Germs of paths on algebraic varieties: a gentle introduction to motivic integration

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Overview I

Overview II

Milnor fiber
Ax-Kochen-Eršov Principle for integrals

Algebraic varieties I

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The set of k-points of the corresponding (affine) algebraic variety X_F is the set of points in k^N which are common zeroes of the polynomials f_i, that is,

$$X_F(k) = \{x = (x_1, \ldots, x_n) \in k^N \colon (\forall i)(f_i(x_1, \ldots, x_N) = 0)\}.$$

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For any ring K containing k, we can also consider the set of K-points

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In particular, if $r = 0$, we get the affine space $\mathbb{A}^N$ with $\mathbb{A}^N(K) = K^N$ for every K containing k.

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of formal germs of arcs on X_F may be extremely powerful.

Subvariety, complement

If $F' = F \cup_{i \in I} \{g_i\}$, we have

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for all K. We write $X_{F'} \subset X_F$ and we say $X_{F'}$ is a subvariety of X_F.

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for all K, so that we can set $X_{\tilde{F}} = X_F \setminus X_{F'}$?

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Yes, add new variables U_i for each g_i and set $\tilde{F} = (f_i, g_i U_i - 1)$.

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If we have two families $F = (f_i(T_1, \ldots, T_N))$ and $F' = (f_i'(S_1, \ldots, S_{N'}))$, with non variable in common, we may set

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There is a natural notion of morphisms between algebraic varieties. Essentially they are induced by "polynomial transformations". In particular, there is a notion of isomorphism of algebraic varieties. For instance, $T \mapsto (T^2, T^3)$ induces an isomorphism between $\mathbb{A}^1 \setminus \{0\}$ and the variety defined by $X_1^3 - X_2^2 = 0, X_1 X_3 - 1 = 0$.

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- Additivity: If X' is a subvariety of X,

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- Multiplicativity:

$$\lambda(X \times X') = \lambda(X) \cdot \lambda(X').$$

Euler characteristic I

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It can be shown to be independent from the cell decomposition.

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Proposition

$X \mapsto \mathrm{Eu}(X)$ is an additive $\mathbb{Z}$-valued invariant.

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If $k = \mathbb{F}_q$ and X is a k-algebraic variety, we set

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Clearly, $X \mapsto N_{q^e}(X)$ is an additive invariant.

Euler characteristics via counting I

When $k = \mathbb{Q}$, and X is a variety over k, we may at the same time consider $\mathrm{Eu}(X)$ and reduce the equations of X mod p, for p not dividing the denominators of the equations of f, in order to get a variety X_p over $\mathbb{F}_p$.

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Is there any relation between $N_{p^e}(X_p)$ and $\mathrm{Eu}(X)$?

Euler characteristics via counting II

The following is a consequence of results by A. Grothendieck going back to the 60's:

Theorem (Crude Form)

Given a X, for almost all p,

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More precisely:

Euler characteristics via counting III

Theorem (Precise Form)

Given a X, for almost all p, there exists finite families of complex numbers α_i, $i \in I$, and β_j, $j \in J$, depending only on X and p, such that

$$N_{p^e}(X_p) = \sum_I \alpha_i^e - \sum_J \beta_j^e$$

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So, Euler characteristics may be computed by counting in finite fields!

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Similarly, the field of p-adic numbers $\mathbb{Q}_p$ is the set of series $\sum_{i \geq -\alpha} a_i p^i$, with a_i in $\{0, \ldots, p-1\}$ and $\alpha \geq 0$.

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However, they are very much the same: by the Ax-Kochen-Eršov principle, we shall discuss later in the talk, they are asymptotically, that is, for $p \gg 0$, the same.

Birational Geometry

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We shall furthermore assume F is equal to the union of smooth connected hypersurfaces (= of complex codimension 1) E_i, $i \in A$, of Y, which are furthermore mutually transverse.

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For $I \subset A$, we set

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Theorem (Denef and Loeser)

Let $h : Y \to X$ be a DNC *modification. Then we have*

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Remark

The result also holds in the complex analytic setting.

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Challenging problem: Find a direct proof ...

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"Birational Calabi-Yau have the same Betti numbers."

In 1995, this was proved by V. Batyrev by using p-adic integrals in a way similar to Denef and Loeser and the part of the Weil conjectures proved by Deligne (which allows for projective varieties to deduce not only Euler characteristics, but also Betti numbers, from counting in finite fields).

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Motivic integration was born ...

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$$X \mapsto [X]$$

from the category of algebraic varieties over k to some universal ring M_k.

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It is characterized by the fact that for every non degenerate invariant λ with values in a ring R,
there exists a unique ring morphism $\alpha : M_k \to R$ such that

$$\lambda(X) = \alpha([X]),$$

for every X.

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More generally, one can integrate functions in a way very similar to the *p*-adic case.

In fact a very general theory of motivic integration within the framework of "Constructible motivic functions" has been recently developed by Cluckers and Loeser. It allows to consider integrals with parameters and avoids using a completion M_k.

Construction of the motivic measure I

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The basic idea is to use the truncation morphisms:

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X),$$

with $\mathcal{L}_n(X)$ defined similarly as $\mathcal{L}(X)$ with $K[[t]]$ replaced by $K[[t]]/t^{n+1}$.

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Note that $\mathcal{L}_n(X)$ is finite dimensional and $[\mathbb{A}^1]$ is invertible in M_k.

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If $h : Y \to X$ is a birational morphism, one can express the motivic volume of $\mathcal{L}(X)$ as a motivic integral on $\mathcal{L}(Y)$ involving the order of vanishing of the jacobian.

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If $h : Y \to X$ is a birational morphism, one can express the motivic volume of $\mathcal{L}(X)$ as a motivic integral on $\mathcal{L}(Y)$ involving the order of vanishing of the jacobian. This is achieved by using an analogue of the "change of variables formula" in this setting.

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This is the reason why measure theoretic arguments seem to be so well adapted to birational geometry.

Birational Geometry III

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But there exists one at the level of arc spaces!

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with $\mathrm{Conj}\, G(x)$ the set of conjugacy classes in $G(x)$ and B a space of infinite codimension in $\mathcal{L}(X)_x$.

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Work of Batyrev, Kontsevich, Denef-Loeser, Yasuda, . . .

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Here $B(a, r)$ denotes the closed ball of center a and radius r.

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$$f(\varphi(t)) = t^n + \text{(higher order terms)}.$$

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Theorem (Denef-Loeser)

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In fact, the spaces $\mathcal{X}_n$ do contain much more information about the Milnor fiber and the monodromy:

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Denef and Loeser proved that the series

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Work of Denef-Loeser, Guibert, Bittner, Guibert-Loeser-Merle, etc.

Classical Ax-Kochen-Eršov I

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A (first order ring) sentence is a formula with no free variable.

Theorem (Ax-Kochen-Eršov)

Let φ be a first order sentence. For almost all prime number p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.

Classical Ax-Kochen-Eršov II

Let d be a positive integer. A field k is called $C_2(d)$ if every homogenous polynomial in $n > d^2$ variables with coefficients in k has a non trivial solution in k^n.

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Question (E. Artin): Does $\mathbb{Q}_p$ have the $C_2(d)$ property?

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Indeed, d being given, for a field to be $C_2(d)$ is expressible by a sentence.

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Theorem (Denef-Loeser)

Let φ be a formula in the valued ring language. Then, for almost all p, the sets $X_\varphi(\mathbb{Q}_p)$ and $X_\varphi(\mathbb{F}_p((t)))$ have the same volume.

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There is a similar statement for integrals.

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Theorem (Cluckers-Loeser)

For almost all p, an equality of definable integrals depending on parameters holds for $\mathbb{Q}_p$ if and only if it holds for $\mathbb{F}_p((t))$.

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As a conclusion, let us mention that the above results of Cluckers and Loeser may be generalized in order to deal with integrals involving exponential functions.