > 1$ and V_g an eigenspace of g containing a regular vector with eigenvalue ξ (which has order d by 2.2). Put $r = \dim V_g$ and let $b_1 \leq b_2 \leq \cdots \leq b_r$ be the coexponents of the centralizer C_g of g considered as a reflection group in V_g. Our next goal is to show Theorem 2.8 that the b_i are exactly the c_j which are $\equiv 1 \bmod d$.

Lemma 2.7.1. *With the above notation we have*:

(i) *The eigenvalues of g are the ξ^{c_i}, $i = 1, \ldots, n$.*
(ii) *There are exactly r values of i with $c_i \equiv 1 \bmod d$.*
(iii) *$b_i \equiv 1 \bmod d$, for $i = 1, \ldots, r$.*

Proof. (i) Apply Proposition 4.5 of [21] with ρ the irreducible components of the action of G on V^* (= the dual of V).

(ii) Follows directly from (i) because there are exactly r eigenvalues of g which equal ξ.

(iii) By (i) with G replaced by C_g and V by V_g, we have $\xi^{b_i} = \xi$ for $i = 1, \ldots, r$. $\square$

The above lemma implies that we may assume that $c_1 \leq c_2 \leq \ldots \leq c_r$ are the coexponents of G which are $\equiv 1 \bmod d$.

Lemma 2.7.2. *With the above notation we have:*

(i) *The coexponents of C_g only depend on the degree d of g but not on g.*

(ii) $b_i \leq c_i$ *for* $i = 1, \ldots, r$.

(iii) *Let e be a regular number which is divisible by d. Then the sequence $(b_i \bmod e)_{i=1,\ldots,r}$ coincides with the sequence $(c_i \bmod e)_{i=1,\ldots,r}$ up to a permutation. In particular if $e \geq c_r$ then $b_i = c_i$ for $i = 1, \ldots, r$.*

(iv) *Let γ be the least common multiple of the regular numbers which are divisible by d.*
Then

$$\sum_{i=1}^{r} b_i \equiv \sum_{i=1}^{r} c_i \bmod \gamma.$$

In particular if $\gamma \geq \sum_{i=1}^{r} c_i$ then $b_i = c_i$ for $i = 1, \ldots, r$.

Proof. (i) Let $g' \in G$ be another regular element of order d and $V_{g'}$ an eigenspace of g' containing a regular vector with eigenvalue ξ'. From Proposition 3.2 of [21] it follows that

$$\bigcup_{h \in G} V(h, \xi) = \bigcup_{h \in G} V(h, \xi'),$$

where $V(h, \xi)$ denotes the eigenspace of h with eigenvalue ξ. Hence there exists $h \in G$ such that

$$V_{g'} = V(g', \xi') = V(h, \xi),$$

because $\dim V_{g'} = r \geq \dim V(h, \xi)$ by 2.2.(ii) and [21, Theorem 3.4]. Then h is conjugate to g, by Theorem 2.2.(iv). Thus it suffices to show that $C_{g'}$ in $V_{g'}$ and C_h in $V(h, \xi)$ have the same coexponents. But this is clear because their reflection arrangements coincide, both being equal to the intersection of $V_{g'} = V(h, \xi)$ with $\mathcal{A}_G$, by Theorem 2.5.

(ii) Choose an Hermitian scalar product on V which is preserved by G and an orthonormal basis for V consisting of eigenvectors of g. Let $x = (x_1, \ldots, x_n)$ be the corresponding coordinate system with V_g the locus of $x_{r+1} = \cdots = x_n = 0$. Let R_g be the $\mathbb{C}$-algebra of C_g-invariant polynomial functions on V_g. By definition of the coexponents c_i, there exists a basis

$$\sum_{j=1}^{n} f_{ij} \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, n,$$

for the $\mathbb{C}[V]^G$ module of G-invariant polynomial vector fields on V, with f_{ij} homogeneous of degree c_i.

Claim. The f_{ij} are zero on V_g for $i = 1, \ldots, r$ and $j = r+1, \ldots, n$.

It is well known that $\det(f_{ij})_{i,j=1,\dots,n}$ equals the product of the linear forms defining the reflection hyperplanes of G up to a factor in $\mathbb{C}$, see [19, p. 238] or [17, (2.11)]. Since V_g contains a regular vector we see that the above determinant is not identically zero on V_g. Hence the claim implies that $\det(f_{ij})_{i,j=1,\dots,r}$ is not identically zero on V_g. Thus the

$$\Gamma_i := \sum_{j=1}^{r} f_{ij}\,|_{V_g} \frac{\partial}{\partial x_j}, \quad \text{for } i = 1, \dots, r,$$

are C_g-invariant polynomial vector fields on V_g which are linearly independent over R_g. By definition of the b_i, there exists a basis

$$\theta_i = \sum_{j=1}^{r} g_{ij} \frac{\partial}{\partial x_j}, \quad i = 1, \dots, r,$$

for the R_g-module of C_g-invariant polynomial vector fields on V_g, with g_{ij} homogeneous of degree b_i. Suppose now that $b_\ell > c_\ell$ for some $\ell \leq r$. Since

$$\deg \theta_r \geq \cdots \geq \deg \theta_\ell = b_\ell > c_\ell = \deg \Gamma_\ell \geq \cdots \geq \deg \Gamma_1,$$

we see that $\Gamma_1, \dots, \Gamma_\ell$ are R_g-linear combinations of $\theta_1, \dots, \theta_{\ell-1}$. But this is impossible because the $\Gamma_1, \dots, \Gamma_\ell$ are linearly independent over R_g. Thus $b_i \leq c_i$ for $i = 1, \dots, r$. To finish the proof of (ii) we still have to give the

Proof of the claim. Note that g is given by

$$(x_1, \dots, x_n) \mapsto (\eta_1 x_1, \dots, \eta_n x_n)$$

with $\eta_1 = \eta_2 = \cdots = \eta_r = \xi$ and $\xi \neq \eta_j \in \mathbb{C}$ for all $j > r$. The G-invariance implies that

$$(1) \qquad f_{ij}(\eta_1^{-1} x_1, \dots, \eta_n^{-1} x_n)\eta_j = f_{ij}(x_1, \dots, x_n).$$

Suppose that f_{ij} is not identically zero on V_g, for some $i, j \leq n$. Then f_{ij} contains a monomial in $x_1, \dots, x_r$ of degree c_i. Hence (1) yields that $\xi^{-c_i}\eta_j = 1$. If $1 \leq i \leq r$ then $c_i \equiv 1 \bmod d$ (by definition) and thus $\xi = \eta_j$ which implies $j \leq r$. This proves the claim. Note that the above also shows that f_{ij} is zero on V_g for $i = r+1, \dots, n$ and $j = 1, 2, \dots, r$.

(iii) Let $h \in G$ be a regular element of order e and V_h an eigenspace of h containing a regular vector with eigenvalue say η. Put $\xi' = \eta^{e/d}$, $g' = h^{e/d}$ and let $V_{g'}$ be the eigenspace of g' with eigenvalue ξ'. Since $V_h \subset V_{g'}$, the element g' is regular of order d. Because of (i) we may suppose that $g = g'$, $\xi = \xi'$, and $V_g = V_{g'}$. Note that $h \in C_g$. By Lemma 2.7.1(i) with g replaced by h we see that the eigenvalues of h on V are the η^{c_i} for $i = 1, \dots, n$. At the other hand, by the same lemma with G replaced by C_g, V by V_g, and g by h, we obtain that the eigenvalues of h on V_g are the η^{b_i} for $i = 1, \dots, r$. Thus the sequence $(b_i \bmod e)_{i=1,\dots,r}$ is a subsequence of $(c_i \bmod e)_{i=1,\dots,n}$ up to permutation. Apply now Lemma 2.7.1(ii), and (iii) to obtain the first assertion of (iii). The second assertion follows directly from the first by (ii).

(iv) Follows directly from (iii) and (ii). □

Theorem 2.8. *Let G be a finite reflection group and $g \in G$ a regular element of order d. The coexponents of the centralizer C_g of g are the coexponents of G which are $\equiv 1 \bmod d$.*

Proof. It suffices to prove the theorem when G is irreducible. We do this case by case using the tables giving the regular numbers [21, p. 175, 177–178], [5, p. 391, 395, and 412] (take all the divisors > 1 of the regular degrees in [5]) and the coexponents [17, p. 92], [19, p. 287] of each irreducible G. For example the coexponents of E_8 are $1, 7, 11, 13, 17, 19, 23, 29$ and the regular numbers are the divisors > 1 of 30, 24 or 20. We may suppose that $n \geq 3$ and $d \geq 2$. Looking through these tables one verifies immediately that the largest regular number which is divisible by d is larger than the largest coexponent of G which is $\equiv 1 \bmod d$, except in the following 'bad' cases:

(1) $d = 4$ in E_8,
(2) $d = 4$ in H_4,
(3) the monomial groups $G(m, p, n)$ with $m \geq 2$, $1 < p < m$, $n \geq 3$,
(4) the monomial groups $G(m, m, n)$ with $m \geq 2$, $n \geq 3$, $d \mid n$, $d \nmid m$.

Thus Lemma 2.7.2(iii) yields the theorem except in the 'bad' cases (1), (2), (3) and (4). Case (1) and (2) follow directly from Lemma 2.7.2(iv). Case (3) follows from the theorem for $G(m, 1, n)$ (which is not a 'bad' case!) because $G(m, 1, n)$ and $G(m, p, n)$ have the same reflection arrangement when $p < m$ and because of Theorem 2.5. Indeed the reflection arrangement determines the coexponents, see 2.6. Finally case (4) follows by an explicit calculation which shows that the arrangement of V_g equals the arrangement of $G(md/e, 1, ne/d)$, where $e = \gcd(d, m)$. □

Corollary 2.9. *If G is irreducible then C_g is also irreducible.*

Proof. By Schur's lemma $(V^* \otimes V)^G = \mathrm{Hom}_G(V^*, V^*)$ has dimension 1 over $\mathbb{C}$. Hence, since G does not act trivially on V, exactly one coexponent of G equals 1. Thus by Theorem 2.8 exactly one coexponent of C_g equals 1. This implies that C_g is irreducible because each irreducible component of C_g would contribute a coexponent 1. □

3. THE LEFSCHETZ NUMBERS OF LOCAL MONODROMY

3.1. Let $G \subset GL(\mathbb{C}^n)$ be a finite reflection group and let δ, Δ be as in 1.1. The zeta function $Z(T, G)$ of local monodromy of Δ at 0 can be written as

$$(3.1.1) \quad Z(T, G) = \prod_i (1 - T^{|n_i|})^{-\mathrm{sign}(n_i)},$$

with i running over a finite index set, and $n_i \in \mathbb{Z} \setminus \{0\}$, $n_i + n_j \neq 0$. (Thus the n_i are unique up to order.) The sequence of numbers m_i in 1.5.1 is obtained from the n_i by the following rule: Replace each odd n_i by $2n_i$ and $-n_i$, delete any pair of opposite numbers, and finally multiply all numbers with $(-1)^{n-1}$. Conversely the n_i are obtained from the m_i by the same rule. Since δ is homogeneous, n_i divides $\deg(\delta)$. Hence when $\deg(\delta)$ is even, also m_i divides $\deg(\delta)$. It is well known [19, p. 231 and p. 238] that

$$(3.1.2) \quad \deg(\delta) = \sum_i (d_i - 1) + \sum_i c_i$$

where the d_i are the degrees and the c_i the coexponents of G. For any $a \in \mathbb{N}$, the *Lefschetz number* $\Lambda(a)$ of the local monodromy to the power a of Δ at 0 is defined by

$$\Lambda(a) = \sum_i (-1)^i \operatorname{Tr}(M^a, H^i(F_0, \mathbb{C})) \in \mathbb{Z},$$

where M is the monodromy automorphism and F_0 the Milnor fiber of Δ at 0. It is well known (see e.g. [15, p. 77]) that the $\Lambda(a)$ completely determine $Z(T, G)$. Indeed the n_i are uniquely determined by $n_i \mid \deg(\delta)$ and

$$(3.1.3) \quad \Lambda(a) = \sum_{n_i \mid a} n_i, \quad \text{for all } a \text{ dividing } \deg(\delta).$$

By Möbius inversion one gets

$$(3.1.4) \quad Z(T, G) = \prod_{d \mid \deg(\delta)} (1 - T^d)^{\alpha(d)},$$

where $\alpha(d) = d^{-1} \sum_{a \mid d} \mu(d/a) \Lambda(a)$ and μ denotes the Möbius function. The proof of the following theorem is based on Theorems 2.5 and 2.6.1.

Theorem 3.2. *Let $G \subset GL(\mathbb{C}^n)$ be a finite reflection group and $d \in \mathbb{N}$ with $d \mid \deg(\delta)$. If G has a regular element g with order d, then*

$$\Lambda\left(\frac{\deg(\delta)}{d}\right) = \frac{\deg(\delta)}{\prod_i d_i(g)} \prod_{i \neq 1} (1 - c_i(g)),$$

where $d_1(g) \leq d_2(g) \leq \cdots$, resp. $c_1(g) \leq c_2(g) \leq \cdots$, are the degrees, resp. coexponents, of the centralizer C_g of g. If G has no regular element with order d, then

$$\Lambda\left(\frac{\deg(\delta)}{d}\right) = 0.$$

Proof. Put $d' = (\deg \delta)/d$. Because δ is homogeneous the map

$$h : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto e^{2\pi i/\deg \delta} x$$

induces the local monodromy M. Hence by [15, Lemma 9.5] we have

$$\Lambda(d') = \chi(\{x \in \mathbb{C}^n/G \mid \delta(x) = 1,\ h^{d'}(x) = x \bmod G\})$$

$$(3.2.1) \quad = \frac{1}{\sharp G} \chi(\{x \in \mathbb{C}^n \mid \delta(x) = 1,\ \exists w \in G:\ w(x) = e^{2\pi i/d} x\})$$

$$(3.2.2) \quad = \frac{\deg(\delta)}{\sharp G} \chi\left(\left\{x \in \mathbb{P}^{n-1} \mid \delta(x) \neq 0,\ x \in \bigcup_{w \in G} \mathbb{P}(V(w, e^{2\pi i/d}))\right\}\right)$$

where for any $w \in G$ and $\xi \in \mathbb{C}$ we denote by $\mathbb{P}(V(w, \xi))$ the projectivization of the affine space $V(w,\xi) := \{x \in \mathbb{C}^n \mid w(x) = \xi x\}$. Note that $x \in \mathbb{C}^n$ is regular if and only if $\delta(x) \neq 0$. Hence $\Lambda(d') = 0$ when G has no regular element with order d, because of (3.2.2) and 2.2(i). Suppose now that G has a regular element g with order d. Then g has a regular eigenvector with eigenvalue a primitive dth root of unity ξ. By [21, Proposition 3.2] we have

$$\bigcup_{w\in G} V(w, e^{2\pi i/d}) = \bigcup_{w\in G} V(w,\xi).$$

Hence

$$\Lambda(d') = \frac{\deg(\delta)}{\sharp G}\,\chi\left(\left\{x \in \mathbb{P}^{n-1} \mid \delta(x) \neq 0,\ x \in \bigcup_{w\in G} \mathbb{P}(V(w,\xi))\right\}\right).$$

By Theorem 2.2, the set of $w \in G$, for which $V(w,\xi)$ contains a regular vector, equals the conjugacy class of g and so has $\sharp G/\sharp C_g$ elements. Thus

$$\Lambda(d') = \frac{\deg(\delta)}{\sharp G}\,\chi(\{x \in \mathbb{P}^{n-1} \mid \delta(x) \neq 0,\ x \in \mathbb{P}(V(g,\xi))\}),$$

because $V(w,\xi) \cap V(w',\xi)$ does not contain any regular vector whenever $w \neq w'$. Theorem 3.2 follows now directly from Theorems 2.5, and 2.6.1, and Proposition 5.1 of [19]. □

Corollary 3.3. *If G is not irreducible but essential (i.e. 0 is the only G-invariant vector), then $Z(T,G) = 1$.*

Proof. In this case there are at least two codegrees equal to 1. Hence by Theorem 3.2 all Lefschetz numbers are zero.

We also give a more direct argument. Write $G = G_1 \oplus G_2$, with G_1 and G_2 essential and denote the δ of G_1, resp. G_2, by δ_1, resp. δ_2. Thus $\delta = \delta_1\delta_2$. Exploiting the homogeneity of δ_1 and δ_2 one verifies that the map $x \mapsto \delta_1(x)$ induces a locally trivial fibration of the space appearing in (3.2.1) onto $\mathbb{C}\setminus\{0\}$. Hence the Euler characteristic of that space is zero and $\Lambda(d') = 0$. This second proof of the corollary does not depend on Theorems 2.5 and 2.6.1. □

Corollary 3.4. *Let G be an irreducible finite reflection group and $d \in \mathbb{N}$, $d > 1$. Then the following assertions are equivalent*

(i) *d is a regular number for G,*
(ii) *$d \mid \deg(\delta)$ and $\Lambda((\deg\delta)/d) \neq 0$,*
(iii) *d divides some $(\deg\delta)/n_i$.*

Moreover when $\deg(\delta)$ *is even these assertions are also equivalent with*

(iv) *d divides some $(\deg\delta)/m_i$.*

Proof. Suppose d is a regular number. Then there is a regular element $g \in G$ with order d and a regular vector v such that $g(v) = \xi v$, with ξ a primitive dth root of unity. Hence $\delta(v) = \delta(\xi v) = \xi^{\deg\delta}\delta(v)$. Since $\delta(v) \neq 0$ we have $\xi^{\deg\delta} = 1$ and $d \mid \deg\delta$. The equivalence of (i) and (ii), follows now from Theorem 3.2, Corollary 2.9, and the fact that an irreducible nontrivial finite reflection group has only one coexponent equal to 1 (cf. the proof of 2.9). The implication

(ii) $\Rightarrow$ (iii) follows directly from (3.1.2). Next assume (iii). We want to show that d is a regular number. Take an n_j with minimal absolute value such that $n_j \mid n_i$. Then $\Lambda(|n_j|) \neq 0$ by (3.1.3). Hence $(\deg \delta)/|n_j|$ is a regular number (or 1) by the equivalence of (i) with (ii). But $d \mid (\deg \delta)/n_i \mid (\deg \delta)/n_j$. Thus d is regular because it divides a regular number. Finally (iii) $\Leftrightarrow$ (iv) because of the rule to obtain the m_i from the n_i. Indeed we have only to consider the n_i which are minimal with respect to divisibility. □

Remark. For the groups G satisfying Theorem 1.6 the regular numbers are thus the divisors > 1 of the m_i. Note however that it is not always sufficient to only take the divisors of the primitive degrees, e.g. E_6 has primitive degrees 12, 9 but 8 is a regular number.

3.5. Let G be a Shephard group, i.e. the symmetry group of a regular complex polytope, and W the associated finite Coxeter group, see [18], [19, p. 265–268]. Denote by κ half the smallest degree of G. It is known [18] that the d_i, resp. $\deg(\delta)$, of G are obtained from the ones of W by multiplying with κ. Moreover the discriminant Δ of G equals the discriminant of W. Hence G and W have the same n_i and m_i. Since G is irreducible, Corollary 3.4 directly implies

Corollary 3.5.1. *In the above situation the regular numbers of G which are maximal with respect to divisibility are the ones of W multiplied with κ.*

Corollary 3.6. *Let $G \subset GL(\mathbb{C}^n)$ be an irreducible finite reflection group which can be generated by n reflections (i.e. a duality group [17]). Then $-n$ is the m_i with smallest absolute value and appears only once among the m_i.*

Proof. Let h be the largest degree of G. Then $\deg(\delta) = nh$ and h divides only one d_i, see [17, Theorem 5.5]. Moreover one verifies in the tables that h is a regular number. The corollary follows now directly from 3.1.3 and Theorem 3.2 for $d = h$ and for d a multiple of h. □

4. CALCULATION OF THE LOCAL MONODROMY

Theorem 4.1. *For the Coxeter groups A_n the m_i in 1.5.1 are $n+1, -n$. For the monomial groups $G(m,p,n)$ the m_i are $2(1+p(n-1)), -(1+p(n-1))$ when $p < m$, and are $m(n-1), -n$ when $p = m$.*

Proof. We only treat the monomial groups:

Case $p < m$: The degrees are $m, 2m, \ldots, (n-1)m, nm/p$, the coexponents $1, m+1, \ldots, (n-1)m+1$ and the regular numbers are the divisors > 1 of mn/p. Hence $\deg \delta = nm(1+p(n-1))/p$. Using Theorem 3.2 one verifies that

$$\Lambda\left(\frac{\deg \delta}{d}\right) = -(1+p(n-1))(-1)^{n\,\gcd(d,m)/d},$$

when $d \mid nm/p$, and zero otherwise. If $d \mid nm$, then $n\gcd(d,m)/d$ is odd when n is odd, and has the same parity as nm/d when n is even. Using 3.1.3 it is easy to see that the n_i are $1+p(n-1)$ if n is odd, $-(1+p(n-1))$ if both n and p are even, and $-2(1+p(n-1)), 1+p(n-1)$ if n is even and p odd.

Case $p=m$: The degrees are $m, 2m, \ldots, (n-1)m, n$, the coexponents 1, $m+1, \ldots, (n-2)m+1, (n-1)(m-1)$ and the regular numbers are the divisors of $(n-1)m$ or n. Hence $\deg \delta = mn(n-1)$. Using Theorem 3.2 one verifies that $\Lambda((\deg \delta)/d) = \Lambda_1 + \Lambda_2$ with $\Lambda_1 = -n(-1)^{(n-1)\gcd(d,m)/d}$ if $d \mid (n-1)m$, and 0 otherwise, $\Lambda_2 = -m(n-1)(-1)^{n\gcd(d,m)/d}$ if $d \mid n$, and 0 otherwise. One proceeds now as in the previous case. □

Note that the Coxeter groups B_n, D_n, G_2 and $I_2(n)$ equal respectively $G(2,1,n)$, $G(2,2,n)$, $G(6,6,2)$, and $G(n,n,2)$.

4.2. It remains to determine the m_i in 1.5.1 for the *exceptional* irreducible finite reflection groups in $\mathbb{C}^n$. In view of 3.5 we have only to care about Coxeter groups and non-Shephard groups. All these are listed in table 4.2.1 below. Here G_j denotes the reflection group with Shephard–Todd number j, see [20, table VII]. The groups in the table who satisfy the hypothesis of Theorem 1.6 are indicated by a $*$ in the last column. The m_i are calculated case by case using Theorems 3.2, 2.2, 2.8 and the tables (mentioned in the proof of 2.8) giving the regular numbers, degrees and coexponents.

Table 4.2.1. Exceptional Coxeter groups and exceptional non-Shephard groups

Group	n	m_i	
G_7	2	$6, -3$	
G_{11}	2	$6, -3$	
G_{12}	2	$12, 3, -6, -4$	
G_{13}	2	$18, 3, -9, -6$	
G_{15}	2	$10, -5$	
G_{19}	2	$6, -3$	
G_{22}	2	$30, 5, 3, -15, -10, -6$	
$G_{23} = H_3$	3	$10, 6, -5, -3$	$*$
G_{24}	3	$14, 6, -7, -3$	$*$
G_{27}	3	$30, 6, -15, -3$	$*$
$G_{28} = F_4$	4	$12, 8, -6, -4$	$*$
G_{29}	4	$20, -4$	$*$
$G_{30} = H_4$	4	$30, 20, 12, -10, -6, -4$	$*$
G_{31}	4	$30, 5, -15, -6$	
G_{33}	5	$18, 10, -9, -5$	$*$
G_{34}	6	$42, -6$	$*$
$G_{35} = E_6$	6	$12, 9, -8, -6$	$*$
$G_{36} = E_7$	7	$18, 14, -9, -7$	$*$
$G_{37} = E_8$	8	$30, 24, 20, -12, -10, -8$	$*$

Remark 4.3. We see there are as many positive as negative m_i. This holds be-

cause -1 is no eigenvalue of monodromy since the Bernstein polynomial of Δ has no root $\equiv \frac{1}{2} \bmod \mathbb{Z}$ by 4.9.2, cf. 4.6.

4.4. The only groups satisfying the hypothesis of Theorem 1.6 are the finite Coxeter groups, the monomial groups $G(m,m,n)$ and $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$. Theorem 1.6 follows from 4.1 and table 4.2.1 by using the tables for the parabolic subgroups [19, p. 189–300]. In the Coxeter case the parabolic subgroups of G correspond (up to conjugation) with full subgraphs of the Coxeter diagram of G.

Finally we mention that the recursion 1.3 does not generalize to all groups in 1.6.

4.5. We denote by $b_G(s)$ the Bernstein polynomial of the discriminant Δ of G. When G is a Coxeter group, Opdam [16] proved

$$b_G(s) = \prod_{i=1}^{n} \prod_{1 \le \ell < d_i} \left(s + \frac{1}{2} + \frac{\ell}{d_i} \right).$$

This was conjectured by Yano and Sekiguchi [25]. We will determine the roots of $b_G(s)$ for any G in 4.10.

4.6. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial map. It is known [6, Lemma 4.6] that ξ is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0)$ if and only if ξ is a zero or pole of the zeta function of local monodromy of f at some (possibly different) point of $f^{-1}(0)$. Thus ξ is an eigenvalue of the local monodromy of the discriminant Δ of G at some point of $\Delta^{-1}(0)$ if and only if ξ is a zero or pole of $Z(P,T)$ for some parabolic subgroup P of G. Using the tables in [19, p. 189–300] and 4.2.1 we can easily determine these ξ. By [14] these eigenvalues ξ are precisely the numbers $e^{2\pi i s}$ with $b_G(s) = 0$.

4.7. Let G be irreducible. The *Yano number* $H(G)$ of G is defined in [22] by

$$\text{(4.7.1)} \quad \frac{1}{2} + \frac{1}{H(G)} = \frac{\sum_{i=1}^{n} d_i}{\deg(\delta)}.$$

Since $\sum_i (d_i - 1)$ equals the number of reflections in G, we have $H(G) \ge 2$ with equality only if $n = 1$. Using the tables one verifies that

$$\text{(4.7.2)} \quad H(P) < H(G)$$

for any proper irreducible parabolic subgroup P of G, considering P as a reflection group in $\mathbb{C}^n/(\mathbb{C}^n)^G$. In 4.11 below we explain why $H(G)$ is integral and equal to the m_i with largest absolute value. The following theorem was conjectured by Yano in [22].

Theorem 4.8. *Let G be an irreducible finite reflection group. Then the largest zero of the Bernstein polynomial $b_G(s)$ equals $-1/2 - 1/H(G)$.*

Proof. We may suppose $n > 1$, hence $-1/2 - 1/H(G) > -1$. Consider the embedded resolution of singularities of the reflection arrangement of G obtained by blowing up first the origin, secondly all 1-dimensional intersections of reflection hyperplanes, next all 2-dimensional intersections, and so on (see [11, §7]). Use this resolution to find the candidate poles (see [3]) of the integral

$$\int_{\mathbb{C}^n/G} \varphi|\Delta|^{2s}|dy \wedge d\bar{y}| = \frac{1}{\#G} \int_{\mathbb{C}^n} \varphi \prod_H |\ell_H|^{2(e(H)s+e(H)-1)}|dx \wedge d\bar{x}|,$$

where φ is a nonnegative real valued C^∞ function on $\mathbb{C}^n/G = \mathbb{C}^n$ with $\varphi(0) > 0$ and compact support, and where the product is over all reflection hyperplanes H, see 1.1. Because of 4.7.2, the largest candiate pole equals $-1/2 - 1/H(G)$ and is really a pole since it is > -1 (compare with [2, §7.3 Theorem 5]). Apply now 3.11 of [23]. □

Prof. Yano has announced the following result:

Theorem 4.9 (Yano [24]). *Let G be any finite reflection group. Then*

(4.9.1) $b_G(-2-s) = \pm b_G(s)$.

Combining Theorems 4.8 and 4.9 we obtain

(4.9.2) $-\frac{3}{2} < s < -\frac{1}{2}$ if $b_G(s) = 0$.

This inequality together with 4.6 implies:

Corollary 4.10. *The zeros of the Bernstein polynomial $b_G(s)$ are $-1/2 - \ell/k$ where $1 \leq \ell < k$, $\gcd(\ell, k) = 1$, and k is the order (as root of unity) of a zero or pole of $Z(-T, P)$ with P a parabolic subgroup of G.*

Using the tables for the parabolic subgroups and the values of the m_i one determines very easily the values of k:
Example. For G_{27} the k are $30, 10, 6, 5, 4, 3, 2$, the degrees are $30, 12, 6$ and the coexponents $25, 19, 1$.

Remark 4.11. Let G be irreducible. Clearly 4.8 and 4.10 imply that $H(G)$ is the largest k. Hence $H(G)$ is integral and equal to the largest $|m_i|$ of G, because of 4.7.2. Moreover using the perversity of the complex of nearby cycles one deduces (as in the proof of Lemma 4.6 in [6]) that $H(G)$ equals the m_i of G with largest absolute value.

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