Definability in non-archimedean geometry

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Abstract. We discuss several situations involving valued fields for which the model-theoretic notion of definability plays a central role. In particular, we consider applications to p-adic integration, diophantine geometry and topology of non-archimedean spaces.

Mathematics Subject Classification (2010). Primary 03C10, 03C65, 03C98,12J10, 14G22, 22E35; Secondary 03C64, 03C68, 11S80, 11F85, 14G20, 14T05, 20G25, 22E50.

Keywords. Valued fields, p-adic integration, non-archimedean geometry, motivic integration, Berkovich spaces.

1. Introduction

After the groundbreaking work of Ax-Kochen [2] and Eršov [25] in the sixties and of Denef [16] in the eighties, a wide array of applications of model theory of valued fields is now flourishing, ranging over topics as diverse as counting subgroups, the Langlands program and singularity theory. In all these applications the concept of definability in first order logic is central. In this survey, we shall focus on three such applications, each using the notion of definability in the context of valued fields in an essential way.

We start by presenting several transfer theorems for p-adic integrals. Such results allow to transfer statements over $\mathbb{Q}_p$ to statements over $\mathbb{F}_p((t))$ and vice versa. A first result, obtained in collaboration with R. Cluckers deals with identities between integrals with parameters. In work with R. Cluckers and T. Hales it was shown how it can used for the integrals occuring in the fundamental lemma. We shall also present more recent results obtained by R. Cluckers, J. Gordon and I. Halupczok on transfering local integrability or uniform boundedeness statements and some of their applications to p-adic harmonic analysis. In the next section, we shall explain how by working in a definable setting one can deduce global bounds from local bounds on differentials, despite the totally disconnected nature of non-archimedean valued fields and present some diophantine applications. This is recent joint work with R. Cluckers and G. Comte. The last section is about the topology of non-archimedean spaces. We shall present our work with E. Hrushovski on stable completion of algebraic varieties over a valued field, a model-theoretic analogue of the Berkovich analytification. A fundamental statement is that the stable completion of an algebraic variety is pro-definable. We shall explain how using this approach one can prove new tameness results for the topology of Berkovich spaces.

The present overview is far from being exhaustive, for instance it completely leaves out important work of Hrushovski and Kazhdan on motivic integration [29, 30], and some of its

Proceedings of the International Congress of Mathematicians, Seoul, 2014

recent applications [32, 35].

2. Definability and integration

2.1. Definable sets. A language $\mathcal{L}$ is a set consisting of symbols for constants (= 0-ary functions), n-ary functions and n-ary relations. Basic examples are the ring language $\{0, 1, +, -, \times, =\}$, the order language $\{<, =\}$, or the ordered abelian group language $\{<, 0, +, -, =\}$.

An $\mathcal{L}$-structure consists of a set M together with interpretations for symbols in $\mathcal{L}$. One requires that $=$ is interpreted by equality in M. A subset of M^n is said to be definable if it is of the form

$$\{(a_1, \cdots, a_n) \in M^n : \varphi(a_1, \cdots, a_n) \text{ holds}\}$$

with φ a first-order formula in $\mathcal{L}$ with n-free variables. When the formula φ involves parameters running over some $A \subset M$, one says the subset is A-definable. A map between A-definable sets is said to be A-definable if its graph is. In this way one defines the category Def_A of A-definable sets. All these notions extend naturally to many-sorted languages.

2.2. p-adic integrals. In his breakthrough paper [16] on the rationality of the Poincaré series associated to the p-adic points on a variety, Denef proved the following general rationality result for p-adic integrals:

Theorem 2.3. *Let X be a definable subset of $\mathbb{Q}_p^n$ and $g : X \to \mathbb{Q}_p$ be a bounded definable function. Then the integral*

$$\int_X |g|^s |dx|$$

is a rational function of p^{-s}.

Here definability refers to the ring language with parameters in $\mathbb{Q}_p$ (or, which amounts to the same here, any standard valued ring language, for instance $\mathcal{L}_{k,\Gamma}$ considered in 4.8). The proof relies on Macintyre's quantifier elimination theorem [36] for $\mathbb{Q}_p$.

For X a definable subset of $\mathbb{Q}_p^n$, denote by $\mathcal{C}_p(X)$ the $\mathbb{Q}$-algebra generated by functions of the form $|g|$ and $\mathrm{val}(g)$ with $g : X \to \mathbb{Q}_p$ definable. In the paper [17] in which he extended his rationality result to the setting of integrals with parameters, Denef proved the following result about stability under integration for functions in $\mathcal{C}_p$.

Theorem 2.4. *Let X be a definable subset of $\mathbb{Q}_p^n$. Let $\varphi \in C_p(X \times \mathbb{Q}_p^m)$. Assume for any $x \in X$, the function $\varphi_x : \lambda \mapsto \varphi(x, \lambda)$ is integrable. Then the function $x \mapsto \int_{\mathbb{Q}_p^m} \varphi_x |d\lambda|$ belongs to $\mathcal{C}_p(X)$.*

In [18], Denef proved a general cell decomposition theorem for $\mathbb{Q}_p$-definable sets, providing direct proofs of Theorems 2.3 and 2.4 and also of Macintyre's quantifier elimination theorem. The natural question of uniformity in p in Denef's Theorem 2.3 has been addressed by Pas in [39] and by Macintyre in [37]. In the paper [39] a three sorted language has been introduced, nowadays called the Denef-Pas language $\mathcal{L}_{\mathrm{DP}}$. In this language, there are three sorts of variables:

- variables running over the valued field for which the language is the ring language

- variables running over the residue field sort for which the language is the ring language
- variables running over the value group sort for which the language is the language of ordered groups.

and two additional symbols ac and val from the valued field sort to the residue field and value group sort, respectively. For $\mathbb{Q}_p$ or $k((t))$ the angular component map ac is interpreted as the first non zero coefficient in the p-adic, resp. t-adic, expansion and val as the valuation. In this setting, Pas proved a cell decomposition theorem which is uniform in p in [39]. In particular, this provides a new elementary proof of the following version of the classical result of Ax-Kochen-Eršov.

Theorem 2.5. *Let φ be a sentence (that is, a formula with no free variable) in the language $\mathcal{L}_{DP}$. For all but finitely prime numbers p, φ is satisfied in $\mathbb{F}_p((t))$ if and only if it is satisfied in $\mathbb{Q}_p$.*

2.6. Motivic integrals. In the series of papers [10] and [11] in collaboration with Raf Cluckers we have developed a general framework for motivic integration on definable sets in the Denef-Pas language. More precisely let k be a field of characteristic zero and set $K = k((t))$. We consider K as a structure for the Denef-Pas language. For any definable subset S of K^m (or more generally of $K^m \times k^n \times \mathbb{Z}^r$), we define in [10] an algebra $\mathcal{C}(S)$ of "constructible motivic functions" on S. For such functions one defines inductively the notion of being integrable and the value of the integral, using the cell decomposition theorem of Pas [39], and one proves an analogue of Theorem 2.4 in this context. Working in a relative setting is essential here. One of the main advantage of working in the definable setting over previous constructions as those in [19] or [20], is that there is no need anymore to consider completions of Grothendieck rings. Also, we are able to state and prove Fubini and change of variables theorems in full generality, and to deal with integrals with parameters. For more detailed, though accessible, presentations of this theory, we refer to the introduction of [10] and to the paper [12].

2.7. Transfer theorems for constructible motivic functions. Let $\mathbf{F}$ be a number field with ring of integers $\mathcal{O}$. Let $\mathcal{C}_\mathcal{O}$ denote the collection of triples (F, ι, ϖ), where F is non-archimedean local field, $\iota : \mathcal{O} \to F$ a ring homomorphism and ϖ a uniformizer in F. We denote by k_F the residue field of F and by q_F the cardinality of k_F. For $M > 0$, we denote by $\mathcal{C}_{\mathcal{O},M}$ the subcollection of triples (F, ι, ϖ) with F of residue characteristic $> M$.

Assume now $k = \mathbf{F}$, and fix a definable subset S of K^n. For some M large enough, for any (F, ι, ϖ) in $\mathcal{C}_{\mathcal{O},M}$ one may consider the specialization S_F of S in F^n obtained by specializing the formulas defining S using ι and sending t to ϖ. Similarly, for M large enough, a function φ in $\mathcal{C}(S)$ may be specialized to a function on S_F which we shall denote by φ_F.

In [11], we prove the following:

Theorem 2.8. *Let $\psi \in \mathcal{C}(S \times K^m)$ and $\psi' \in \mathcal{C}(S \times K^{m'})$. Then, there exists $M > 0$ such that, for every F_1 and F_2 in $\mathcal{C}_{\mathcal{O},M}$ such that $k_{F_1} \simeq k_{F_2}$,*

$$\int_{F_1^m} (\psi_{F_1})_x |d\lambda| = \int_{F_1^{m'}} (\psi'_{F_1})_x |d\lambda'|$$

for every $x \in S_{F_1}$ if and only if

$$\int_{F_2^m} (\psi_{F_2})_x |d\lambda| = \int_{F_2^{m'}} (\psi'_{F_2})_x |d\lambda'|$$

for every $x \in S_{F_2}$.

In particular, when $\mathbf{F} = \mathbb{Q}$, we get that, for almost all p, the identity

$$\int_{\mathbb{Q}_p^m} (\psi_{\mathbb{Q}_p})_x |d\lambda| = \int_{\mathbb{Q}_p^{m'}} (\psi'_{\mathbb{Q}_p})_x |d\lambda'|$$

holds for every $x \in S_{\mathbb{Q}_p}$ if and only if

$$\int_{\mathbb{F}_p((t))^m} (\psi_{\mathbb{F}_p((t))})_x |d\lambda| = \int_{\mathbb{F}_p((t))^{m'}} (\psi'_{\mathbb{F}_p((t))})_x |d\lambda'|$$

holds for every $x \in S_{\mathbb{F}_p((t))}$. Note that Theorem 2.5 can be viewed as a special case of Theorem 2.8 when $m = m' = 0$ and S is the definable subset of K^0 defined by the sentence φ.

In work with Cluckers and Hales [12] we have shown that Theorem 2.8 applies in particular to the integrals occurring in the fundamental lemma, both in the unweighted and weighted case. This is performed by representing all the data entering into the fundamental lemma within the general framework of identities of motivic integrals of constructible functions. This provides alternative proofs of results of Waldspurger in [46] and [47] and is of special interest in view of Ngô's proof of the fundamental lemma over local fields of positive characteristic [38]. One advantage of our approach is that it may be applied quite directly to other versions of the fundamental lemma, as in [50].

Another important property of motivic constructible functions is that they satisfy strong uniform boundedness statements, as proved by Cluckers, Gordon, Halupczok in the appendix B of [44]:

Theorem 2.9. *Let S be a definable set and let $\varphi \in \mathcal{C}(S \times \mathbb{Z}^n)$.*

(1) *There exist integers a and b, M, such that for every F in $\mathcal{C}_{\mathcal{O},M}$, if there exists a set-theoretical function $\alpha : \mathbb{Z}^n \to \mathbb{R}$ such that $|\varphi_F(s,\lambda)|_{\mathbb{R}} \leq \alpha(\lambda)$ on $S_F \times \mathbb{Z}^n$, then $|\varphi_F(s,\lambda)|_{\mathbb{R}} \leq q_F^{a+b||\lambda||}$ on $S_F \times \mathbb{Z}^n$, with $||\lambda|| = \sum_i |\lambda_i|$.*

(2) *Given integers a and b, there exists M, such that whether the bound*

$$|\varphi_F(s,\lambda)|_{\mathbb{R}} \leq q_F^{a+b||\lambda||}$$

holds or not on the whole of $S_F \times \mathbb{Z}^n$ depends only on k_F, for F in $\mathcal{C}_{\mathcal{O},M}$.

In the same paper they show this result may be applied to provide uniform bounds for orbitals integrals that are used in an essential way in the paper [44].

2.10. Transfer theorems for exponential constructible motivic functions. In [11], we extend the construction of algebras constructible motivic functions $\mathcal{C}(S)$, to take in account motivic versions of exponential functions, by constructing the algebra $\mathcal{C}^{exp}(S)$ of exponential constructible motivic functions on S for any definable set S. The formalism developed in [10] for $\mathcal{C}(S)$ carries over to $\mathcal{C}^{exp}(S)$.

Given an non-archimedean field F, one denotes by $\mathcal{D}_F$ the set of additive characters on F that are trivial on the maximal ideal and nontrivial on the valuation ring. Now, given φ in $\mathcal{C}^{exp}(S)$, for any F in $\mathcal{C}_{\mathcal{O},M}$ and any character θ in $\mathcal{D}_F$, one may specialize φ to a function $\varphi_{F,\theta}$ on S_F.

In this setting, Theorem 2.8 may be generalized as follows:

Theorem 2.11. *Let $\psi \in \mathcal{C}^{exp}(S \times K^m)$ and $\psi' \in \mathcal{C}^{exp}(S \times K^{m'})$. Then, there exists $M > 0$ such that, for every F_1 and F_2 in $\mathcal{C}_{\mathcal{O},M}$ such that $k_{F_1} \simeq k_{F_2}$,*

$$\int_{F_1^m} (\psi_{F_1,\theta})_x |d\lambda| = \int_{F_1^{m'}} (\psi'_{F_1,\theta})_x |d\lambda'|$$

for every $x \in S_{F_1}$ and any $\theta \in \mathcal{D}_{F_1}$ if and only if

$$\int_{F_1^m} (\psi_{F_1,\theta})_x |d\lambda| = \int_{F_1^{m'}} (\psi'_{F_1,\theta})_x |d\lambda'|$$

for every $x \in S_{F_1}$ and any $\theta \in \mathcal{D}_{F_1}$.

In the paper [13], Cluckers, Gordon, Halupczok prove the following remarkable transfer theorem for (local) integrability and boundedness:

Theorem 2.12. *Let S be a definable subset of K^m and let $\varphi \in \mathcal{C}^{exp}(S)$. There exists $M > 0$ such that, for fields F in $\mathcal{C}_{\mathcal{O},M}$, the validity of the statement that $\varphi_{F,\theta}$ is (locally) integrable, resp. (locally) bounded, for all $\theta \in \mathcal{D}_F$ depends only on the isomorphism class of k_F.*

Using Theorem 2.12, Cluckers, Gordon, Halupczok have been able in [14] to transfer Harish-Chandra's theorems on local integrability of characters of irreducible admissible representations of connected reductive p-adic groups from characteristic zero to (large) positive characteristic. An important ingredient in their approach is the definability of the Moy-Prasad filration subgroups, which they have proved in a number of important special cases.

3. Definability and non-archimedean diophantine geometry

3.1. Lipschitz functions. A C^1-function on an interval in $\mathbb{R}$ which has bounded derivative is automatically Lipschitz continuous. It is well known that such a result cannot hold for general C^1-functions over the p-adics since $\mathbb{Q}_p$ is total disconnectedness. However, under some definability conditions it is still possible to get results of this kind, as we shall explain now.

Let K be a field endowed with a discrete valuation for which it is complete. In this section, by definable we shall mean definable in the ring language L_K with parameters in K (in this case definable sets are also called semi-algebraic sets), or in the analytic language L_K^{an} which is obtained by adding to L_K a symbol for each restricted power series f in $K\{x_1, \dots, x_m\}$, for $m \geq 1$. Such a symbol is interpreted as the function $K^m \to K$ which is zero outside $\mathcal{O}_K^m$ and given by $x \mapsto f(x)$ for $x \in \mathcal{O}_K^m$. In this case definable sets are also called subanalytic sets.

Let X be a subset of K^m. We say a function $f : X \to K$ is C-Lipschitz if for every x and y in X, $|f(x) - f(y)| \leq C|x - y|$. We say it is locally C-Lipschitz if for each point x_0 in X, the restriction of f to some neighborhood of x_0 is C-Lipschitz.

In the paper [7] with Cluckers and Comte we prove the following:

Theorem 3.2. *Let X be a definable subset of $\mathbb{Q}_p^m$ and let $f : X \to \mathbb{Q}_p$ be a definable map. Assume f is locally C-Lipschitz. Then there exists a finite partition of X into definable sets X_i and C' such that the restriction of f to each X_i is C'-Lipschitz.*

This statement is a p-adic analogue of a theorem of Kurdyka for real subanalytic sets [34]. In [9] Cluckers and Halupczok proved that it is in fact always possible to take $C' = C$.

3.3. A p-adic analogue of the Yomdin-Gromov lemma. A very efficient tool in diophantine geometry is the so-called determinant method which was developed by Bombieri and Pila in the influential paper [6] about the number of integral points of bounded height on affine algebraic and transcendental plane curves. Basically, the method consists in using a determinant of a suitable set of monomials evaluated at the integral points, in order to construct a family of auxiliary polynomials vanishing at all integral points on the curve within a small enough box. Building on the estimates in [6] for algebraic curves, Pila proved in [40] bounds on the number of integral (resp. rational) points of bounded height on affine (resp. projective) algebraic varieties of any dimension, improving on previous results by S. D. Cohen using the large sieve method [15].

In [41], Pila and Wilkie proved a general estimate for the number of rational points on the transcendental part of sets definable in an o-minimal structure; this has been used in a spectacular way by Pila to provide an unconditional proof of some cases of the André-Oort Conjecture [42]. Lying at the heart of Pila and Wilkie's approach is the possibility of having uniform - in terms of number of parametrizations and in terms of bounds on the partial derivatives - C^k-parametrizations. These parametrizations are provided by an o-minimal version of Gromov's algebraic parametrization Lemma [26], itself a refinement of a previous result of Yomdin [48],[49]. Such C^k-parametrizations enter the determinant method via Taylor approximation.

In the work [8] with Cluckers and Comte we provide a version of the Yomdin-Gromov lemma and the Pila-Wilkie theorem valid over $\mathbb{Q}_p$. At first sight one may have doubts such a statement could exist, since there seem there is no way for a global Taylor formula to make sense in this framework. However Theorem 3.2 which provides a version of first-order Taylor approximation, piecewise globally, in the definable p-adic setting is an encouraging sign. In [8], instead of generalizing this result to higher order, we show directly the existence of uniform C^k-parametrizations that do satisfy Taylor approximation, which is enough for our purpose.

Our p-adic analogue of the Yomdin-Gromov lemma is the following statement:

Theorem 3.4. *Let $n \geq 0$, $m \geq 0$ and $r \geq 0$ be integers and let $X \subset \mathbb{Z}_p^n$ be a subanalytic set of dimension m. Then there exists a finite collection of subanalytic functions $g_i : P_i \subset \mathbb{Z}_p^m \to X$ such that the union of the $g_i(P_i)$ equals X, the g_i have C^r norm bounded by 1, and the g_i may be approximated by Taylor polynomials of degree $r-1$ with remainder of order r, globally on P_i.*

For the precise definition of the C^r norm and of approximation by Taylor polynomials of certain degree and with certain error we refer to [8].

3.5. A p-adic analogue of the Pila-Wilkie theorem. For X a subset of $\mathbb{Q}_p^n$ and $T > 1$ a real number, write $X(\mathbb{Q}, T)$ for the set consisting of points $(x_1, \cdots, x_n)$ in $X \cap \mathbb{Q}^n$ such that one can write x_i as a_i/b_i where a_i and $b_i \neq 0$ are integers with $|a_i|_{\mathbb{R}} \leq T$ and $|b_i|_{\mathbb{R}} \leq T$.

For X a subset of $\mathbb{Q}_p^n$, write X^{alg} for the subset of X consisting of points x such that there exists an algebraic curve $C \subset \mathbb{A}^n_{\mathbb{Q}_p}$ such that $C(\mathbb{Q}_p) \cap X$ is locally at x of dimension 1.

We prove in [8] the following p-adic analogue of the Pila-Wilkie theorem:

Theorem 3.6. *Let $X \subset \mathbb{Q}_p^n$ be a subanalytic set of dimension m with $m < n$. Let $\varepsilon > 0$ be given. Then there exist an integer $C = C(\varepsilon, X) > 0$ and a semialgebraic set $W = W(\varepsilon, X) \subset \mathbb{Q}_p^n$ such that $W \cap X$ lies inside X^{alg}, and such that for each T, one has*

$$\#(X \setminus W)(\mathbb{Q}, T) \leq CT^{\varepsilon}.$$

3.7. Results over $\mathbb{C}[\![t]\!]$. In the paper [8] we also obtain results when $K = \mathbb{C}(\!(t)\!)$. For instance a version of Theorem 3.2 still holds over $\mathbb{C}(\!(t)\!)$ (with $C' = C$), if one replaces "a finite partition of X" by "a partition parametrized by $\mathbb{C}^r$, for some r". For this to make sense one has to enlarge the language to have (higher) angular components maps à la Denef-Pas, see [8] for more details. Similarly, a version of Theorem 3.4 over $\mathbb{C}(\!(t)\!)$ is also proved in [8]. We end this section by stating a diophantine application of this result.

For each positive integer r one denotes by $\mathbb{C}[t]_{<r}$ the set of complex polynomials of degree $< r$. When A is a subset of $\mathbb{C}(\!(t)\!)^n$, one denotes by A_r the set $A \cap (\mathbb{C}[t]_{<r})^n$ and by $n_r(A)$ the dimension of the Zariski closure of A_r in $(\mathbb{C}[t]_{<r})^n \simeq \mathbb{C}^{nr}$.

Let X be an algebraic subvariety of $\mathbb{A}^n_{\mathbb{C}(\!(t)\!)}$ of dimension m. One can prove that for any $r > 0$, $n_r(X) \leq rm$. When X is linear this "trivial" estimate is the best possible. However, we prove in [8] that as soon as X has degree $d \geq 2$, the following non-trivial bound holds:

Theorem 3.8. *Let X be an irreducible subvariety of $\mathbb{A}^n_{\mathbb{C}(\!(t)\!)}$ of dimension m and degree $d \geq 2$. Then, for every positive integer r, one has*

$$n_r(X) \leq r(m-1) + \left\lceil \frac{r}{d} \right\rceil.$$

This result is a geometric analogue of a result of Pila in [40] on the number of integral (resp. rational) points of bounded height on affine (resp. projective) algebraic varieties of any dimension. Pila's proof proceeds by reducing to the case of curves which was considered by Bombieri and Pila in [6].

4. Definability and topology

In this section we present a model-theoretic approach to proving topological tameness properties in non-archimedean geometry which we developed in collaboration with Ehud Hrushovski [31].

4.1. o-minimality. It is by now quite well known that o-minimal geometry provides an efficient framework for the study of topology arising from an ordered structure, in particular in the context of ordered fields. Let us recall that an infinite structure M which is totally ordered by a binary relation $<$ is said to be o-minimal if every definable subset $X \subset M$, with parameters in M, is a finite union of intervals and points. Sets definable in a o-minimal structure have nice topological properties. For instance, for o-minimal expansions of the field $\mathbb{R}$ of real numbers, and $n \in \mathbb{N}$, definable subsets of $\mathbb{R}^n$ have a finite number of connected components which furthermore are definable, they are locally contractible and triangulable;

in particular they have the homotopy type of a finite simplicial complex. Classical examples of subsets of $\mathbb{R}^n$ definable in a o-minimal structure include semi-algebraic sets, subanalytic sets, or sets definable in the language of ordered rings with an exponential function. Another class of examples of o-minimal structures, playing an important role in our work, is provided by divisible ordered abelian groups Γ. In this last setting definable subsets of Γ^n essentially correspond to piecewise linear sets. An important feature of this model-theoretic framework for tameness is that it is particularly well adapted to proving uniformity statements for the topology of definable sets varying in definable families, for instance finiteness of homotopy types occuring in a given such family.

4.2. Valued fields. By a valued field we mean a field K, together with a surjective multiplicative map $\mathrm{val} : K^\times \to \Gamma$, with $\Gamma = (\Gamma, 0, +, <)$ an ordered abelian group such that $\mathrm{val}(x+y) \geq \min(\mathrm{val}(x), \mathrm{val}(y))$. We extend val to a map $\mathrm{val} : K \to \Gamma_\infty$, with Γ_∞ the disjoint union of Γ with a distinguished element ∞ which is larger than any element of Γ and absorbing for the addition. We shall denote by $\mathcal{O}_K$ the valuation ring of K and by $\mathcal{M}_K$ the maximal ideal of K.

4.3. Berkovich spaces. Let K be a valued field such that Γ is a subgroup of $(\mathbb{R}, +)$. Then $x \mapsto |x| = e^{-\mathrm{val}(x)}$ defines an absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. One says K is ultrametric if it is complete for this norm.

In [3], Berkovich introduced a general notion of analytic spaces over an ultrametric field K. In particular, for any algebraic variety V over K one may consider its Berkovich analytification V^{an}. In case V is affine with ring of regular functions $K[V]$, let us define V^{an} as a topological space. As a set V^{an} is the set of multiplicative seminorms on $K[V]$ extending the absolute value on K. There is a natural embedding $V^{an} \subset \mathbb{R}^{K[V]}$ and one endows V^{an} with the topology induced by the product topology on $\mathbb{R}^{K[V]}$. For an arbitrary algebraic variety V over K, one defines V^{an} by glueing. This construction is functorial: any morphism of algebraic variety $f : V \to W$ gives rise to a morphism $f^{an} : V^{an} \to W^{an}$. Note that $V(K)$ may be naturally identified with a subset of V^{an}. When V is affine, this is done by assigning to a point a in $V(K)$ the seminorm $f \mapsto |f(a)|$.

4.4. Some previously known topological properties of Berkovich spaces. Already in [3] Berkovich proved that general analytic spaces (including analytifications of algebraic varieties) have excellent general topological properties, in particular they are locally compact and locally path-connected.

More recently, in his paper [4], Berkovich proved that the general fibre of any polystable formal scheme admits a strong deformation retraction to a finite polyhedron, and using de Jong's results on alterations he deduced that any smooth analytic space is locally contractible.

On the other hand, Ducros proved in [21] that semi-algebraic subsets of V^{an}, i.e. subsets which are Zariski locally boolean combinations of subsets defined by inequalities $|f| \bowtie \lambda |g|$ with f, g in $K[V]$ and $\lambda \in \mathbb{R}_{\geq 0}$, where $\bowtie \in \{<, >, \leq, \geq\}$, have only a finite number of connected components, each of them semi-algebraic.

Another statement with an o-minimal flavour us the following. Let X be a compact analytic space and let f be an analytic function on X. For every $\varepsilon \geq 0$, let X_ε denote the set of points x in X such that $|f(x)| \geq \varepsilon$. According to Abbes and Saito under the assumption that f is invertible [1] and to Poineau in general [43], there is a finite partition of $\mathbb{R}_{\geq 0}$ into intervals such that on each of these intervals the natural map $\pi_0(X_{\varepsilon'}) \to \pi_0(X_\varepsilon)$

is a bijection whenever $\varepsilon \leq \varepsilon'$.

4.5. Statement of results. The results recalled in 4.4 provide rather strong evidence that there should exist general tameness results for the topology of non-archimedean spaces, quite analogous to the ones available in the o-minimal world. In the paper [31], we prove the following general statements on the topology of analytifications of algebraic varieties:

Theorem 4.6. *Let K be an ultrametric[1] field. Let V be a quasi-projective variety over K and let X be a semi-algebraic subset of V^{an}.*

(1) *There exists a strong homotopy retraction $h : [0,1] \times X \to X$ onto a closed subset of X which is homeomorphic to a compact finite polyhedral complex.*
(2) *The space X is locally contractible (one may drop the assumption V quasi-projective here).*
(3) *Let $f : V \to W$ be a morphism of algebraic varieties over K. Then the set of homotopy types of fibers of the map $f^{an}|_X : X \to W^{an}$ is finite.*
(4) *Let $f : V \to \mathbb{A}^1_K$ a morphism. For every $\varepsilon \geq 0$, let X_ε denote the set of points x in X such that $|f(x)| \geq 0$. Then there exists a finite partition of $\mathbb{R}_{\geq 0}$ into intervals such that the natural map $X_{\varepsilon'} \hookrightarrow X_\varepsilon$ is a homotopy equivalence whenever $\varepsilon \leq \varepsilon'$ belong to the same interval.*

4.7. Model-theoretic preliminaries. We shall deal with a complete theory T having quantifier elimination and work in a fixed universe $\mathbb{U}$, by which we mean a large very saturated and homogeneous model. All models M (and parameter sets A) we shall consider will be small substructures (resp. subsets) of $\mathbb{U}$.

If A is a small subset of $\mathbb{U}$, the definable closure $\mathrm{dcl}(A)$ is the set of all elements c in $\mathbb{U}$ such that there exists a formula $\varphi(x)$ with one free variable and parameters in A such that c is the only element of $\mathbb{U}$ such that $\varphi(c)$ holds. If X is a C-definable set and $C \subset A$, we write $X(A)$ for $X(\mathbb{U}) \cap \mathrm{dcl}(A)$.

A basic notion we shall use is that of a definable type. Let assume for simplicity of notation that there is only one sort. Let B be a set of parameters. Let $c = (c_1, \cdots, c_n)$ be a finite tuple of elements of $\mathbb{U}$. The set of all B-formulas satisfied by c in some model of T containing the c_i's is denoted by $\mathrm{tp}(c/B)$ and called the type of c over B. Such a set of formulas is called an n-type over B. In the special case where all c_i's already belong to B one says the type is realized (over B). Let $A \subset M$. We say an n-type p over M is A-definable if for every formula $\varphi(x_1, \cdots, x_n, y_1, \cdots, y_m)$ without parameters, there exists a formula $\varphi_p(y_1, \cdots, y_m)$ with parameters in A, such that for any $(b_1, \cdots, b_m)$ in M^m, $\varphi(x_1, \cdots, x_n, b_1, \cdots, b_m)$ belongs to p if and only if $\varphi_p(b_1, \cdots, b_m)$ holds in M. The mapping $\varphi \mapsto \varphi_p$ is called a defining scheme for p. If p is such an A-definable type over M, for any model M' containing M one can extend p to an A-definable type over M', by using the same defining scheme. Thus, we will not care about a specific M anymore when dealing A-definable types. Note that a realized type over A is always A-definable. These definitions extend naturally to many-sorted languages.

Let X be a C-definable set with $C \subset A$. We say that an A-definable type p is on X if the formula expressing that $x \in X$ belongs to the type p. We denote by $S_{X,def}(A)$ the set of A-definable types on X and set $S_{X,def} = \cup_A S_{X,def}(A)$. Any C-definable map

[1] In fact the completeness hypothesis on K plays no role here.

$f : X \to Y$ between C-definable sets induces a natural push-forward maps

$$f_* : S_{X,def}(A) \to S_{Y,def}(A) \text{ and } f_* : S_{X,def} \to S_{Y,def}.$$

4.8. The language. Classically, to study valued fields one considers a 3-sorted language $\mathcal{L}_{k,\Gamma}$ (or one of its variants) with sorts VF, Γ and $\mathbf{k}$ for the valued field, value group and residue field sorts, with respectively the ring, ordered abelian group and ring language, and additional symbols for the valuation val and the map $\mathrm{Res} : \mathrm{VF}^2 \to k$ sending (x, y) to the residue of xy^{-1} if $\mathrm{val}(x) \geq \mathrm{val}(y)$ and $y \neq 0$ and to 0 otherwise. We consider ACVF, the theory of algebraically closed fields with non trivial valuation such that val is surjective in this language. This theory become complete once the characteristic of the valued field and of its residue field are both fixed. It is a classical result of A. Robinson that ACVF admits quantifier elimination. Note that this result has already nice consequences in non-archimedean geometry. For instance in the paper of Ducros [23] it is used to give an alternate proof of the Bieri-Groves theorem [5].

We shall use an expansion $\mathcal{L}_\mathcal{G}$ of this language introduced by Haskell, Hrushovski and Macpherson in [27]. It has additional sorts S_n and T_n for $n \geq 1$, coding respectively n-dimensional lattices over the valuation ring, and elements in the reduction modulo the maximal ideal of such lattices. The main result of [27] is that ACVF has elimination of imaginaries in the language $\mathcal{L}_\mathcal{G}$ (which was not the case in the original language $\mathcal{L}_{k,\Gamma}$). A theory T is said to have elimination of imaginaries in a given language if all quotients of definable sets by definable equivalence relations are representable by definable sets. It is also proved in [27] that ACVF still has elimination of quantifiers in $\mathcal{L}_\mathcal{G}$.

One should note that expanding the language from $\mathcal{L}_{k,\Gamma}$ to $\mathcal{L}_\mathcal{G}$ does not create new definable sets in the sorts VF, Γ and $\mathbf{k}$. If V is an algebraic variety over a valued field, we may define definable subsets of V by requiring that their intersection with any affine open is a definable set.

Given a valued field F, a in F and α in $\mathrm{val}(F)$, resp. α in $\mathrm{val}(F^\times)$, one denotes by $B(a, \alpha)$ and $B^o(a, \alpha)$ respectively the closed and open ball of center a and valuative radius α. They are definable sets defined respectively by the formulas $\mathrm{val}(x - a) \geq \alpha$ and $\mathrm{val}(x - a) > \alpha$. If B is a ball defined over a model K of ACVF, the type expressing that $x \in B$ and $x \notin B'$ for every K-definable ball B' strictly contained in B is a K-definable type, called the generic type of B, and denoted by p_B.

Remark 4.9. Note that the set of all closed balls for K running over all models of ACVF (contained in $\mathbb{U}$) is definable in $\mathcal{L}_\mathcal{G}$ (without parameters). Indeed, it suffices to prove that the set of all closed balls of finite valuative radius is definable in $\mathcal{L}_\mathcal{G}$, and this follows from the following observation: given a, a' in K and b, b' in $K^\times$, the balls $B(a, \mathrm{val}(b))$ and $B(a', \mathrm{val}(b'))$ are equal if and only if the two-dimensional $\mathcal{O}_K$-lattices generated by $((b, 0), (a, b))$ and by $((b', 0), (a', b'))$ are equal. More precisely, there exists a definable set D in $\mathcal{L}_\mathcal{G}$ such that for any $A \subset \mathbb{U}$, $D(A)$ is in natural bijection with the set of A-definable closed balls.

4.10. Stably dominated types. In [28], Haskell, Hrushovski and Macpherson introduced within a general model-theoretic framework the notion of stably dominated types. Roughly speaking, a stably dominated type is a definable type which is "controlled by its stable part". In ACVF, stable domination is equivalent to being orthogonal to Γ in the following sense. Let X be a C-definable set and let $p \in S_{X,def}(A)$, for $C \subset A$. We shall say that p is orthogonal to Γ if for every model M of ACVF containing A, every tuple c such that $p = \mathrm{tp}(c/M)$, and every M-definable map $f : X \to \Gamma_\infty$, $f(c) \in \mathrm{val}(M)$. We denote by $\widehat{X}(A)$ the set of A-definable types on X that are orthogonal to Γ and by $\widehat{X}$ the union of all the sets $\widehat{X}(A)$, for $A \subset \mathbb{U}$. We call $\widehat{X}$ the stable completion of X.

Examples 4.11.

1. Realized types are stably dominated, i.e. for any definable set X there is a natural inclusion $\iota : X \to \widehat{X}$.
2. A type over Γ_∞^n is stably dominated if and only it is realized, i.e. $\iota : \Gamma_\infty^n \to \widehat{\Gamma_\infty^n}$ is a bijection.
3. The generic type of a ball is stably dominated if and only if the ball is closed.

It follows from Remark 4.9 and Example 4.11 (3) that, given a valued field F, there is a natural bijection ϑ between $\widehat{\mathbb{A}_F^1}$ and a definable set D, inducing, for any $A \subset \mathbb{U}$, a bijection between $\widehat{\mathbb{A}_F^1}(A)$ and $D(A)$. This is a special case of Theorem 4.14, but before going any further, we should introduce the notion of a pro-definable set. One defines the category ProDef_C of pro-definable sets over C as the category of pro-objects in the category of C-definable sets indexed by a small directed partially ordered set. Thus, if $X = (X_i)_{i \in I}$ and $Y = (Y_j)_{i \in J}$ are two such pro-objects

$$\mathrm{Hom}_{\mathrm{ProDef}_C}(X, Y) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathrm{Def}_C}(X_i, Y_j).$$

Elements of $\mathrm{Hom}_{\mathrm{ProDef}_C}(X, Y)$ will be called C-pro-definable morphisms between X and Y. By a result of Kamensky [33], the functor of “taking $\mathbb{U}$-points” induces an equivalence of categories between the category ProDef_C and the sub-category of the category of sets whose objects and morphisms are inverse limits of $\mathbb{U}$-points of definable sets indexed by a small directed partially ordered set. By pro-definable, we mean pro-definable over some C. We shall thus freely identify a pro-definable set $X = (X_i)_{i \in I}$ with the set $X(\mathbb{U}) = \varprojlim_i X_i(\mathbb{U})$. For any set B with $C \subset B \subset \mathbb{U}$, we set $X(B) = X(\mathbb{U}) \cap \mathrm{dcl}(B) = \varprojlim_i X_i(B)$.

Definition 4.12. Let X be a pro-definable set.

(1) X is called strict pro-definable if it can be written as a pro-definable set with surjective transition morphisms.

(2) X is called iso-definable if it is in pro-definable bijection with a definable set.

(3) $Y \subset X$ is called relatively definable if there exists $i \in I$ and a definable subset W of X_i such that $Y = \pi_i^{-1}(W)$, with π_i the canonical projection $X \to X_i$.

Theorem 4.13. *Let X be a B-definable set. Then $\widehat{X}$ may be canonically endowed with the structure of a strict B-pro-definable set. In particular, there exists a strict B-pro-definable set E such that for any $B \subset A$, there is a canonical identification $\widehat{X}(A) = D(A)$.*

For curves we have the following stronger statement:

Theorem 4.14. *Let C be an algebraic curve over a valued field K and let X be a definable subset of C. Then $\widehat{X}$ is iso-definable.*

For $C = \widehat{\mathbb{P}^1}$ the result follows from the description of $\widehat{\mathbb{A}^1}$ in terms of closed balls given above. The proof in the general case uses Riemann-Roch and Theorem 4.13.

Remark 4.15. The previous statement is optimal since one can show that, for X a definable subset of K^n, $\widehat{X}$ is iso-definable if and only the dimension of the Zariski closure of X is ≤ 1.

Lemma-Definition 4.16. Let $f : X \to Y$ be a map between B-definable sets. Then the map $f_* : S_{X,def} \to S_{Y,def}$ restricts to a B-pro-definable map $\widehat{f} : \widehat{X} \to \widehat{Y}$. In this way we have a functor from the category of B-definable sets to the category of B-pro-definable sets.

Let X be a definable subset. If Y is a definable subset of X, then $\widehat{Y}$ is a relatively definable subset of $\widehat{X}$. The set of realized types in $\widehat{X}$, which can be identified with $X(\mathbb{U})$ is iso-definable and relatively definable in $\widehat{X}$. Its points are called simple points of $\widehat{X}$.

4.17. $\widehat{V}$ as a topological space. We endow $\widehat{\mathbb{A}^n}$ with the coarsest topology such that for every polynomial $F \in \mathbb{U}[x_1, \cdots, x_n]$, the map $\widehat{\mathrm{val} \circ F} : \widehat{\mathbb{A}^n} \to \Gamma_\infty$ is continuous, where the topology on Γ_∞ is the order topology. For any definable subset X of $\mathbb{A}^n$, we endow $\widehat{X}$ with the induced topology. If V is an algebraic variety over a valued field K, we define the topology on $\widehat{V}$ by gluing: it is the unique topology inducing the previous topology on $\widehat{U}$ for U an affine open in V. If X is a definable subset of V, we endow the relatively definable subset $\widehat{X}$ with the induced topology.

We have the following basic properties:

Proposition 4.18. *Let V be an algebraic variety defined over a valued field K. Then:*

(1) *The topology on $\widehat{V}$ is pro-definable in the following sense: there exists a small set I, and for each $i \in I$, a K-definable family $U_i = (U_{i,b})_{b \in \mathbb{U}}$ of relatively definable subsets of $\widehat{V}$, such that the sets $U_{i,b}$, for $b \in \mathbb{U}$ and $i \in I$ generate the topology on $\widehat{V}$.*

(2) *The topology on $\widehat{V}$ is Hausdorff.*

(3) *The subset of simple points is dense in $\widehat{V}$.*

(4) *The induced topology on the set of simple points is the valuation topology.*

In general, we shall call pro-definable sets with a pro-definable topology, pro-definable spaces.

More generally, consider the map $\pi : V \times \mathbb{A}^m \to V \times \Gamma_\infty^m$ which is the identity on the V factor and val on the remaining ones. It induces a map $\widehat{\pi} : \widehat{V \times \mathbb{A}^m} \to \widehat{V \times \Gamma_\infty^m}$ and we endow $\widehat{V \times \Gamma_\infty^m}$ with the direct image topology, making it a pro-definable space. One shows that the canonical map $\widehat{V \times \Gamma_\infty^m} \to \widehat{V} \times \widehat{\Gamma_\infty^m} = \widehat{V} \times \Gamma_\infty^m$ is an homeomorphism.

4.19. Definable compactness. The usual notion of compactness is not well suited to the present setting as shown by the following example. Let K be a valued field with $\mathrm{val}(K^\times) = \mathbb{Q}$. Fix $\varepsilon \in \mathrm{val}(\mathbb{U}^\times)$ such that $0 < \varepsilon < \alpha$ for every positive α in $\mathbb{Q}$. Let C be set defined by the formula $0 \leq \mathrm{val}(x) \leq 1$. For $\alpha \in \mathbb{Q} \cap [0,1]$ let U_α be defined by $\alpha - \varepsilon < \mathrm{val}(x) < \alpha + \varepsilon$. The family of open sets $\widehat{U_\alpha}$ is a cover of $\widehat{C}$ with no finite subcover.

To remidy this we shall introduce the notion of definable compactness for pro-definable spaces. Let us note that the definition we gave of a definable type still makes sense on pro-definable set.

Definition 4.20. Let X be a pro-definable space.

(1) Let p be a definable type on X. We say $a \in X$ is a limit of p if for every relatively definable neighborhood W of a, the formula expressing $x \in W$ belongs to p.

(2) We say X is definably compact if every definable type on X has a limit.

Note that if X is Hausdorff, limits are unique when they exist.

Let V be a closed subvariety of $\mathbb{A}^m$. A subset $X \subset V$ is said to be bounded in V if it is contained in a product of closed balls. For an arbitrary variety V, a definable subset $X \subset V$ is said to be bounded, if one may write $V = \cup_{i=1}^n V_i$ with V_i open and affine and $X = \cup_{i=1}^n X_i$, with X_i bounded in V_i. A subset of $V \times \Gamma_\infty^m$ will be said to be bounded if its preimage in $V \times \mathbb{A}^m$ is. Finally, a pro-definable subset $X \subset \widehat{V} \times \Gamma_\infty^m$ will be said to be bounded if there exists a bounded definable subset W of $V \times \Gamma_\infty^m$ such that $X \subset \widehat{W}$.

Theorem 4.21. *Let X be a pro-definable subset of $\widehat{V} \times \Gamma_\infty^m$. Then X is definably compact if and only if it is closed and bounded.*

Corollary 4.22. *A variety V over a valued field is complete if and only if $\widehat{V}$ is definably compact.*

4.23. Γ-internality. We shall now define an important class of subsets of $\widehat{V} \times \Gamma_\infty^m$ which "look like o-minimal sets".

Definition 4.24. A subset Z of $\widehat{V} \times \Gamma_\infty^m$ is said to be Γ-internal if it is iso-definable and there is a definable subset D of some Γ_∞^n and a surjective pro-definable map $D \to Z$.

The iso-definability condition is crucial here, and cannot be replaced by just requiring pro-definability. This definition is purely definable and does not say anything a priori about the topology of Z. The following embedding result shows that being Γ-internal imposes strong restrictions on the topology:

Theorem 4.25. *Let Z be a Γ-internal subset of $\widehat{V} \times \Gamma_\infty^m$. Then there exists an injective continuous definable map $f : Z \hookrightarrow \Gamma_\infty^n$ for some n. If Z is definably compact, such an f is an homeomorphism.*

If V and Z are defined over some set of parameters A, one cannot in general expect such an f to be defined, because it should be respect the Galois action. However the following holds:

Proposition 4.26. *Assume V and Z are defined over some set of parameters A in the* VF *and* Γ *sorts. Then there exists a finite A-definable set w and an injective continuous A-definable map $f : Z \hookrightarrow \Gamma_\infty^w$.*

4.27. Paths and definable connectedness. The mapping $[0, \infty] \to \widehat{\mathbb{P}^1}$ sending t to the generic type of the ball $B(0, t)$ may be seen as a path connecting 0 and the generic type $p_{\mathcal{O}}$ of the closed unit ball. Similarly the mapping $[0, \infty] \to \widehat{\mathbb{P}^1}$ sending t to the generic type of the ball $B(1, t)$ connects 1 and $p_{\mathcal{O}}$. By composing these paths one connects the point 0 and 1. However a technical issue occurs here. Since multiplication is not part of the structure Γ_∞, there is no way to identify the space obtained by gluing two copies of $[0, \infty]$ at 0 with an interval. We are thus led to consider generalized intervals, that is spaces obtained by concatening a finite number of closed intevals in Γ_∞ either with the order from Γ_∞ or with the reverse order.

We denote by $I = [i_I, e_I]$ such a generalized interval. A path $\gamma : I \to \widehat{V} \times \Gamma_\infty^m$ is a continuous (pro)-definable map.

Let V be an algebraic variety over some valued field. We say a strict pro-definable subset Z of $\widehat{V}$ is definably connected if it contains no clopen strict pro-definable subsets other than $\emptyset$ and Z. We say that Z is definably path connected if for any two points a and b of Z there exists a definable path in Z connecting a and b. Clearly definable path connectedness implies definable connectedness. When V is quasi-projective and $Z = \widehat{X}$ with X a definable subset of V, the reverse implication will eventually follow from Theorem 4.32.

We have the following GAGA type theorem:

Theorem 4.28. *Let V be an algebraic variety over some valued field. Then $\widehat{V}$ is definably connected if and only if V is geometrically connected.*

4.29. Strong retractions for curves. Let $I = [i_I, e_I]$ be a generalized interval. A continuous pro-definable map $H : I \times \widehat{X} \to \widehat{Y}$ is called a definable homotopy between the maps $H_i = H_{|\{i_I\} \times \widehat{X}}$ and $H_e = H_{|\{e_I\} \times \widehat{X}}$, viewed as maps $\widehat{X} \to \widehat{Y}$. A definable homotopy $H : I \times \widehat{X} \to \widehat{X}$ is called a strong deformation retraction onto the set $\Sigma \subset \widehat{X}$ if $H_i = \mathrm{Id}_{\widehat{X}}$, $H(t,x) = x$ for every $t \in I$ and every $x \in \Sigma$ and $H_e(\widehat{X}) = \Sigma$.

There is a canonical strong deformation retraction of $\widehat{\mathbb{P}^1}$ onto the point $p_{\mathcal{O}}$ which is described as follows. Using the two standards affine charts, one may write each point of $\widehat{\mathbb{P}^1}$ as $p_{B(a,\alpha)}$ with $a \in \mathbb{P}^1(\mathbb{U})$ and $\alpha \geq 0$. The homotopy is given by taking $I = [\infty, 0]$ (thus $i_I = \infty$ and $e_I = 0$) and setting $\psi(t, p_{B(a,\alpha)}) = p_{B(a,\min(t,\alpha))}$.

More generally, given any finite subset D in $\mathbb{P}^1(\mathbb{U})$, let C_D be the image of $I \times (D \cup p_{\mathcal{O}})$ under ψ. The set C_D is a closed Γ-internal subset of $\widehat{\mathbb{P}^1}$. Set $\gamma(a) = \max\{t \in I; \psi(t,a) \in C_D\}$. Then $\psi_D : I \times \widehat{\mathbb{P}^1} \to \widehat{\mathbb{P}^1}$ sending (t,a) to $\psi(\max(\gamma(a),t),a)$ is a strong deformation retraction of $\widehat{\mathbb{P}^1}$ onto C_D.

Theorem 4.30. *Let C be an algebraic curve over a valued field K. There exists a strong deformation retraction, defined over K, $H : [0,\infty] \times \widehat{C} \to \widehat{C}$ onto a Γ-internal subset of $\widehat{C}$.*

Let us sketch the proof. A standard outward path on $\widehat{\mathbb{A}^1}$ at $x = p_{B(a,\alpha)}$ is given by $t \mapsto p_{B(a,t)}$ for $t \in (\beta, \alpha]$ for some $\beta < \alpha$. Now if $g : C \to \mathbb{A}^1$ is finite, with C a curve, by an outward path starting at $x \in \widehat{C}$, we mean a continuous definable lifting of a standard outward path starting at $g(x)$. One proves that for any $x \in \widehat{C}$ there exists at least one outward path starting at x and one says that x is branching if there is more than one outward path starting at x. A key lemma states that the number of such branching points is finite. For the proof of the theorem we may assume C is projective and consider $f : C \to \mathbb{P}^1$ finite and generically étale. One considers a finite set $D \subset \mathbb{P}^1$, defined over K, such that f is étale above the complement of D and C_D contains all the branching points, with respect to the restriction of g over both standards affine charts. One concludes the proof by showing that ψ_D lifts to the strong deformation retraction we are looking for.

4.31. The main theorem. We may now state the main result from [31]:

Theorem 4.32. *Let K be a valued field and $A = (K, G)$ with G a subset of Γ containing $\mathrm{val}(K)$. Let V a quasi-projective variety defined over K, X an A-definable subset of V. Assume given finitely many A-definable functions $\xi_i : X \to \Gamma_\infty$ and an action of a finite algebraic group over K on V leaving X globally invariant. Then there exists an A-definable strong deformation $H : I \times \widehat{X} \to \widehat{X}$ onto a Γ-internal subset Υ of $\widehat{X}$ such that:*

(1) *The set Υ embeds homeomorphically into Γ_∞^w for some finite A-definable set.*

(2) *H respects the functions ξ_i and is equivariant with respect to the group action.*

The structure of the proof goes as follows. One uses induction on the dimension of V. One start by reducing to the case where $X = V$ is projective equidimensional. One fixes an hypersurface $D_0 \subset V$ containing the singular locus of V and such that there exists an equivariant étale morphism $V \setminus D_0 \to \mathbb{A}^n$. Some further geometric considerations allow to reduce to the case when there is a morphism $u : V \to U = \mathbb{P}^{n-1}$, whose restriction to D_0 is finite, and a Zariski dense open subset U_0 of U such that, setting $V_0 = u^{-1}(U_0)$, $u|V_0$ factorizes as $q \circ f$ with $f : V_0 \to E_0 = U_0 \times \mathbb{P}^1$ a finite morphism and $q : E_0 \to U_0$ the projection.

Over U_0 the situation is that of a relative curve. Performing the curve construction in this relative setting provides a strong deformation retraction

$$H_{curves} : [0, \infty] \times \widehat{V_0 \cup D_0} \longrightarrow \widehat{V_0 \cup D_0}$$

fixing pointwise $\widehat{D_0}$ and with image a relatively Γ-internal set Υ_{curves}. By using the induction hypothesis (note that even if one starts with V without group action and no ξ_i's, they are needed at this stage of this induction), one constructs a definable homotopy $I \times \widehat{U} \to \widehat{U}$ whose restriction lifts to a strong deformation retraction

$$H_{base} : I \times \Upsilon_{curves} \longrightarrow \Upsilon_{curves}.$$

A third homotopy, which we call "inflation" is used to get out of of the complement of $\widehat{V_0 \cup D_0}$. On $\widehat{\mathbb{A}^n}$ one may consider the standard homotopy given by "increasing the polyradius". Using an appopriate stopping time function one gets another homotopy which we may lift, via the étale map $V \setminus D_0 \to \mathbb{A}^n$, to an homotopy

$$H_{inf} : [0, \infty] \times \widehat{V} \longrightarrow \widehat{V_0 \cup D_0}$$

fixing pointwise $\widehat{D_0}$.

After composing these three homotopies, one gets an homotopy $H' : I' \times \widehat{V} \to \widehat{V}$ that almost does the job, except that because of the use of inflation, we cannot insure that the points of the image of H' are all kept pointwise fixed by H' for all time values. To remedy this issue, we have to construct a fourth homotopy, H_Γ whose construction lies purely in the tropical Γ-internal world, so that the composition $H = H_\Gamma \circ H'$ finally satisfies the conclusion of the theorem.

4.33. Back to Berkovich spaces. A type $p = \mathrm{tp}(c/A)$ is said to be almost orthogonal to Γ is $\Gamma(Ac) = \Gamma(A)$.

Let F be a valued field with $\mathrm{val}(F^\times) \subset \mathbb{R}$. We consider the structure $\mathbb{F} = (F, \mathbb{R})$, where $\mathbb{R}$ belongs to the Γ-sort. Let V be a variety defined over F and X an $\mathbb{F}$-definable subset of V.

One defines $B_X(\mathbb{F})$ as the set of types over $\mathbb{F}$ lying on X and almost orthogonal to Γ. Similarly as for the Berkovich analytification and the stable completion, one endows $B_X(\mathbb{F})$ with a topology coming from the topology on $\mathbb{R}$. When F is complete, $B_V(\mathbb{F})$ and V^{an} are canonically homeomorphic.

By a result of Kaplansky, there exists a unique field F^{max}, up to $\mathbb{F}$-automorphism, which is a maximally complete algebraically closed non trivially valued field containing F, and has value group $\mathbb{R}$ and residue field the algebraic closure of the residue field of F.

The following proposition provides the link allowing to deduce the results about Berkovich spaces stated in Theorem 4.6 from Theorem 4.32 and its relative variants.

Proposition 4.34. *Let X be an $\mathbb{F}$-definable subset of some F-variety. Restriction of types induces a continuous, surjective and closed map $\pi : \widehat{X}(F^{max}) \to B_X(\mathbb{F})$.*

(1) *Let $f : \widehat{X} \to \widehat{Y}$ be a continuous $\mathbb{F}$-pro-definable map, with Y an $\mathbb{F}$-definable subset of some F-variety. Then there exists a unique continuous map $\tilde{f} : B_X(\mathbb{F}) \to B_Y(\mathbb{F})$ such that $\pi \circ f = \tilde{f} \circ \pi$.*

(2) *Let $H : I \times \widehat{X} \to \widehat{X}$ be a definable strong deformation retraction. Then $\tilde{H} : I(\mathbb{R}_\infty) \times B_X(\mathbb{F}) \to B_X(\mathbb{F})$ is a strong deformation retraction.*

(3) *$B_X(\mathbb{F})$ is compact if and only if $\widehat{X}$ is definably compact.*

Acknowledgements. The research of the author has been partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) ERC Grant agreement no. 246903/NMNAG. The results presented in this survey are the outcome of long term collaborations with Raf Cluckers, Georges Comte and Ehud Hrushovski. The author would like to express them his heartfelt gratitude.

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