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Trace Formulas for Motivic Volumes

François Loeser[1]

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Abstract We present some recent trace formulas for varieties over valued fields which can be seen as analogues of Grothendieck's Lefschetz trace formula for varieties over finite fields. This involves motivic integration and non-archimedean geometry.

Keywords Milnor fiber · Monodromy · Motivic volume

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1 Summary

The aim of this lecture is to present some recent trace formulas obtained in [11, 17, 27, 28] for varieties over valued fields using motivic integration. We start by recalling in Section 2 the trace formula of Grothendieck that provides a cohomological expression for the number of points of varieties over finite fields. We also review Grothendieck's function-sheaf dictionary and briefly present some applications.

Section 3 is devoted to introducing the motivic Serre invariant from Loeser and Sebag [22], which can be considered as a right substitute for counting point over finite fields; indeed, for smooth varieties over a valued field with perfect residue field, it provides a measure of the set of unramified points. The construction proceeds in analogy with an invariant that Serre defined in [33] for locally analytic p-adic varieties, which we recall first.

We are then in position to state in Section 4 the trace formula of Nicaise and Sebag [28] that provides a cohomological interpretation for the Euler characteristic of the motivic

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✉ François Loeser
Francois.Loeser@upmc.fr

1 Institut Mathématique de Jussieu, Sorbonne Universités, UPMC Univ Paris 06, UMR 7586 CNRS, F-75005 Paris, France

Serre invariant. In the remaining of the section, we explain the connection with monodromy and the Milnor fiber as expressed in the original trace formula of Denef and Loeser [11]. We then explain the strategy of proof, relying on explicit computations on resolutions and motivic integration.

In Section 5, we shall present another approach due to Hrushovski and Loeser [17]. It is based on non-archimedean geometry and avoids explicit computations on resolutions. It uses the version of motivic integration of Hrushovski and Kazhdan [16], that we review, and ultimately relies on a classical form of the Lefschetz fixed point formula.

2 Trace Formulas over Finite Fields and Grothendieck's Function-Sheaf Dictionary

2.1 The Lefschetz Trace Formula

The classical Lefschetz trace formula expresses the number of fixed points of an endomorphism of a topological space in terms of the traces of the corresponding endomorphisms in the cohomology groups. In its original form [21], it states the following:

Theorem 2.1.1 (Lefschetz, [21]) *Let X be a connected orientable n-dimensional compact topological manifold or an n-dimensional finite cell complex, let $f : X \to X$ be a continuous mapping and let*

$$\Lambda(f) = \mathrm{Tr}(f; H^\bullet(X, \mathbb{Q})) = \sum_{i\geq 0}(-1)^i \,\mathrm{Tr}(f; H^i(X, \mathbb{Q})) \tag{2.1.1}$$

be the Lefschetz number of f. Assume that all fixed points of the mapping are isolated. For each fixed point x of f, denote by $\iota(x)$, the local degree of f in a neighborhood of x. Then

$$\sum_{f(x)=x} \iota(x) = \Lambda(f). \tag{2.1.2}$$

Note that, it follows in particular that if $\Lambda(f) \neq 0$, then f has at least one fixed point, which generalizes the Brouwer fixed point theorem.

2.2 Grothendieck's Trace Formula

Let X_0 be an algebraic variety over the finite field $\mathbb{F}_q$. Fix an algebraic closure $\overline{\mathbb{F}_q}$ and set $X = X_0 \otimes \overline{\mathbb{F}_q}$. Let $\mathrm{Frob} : X \to X$ be the geometric Frobenius (which corresponds to raising the coordinates to the q-th power). Then the fixed points of Frob are exactly the $\mathbb{F}_q$-rational points, and more generally, for any $m \geq 1$, the set of fixed points of Frob^m is exactly $X_0(\mathbb{F}_{q^m})$. In this context, Grothendieck proves a Lefschetz trace formula using ℓ-adic cohomology.

Fix a prime ℓ not equal to the characteristic of $\mathbb{F}_q$.

Theorem 2.2.1 (Grothendieck trace formula, version 1) *For any $m \geq 1$,*

$$\mathrm{Tr}(\mathrm{Frob}^m, H_c^\bullet(X, \mathbb{Q}_\ell)) = \#(X_0(\mathbb{F}_{q^m})), \tag{2.2.1}$$

where Frob *denotes the (geometric) Frobenius and $H_c^\bullet(X, \mathbb{Q}_\ell))$ denotes ℓ-adic cohomology with compact supports.*

More generally, let $\mathcal{F}_0$ be a constructible ℓ-adic sheaf on X_0, or an object of $D_c^b(X_0, \overline{\mathbb{Q}_\ell})$, the bounded derived category of constructible $\overline{\mathbb{Q}_\ell}$-sheaves on X_0. Denote by $\mathcal{F}$ the pullback of $\mathcal{F}_0$ to X. For any $x \in X_0(\mathbb{F}_{q^m})$, one may consider $\mathrm{Tr}(\mathrm{Frob}^m, \mathcal{F}_{\overline{x}})$, with $\overline{x}$ the geometric point attached to x. Let $F(X_0(\mathbb{F}_{q^m}), \overline{\mathbb{Q}_\ell})$ be the ring of functions from $X_0(\mathbb{F}_{q^m})$ to $\overline{\mathbb{Q}_\ell}$. One defines this way an additive morphism

$$\mathrm{Tr}(\mathrm{Frob}^m) : D_c^b(X_0, \overline{\mathbb{Q}_\ell}) \longrightarrow F(X_0(\mathbb{F}_{q^m}), \overline{\mathbb{Q}_\ell}) \tag{2.2.2}$$

by assigning to an object $\mathcal{F}_0$ in $D_c^b(X_0, \overline{\mathbb{Q}_\ell})$ the function $x \mapsto \mathrm{Tr}(\mathrm{Frob}^m, \mathcal{F}_{\overline{x}})$. Now if $f : X_0 \to Y_0$ is a morphism of $\mathbb{F}_q$-varieties, we may define a ring morphism $f_! : F(X_0(\mathbb{F}_{q^m}), \overline{\mathbb{Q}_\ell}) \to F(Y_0(\mathbb{F}_{q^m}), \overline{\mathbb{Q}_\ell})$ by sending a function φ on X_0 to the function $y \mapsto \sum_{x \in X_0(\mathbb{F}_{q^m}), f(x)=y} \varphi(x)$. On the other hand, direct image with compact supports provides a functor $f_! : D_c^b(X_0, \overline{\mathbb{Q}_\ell}) \to D_c^b(Y_0, \overline{\mathbb{Q}_\ell})$. This allows to state the following relative form of Grothendieck's trace formula:

Theorem 2.2.2 (Grothendieck trace formula, version 2) *Let $f : X_0 \to Y_0$ be a morphism of $\mathbb{F}_q$-varieties. Then, for any $m \geq 1$, the diagram*

$$\begin{array}{ccc}
D_c^b(X_0, \overline{\mathbb{Q}_\ell}) & \xrightarrow{\mathrm{Tr}(\mathrm{Frob}^m)} & F(X_0(\mathbb{F}_{q^m}), \overline{\mathbb{Q}_\ell}) \\
\big\downarrow f_! & & \big\downarrow f_! \\
D_c^b(Y_0, \overline{\mathbb{Q}_\ell}) & \xrightarrow{\mathrm{Tr}(\mathrm{Frob}^m)} & F(Y_0(\mathbb{F}_{q^m}), \overline{\mathbb{Q}_\ell})
\end{array}$$

is commutative.

The relative version of Grothendieck's trace formula provides the basis for the so-called function-sheaf dictionary. The idea is that in most cases of interest, functions on $X_0(\mathbb{F}_{q^m})$ may be expressed as traces on complexes of sheaves. Thus, if one wants to prove some equality between two such functions f_1 and f_2, one may expect it comes from an isomorphism between two complexes of sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X_0. This is often more tractable than the original problem, since this allows to use geometric tools. For instance, in favorable cases, one may have that $\mathcal{F}_1$ and $\mathcal{F}_2$ belong to the abelian category of perverse sheaves and that a natural morphism $\theta : \mathcal{F}_1 \to \mathcal{F}_2$ is given; instead of proving that f_1 and f_2 are equal, which might be a very difficult combinatorial problem, one may consider to take benefit of geometry to prove that the kernel and cokernel of θ vanish.

Remark 2.2.3 Let $K(X_0, \overline{\mathbb{Q}_\ell})$ be the Grothendieck ring of the category $D_c^b(X_0, \overline{\mathbb{Q}_\ell})$. There is a ring morphism $\iota : K(X_0, \overline{\mathbb{Q}_\ell}) \to \prod_{m \geq 1} F(X_0(\mathbb{F}_{q^m}), \overline{\mathbb{Q}_\ell})$ given componentwise by $\mathrm{Tr}(\mathrm{Frob}^m)$. It is a well-known consequence of Chebotarev's density theorem, cf. Théorème 1.1.2 in [19], that ι is injective.

2.3 Some Examples

2.3.1 Evaluation of Character Sums

A typical situation where the geometric approach is powerful is the evaluation of complicated character sums. Let us illustrate this with an example in which we were involved.

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{R})$ generated by reflections around hyperplanes $\ell_i = 0$, $1 \leq i \leq N$, normalized in such a way that $\|\ell_i\| = \sqrt{2}$. Set $\Delta = (\prod_{i=1}^N \ell_i)^2$ and let d_1,

$\dots, d_n$ denote the degrees of the fundamental invariants of the ring of polynomial invariants of G. In [23], I. Macdonald conjectured the remarkable identity

$$\int_{\mathbb{R}^n} \Delta(x)^s \exp(-(\sum x_i^2)/2) dx = (2\pi)^n \prod_{i=1}^{n} \frac{\Gamma(d_i s + 1)}{\Gamma(s+1)}. \tag{2.3.1}$$

This was later proved by E. Opdam [30] using the existence of so-called hypergeometric shift operators. In [8], Denef and Loeser proved a finite field analogue of the Macdonald identity, using the cohomological interpretation of character sums, together with a detailed study of the monodromy of the discriminant of a finite Coxeter group, Laumon's product formula [19] and Macdonald's integral formula.

2.3.2 The Fundamental Lemma

One of the most spectacular successes of the function-sheaf dictionary is provided by Ngô's celebrated proof of the Fundamental Lemma over a local field F of positive characteristic [26]. Recall that the Fundamental Lemma is an identity between an orbital integral on a reductive group G over F and a stable oribital integral on an endoscopic group of G, up to a transfer factor. These integrals may be reexpressed in purely combinatorial terms involving counting points in some affine Springer fibers over the finite residue field k of F. One of Ngô's main insights was to globalize the problem, switching from local geometry of F to global geometry over a k-curve and using the Hitchin fibration as the global analogue of affine Springer fibers. He can then make full use of geometry, reducing the proof of the Fundamental Lemma to a deep theorem about the support of the perverse cohomology sheaves of the Hitchin fibration.

3 The Motivic Serre Invariant

3.1 A Toy Model: the Serre Invariant

Let K be a locally compact field for the topology defined by a discrete valuation val. The residue field k of K is isomorphic to $\mathbb{F}_q$. For instance, one may take $K = \mathbb{Q}_p$ and then $k = \mathbb{F}_p$. One defines a norm on K by setting $|a| = q^{-\operatorname{val}(a)}$. The valuation ring R of K is equal to the closed unit disk for this norm.

Let U be an open subset of K^n. A function $f : U \to K$ is said to be *locally analytic* if for any point a in U, f can be expressed by a converging power series on some open neighborhood of a. This notion allows to define by glueing the category of locally analytic varieties over K. Such varieties are locally modeled on open subsets of K^n, for some n. Fix $n \geq 1$. By a compact n-variety, we shall mean a non-empty compact locally analytic variety over K which is locally of dimension n at each point. The following result was proved by Serre in [33].

Theorem 3.1.1 (Serre, [33]) (1) *Let X be a compact n-variety. Then, for some integer $s \geq 1$, X is isomorphic to $sR^n = \{sa; a \in R^n\}$.*
(2) *sR^n is isomorphic to $s'R^n$ if and only if $s \equiv s' \mod (q-1)$.*

It follows that the class $s(X)$ of the integer s in Theorem 3.1.1 (1) in $\mathbb{Z}/(q-1)\mathbb{Z}$ depends only on X; we call it the *Serre invariant* of X.

The following statement provides a link with p-adic integration:

Theorem 3.1.2 (Serre, [33]) *Let X be a compact n-variety and let ω be a degree n locally analytic differential form which vanishes nowhere on X (such forms always exist). Then $s(X)$ is equal to the class of the integral*

$$\int_X |\omega| \tag{3.1.1}$$

in $\mathbb{Z}/(q-1)\mathbb{Z}$.

3.2 The Motivic Serre Invariant

Consider now a complete discrete valuation ring R with fraction field K and perfect residue field k.

By analogy with the Serre invariant, Loeser and Sebag introduced in [22] the *motivic Serre invariant* for smooth *rigid analytic* varieties over K.

Rigid analytic varieties were introduced by Tate in [34] to remedy the fact that in the non-archimedean world, because of total disconnectedness, there do not exist satisfactory notions of analytic continuation and connectedness in the context of locally analytic functions. They were later reinterpreted by Raynaud [31] as generic fibers of formal schemes over the valuation ring R. In particular, Raynaud showed that the category of quasi-compact quasi-separated rigid spaces over K is equivalent to the localization of the category of quasi-compact admissible formal schemes over R with respect to admissible formal blow-ups. All the rigid spaces, we shall consider will be assumed to be quasi-compact and quasi-separated.

The construction of the motivic Serre invariant is based on motivic integration. Originally, say for k of characteristic zero, motivic integration assigns to subsets of the arc space $\mathcal{L}(X)$ of an algebraic variety X over k a volume in (a completion of) the localized Grothendieck ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ cf. [9, 10]. Here, $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ is the localization with respect to the class of the affine line of $K_0(\mathrm{Var}_k)$, the Grothendieck ring of algebraic varieties over k. In concrete terms, note that two varieties over k that are stably piecewise isomorphic (i.e., which become isomorphic after being cut into locally closed pieces and stabilization by product with a power of the affine line) define the same class in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$. In general, when k has positive characteristic, one has to replace $K_0(\mathrm{Var}_k)$ and $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ by their quotients $K_0'(\mathrm{Var}_k)$ and $K_0'(\mathrm{Var}_k)_{\mathrm{loc}}$ obtained by trivializing universal homeomorphisms, cf. [29]. Note that when k has characteristic zero $K_0(\mathrm{Var}_k)$ and $K_0'(\mathrm{Var}_k)$ are identical (and so are $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ and $K_0'(\mathrm{Var}_k)_{\mathrm{loc}}$).

Let Y be a smooth rigid K-space of dimension d. In [22], we assign to a gauge form ω on Y, i.e., a nowhere vanishing differential form of degree d on Y, a motivic integral $\int_Y |\omega|$ with value in $K_0'(\mathrm{Var}_k)_{\mathrm{loc}}$. The construction is done by viewing Y as the generic fiber of some formal R-scheme $\mathcal{Y}$. To such a formal R-scheme, by means of the Greenberg functor $\mathcal{Y} \mapsto \mathrm{Gr}(\mathcal{Y})$, one associates a certain k-scheme $\mathrm{Gr}(\mathcal{Y})$. When $R = k[[t]]$, and $\mathcal{Y}$ is the formal completion of $X \otimes k[[t]]$, for X an algebraic variety over k, is nothing else than the arc space $\mathcal{L}(X)$ of X. We may then use the general theory of motivic integration on schemes $\mathrm{Gr}(\mathcal{Y})$ which is developed in [32]. Of course, for the construction to work, one needs to check that it is independent of the chosen model. This is done by using two main ingredients. The first is the theory of weak Néron models for smooth rigid varieties developed in [5] and [6]. A weak Néron model for Y is a smooth quasi-compact formal R-scheme $\mathcal{U}$, whose generic fiber U is an open rigid subspace of Y, and which has the property that the canonical map $\mathcal{U}(R') \to Y(K')$ is bijective for every finite unramified extension R' of R with quotient field K'. The second ingredient is the analogue for schemes of the form $\mathrm{Gr}(\mathcal{Y})$ of the change of variable formula from [9] which is proven in [32].

Definition 3.2.1 Let Y be a smooth rigid K-space of dimension d. One defines its motivic Serre invariant $S(Y)$ in $K_0'(\mathrm{Var}_k)/(\mathbb{L}-1)K_0'(\mathrm{Var}_k)$ as follows (sketch). Assume first Y admits a gauge form. If ω and ω' are two gauge forms on Y, one shows that

$$\int_Y |\omega| - \int_Y |\omega'| \in (\mathbb{L}-1)K_0'(\mathrm{Var}_k)_{\mathrm{loc}}. \tag{3.2.1}$$

Thus, one can define $S(Y)$ as the class of $\int_Y |\omega|$ in

$$K_0'(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbb{L}-1)K_0'(\mathrm{Var}_k)_{\mathrm{loc}} \simeq K_0(\mathrm{Var}_k)/(\mathbb{L}-1)K_0'(\mathrm{Var}_k). \tag{3.2.2}$$

In general, Y admits a gauge form on some non-empty open affinoid, and one reduces to the former case by additivity.

Note that the class of a variety in $K_0'(\mathrm{Var}_k)/(\mathbb{L}-1)K_0'(\mathrm{Var}_k)$ still contains a lot of information about it.

Example 3.2.2 In case Y admits a smooth formal R-model with good reduction, $S(Y)$ is just the class of the fiber of that model. More generally, if $\mathcal{U}$ is a weak Néron model of Y, $S(Y)$ is equal to the class of the special fiber of $\mathcal{U}$ in $K_0'(\mathrm{Var}_k)/(\mathbb{L}-1)K_0'(\mathrm{Var}_k)$. In particular, one obtains that this class is independent of the choice of the weak Néron model $\mathcal{U}$.

Remark 3.2.3 From the above example, one sees that one can interpret $S(Y)$ as a *measure of the set of non ramified points of* Y.

To conclude, let us note that the motivic Serre invariant is a refinement of the classical Serre invariant in the following sense. Let K be a finite extension of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_q$. Counting rational points in k yields a canonical morphism

$$K_0'(\mathrm{Var}_k)/(\mathbb{L}-1)K_0'(\mathrm{Var}_k) \longrightarrow \mathbb{Z}/(q-1)\mathbb{Z}, \tag{3.2.3}$$

and one may show that the image of $S(Y)$ by this morphism is equal to $s(Y(K))$.

Variant 3.2.4 Let $\mathcal{Y}$ be a formal model of Y with special fiber $\mathcal{Y}_s$. For Z, a locally closed subscheme of $\mathcal{Y}_s$, one defines the tube $]Z[$ as the preimage $\mathrm{sp}^{-1}(Z)$ of Z under the specialization morphism $\mathrm{sp} : Y \to \mathcal{Y}_s$. By integrating over $]Z[$ instead of over the whole Y, one defines a localized version $S_Z(Y)$ of $S(Y)$, cf. [28].

4 A Trace Formula for the Motivic Serre Invariant

4.1 The Trace Formula of Nicaise and Sebag

In this section, we will present a trace formula due to Nicaise and Sebag [28], which can be seen as an analogue over the field $\mathbb{C}((t))$ of Grothendieck's trace formula Theorem 2.2.1. For simplicity, we shall assume from now on that $K = \mathbb{C}((t))$ and $R = \mathbb{C}[[t]]$. The algebraic closure of $\mathbb{C}((t))$ is the field of Puiseux series $\mathbb{C}((t))^{\mathrm{alg}} = \cup_{m\geq 1}\mathbb{C}((t^{1/m}))$, we shall denote by $\widehat{\mathbb{C}((t))^{\mathrm{alg}}}$ its completion. The Galois group $G = \mathrm{Aut}(\widehat{\mathbb{C}((t))^{\mathrm{alg}}}/\mathbb{C}((t))) = \mathrm{Aut}(\mathbb{C}((t))^{\mathrm{alg}}/\mathbb{C}((t)))$ is canonically isomorphic to the group $\hat{\mu} = \varprojlim \mu_n$ of roots of unity, namely, $(\zeta_n)_{n\geq 1} \in \hat{\mu}$ sends the series $\sum a_m t^{i/m}$ to $\sum a_m \zeta_m^i t^{i/m}$. Let $\varphi = (\exp 2\pi i/n)_{n\geq 1} \in \hat{\mu}$. It is a topological generator of the profinite group $\hat{\mu}$, and for any $m \geq 1$, $K(m) :=$

$\mathbb{C}((t^{1/m}))$ is the field fixed by φ^m. Thus, φ can be considered as an analogue of the Frobenius element Frob.

Let Y be a smooth rigid K-variety with formal model $\mathcal{Y}$. For any $m \geq 1$, we can consider the variety $Y(m)$ obtained by extending the scalars from K to $K(m)$. Thus, $Y(m)$ has a formal model $\mathcal{Y}(m) := \mathcal{Y}\widehat{\otimes}_R \mathbb{C}[[t^{1/m}]]$.

The Euler characteristic with compact supports for complex algebraic varieties factorizes as a ring morphism

$$\chi_c : K_0(\mathrm{Var}_{\mathbb{C}})/(\mathbb{L}-1)K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}. \tag{4.1.1}$$

Theorem 4.1.1 (Nicaise-Sebag, [28]) *Assume X is a generically smooth flat algebraic variety over R. Denote by $\mathcal{Y}$ its t-adic completion with rigid generic fiber Y. Let Z be a proper subvariety of the special fiber of $\mathcal{Y}$. Then, for any $m \geq 1$,*

$$\chi_c(S_Z(Y(m))) = \mathrm{Tr}(\varphi^m, H^\bullet(]\overline{Z}[^{an}, \mathbb{Q}_\ell)). \tag{4.1.2}$$

In this statement, $]\overline{Z}[^{\mathrm{an}}$ stands for $]Z[^{\mathrm{an}}\widehat{\otimes}\widehat{\mathbb{C}((t))^{\mathrm{alg}}}$ with $]Z[^{\mathrm{an}}$ the Berkovich analytification of $]Z[$, and $H^\bullet(]\overline{Z}[^{\mathrm{an}}, \mathbb{Q}_\ell))$ for the corresponding ℓ-adic étale cohomology groups as defined by V. Berkovich in [4].

Theorem 4.1.1 can be considered as a satisfactory analogue of Theorem 2.2.1. Assume for simplicity, X is proper and that Z is equal to the special fiber. Then $S_Z(Y(m)) = S(Y(m))$ and $\chi_c(S(Y(m))$ can be seen as a reasonable substitute for the number of rational points in the degree m extension of a finite field.

Remark 4.1.2 We refer to [28] for the original, more general, statement of Theorem 4.1.1. Note also that it has been extended in [27] beyond the algebraic case.

4.2 The Trace of the Monodromy

Let X be a smooth complex algebraic variety of dimension d and let $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a non-constant morphism to the affine line. Let x be a closed point of $f^{-1}(0)$.

Fix a distance function δ on an open neighborhood of x induced from a local embedding of this neighborhood in some complex affine space. For $\varepsilon > 0$ small enough, one may consider the corresponding closed ball $B(x, \varepsilon)$ of radius ε around x. For $\eta > 0$, we denote by D_η the closed disk of radius η around the origin in $\mathbb{C}$.

By Milnor's local fibration Theorem (see [12, 25]), there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there exists $0 < \eta < \varepsilon$ such that the morphism f restricts to a fibration, called the Milnor fibration,

$$B(x, \varepsilon) \cap f^{-1}(D_\eta \setminus \{0\}) \longrightarrow D_\eta \setminus \{0\}. \tag{4.2.1}$$

The Milnor fiber at x,

$$F_x = f^{-1}(\eta) \cap B(x, \varepsilon), \tag{4.2.2}$$

has a diffeomorphism type that does not depend on δ, η and ε. The characteristic mapping of the fibration induces on F_x an automorphism which is defined up to homotopy, the monodromy M_x. In particular, the singular cohomology groups $H^i(F_x, \mathbb{Q})$ are endowed with an automorphism M_x, and for any integer m, one can consider the Lefschetz numbers

$$\Lambda(M_x^m) = \mathrm{Tr}(M_x^m; H^\bullet(F_x, \mathbb{Q})) = \sum_{i \geq 0} (-1)^i \,\mathrm{Tr}(M_x^m; H^i(F_x, \mathbb{Q})). \tag{4.2.3}$$

Theorem 4.2.1 (A'Campo, [1]) *Assume x is a singular point of $f^{-1}(0)$, that is, $df(x) = 0$. Then*

$$\Lambda(M_x) = 0. \tag{4.2.4}$$

A'Campo's initial proof uses resolution of singularities, but in [20], Lê Dũng Tráng constructed geometrically a representative of the monodromy with no fixed points, providing a proof without resolution of Theorem 4.2.1.

This was later generalized by Deligne to the statement

Theorem 4.2.2 (Deligne, cf. [2]) *Let μ denote the multiplicity of f at x. Then*

$$\Lambda(M_x^m) = 0 \quad \textit{for} \quad 0 < m < \mu. \tag{4.2.5}$$

In [11], Denef and Loeser proved that $\Lambda(M_x^m)$ can be expressed in terms of Euler characteristics of arc spaces as follows. For any integer $m \geq 0$, let $\mathcal{L}_m(X)$ denote the space of arcs modulo t^{m+1} on X: a $\mathbb{C}$-rational point of $\mathcal{L}_m(X)$ corresponds to a $\mathbb{C}[t]/t^{m+1}$-rational point of X, cf. [9]. Consider the locally closed subset $\mathcal{X}_{m,x}$ of $\mathcal{L}_m(X)$

$$\mathcal{X}_{m,x} = \{\varphi \in \mathcal{L}_m(X);\ f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x\}. \tag{4.2.6}$$

Note that $\mathcal{X}_{m,x}$ can be viewed in a natural way as the set of closed points of a complex algebraic variety.

Theorem 4.2.3 (Denef-Loeser, [11]) *For every $m \geq 1$,*

$$\chi_c(\mathcal{X}_{m,x}) = \Lambda(M_x^m). \tag{4.2.7}$$

Note that one recovers Theorem 4.2.2 as a corollary since $\mathcal{X}_{m,x}$ is empty for $0 < m < \mu$.

Remark 4.2.4 The statement of Theorem 4.2.3 emerged at a time F. Loeser had discussions with P. Seidel on the analogy between the use of arcs in motivic integration and that of symplectic disks in Floer theory. In particular, P. Seidel noted the close analogy between Theorem 4.2.3 and the fact that, in symplectic Floer homology, the Lefschetz number of a symplectomorphism is equal to the Euler characteristic of the corresponding Floer homology groups, cf. [13].

4.3 Sketch of Proof of Theorem 4.2.3

By a log-resolution $h : Y \to X$ of $(X, f^{-1}(0))$, we mean a proper morphism $h : Y \to X$ with Y smooth such that the restriction of h: $Y \setminus h^{-1}(f^{-1}(0)) \to X \setminus f^{-1}(0)$ is an isomorphism, and $h^{-1}(f^{-1}(0))$ is a divisor with simple normal crossings. We denote by E_i, i in A, the set of irreducible components of the divisor $h^{-1}(f^{-1}(0))$. Hence, by definition, the E_i's are smooth and intersect transversally. For $I \subset A$, we set $E_I := \bigcap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus \bigcup_{j \notin I} E_j$. We denote by N_i the order of vanishing of $f \circ h$ along E_i, and we define the log discrepancies ν_i by the equality of divisors

$$K_Y = h^* K_X + \sum_{i \in A} (\nu_i - 1) E_i. \tag{4.3.1}$$

We shall use ideas coming from motivic integration to compute the left hand side of (4.2.7) on a log-resolution. Let us denote by $\mathcal{L}(X)$ the arc space of X. As a scheme $\mathcal{L}(X) = \varprojlim \mathcal{L}_m(X)$ and there is a natural bijection between $\mathbb{C}$-rational points of $\mathcal{L}(X)$ and $\mathbb{C}[\![t]\!]$-rational points of X. The following statement is a special case of Lemma 3.4 of

[9] (corresponding to the case when X is smooth), which is the key geometric statement underlying the change of variables formula for motivic integration proved in [9].

Proposition 4.3.1 (Denef-Loeser, [9]) *Let X be a smooth complex algebraic variety of dimension d. Let $h : Y \to X$ be a proper birational morphism with Y smooth. For e in $\mathbb{N}$, set*

$$\Delta_e := \left\{\varphi \in \mathcal{L}(Y) \,\middle|\, \operatorname{ord} h^*(\Omega^d_X)(\varphi) = e\right\}, \tag{4.3.2}$$

where $\operatorname{ord} h^*(\Omega^d_X)(\varphi)$ denotes the order of vanishing of the Jacobian of h at φ. Then, for $n \geq 2e$, the image $\Delta_{e,n}$ of Δ_e in $\mathcal{L}_n(Y)$ is a union of fibers of h_n, the morphism induced by h, and the morphism $h_n : \Delta_{e,n} \to h_n(\Delta_{e,n})$ is a piecewise Zariski fibration with fiber $\mathbf{A}^e_{\mathbb{C}}$.

Consider a log-resolution $h : Y \to X$ of $f^{-1}(0)$ such that $h^{-1}(x)$ is a union of components E_i, $i \in A_0$. It is not difficult to deduce from Proposition 4.3.1, the following equality in $K_0(\mathrm{Var}_{\mathbb{C}})$:

$$[\mathcal{X}_{m,x}] = \mathbb{L}^{md} \sum_{I \cap A_0 \neq \emptyset} (\mathbb{L}-1)^{|I|-1} [\widetilde{E}^\circ_I] \Bigg(\sum_{k_i \geq 1, i \in I, \sum_I k_i N_i = m} \mathbb{L}^{-\sum k_i \nu_i} \Bigg) \tag{4.3.3}$$

with $\widetilde{E}^\circ_I \to E^\circ_I$ an étale cover of degree $\gcd(N_i)_{i \in I}$.

Taking χ_c of both sides, all terms with $|I| \geq 2$ cancel out, and one gets

$$\chi_c(\mathcal{X}_{m,x}) = \sum_{N_i | m, i \in A_0} N_i \, \chi_c(E^0_{\{i\}}). \tag{4.3.4}$$

On the other hand, by a result of A'Campo [2], which is an easy consequence of the Leray-Serre spectral sequence associated to the direct image of nearby cycles, the right hand side of (4.3.4) is equal, for $m \geq 1$, to the Lefschetz number $\Lambda(M^m_x)$. In conclusion, the proof of Theorem 4.2.3 in [11] consists in computing explicitly both sides of (4.3.4) and checking both quantities are equal. Such a proof is not very enlightening. In Section 5, we shall present another approach, based on non-archimedean geometry, that is more conceptual, avoids explicit computations on resolutions and allows to see Theorem 4.2.3 as a consequence of some Lefschetz fixed point formula.

4.4 Back to Non-Archimedean Geometry

We shall now explain how one can see Theorem 4.2.3 as a special case of Theorem 4.1.1.

Let X be a smooth complex algebraic variety of dimension d and let $f : X \to \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$ be a non-constant morphism to the affine line. Via f, we can see X as a $\mathbb{C}[t]$-scheme. Let $\mathcal{Y}$ be its t-adic completion and Y the corresponding rigid space. For any closed point x of $f^{-1}(0)$, we can consider the tube $]x[$ in Y. The authors of [28] call this tube the analytic Milnor fiber at x and denote it by $\mathcal{F}_x$.

As in Section 4.1, we shall write $\overline{\mathcal{F}_x}^{\mathrm{an}}$ for $\mathcal{F}^{\mathrm{an}}_x \widehat{\otimes} \widehat{\mathbb{C}((t))^{\mathrm{alg}}}$ and $H^\bullet(\overline{\mathcal{F}_x}^{\mathrm{an}}, \mathbb{Q}_\ell)$ for the corresponding ℓ-adic étale cohomology groups in the Berkovich sense. Note that the Galois element φ acts on $H^\bullet(\overline{\mathcal{F}_x}^{\mathrm{an}}, \mathbb{Q}_\ell)$.

Using a comparison theorem of Berkovich, Nicaise, and Sebag show in [28] that, for every $i \geq 0$, there is an isomorphism

$$H^i(\overline{\mathcal{F}_x}^{\mathrm{an}}, \mathbb{Q}_\ell) \simeq H^i(F_x, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \tag{4.4.1}$$

such that the action of φ on the left hand side corresponds to the action of the monodromy M_x on the right hand side. Thus, it follows that the right-hand sides of (4.1.2) and (4.2.7) are equal, for $Z = \{x\}$.

It remains to see the connection between $S_{\{x\}}(Y(m))$ and $\mathcal{X}_{m,x}$. For this, let us observe that a $\mathbb{C}((t^{1/m}))$-point of $\mathcal{F}_x$ is nothing but a morphism $\psi : \mathrm{Spec}\,\mathbb{C}[[t^{1/m}]] \to X$ such that $f(\psi) = t$ and $\psi(0) = x$. Since there is no harm in reparameterizing ψ by t, we may identify $\mathcal{F}_x(\mathbb{C}((t^{1/m})))$ with the set of arcs $\psi : \mathrm{Spec}\,\mathbb{C}[[t]] \to X$ such that $f(\psi) = t^m$ and $\psi(0) = x$. One can check that the motivic volume of this set, taken in an appropriate Greenberg scheme, is equal to the class $\mathcal{X}_{m,x}$ in $K_0(\mathrm{Var}_{\mathbb{C}})$ up to a power of $\mathbb{L}$, from which it follows that in this case the left-hand sides of (4.1.2) and (4.2.7) are equal too.

Remark 4.4.1 The proof of Theorem 4.1.1 follows similar lines to that of Theorem 4.2.3, relying ultimately on explicit computations on a resolution.

5 Trace Formulas via Non-Archimedean Geometry

In this section, we present the new approach to the previous results developed with E. Hrushovski in the paper [17].

5.1 Hrushovski-Kazhdan Motivic Integration

For the convenience of the reader, we shall present now some of the main results of [16], in a simplified form adapted to the needs of the present survey. We shall work in the setting of valued fields that is of fields L endowed with a valuation $\mathrm{val} : L^\times \to \Gamma(L)$ with $\Gamma(L)$ an ordered abelian group. Setting $\mathrm{val}(0) = \infty$, one extends val to $\mathrm{val} : L \to \Gamma_\infty(L) := \Gamma(L) \cup \{\infty\}$. We will mostly consider $K = \mathbb{C}((t))$ and $\overline{K} = \mathbb{C}((t))^{\mathrm{alg}}$ with their standard valuation satisfying $\mathrm{val}(t^\gamma) = \gamma$, thus there is an inclusion $\Gamma(K) = \mathbb{Z} \subset \Gamma(\overline{K}) = \mathbb{Q}$.

We define semi-algebraic subsets of $\overline{K}^n$ as elements of the Boolean algebra generated by subsets of $\overline{K}^n$ defined by conditions of the form $\mathrm{val}(f) \geq \mathrm{val}(g)$ with f and g polynomials with coefficients in K. More generally, if X is a K-algebraic variety, a subset Z of $X(\overline{K})$ is called semi-algebraic if there is a cover by affine K-varieties U_i, such that $Z \cap U_i(\overline{K})$ is semi-algebraic for each i. We define a category VF_K whose objects are semi-algebraic subsets of some K-algebraic variety, morphisms being functions whose graph is semi-algebraic. In the terminology of [16], VF_K is (equivalent to) the category of K-definable sets in the VF-sort.

Let L be a valued field. Denote by M_L the maximal ideal of its valuation ring. The quotient $\mathrm{RV}(L) := L^\times/1 + M_L$ plays a central role in the Hrushovski-Kazhdan approach. It fits in a short exact sequence

$$1 \longrightarrow k(L)^\times \longrightarrow \mathrm{RV}(L) \longrightarrow \Gamma(L) \longrightarrow 0 \qquad (5.1.1)$$

with $k(L)$ the residue field of L. We denote by $\mathrm{rv} : L^\times \to \mathrm{RV}(L)$, and more generally $\mathrm{rv} : (L^\times)^n \to \mathrm{RV}(L)^n$, the quotient morphism and by $\mathrm{val} : \mathrm{RV}(L) \to \Gamma(L)$ the morphism induced by val.

We will say a subset of $\Gamma(\overline{K})^n$ is semi-algebraic if it belongs to the Boolean algebra generated by subsets of $\Gamma(\overline{K})^n$ of the form $\sum_{i=1}^n a_i x_i + b \geq 0$ with a_i in $\mathbb{Z}$ and $b \in \Gamma(\overline{K})$. Semi-algebraic subsets of $\Gamma(\overline{K})^n$, for variable n, form a category that we denote by Γ_K. For $n \geq 0$, we denote by $\Gamma_K[n]$ the subcategory of semi-algebraic subsets of $\Gamma(\overline{K})^n$.

Similarly, one may define a notion of semi-algebraic subsets of $\mathrm{RV}(\overline{K})^n$ (K-definable sets in the RV-sort in the terminology of [16]). We will not give a precise definition here, but here are some properties that should allow to get some feeling about them:

(a) If X is a semi-algebraic subset $\mathrm{RV}(\overline{K})^n$, its projection to $\Gamma(\overline{K})^n$ is semi-algebraic and its intersection with $(k(\overline{K})^\times)^n$ is the set of $\mathbb{C}$-points of a $\mathbb{C}$-constructible set.
(b) The image, resp. preimage, under $\mathrm{rv} : (L^\times)^n \to \mathrm{RV}(L)^n$ of a semi-algebraic set is semi-algebraic.

Semi-algebraic subsets of $\mathrm{RV}(\overline{K})^n$, for variable n, form a category that we denote by RV_K. For $n \geq 0$, we denote by $\mathrm{RV}_K[n]$ the category of morphisms $f : X \to \mathrm{RV}(\overline{K})^n$ in RV_K with finite fibers.

For each of the category VF_K, RV_K, ..., one denotes by $K(\mathrm{VF}_K)$, $K(\mathrm{RV}_K)$, ..., the corresponding Grothendieck ring. It is the free abelian group on isomorphism classes of objects modulo the cut and paste relation.

Given a semialgebraic subset X of $\mathrm{RV}(\overline{K})^n$, one may consider its preimage $\mathrm{rv}^{-1}(X)$ in $\mathrm{VF}(\overline{K})^n$, which one denotes by $\mathbf{L}(X)$ ($\mathbf{L}$ stands for "lifting"). More generally, given $f : X \to \mathrm{RV}(\overline{K})^n$ in RV_K with finite fibers, one may consider the set $\mathbf{L}(X) = \{(x, y) \in X \times \mathrm{VF}(\overline{K})^n; f(x) = \mathrm{rv}(y)\}$. With some thought, one may identify $\mathbf{L}(X)$ with an object of VF_K well defined up to isomorphism. The assignment $X \to \mathbf{L}(X)$ gives rise to a morphism

$$\mathbf{L} : K(\mathrm{RV}_K[n]) \longrightarrow K(\mathrm{VF}_K). \tag{5.1.2}$$

Setting $K(\mathrm{RV}_K[*]) = \oplus_n K(\mathrm{RV}_K[n])$, one gets a morphism

$$\mathbf{L} : K(\mathrm{RV}_K[*]) \longrightarrow K(\mathrm{VF}_K). \tag{5.1.3}$$

Hrushovski and Kazhdan proved the following remarkable result:

Theorem 5.1.1 (Hrushovski-Kazhdan, [16]) *The morphism* (5.1.3) *is surjective.*

The morphism (5.1.3) is not injective. Indeed, if $[1]_n$ stands for the class of a point embedded in $\mathrm{RV}(\overline{K})^n$, $\mathbf{L}([1]_0)$ is equal to the class of a point, while $\mathbf{L}([1]_1)$ is equal to the class of the open ball $1 + M_{\overline{K}}$. On the other hand, if $[\mathrm{RV}^{>0}]_1$ denotes the class of the subset defined by the condition $\mathrm{val}(x) > 0$ in $\mathrm{RV}(\overline{K})$, one notices that $\mathbf{L}([\mathrm{RV}^{>0}]_1)$ is the class of the punctured open ball $M_{\overline{K}} \setminus \{0\}$. Thus, $[\mathrm{RV}^{>0}]_1 + [1]_0 - [1]_1$ belongs to the kernel of $\mathbf{L}$. An important result of Hrushovski and Kazhdan states that this is the only relation, namely:

Theorem 5.1.2 (Hrushovski-Kazhdan, [16]) *The kernel of the morphism* (5.1.3) *is exactly the ideal I generated by* $[\mathrm{RV}^{>0}]_1 + [1]_0 - [1]_1$.

Putting together Theorem 5.2.1 and Theorem 5.1.2 and inverting $\mathbf{L}$, one gets an isomorphism

$$\mathrm{HK} : K(\mathrm{VF}_K) \longrightarrow K(\mathrm{RV}_K[*])/I. \tag{5.1.4}$$

Remark 5.1.3 In fact, the aforementioned results of Hrushovski-Kazhdan are already true at the semi-ring level, cf. [16].

5.2 Construction of a Morphism $K(\mathrm{VF}_K) \to K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/(\mathbb{L} - 1)K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$

From the short exact sequence (5.1.1), one may expect that $K(\mathrm{RV}_K[*])$ can be expressed in terms of data defined over Γ and over the residue field. This is in fact the case, but to achieve this one needs to introduce the subcategory RES_K of RV_K consisting of semi-algebraic subsets of some $\mathrm{RV}(\overline{K})^n$ whose image in $\Gamma(\overline{K})^n$ under val is finite. Objects in RES_K correspond to twisted forms of $\mathbb{C}$-constructible sets, like, for instance, the set defined by the equation $2\,\mathrm{val}(x) = \mathrm{val}(t)$. For $n \geq 0$, we denote by $\mathrm{RES}_K[n]$ the subcategory of $\mathrm{RV}_K[n]$ consisting of morphisms $f : X \to \mathrm{RV}(\overline{K})^n$ with finite fibers, with X in RES_K.

The last category we will have to consider is the subcategory $\Gamma_K^{\mathrm{fin}}[n]$ of $\Gamma_K[n]$ consisting of finite semi-algebraic sets. Note that we have a functor $\Gamma_K^{\mathrm{fin}}[n] \to \mathrm{RES}_K[n]$ sending a finite set $S \subset \Gamma^n$ to $\mathrm{val}^{-1}(S)$ in $\mathrm{RES}_K[n]$. We set $K(\mathrm{RES}_K[*]) = \oplus_n K(\mathrm{RES}_K[n])$ and $K(\Gamma_K^{\mathrm{fin}}[*]) = \oplus_n K(\Gamma_K^{\mathrm{fin}}[n])$. Thus, $K(\mathrm{RES}_K[*])$ is endowed with the structure of a graded $K(\Gamma_K^{\mathrm{fin}}[*])$-algebra induced by the previous functor.

Theorem 5.2.1 (Hrushovski-Kazhdan, [16]) *The morphism*

$$\Psi : K(\mathrm{RES}_K[*]) \otimes_{K(\Gamma_K^{\mathrm{fin}}[*])} K(\Gamma_K[*]) \longrightarrow K(\mathrm{RV}_K[*]) \tag{5.2.1}$$

sending $[X] \otimes [Y]$ *to* $[X \times \mathrm{val}^{-1}(Y)]$ *is an isomorphism.*

Thus, one can rewrite the isomorphism HK from (5.1.4) as an isomorphism $K(\mathrm{VF}_K) \to K(\mathrm{RES}_K[*]) \otimes_{K(\Gamma_K^{\mathrm{fin}}[*])} K(\Gamma_K[*])/I'$ with I' the ideal corresponding to I under the isomorphism Ψ. In [17], we made use of the combinatorial Euler characteristic on Γ_K to "kill" the Γ-part and to get a morphism

$$\Theta : K(\mathrm{VF}_K) \longrightarrow K(\mathrm{RES}_K)/I'' \tag{5.2.2}$$

with I'' some explicit ideal of $K(\mathrm{RES}_K)$.

In fact, the quotient $K(\mathrm{RES}_K)/I''$ can be reinterpreted in terms of varieties with $\hat{\mu}$-action. Let us say a $\hat{\mu}$-action on a complex quasi-projective is good if it factorizes through some μ_n-action, for some $n \geq 1$. We denote by $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ the quotient of the abelian group generated by isomorphism classes of complex quasi-projective varieties with a good $\hat{\mu}$-action by the standard cut and paste relations and the following additional relations: for every complex quasi-projective variety X with good $\hat{\mu}$-action, for every finite dimensional complex vector space V endowed with two good linear actions ϱ and ϱ', the class of $X \times (V, \varrho)$ is equal to the class of $X \times (V, \varrho')$.

Proposition 5.2.2 (Hrushovski-Loeser, [17]) *There is a canonical isomorphism between* $K(\mathrm{RES}_K)/I''$ *and* $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/(\mathbb{L}-1)K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$*. Here,* $\mathbb{L}$ *is the class of the affine line, say with trivial* $\hat{\mu}$*-action.*

Putting (5.2.2) and Proposition 5.2.2 together, we get a canonical morphism

$$\mathrm{EU}_{\Gamma} : K(\mathrm{VF}_K) \longrightarrow K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/(\mathbb{L}-1)K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \tag{5.2.3}$$

which plays a major role in our approach.

5.3 Equivariant Euler Characteristics

Let X be a K-algebraic variety of dimension d. We denote by X^{an} its Berkovich analytification. Now let U be a semi-algebraic subset of $X(\overline{K})$. It is defined Zariski locally by some finite Boolean combination of inequalities between valuations of functions, with data defined over K. We denote by U^{an} the subset of X^{an} defined by the same conditions. We set

$\overline{X}^{\rm an} = X^{\rm an}\widehat{\otimes}\widehat{\mathbb{C}((t))^{\rm alg}}$ and we denote by $\overline{U}^{\rm an}$ the preimage of $U^{\rm an}$ in $\overline{X}^{\rm an}$ under the canonical morphism $\overline{X}^{\rm an} \to X^{\rm an}$.

When $U^{\rm an}$ is locally closed in $X^{\rm an}$, the theory of germs in [4] allows to define cohomology groups $H^i_c(\overline{U}^{\rm an}, \mathbb{Q}_\ell)$ endowed with an action of the Galois group $\mathrm{Aut}(\widehat{\mathbb{C}((t))^{\rm alg}}/\mathbb{C}((t))) = \hat{\mu}$. Furthermore, Florent Martin proved in [24] that they are finite dimension $\mathbb{Q}_\ell$-vector spaces and that they are zero for $i > 2d$.

Let $K(\hat{\mu}{-}\mathrm{Mod})$ be the Grothendieck ring of the category of $\mathbb{Q}_\ell[\hat{\mu}]$-modules that are finite dimensional as $\mathbb{Q}_\ell$-vector spaces. When $U^{\rm an}$ is locally closed in $X^{\rm an}$, one defines $\mathrm{EU}_{\text{ét}}(U)$ as the class of

$$\sum_i (-1)^i [H^i_c(\overline{U}^{\rm an}, \mathbb{Q}_\ell)] \tag{5.3.1}$$

in $K(G{-}\mathrm{Mod})$.

Using further results from [24], one proves the existence of a unique morphism

$$\mathrm{EU}_{\text{ét}} \colon K(\mathrm{VF}_K) \longrightarrow K(\hat{\mu}{-}\mathrm{Mod}) \tag{5.3.2}$$

extending the previous construction.

Let now Y be a complex quasi-projective variety endowed with a $\hat{\mu}$-action factoring for some n through a μ_n-action. The ℓ-adic étale cohomology groups $H^i_c(Y, \mathbb{Q}_\ell)$ are endowed with a $\hat{\mu}$-action, and we may consider the element

$$\mathrm{eu}_{\text{ét}}(Y) := \sum_i (-1)^i [H^i_c(Y, \mathbb{Q}_\ell)] \tag{5.3.3}$$

in $K(\hat{\mu}{-}\mathrm{Mod})$. Note that $\mathrm{eu}_{\text{ét}}([V, \varrho]) = 1$ for any finite dimensional $\mathbb{C}$-vector space V endowed with a $\hat{\mu}$-action factoring for some n through a linear μ_n-action. Thus, $\mathrm{eu}_{\text{ét}}$ factors give rise to a morphism

$$\mathrm{eu}_{\text{ét}} \colon K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/(\mathbb{L}-1)K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \longrightarrow K(\hat{\mu}{-}\mathrm{Mod}). \tag{5.3.4}$$

We have the following fundamental compatibility property between $\mathrm{EU}_{\text{ét}}$ and $\mathrm{eu}_{\text{ét}}$.

Theorem 5.3.1 (Hrushovski-Loeser, [17]) *The diagram*

$$\begin{array}{ccc} K(\mathrm{VF}) & \xrightarrow{\ \mathrm{EU}_\Gamma\ } & K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/(\mathbb{L}-1)K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \\ & {\scriptstyle \mathrm{EU}_{ét}} \searrow \qquad \swarrow {\scriptstyle \mathrm{eu}_{ét}} & \\ & K(\hat{\mu}{-}\mathrm{Mod}) & \end{array} \tag{5.3.5}$$

is commutative.

5.4 A Fixed Point Formula

We shall use the following version of the Lefschetz fixed point theorem. It is classical and follows in particular from Theorem 3.2 of [7]:

Proposition 5.4.1 *Let Y be a quasi-projective variety over an algebraically closed field of characteristic zero. Let T be a finite order automorphism of Y. Let Y^T be the fixed point set of T and denote by $\chi_c(Y^T, \mathbb{Q}_\ell)$ its ℓ-adic Euler characteristic with compact supports. Then*

$$\chi_c(Y^T, \mathbb{Q}_\ell) = \mathrm{Tr}(T; H^\bullet_c(Y, \mathbb{Q}_\ell)). \tag{5.4.1}$$

Remark 5.4.2 In general, one cannot expect to have a fixed point theorem for non proper varieties without a good control of the behaviour of the automorphism at infinity. Thus, in the above statement, the condition that T is of finite order is crucial.

5.5 A Proof of Theorem 4.2.3 using Non-Archimedean Geometry

We are now in position to explain the proof of Theorem 4.2.3 given in [17].

Fix $m \geq 1$. By (4.4.1), one may write

$$\Lambda(M_x^m) = \mathrm{Tr}(\varphi^m; H^\bullet(\overline{\mathcal{F}_x}^{\mathrm{an}}, \mathbb{Q}_\ell)). \tag{5.5.1}$$

One deduces easily from Poincaré Duality that

$$\mathrm{Tr}(\varphi^m; H^\bullet(\overline{\mathcal{F}_x}^{\mathrm{an}}, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^m; H_c^\bullet(\overline{\mathcal{F}_x}^{\mathrm{an}}, \mathbb{Q}_\ell)). \tag{5.5.2}$$

By definition, we have

$$\mathrm{Tr}(\varphi^m; H_c^\bullet(\overline{\mathcal{F}_x}^{\mathrm{an}}, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^m; \mathrm{EU}_{\text{ét}}([X_{t,x}])), \tag{5.5.3}$$

with $X_{t,x}$ the semi-algebraic subset of $X(\mathcal{O}_K)$ defined by $f(\varphi) = t$ and $\varphi(0) = x$. On the other hand, by Theorem 5.3.1,

$$\mathrm{Tr}(\varphi^m; \mathrm{EU}_{\text{ét}}([X_{t,x}])) = \mathrm{Tr}(\varphi^m; \mathrm{eu}_{\text{ét}} \circ \mathrm{EU}_\Gamma([X_{t,x}])). \tag{5.5.4}$$

Using the Lefschetz fixed point formula provided by Proposition 5.4.1, we get

$$\mathrm{Tr}(\varphi^m; \mathrm{eu}_{\text{ét}} \circ \mathrm{EU}_\Gamma([X_{t,x}])) = \chi_c(\mathrm{EU}_\Gamma([X_{t,x}])^{\varphi^m}), \tag{5.5.5}$$

where $\mathrm{EU}_\Gamma([X_{t,x}])^{\varphi^m}$ denotes the fixed point set of φ^m acting on the virtual object $\mathrm{EU}_\Gamma([X_{t,x}])$. Finally, one proves that $\mathrm{EU}_\Gamma([X_{t,x}])^{\varphi^m})$ and $\mathcal{X}_{m,x}$ have the same class in $K_0(\mathrm{Var}_\mathbb{C})/(\mathbb{L} - 1)K_0(\mathrm{Var}_\mathbb{C})$. In particular,

$$\chi_c(\mathrm{EU}_\Gamma([X_{t,x}])^{\varphi^m}) = \chi_c(\mathcal{X}_{m,x}), \tag{5.5.6}$$

which finishes the proof of Theorem 4.2.3.

Remark 5.5.1 As explained in Section 7 of [17], one can give by a similar argument a new proof of Theorem 4.1.1 when the algebraic generic fiber X_K is proper. This is proved in [17] when Z is the entire special fiber, but the proof should extend to any Z. In fact, with the notation of Theorem 4.1.1, we prove that the Serre invariant $S(Y(m))$ and $\mathrm{EU}_\Gamma([X_K])^{\varphi^m}$ have the same class in $K_0(\mathrm{Var}_\mathbb{C})/(\mathbb{L}-1)K_0(\mathrm{Var}_\mathbb{C})$. This provides a geometric interpretation of the motivic Serre invariant as a fixed point set.

5.6 Final Remarks

In view of Grothendieck's function-sheaf dictionary recalled in Section 2, it is natural to expect that there should exist an appropriate categorisation of the isomorphism HK of (5.1.4). Such a categorisation should provide a lift of the isomorphism HK between Grothendieck rings to an equivalence between actual localized (higher?) categories of objects over VF and RES. [In fact, Hrushovski and Kazhdan also construct a version of HK involving volume forms and it would certainly be also very interesting to try to categorify that version.] Note that although we consider here only valued fields of equicharacteristic zero, this might already provide interesting results for local fields of large residue characteristic.

We now briefly present some recent partial results going into the direction of categorising the isomorphism HK. In our work with Hrushovski [18], to any algebraic variety X over a valued field K, with no restriction on the rank of the valuation nor on characteristics, we assign its stable completion $\widehat{X}$ which is defined as the space of stably dominated types on X in the sense of [15]. It is endowed with a topology coming from the value group Γ. When K is of rank 1, it is closely related to the Berkovich analytification of X. A key feature in our approach is the fact that $\widehat{X}$ can in a natural way be endowed with the structure of a pro-definable set in the geometric language of [14]. The main result of [18] is the existence, for any algebraic variety or definable set X over a valued field, of a strong deformation retraction of $\widehat{X}$ onto a definable set Υ sitting inside $\widehat{X}$ which is definably homeomorphic to a definable set in the sort Γ. This can be rephrased as an equivalence between a homotopy category HC_{VF} of definable objects in the VF-sort whose construction involves the stable completion and a homotopy category HC_{Γ} of definable sets in the Γ-sort. Thus, keeping in mind that RV roughly consists of a Γ-part and a RES-part, this equivalence solves "one half" of the categorisation problem, namely the Γ-part.

In [3], Ayoub extends the theory of motives to the framework of rigid geometry in the sense of Tate and Raynaud. For any complete non-archimedean field K, he constructs a stable homotopy category RigSH(K) using the Nisnevich site of smooth rigid varieties with interval the closed unit ball. Under the assumption that $K = k(\!(t)\!)$, with k a field of characteristic zero, he proves that RigSH(K) can be naturally identified with a full subcategory of the category SH$(\mathbb{G}_{m,k})$, the stable homotopy category of schemes over the torus $\mathbb{G}_{m,k}$. This is quite suggestive since varieties over $\mathbb{G}_{m,k}$ are closely related to varieties with $\hat{\mu}$-action, that are connected to twisted varieties in the RES-sort.

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