MOTIVIC INTEGRATION AND MCKAY CORRESPONDENCE

FRANÇOIS LOESER

These notes correspond to a series of 5 lectures given at ICTP in September 2002. We tried to keep as much as possible from the informal style of exposition we adopted during the lectures. We thank Magda Sebestean for providing us her set of notes which showed very useful for the present write-up.

1. Arc spaces

1.1. **Arc spaces.** We shall assume throughout these notes that k is a field of characteristic 0. Many of the results presented in these lectures do not hold anymore or become unknown in positive characteristic. By a variety X over k will shall always mean a separated and reduced scheme, of finite type over k.
For $n \geq 0$, we introduce the space of n-arcs on X, denoted by $\mathcal{L}_n(X)$. It is a k-scheme of finite type which represents the functor:

$$k-\text{algebras} \longrightarrow \text{Sets}$$

$$R \mapsto \text{Hom}_{k-\text{schemes}}(\text{Spec}(R[t]/(t^{n+1}), X) := X(R[t]/(t^{n+1})).$$

For example, when X is an affine variety with equations $f_i(\vec{x}) = 0$, $i = 1, \cdots, m$, $\vec{x} = (x_1, \cdots, x_r)$, then $\mathcal{L}_n(X)$ is given by the equations, in the variables $\vec{a}_0, \cdots, \vec{a}_n$, expressing that $f_i(\vec{a}_0 + \vec{a}_1 t + \cdots + \vec{a}_n t^n) \equiv 0 \mod t^{n+1}, i = 1, \cdots, m$.
We have canonical isomorphisms $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X) = TX$, where TX denotes the tangent space of the variety X.
For $m \geq n$, there are canonical morphisms $\theta_m^n : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$. In general, when X is not smooth, they need nor to be surjective. When X is smooth of dimension d, θ_m^n is a locally trivial fibration for the Zariski topology with fiber $\mathbf{A}_k^{(m-n)d}$.
Taking the projective limit of these algebraic varieties $\mathcal{L}_n(X)$, we obtain the arc space $\mathcal{L}(X)$ of X. A priori this is just a pro-scheme, but since the transition maps θ_m^n are affine it is indeed a k-scheme.
In general, $\mathcal{L}(X)$ is not of finite type over k. The K-rational points of $\mathcal{L}(X)$ are the $K[[t]]$-rational points of X. These are called K-arcs on X. For example when X is an affine variety with equations $f_i(\vec{x}) = 0, i = 1, \cdots m, \vec{x} = (x_1, \cdots, x_r)$, then the K-rational points of $\mathcal{L}(X)$ are the sequences $(\vec{a}_0, \vec{a}_1, \vec{a}_2, \cdots) \in (K^n)^{\mathbf{N}}$ satisfying $f_i(\vec{a}_0 + \vec{a}_1 t + \vec{a}_2 t^2 + \cdots) = 0$, for $i = 1, \cdots, m$. For every n we have natural morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

obtained by truncation. For any arc γ on X (i.e. a K-arc for some field K containing k), we call $\pi_0(\gamma)$ the origin of the arc γ.
One can easily check that $\mathcal{L}(X)$ represents the functor

$$k-\text{algebras} \longrightarrow \text{Sets}$$

$$R \mapsto \mathrm{Hom}_{k-\text{schemes}}(\mathrm{Spec}(R[[t]]), X) := X(R[[t]]).$$

It also represents the functor

$$k-\text{schemes} \longrightarrow \text{Sets}$$

$$S \mapsto \mathrm{Hom}_{\text{locally ringed spaces}}((S, \mathcal{O}_S[[t]]), X).$$

1.2. **Kolchin's theorem.** Not so many non trivial results about the arc space $\mathcal{L}(X)$ are known. Here is one.

1.2.1. Theorem (Kolchin). *If X is an integral scheme, then $\mathcal{L}(X)$ is irreducible.*

We shall sketch a geometric proof of this result, after Ishii and Kollár. Let us remark the result is quite clear if X is smooth of dimension d, using the fact that the maps $\mathcal{L}_n(X) \to X$ are Zariski fibrations with fiber $\mathbf{A}_k^{nd}$. The key point is the following statement (Lemma 2.12 of [16]):

1.2.2. Key-Point (Ishii-Kollár). *Every arc passing through the singular locus* $\mathrm{Sing}(X)$ *of X is the specialization of some arc whose origin lies on* $\mathrm{Sing}(X)$ *and whose generic point lies on* $X \setminus \mathrm{Sing}(X)$.

Sketch of proof. We may assume X is affine. Take $\phi : \mathrm{Spec} k'[[t]] \to X$ an arc with $\phi(0)$ in $\mathrm{Sing}(X)$. We denote by Y the Zariski closure of the image of ϕ. We may assume k' is algebraically closed.
First step: We construct $\Phi : \mathrm{Spec} K[[t]] \to Y \subset X$ with $\Phi(0)$ the generic point of Y such that ϕ is a specialization of Φ. This is done by considering the embedding

$$k'[[t]] \hookrightarrow k'[[T, U]]$$

given by $t \mapsto T + U$. We define Φ to be the arc given by the composition of $\phi : \mathcal{O}_Y \to k'[[t]]$ with

$$k'[[t]] \hookrightarrow k'[[T, U]] \hookrightarrow k'((U))[[T]] \hookrightarrow K[[T]],$$

with K the algebraic closure of $k'((U))$. Since the pull back of (T) to $k'[[t]]$ is the zero ideal, the closed point of $\mathrm{Spec} K[[T]]$ maps to the generic point of Y.
Second step: By cutting with hypersurfaces containing Y, we may assume $Y \subset Z \subset X$ with $\dim Z = \dim Y + 1$ and X smooth at the generic point of Z. Now consider the normalization $n : \overline{Z} \to Z$ and set $\overline{Y} := n^{-1}(Y)$ (with the reduced scheme structure). The morphism $\overline{Y} \to Y$ being generically étale, Φ may be lifted to some arc $\overline{\Phi} : \mathrm{Spec} K[[T]] \to \overline{Y}$. But $\overline{Z}$ is smooth at the generic point of $\overline{Y}$; so $\overline{\Phi}$ is the specialization of an arc through $\overline{Y}$ whose generic point maps to the generic point of $\overline{Z}$. Now the image of this arc by n does the job. □

Now to finish the proof of Kolchin's Theorem, it is enough to prove that every arc whose generic point lies in the smooth locus of X is the specialization of an arc entirely contained in the smooth locus. This can be done directly (e.g. using a desingularization of X) or by applying again the deformation trick we used in the first step.

1.3. **Nash Problem.** Suppose X is singular. The original idea of Nash was try to use the arc space $\mathcal{L}(X)$ to get some information about resolutions of singularities of X. Let us introduce some terminology concerning resolutions of singularities.
A resolution of singularities of a singular variety X is a morphism $\pi : Y \to X$ such that Y is smooth, π is proper and is an isomorphism away from the singular locus of X, i.e. π induces an isomorphism between $Y \setminus \pi^{-1}(\mathrm{Sing}(X))$ and $X \setminus \mathrm{Sing}(X)$. A resolution of singularities is called a divisorial resolution if, moreover, its exceptional set, i.e. the locus where π is not a local isomorphism, is a divisor in Y.
In the following we shall assume X is a normal variety. Take $g : X_1 \to X$ to be a proper, birational morphism and let E denote an irreducible exceptional divisor of g. If $f : X_2 \to X$ is another proper, birational morphism, one gets a birational map $f^{-1} \circ g : X_1 \dashrightarrow X_2$. We say E appears in f if $f^{-1} \circ g$ is a local isomorphism at the generic point ξ of E. This allows to identify E and the closure of ξ in X_2. We say an exceptional divisor E in a resolution of X is an essential divisor (resp. an essential component) if it appears on each resolution (resp. divisorial resolution) of the X.
Nash's idea is that there should be a relation beween essential components and arcs. Fix a point x in the singular locus of X. Denote by $\mathcal{N}_x(X)$ the set of arcs in $\mathcal{L}(X)$ with origin x but no contained in $\mathrm{Sing}(X)$. Using Hironaka's Theorem on resolution of singularities he proved:

1.3.1. Theorem (Nash [22])**.** *The number of irreducible components of $\mathcal{N}_x(X)$ is bounded by the number of the essential components above x.*

More precisely Nash constructed an injective mapping N from the set of irreducible components of $\mathcal{N}_x(X)$ to the set of essential components above x, and he asked whether N is a bijection between these two sets.
The answer is known only in very special cases. Let us give one of the examples in the original paper of Nash [22]: the surface singularity $x^2 + y^2 + z^{n+1} = 0$ at 0 has a chain of n rational curves as an exceptional divisor in its minimal resolution and the space $\mathcal{N}_0(X)$ has n components of the form $x = \alpha t^\nu + \dots$, $y = \beta t^{n+1-\nu} + \dots$, $z = \gamma t + \dots$, with $\nu = 1, \dots, n$ and $\alpha\beta = \gamma^{n+1}$ which correspond naturally to the n rational curves.
The answer to Nash question is yes for certain rational surface singularities [26] and toric singuarities by a joint work of Ishii and Kollár [16]. In the same paper, Ishii and Kollár gave the following example for X of dimension 4 where the answer is no: they proved that, for the hypersurface $x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$, $\mathcal{N}_0(X)$ has only one irreducible component but that there are two essential components above the origin.

2. Additive invariants of algebraic varieties

2.1. Let R be a ring. We denote by Var_R the category of algebraic varieties over R. An additive invariant

$$\lambda : \mathrm{Var}_R \longrightarrow S,$$

with S a ring, assigns to any X in Var_R an element $\lambda(X)$ of S such that

$$\lambda(X) = \lambda(X')$$

for $X \simeq X'$,

$$\lambda(X) = \lambda(X') + \lambda(X \setminus X'),$$

for X' closed in X, and

$$\lambda(X \times X') = \lambda(X)\lambda(X')$$

for every X and X'.
Let us remark that additive invariants λ naturally extend to take their values on constructible subsets of algebraic varieties. Indeed a constructible subset W may be written as a finite disjoint union of locally closed subvarieties Z_i, $i \in I$. One may define $\lambda(W)$ to be $\sum_{i \in I} \lambda(Z_i)$. By the very axioms, this is independent of the decomposition into locally closed subvarieties.

2.2. Examples.

2.2.1. *Euler characteristic.* Here $R = k$ is a field. When k is a subfield of $\mathbf{C}$, the Euler characteristic $\mathrm{Eu}(X) := \sum_i (-1)^i \mathrm{rk} H_c^i(X(\mathbf{C}), \mathbf{C})$ give rise to an additive invariant $\mathrm{Eu} : \mathrm{Var}_k \to \mathbf{Z}$. For general k, replacing Betti cohomology with compact support by ℓ-adic cohomology with compact support, $\ell \neq \mathrm{char} k$, one gets an additive invariant $\mathrm{Eu}_\ell : \mathrm{Var}_k \to \mathbf{Z}$, which does not depend on ℓ.

2.2.2. *Hodge polynomial.* Let us assume $R = k$ is a field of characteristic zero. Then it follows from Deligne's Mixed Hodge Theory that there is a unique additive invariant $H : \mathrm{Var}_k \to \mathbf{Z}[u, v]$, which assigns to a smooth projective variety X over k its usual Hodge polynomial

$$H(u, v) := \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q,$$

with $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$ the (p, q)-Hodge number of X.

2.2.3. *Virtual motives.* More generally, when $R = k$ is a field of characteristic zero, there exists by Gillet and Soulé [12], Guillen and Navarro-Aznar [14], a unique additive invariant $\chi_c : \mathrm{Var}_k \to K_0(\mathrm{CHMot}_k)$, which assigns to a smooth projective variety X over k the class of its Chow motive, where $K_0(\mathrm{CHMot}_k)$ denotes the Grothendieck ring of the category of Chow motives over k (with rational coefficients).

2.2.4. *Counting points.* Counting points also yields additive invariants. Assume $k = \mathbf{F}_q$, then $N_n : X \mapsto |X(\mathbf{F}_{q^n})|$ gives rise to an additive invariant $N_n : \mathrm{Var}_k \to \mathbf{Z}$. Similarly, if R is (essentially) of finite type over $\mathbf{Z}$, for every maximal ideal $\mathfrak{P}$ of R with finite residue field $k(\mathfrak{P})$, we have an additive invariant $N_{\mathfrak{P}} : \mathrm{Var}_R \to \mathbf{Z}$, which assigns to X the cardinality of $(X \otimes k(\mathfrak{P}))(k(\mathfrak{P}))$.

2.3. There exists a universal additive invariant $[\_] : \mathrm{Var}_R \to K_0(\mathrm{Var}_R)$ in the sense that composition with $[\_]$ gives a bijection between ring morphisms $K_0(\mathrm{Var}_R) \to S$ and additive invariants $\mathrm{Var}_R \to S$. The construction of $K_0(\mathrm{Var}_R)$ is quite easy: take the free abelian group on isomorphism classes $[S]$ of objects of Var_R and mod out by the relations $[S] = [S'] + [S \setminus S']$ for S' closed in S. The product is now defined by $[S][S'] = [S \times S']$.
We shall denote by $\mathbf{L}$ the class of the affine line $\mathbf{A}^1_R$ in $K_0(\mathrm{Var}_R)$. An important role will be played by the ring $\mathcal{M}_R := K_0(\mathrm{Var}_R)[\mathbf{L}^{-1}]$ obtained by localization with respect to the multiplicative set generated by $\mathbf{L}$. This construction is analogous to the construction of the category of Chow motives from the category of effective Chow motives by localization with respect to the Lefschetz motive. (Remark that the morphism χ_c of 2.2.3 sends $\mathbf{L}$ to the class of the Lefschetz motive.)
One should stress that very little is known about the structure of the rings $K_0(\mathrm{Var}_R)$ and $\mathcal{M}_R$ even when R is a field. Let us just quote a result by Poonen [25] saying that when k is a field of characteristic zero the ring $K_0(\mathrm{Var}_k)$ is not a domain (we shall explain this result with more details in §2.6). For instance, even for a field k, it is not known whether the localization morphism $K_0(\mathrm{Var}_k) \to \mathcal{M}_k$ is injective or not (although the whole point of §2.7 relies on the guess it should not). We shall denote by $\bar{\mathcal{M}}_k$ the image of $K_0(\mathrm{Var}_k)$ in $\mathcal{M}_k$.

2.3.1. *Remark.* In fact, the ring $K_0(\mathrm{Var}_k)$ as well as the canonical morphism $\chi_c : K_0(\mathrm{Var}_k) \to K_0(\mathrm{CHMot}_k)$, were already considered by Grothendieck in a letter to Serre dated August 16, 1964, cf. p. 174 of [13].

2.4. We shall need in §5.5 the following generalisation of $K_0(\mathrm{Var}_R)$ to $K_0(\mathrm{Var}_X)$ when X is a variety over R. The definition is just the same using the category of varieties over X instead of the category of varieties over R. Recall that objects in this category are arrows $f : Y \to X$ in Var_R and that a morphism between $f : Y \to X$ and $f' : Y' \to X$ is just a morphism $g : Y \to Y'$ such that $f = f' \circ g$. One also defines $\mathcal{M}_X$ by inverting the class $\mathbf{L}$ of $\mathbf{A}^1_X \to X$ in $K_0(\mathrm{Var}_X)$. We shall write $[Y/X]$ for the class of $f : Y \to X$.

2.5. **Stable birational invariants.** It is a rather straightforward consequence of Hironaka's theorem that $K_0(\mathrm{Var}_k)$ is generated by classes of smooth irreducible proper varieties. More subtle is the following presentation by generators and relations of $K_0(\mathrm{Var}_k)$ due to F. Bittner [6]. We denote by $K_0^{\mathrm{bl}}(\mathrm{Var}_k)$ the quotient of the free abelian group on isomorphism classes of irreducible smooth projective varieties over k by the relations

$$[\mathrm{Bl}_Y X] - [E] = [X] - [Y],$$

for Y and X irreducible smooth projective over k, Y closed in X, $\mathrm{Bl}_Y X$ the blowup of X with center Y and E the exceptional divisor in $\mathrm{Bl}_Y X$. As for $K_0(\mathrm{Var}_k)$, cartesian product induces a product on $K_0^{\mathrm{bl}}(\mathrm{Var}_k)$ which endowes it with a ring structure. There is a canonical ring morphism $K_0^{\mathrm{bl}}(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$, which sends $[X]$ to $[X]$.

2.5.1. Theorem (Bittner [6]). *Assume k is of characteristic zero. The canonical ring morphism*

$$K_0^{\mathrm{bl}}(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$$

is an isomorphism.

The proof is based on Hironaka's resolution of singularities and the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [1] in the following form:

2.5.2. Theorem (Weak factorization theorem). *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper smooth irreducible varieties. Let $U \subset X_1$ be the largest open subset on which ϕ is an isomorphism. Then ϕ can be factored into a sequence of blowing ups and blowing down with smooth centers disjoint from U: $\phi_i : V_{i-1} \dashrightarrow V_i$, $i = 1, \ldots, \ell$, with $V_0 = X_1$, $V_\ell = X_2$, with ϕ_i or ϕ_i^{-1} blowing ups with smooth centers away from U. Moreover there exists i_0 such that $V_i \dashrightarrow X_1$ is defined everywhere and projective for $i \leq i_0$ and $V_i \dashrightarrow X_2$ is defined everywhere an projective for $i \geq i_0$.*

Theorem 2.5.1 is a very efficient tool to provide additive invariants. Indeed, it is enough to know the invariant for smooth projective varieties and to check it behaves properly for blowing ups with smooth centers. In particular it is now a straightforward consequence of Theorem 2.5.1 (but using the full strenght of weak factorization) that the Hodge-Deligne polynomial of 2.2.2 and the virtual motives of 2.2.3 are well-defined additive invariants.
One also deduces easily the following result, first proved by Larsen and Lunts [19].

2.5.3. Corollary (Larsen and Lunts [19]). *Let us assume k is algebraically closed of characteristic zero. Let A be the monoid of isomorphism classes of smooth projective irreducible varieties over k and let $\Psi : A \to G$ be a morphism of commutative monoids such that*

1) *If X and Y are birationally equivalent smooth projective irreducible varieties over k, then $\Psi([X]) = \Psi([Y])$.*
2) $\Psi([\mathbf{P}_k^n]) = 1$.

Then there exists a unique morphism a rings

$$\Phi : K_0(\mathrm{Var}_k) :\longrightarrow \mathbf{Z}[G]$$

such that $\Phi([X]) = \Psi([X])$ when X is smooth projective irreducible.

We assume from now on that k is algebraically closed of characteristic zero. We denote by SB the monoid of equivalence classes of smooth projective irreducible varieties over k under stably birational equivalence[1]. It follows from Corollary 2.5.3 that there exists a universal stable birational invariant

$$\Phi_{\mathrm{SB}} : K_0(\mathrm{Var}_k) :\longrightarrow \mathbf{Z}[\mathrm{SB}].$$

2.5.4. Proposition (Larsen and Lunts [19]). *The kernel of the morphism Φ_{SB} : $K_0(\mathrm{Var}_k) :\to \mathbf{Z}[\mathrm{SB}]$ is the principal ideal generated by $\mathbf{L} = [\mathbf{A}_k^1]$.*

[1] X and Y are called stably birational if $X \times \mathbf{P}_k^r$ is birational to $Y \times \mathbf{P}_k^s$ for some $r, s \geq 0$.

Sketch of proof. It is clear that $\mathbf{L}$ lies in the kernel of Φ_{SB}. Conversely, take $\alpha = \sum_{1\leq i\leq r}[X_i]-\sum_{1\leq j\leq s}[Y_j]$ in the kernel of Φ_{SB}, with X_i and Y_j smooth, projective and irreducible. Since $\sum_{1\leq i\leq r}[X_i] = \sum_{1\leq j\leq s}[Y_j]$ in $\mathbf{Z}[\mathrm{SB}]$, $r = s$ and, after renumbering the X_i's, we may assume X_i is stably birational to Y_i for every i. Hence it is enough to show that if X and Y are smooth, projective and irreducible stably birationally equivalent, then $[X] - [Y]$ belongs to $\mathbf{L}K_0(\mathrm{Var}_k)$. Since $[X] - [\mathbf{P}^r_k \times X]$ belongs to $\mathbf{L}K_0(\mathrm{Var}_k)$, we can even assume X and Y are birationally equivalent and then the result follows easily from the weak factorization Theorem. □

2.6. **Back to Poonen's result.** As promised, we shall now give some explanations concerning the proof of Poonen's Theorem 2.6.3.

2.6.1. Key-Lemma (Poonen [25])**.** *Let k be a field of characteristic zero. There exists abelian varieties A and B over k such that $A \times A$ is isomorphic to $B \times B$ but $A_{\overline{k}} \not\cong B_{\overline{k}}$.*

The proof relies on the following lemma:

2.6.2. Lemma (Poonen [25])**.** *Let k be a field of characteristic zero. There exists an abelian variety A over k such that $\mathrm{End}_k(A) = \mathrm{End}_{\overline{k}}(A) \cong \mathcal{O}$, with $\mathcal{O}$ the ring of integers of a number field of class number 2.*

When $k = \mathbf{C}$, one may take A an elliptic curve with complex multiplication by $\mathbf{Z}[\sqrt{-5}]$. The general case is much more involved and necessitates the use of modular forms and Eichler-Shimura Theory as well as some table checking, see [25].
Let us now explain how Poonen deduces from the Key-Lemma the following:

2.6.3. Theorem. *The ring $K_0(\mathrm{Var}_k)$ is not a domain, for k a field of characteristic zero.*

Proof. Take A and B as in the Key-Lemma. We have $([A] + [B])([A] - [B]) = 0$ in $K_0(\mathrm{Var}_k)$. To check that $[A] + [B]$ and $[A] - [B]$ are nonzero in $K_0(\mathrm{Var}_k)$, it is enough to check that they have a nonzero image under the composition

$$K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_{\overline{k}}) \to \mathbf{Z}[\mathrm{SB}_{\overline{k}}] \to \mathbf{Z}[\mathrm{AV}_{\overline{k}}],$$

where $\mathrm{AV}_{\overline{k}}$ is the monoid of isomorphism classes of abelian varities over $\overline{k}$ and the last morphism is induced by the Albanese functor assigning to a smooth irreducible variety its Albanese variety (which is indeed a stable birational invariant). To conclude we just have to remark that the Albanese variety of an abelian variety is equal to itself. □

2.6.4. *Remark.* Poonen's proof does not tell us anything about zero divisors in $\mathcal{M}_k$. Indeed, it relies on the use of stable birational invariants, and after inverting $\mathbf{L}$ no (non trivial) such invariant is left.

2.6.5. *Remark.* It follows also from Poonen's construction that the morphism $\chi_c : \mathrm{Var}_k \to K_0(\mathrm{CHMot}_k)$ is not injective. Indeed the abelian varieties A and B in loc. cit. are isogenous, which implies they have the same Chow motive. Hence the non zero element $[A]-[B]$ in $K_0(\mathrm{Var}_k)$ has a zero image in $K_0(\mathrm{CHMot}_k)$. On the other hand, it is still unknown whether $\chi_c : \mathcal{M}_k \to K_0(\mathrm{CHMot}_k)$ is injective or not.

2.7. **A motivic zeta function of Hasse-Weil type.** Though is not directly related to the main topic of these notes, we cannot refrain from mentioning the following construction. Let k be a field and let X be a variety over k. For $n \geq 0$, we denote bt $X^{(n)}$ the n-fold symmetric product of X, i.e. the quotient of the cartesian product X^n by the symmetric group of n elements. Note that $X^{(0)}$ is isomorphic to Spec$k$.
Following Kapranov [17], we define the motivic zeta function of X as the power series

$$Z_{\mathrm{mot}}(T) := \sum_{n=0}^{\infty} [X^{(n)}]\, T^n$$

in $K_0(\mathrm{Var}_k)[[T]]$.
Also, when $\alpha : K_0(\mathrm{Var}_k) \to A$ is a morphism of rings, we denote by $Z_{\mathrm{mot},\alpha}(T)$ the power series $\sum_{n=0}^{\infty} \alpha([X^{(n)}])\, T^n$ in $A[[T]]$. We shall write $\mathbf{L}$ for $\alpha(\mathbf{L})$.
Let X be a variety over $\mathbf{F}_q$. The Hasse-Weil zeta function of X is defined as the formal series

$$Z_{\mathrm{HW}}(T) := \exp\Big(\sum_{n\geq 1} \frac{N_n}{n} T^n\Big),$$

with $N_n := |X(\mathbf{F}_{q^n})|$, for $n \geq 1$. By a celebrated result of Dwork [11], $Z_{\mathrm{HW}}(T)$ is rational function of T.

2.7.1. Proposition. *If $k = \mathbf{F}_q$, and we write $N(S) = N_1(S) = |S(k)|$ for S a variety over k (cf. 2.2.4), then $Z_{\mathrm{mot,N}}(T)$ is equal to the Hasse-Weil zeta function $Z_{\mathrm{HW}}(T)$.*

Proof. Rational points of $X^{(n)}$ over k correspond to degree n effective zero cycles of X, hence the result follows from the usual inversion formula between the number of effective zero cycles of given degree on X and the number of rational points of X over finite extensions of k. □

Kapranov proved the rationality of Z_{mot} when X is a smooth projective curve (cf. [17], [20]) and conjectured ratinality of Z_{mot} in general. Recently, Larsen and Lunts [19] [20] gave examples of surfaces for which Z_{mot} is not rational in $K_0(\mathrm{Var}_k)$. However these counterexamples use in a crucial way stable birational invariants, hence they are not counterexamples to the rationality of Z_{mot} in $\mathcal{M}_k$ and it still makes sense to conjecture that Z_{mot} is rational as a series in $\mathcal{M}_k[[T]]$ (cf. [10]).

3. Motivic integration

3.1. **Completing $\mathcal{M}_k$.** We want to assign a measure to subsets of $\mathcal{L}(X)$. This measure will take values in a ring related to $K_0(\mathrm{Var}_k)$. In the analogy with p-adic integration, $K_0(\mathrm{Var}_k)$ is the analogue of $\mathbf{Z}$ and $\mathcal{M}_k$ is the analogue of $\mathbf{Z}[p^{-1}]$ (the number of rational points of the affine line over $\mathbf{F}_p$ is p). Since in $\mathbf{R}$, p^{-i} has limit 0 as $i \mapsto \infty$, we should complete $\mathcal{M}_k$ is such a way that $\mathbf{L}^{-i}$ has limit 0 as

$i \mapsto \infty$. This is achieved in the following way: we define $F^m\mathcal{M}_k$ to be the subgroup of $\mathcal{M}_k$ generated by elements of the form $[S]\mathbf{L}^{-i}$, with $\dim S - i \leq -m$. We have $F^{m+1} \subset F^m$, $\mathbf{L}^{-m} \in F^m$ and $F^nF^m \subset F^{n+m}$. We denote by $\widehat{\mathcal{M}}_k$ the completion of $\mathcal{M}_k$ with respect to that filtration.
A minor technical issue shows up here, since it is not known whether the canonical morphism $\mathcal{M}_k \to \widehat{\mathcal{M}}_k$ is injective or not. Nevertheless, this is not much a problem by the following:

3.1.1. Proposition. *Invariants* Eu : $\mathcal{M}_k \to \mathbf{Z}$ *(Euler number) and* $H : \mathcal{M}_k \to \mathbf{Z}[u,v,(uv)^{-1}]$ *(Hodge polynomial) factor through the image* $\bar{\mathcal{M}}_k$ *of* $\mathcal{M}_k$ *in* $\widehat{\mathcal{M}}_k$.

Proof. Since Eu $= H(1,1)$, it is enough to prove the result for H. But if a is in $F^m\mathcal{M}_k$, the total degree of $h(a)$ is $\leq -2m$, so if a belongs to the kernel $\cap_m F^m\mathcal{M}_k$ of $\mathcal{M}_k \to \widehat{\mathcal{M}}_k$, $h(a)$ should be zero. □

3.2. **Measurable sets.** For more details about this section, see the appendix to [9]. Let X be an algebraic variety over k of dimension d, maybe singular. By a cylinder in $\mathcal{L}(X)$, we mean a subset A of $\mathcal{L}(X)$ of the form $A = \pi_n^{-1}(C)$ with C a constructible subset of $\mathcal{L}_n(X)$, for some n. We say A is stable (at level n) if furthermore $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ is a piecewise Zariski fibration over $\pi_m(A)$ with fiber $\mathbf{A}_k^d$ for all $m \geq n$. By a piecewise Zariski fibration over $\pi_m(A)$ we mean that there exists a finite partition of $\pi_m(A)$ into locally closed subsets of $\mathcal{L}_m(X)$ over which the morphism is a locally trivial fibration for the Zariski topology.
If A is a stable cylinder at level n, we set

$$\tilde{\mu}(A) := [\pi_n(A)]\mathbf{L}^{-(n+1)d}$$

in $\mathcal{M}_k$. Remark that the stability condition insures that we would get the same value by viewing A as a stable cylinder at level m, $m \geq n$. Also, it can be proved that if X is smooth, all cylinders are stable. In particular, in this case $\mathcal{L}(X)$ itself is a stable cylinder and $\tilde{\mu}(\mathcal{L}(X)) = [X]\mathbf{L}^{-d}$.
In general, we can assign to any cylinder A in $\mathcal{L}(X)$ a measure $\mu(A)$ in $\widehat{\mathcal{M}}_k$ by a limit process as follows: for $e \geq 0$, set $\mathcal{L}^{(e)}(X) := \mathcal{L}(X) \setminus \pi_e^{-1}(\pi_e(\mathcal{L}(X_{\text{sing}})))$, where X_{sing} denote the singular locus of X and we view $\mathcal{L}(X_{\text{sing}})$ as a subset of $\mathcal{L}(X)$. The set $\mathcal{L}^{(e)}(X)$ should be viewed as $\mathcal{L}(X)$ minus some tubular neighborhood of the singulat locus. It can be proved that $A \cap \mathcal{L}^{(e)}(X)$ is a stable cylinder and that $\tilde{\mu}(A \cap \mathcal{L}^{(e)}(X))$ does have a limit in $\widehat{\mathcal{M}}_k$ as e goes to ∞ which we define to be $\mu(A)$. This apply in particular to $A = \mathcal{L}(X)$ when X is not smooth.
We shall define

$$||\_|| : \widehat{\mathcal{M}}_k :\to \mathbf{R}_{\geq 0}$$

to be given by $||a|| = 2^{-n}$ if $a \in F^n\widehat{\mathcal{M}}_k$ and $a \notin F^{n+1}\widehat{\mathcal{M}}_k$, where $F^\bullet\widehat{\mathcal{M}}_k$ denotes the induces filtration on $\widehat{\mathcal{M}}_k$.
We shall say a subset A of $\mathcal{L}(X)$ is measurable if, for every $\varepsilon > 0$, there exists cylinders $A_i(\varepsilon)$, $i \in \mathbf{N}$, such that $(A \cup A_0(\varepsilon)) \setminus (A \cap A_0(\varepsilon))$ is contained in $\cup_{i\geq 1} A_i(\varepsilon)$, and $||\mu(A_i(\varepsilon))|| \leq \varepsilon$, for every $i \geq 1$. Then one can show (cf. appendix to [9]) that

$\mu(A) := \lim_{\varepsilon\mapsto 0}\mu(A_0(\varepsilon))$ exists and is independent of the choice of the $A_i(\varepsilon)$'s. We say that A is strongly measurable if moreover we can take $A_0(\varepsilon)\subset A$.
Let A be a measurable subset of $\mathcal{L}(X)$ and $\alpha : A\to \mathbf{Z}\cup\{\infty\}$ be a function such that all its fibers are measurable. We shall say $\mathbf{L}^{\alpha}$ is integrable if the series

$$\int_A \mathbf{L}^{-\alpha}d\mu := \sum_{n\in\mathbf{Z}} \mu(A\cap\alpha^{-1}(n))\mathbf{L}^{-n}$$

is convergent in $\widehat{\mathcal{M}}_k$.

3.3. **Semi-algebraic subsets.** An important class of strongly measurable sets is that of semi-algebraic subsets of $\mathcal{L}(X)$. We shall explain here only what are semi-algebraic subsets of $\mathcal{L}(\mathbf{A}^n_k)$, the definition for general X being deduced by using charts from the affine case.
We shall view points of $\mathcal{L}(\mathbf{A}^n_k)$ as n-uplets of formal power series. A semi-algebraic subset of $\mathcal{L}(\mathbf{A}^n_k)$ is a finite boolean combination of subsets defined by conditions of the form

$$\operatorname{ord} f_1(x_1,\dots,x_m)\geq \operatorname{ord} f_2(x_1,\dots,x_m)+L(\ell_1,\dots,\ell_r) \tag{1}$$

$$\operatorname{ord} f_1(x_1,\dots,x_m)\equiv L(\ell_1,\dots,\ell_r)\pmod d \tag{2}$$

and

$$h(\mathrm{ac}(f_1(x_1,\dots,x_m)),\dots,\mathrm{ac}(f_{m'}(x_1,\dots,x_m)))=0, \tag{3}$$

where f_i are polynomials with coefficients in $k[[t]]$, h is a polynomial with coefficients in k, L is a polynomial of degree ≤ 1 over $\mathbf{Z}$, $d\in\mathbf{N}$, $\operatorname{ord}(x)$ is the t-adic valuation of x and $\mathrm{ac}(x)$ is the coefficient of lowest degree in t of x if $x\neq 0$, and is equal to 0 otherwise. Here we use the convention that $\infty+\ell=\infty$ and $\infty\equiv\ell \bmod d$, for all $\ell\in\mathbf{Z}$. In particular the algebraic condition $f(x_1,\dots,x_m)=0$, for f a polynomial over $k[[t]]$, defines a semi-algebraic subset.
The following (consequence of a) quantifier elimination Theorem of J. Pas [24] is of fundamental use in the theory:

3.3.1. Theorem. *Let $\pi : \mathbf{A}^n_k\to\mathbf{A}^{n-1}_k$ be the projection on the $n-1$ first coordinates. If A is a semi-algebraic subset of $\mathcal{L}(\mathbf{A}^n_k)$, then $\pi(A)$ is a semi-algebraic subset of $\mathcal{L}(\mathbf{A}^{n-1}_k)$.*

One can prove that every semi-algebraic subset of $\mathcal{L}(X)$ is strongly measurable. Furthermore, we have the following nice description of $\mu(A)$ in this case, which is an analogue of a p-adic result of Oesterlé [23], cf. [8]:

3.3.2. Theorem. *If A is a semi-algebraic subset of $\mathcal{L}(X)$, with X of dimension d, then $\mu(A)$ is equal to limit of $[\pi_n(A)]\mathbf{L}^{-(n+1)d}$ in $\widehat{\mathcal{M}}_k$.*

Note that $[\pi_n(A)]$ in the above statement makes sense since one can deduce from Pas' Theorem that $\pi_n(A)$ is constructible.

3.4. **Change of variables formula.** We have the following motivic analogue of the p-adic change of variables formula (cf [15]):

3.4.1. Theorem (Change of variables formula)**.** *Let X be an algebraic variety over k of dimension d. Let $h : Y \to X$ be proper birational morphism. We assume Y to be smooth. Let A be a subset of $\mathcal{L}(X)$ such that A and $h^{-1}(A)$ are strongly measurable. Assume $\mathbf{L}^{-\alpha}$ is integrable on A. Then*

$$\int_A \mathbf{L}^{-\alpha} d\mu = \int_{h^{-1}(A)} \mathbf{L}^{-\alpha\circ h - \mathrm{ord} h^*(\Omega^d_X)} d\mu.$$

We should explain what is meant by $\mathrm{ord} h^*(\Omega^d_X)$, the order of the jacobian of h. Firstly, if $\mathcal{I}$ is some ideal sheaf on Y, we denote by $\mathrm{ord}\mathcal{I}$ the function which to a arc φ in $\mathcal{L}(Y)$ assigns $\inf \mathrm{ord} g(\varphi)$ where g runs over local sections of $\mathcal{I}$ at $\pi_0(\varphi)$. We set Ω^d_X to be the d-th exterior power of Ω^1_X, the Kähler differentials. The image of $h^*(\Omega^d_X)$ in Ω^d_Y is of the form $\mathcal{I}\Omega^d_Y$ and we set $\mathrm{ord} h^*(\Omega^d_X) := \mathrm{ord}\mathcal{I}$. Note that this definition of $\mathrm{ord} h^*(\Omega^d_X)$ carries over to the case h is a generically finite morphism.
The proof of Theorem 3.4.1, given in section 3 of [8] (for A semi-algebraic), relies on the following geometric statement (Lemma 3.4 of [8]):

3.4.2. Proposition. *Let X be an algebraic variety over k. Let $h : Y \to X$ be proper birational morphism. We assume Y to be smooth. For e and e' in $\mathbf{N}$, we set*

$$\Delta_{e,e'} := \Big\{ \varphi \in \mathcal{L}(Y) \;\Big|\; \mathrm{ord} h^*(\Omega^d_X)(\varphi) = e \quad \textit{and} \quad h(\varphi) \in \mathcal{L}^{(e)}(X) \Big\}.$$

Then there exists $c > 0$ such that, for $n \geq \sup(2e, e + ce')$,

(1) *The image $\Delta_{e,e',n}$ of $\Delta_{e,e'}$ in $\mathcal{L}_n(Y)$ is a union of fibers of h_n, the morphism induced by h.*
(2) *The morphism $h_n : \Delta_{e,e',n} \to h_n(\Delta_{e,e',n})$ is a piecewise Zariski fibration with fiber $\mathbf{A}^e_k$.*

4. First appplications

4.1. **Arc invariants.** Let X be an algebraic variety over k of pure dimension d and let $h : Y \to X$ be a proper birational morphism with Y smooth. Let W be a closed subvariety of X. Assume that the exceptional locus of h has normal crossings and that the image of $h^*(\Omega^d_X)$ in Ω^d_Y is an invertible subsheaf. Let $E_j, j \in J$, be the k-irreducible components of the exceptional locus of h. For any subset I of J, set $E^\circ_I = (\cap_{i\in I} E_i) \setminus \cup_{j\in J\setminus I} E_j$. For $i \in I$, let $\nu_i - 1$ be the multiplicity along E_i of the divisor associated to $h^*(\Omega^d_X)$.

4.1.1. Theorem. *Under the above assumptions, the following equality holds in $\widehat{\mathcal{M}}_k$:*

$$\mu(\pi_0^{-1}(W)) = \mathbf{L}^{-d} \sum_{I\subset J} [E^\circ_I \cap h^{-1}(W)] \prod_{i\in I} \frac{\mathbf{L} - 1}{\mathbf{L}^{\nu_i} - 1}. \tag{4.1.1}$$

In particular we see that $\mu(\mathcal{L}(X))$ belongs to $\bar{\mathcal{M}}_{k,\mathrm{loc}}$ the ring obtained from $\bar{\mathcal{M}}_k$ by inverting the elements $1+\mathbf{L}+\cdots+\mathbf{L}^i$, for all $i = 1, 2, 3, \cdots$. Note that $1+\mathbf{L}+\cdots+\mathbf{L}^i$ is equal to $[\mathbf{P}^i_k]$.

Sketch of proof. We remark that $\mu(\mathcal{L}(X)) = \int_{\mathcal{L}(Y)} \mathbf{L}^{\mathrm{ord} h^*(\Omega^d_X)} d\mu$, by the change of variables formula. This integral can be easily calculated, since the image of $h^*(\Omega^d_X)$ in Ω^d_Y is locally generated by monomials. □

Since Eu and H factor through $\bar{\mathcal{M}}_k$, they extend to natural maps $\mathrm{Eu} : \bar{\mathcal{M}}_{k,\mathrm{loc}} \to \mathbf{Q}$ and $H : \bar{\mathcal{M}}_{k,\mathrm{loc}} \to \mathbf{Z}[[u,v]][u^{-1},v^{-1}]$. In particular we get new invariants, arc invariants, $\mathrm{Eu}_{\mathrm{arc}}(X) := \mathrm{Eu}(\mu(\mathcal{L}(X)))$ in $\mathbf{Q}$ and $H_{\mathrm{arc}}(X) := H((\mu(\mathcal{L}(X)))$ in $\mathbf{Z}[[u,v]][u^{-1},v^{-1}]$ that reduce respectively to $\mathrm{Eu}(X)$ and $(uv)^{-d}H(X)$ when X is smooth.

4.2. **Euler characteristics and modifications.** Let $h : Y \to X$ be a modification of nonsingular algebraic varieties over k, meaning that h is a proper birational morphism. Assume that the exceptional locus of h has normal crossings, and let J, E_i, E_I° and ν_i be as in Theorem 4.1.1. Because X is nonsingular, $\mu(\mathcal{L}(X)) = [X]\mathbf{L}^{-d}$ and Theorem 4.1.1 yields the following equality in $\bar{\mathcal{M}}_{k,\mathrm{loc}}$:

$$[X] = \sum_{I \subset J} [E_I^\circ] \prod_{i \in I} \frac{\mathbf{L} - 1}{\mathbf{L}^{\nu_i} - 1}. \tag{4.2.1}$$

When $k = \mathbf{C}$, applying the topological Euler characteristic on (4.2.1) yields $\mathrm{Eu}(X) = \sum_{I \subset J} \mathrm{Eu}(E_I^\circ) / \prod_{i \in I} \nu_i$. This surprising formula about the Euler characteristic of modifications was first obtained in [7] using p-adic integration and the Grothendieck-Lefschetz trace formula.

4.3. **Birational Calabi-Yau varieties.** Let X and Y be two Calabi-Yau manifolds, i.e. nonsingular proper complex algebraic varieties that admit a nonvanishing regular differential form of maximal degree, which we denote respectively by ω_X and ω_Y. Kontsevich [18] proved that X and Y have the same Hodge numbers and the same Hodge structure on their cohomology, when X and Y are birationally equivalent. The proof goes as follows: There exists a nonsingular proper complex algebraic variety Z and birational morphisms $h_X : Z \to X$ and $h_Y : Z \to Y$. Note that $(h_Y \circ h_X^{-1})^*(\omega_Y)\mathcal{M}\mathcal{M}$ equals $c\ \omega_X$ for some $c \in \mathbf{C}^\times$ because ω_X has no zeroes. Hence $c\ h_X^*(\omega_X) = h_Y^*(\omega_Y)$. Thus $\mathrm{ord} h^*(\Omega^d_X) = \mathrm{ord} h^*(\Omega^d_X)$ on $\mathcal{L}(Z)$, and by the change of variables formula both $\mu(\mathcal{L}(X))$ and $\mu(\mathcal{L}(Y))$ equal the same integral on $\mathcal{L}(Z)$. Because $\mu(\mathcal{L}(X)) = [X]\mathbf{L}^{-d}$ and $\mu(\mathcal{L}(Y)) = [Y]\mathbf{L}^{-d}$, this implies that $[X] = [Y]$ in $\bar{\mathcal{M}}_k$, which finishes the proof.
Actually Batyrev [3] first proved that X and Y have the same Betti numbers using p-adic integration and the Weil conjectures, and Kontsevich invented motivic integration to prove that X and Y have the same Hodge numbers.

4.4. **Stringy invariants.** Let X be an irreducible normal algebraic variety over k of dimension d. Let us consider the canonical sheaf $\omega_X := j_*(\Omega^d_{X^0})$ where $j : X^0 \to X$ denotes the inclusion of the smooth part. We assume X is Gorenstein, i.e. ω_X is an invertible sheaf. For φ in $\mathcal{L}(X) \setminus \mathcal{L}(\mathrm{Sing}(X))$, viewing φ as a morphism $\varphi : \mathrm{Spec} K[[t]] \to X$, we may write $\varphi^*(\omega_X) = t^n \varphi^*(\Omega^d_X)$, with n in $\mathbf{N}$. We set $\mathrm{ord}(\omega_X)(\varphi) = n$ and we extend by setting $\mathrm{ord}(\omega_X)(\varphi) = \infty$ if φ is in $\mathcal{L}(\mathrm{Sing}(X))$.

Assume furthermore all singularities of X are canonical, i.e. for some (hence for all) resolution $h : Y \to X$, $h^*(\omega_X) \subset \Omega^d_Y$. Then one can show, using the change of variables formula, that $\mathbf{L}^{-\mathrm{ord}(\omega_X)}$ is integrable on X.
For every measurable subset A of $\mathcal{L}(X)$ we define its motivic Gorenstein measure $\mu^{Gor}(A)$ as

$$\mu^{Gor}(A) := \int_A \mathbf{L}^{-\mathrm{ord}\omega_X} d\mu$$

in $\widehat{\mathcal{M}}_k$.
Furthermore we have the following variant of Theorem 4.1.1.
Let $h : Y \to X$ be a proper birational morphism with Y smooth. Let W be a closed subvariety of X. Assume that the exceptional locus of h has normal crossings, and keep the notations of 4.1 except that now $\nu_i^* - 1$ denotes the length of $\Omega^d_Y / h^*\omega_X$ at the generic point of E_i.

4.4.1. Theorem. *Under the above assumptions, the following equality holds in* $\widehat{\mathcal{M}}_k$*:*

$$\mu^{Gor}(\pi_0^{-1}(W)) = \mathbf{L}^{-d} \sum_{I \subset J} [E_I^\circ \cap h^{-1}(W)] \prod_{i \in I} \frac{\mathbf{L} - 1}{\mathbf{L}^{\nu_i^*} - 1}. \tag{4.4.1}$$

Since $\mu^{Gor}(\mathcal{L}(X))$ belongs to $\bar{\mathcal{M}}_{k,\mathrm{loc}}$, we can define $\mathrm{Eu}_{st}(X) := \mathrm{Eu}(\mu^{Gor}(\mathcal{L}(X)))$ and $H_{st}(X) := H(\mu^{Gor}(\mathcal{L}(X)))$ obtaining new invariants, Batyrev's stringy Euler number and stringy Hodge numbers, defined in [4].
Remark that if the resolution is crepant, i.e. $h^*(\omega_X) = \Omega^d_Y$, all ν_i^* are equal to 1, hence $\mu^{Gor}(\pi_0^{-1}(W)) = \mathbf{L}^{-d}[h^{-1}(W)]$.

4.4.2. *Remark.* Everything in §4.4 can be quite easily generalised to the $\mathbf{Q}$-Gorenstein case, replacing $\widehat{\mathcal{M}}_k$ and $\bar{\mathcal{M}}_{k,\mathrm{loc}}$ by $\widehat{\mathcal{M}}_k[\mathbf{L}^{1/d}]$ and $\bar{\mathcal{M}}_{k,\mathrm{loc}}[\mathbf{L}^{1/d}]$ for some integer d. In this setting $\nu_i^* - 1$ is defined as $1/d$ times the length of $(\Omega^d_Y)^{\otimes d} / h^*(\omega_X)^{\otimes d}$ at the generic point of E_i if $(\omega_X)^{\otimes d}$ is invertible. We extend Eu and H to $\bar{\mathcal{M}}_{k,\mathrm{loc}}[\mathbf{L}^{1/d}]$ by setting $\mathrm{Eu}(\mathbf{L}^{1/d}) = 1$ and $H(\mathbf{L}^{1/d}) = (uv)^{1/d}$.

4.4.3. *Remark.* The fact that the right-hand side of (4.1.1) and (4.4.1) are independent of the resolution and the results in 4.2 and 4.3 may also be obtained now as a consequence of the weak factorisation Theorem.

5. McKay Correspondence

5.1. **The basic local case.** Let $d \geq 1$ be an integer and let k be field of characteristic 0 containing all d-th roots of unity. Let G be a finite subgroup of $\mathrm{GL}_n(k)$ of order d. We fix a primitive d-th root of unity ξ in k. We denote by $\mathrm{Conj}(G)$ the set of conjugacy classes in G. We let G act on $\mathbf{A}^n_k$ and we consider the morphism of schemes $h : \tilde{X} = \mathbf{A}^n_k \longrightarrow X = \mathbf{A}^n_k/G$. It is well known and easy to chack that X is Gorenstein with all its singularities canonical. We denote by 0 the origin in $\tilde{X}$ and X. Let $\tilde{\Delta}$ be the closed subvariety of $\tilde{X}$ consisting of the closed points having a nontrivial stabilizer and let Δ be its image in X (the discriminant).

We shall consider ramified arcs on $\tilde{X}$. More precisely we consider the space $\mathcal{L}^{1/d}(\tilde{X})$ defined as $\mathcal{L}(\tilde{X})$ by replacing everywhere t by $t^{1/d}$, so points of $\mathcal{L}(\tilde{X})$ are morphisms $\mathrm{Spec} K[[t^{1/d}]] \to \tilde{X}$. Of course as a scheme $\mathcal{L}^{1/d}(\tilde{X})$ is isomorphic to $\mathcal{L}(\tilde{X})$. We denote by $\mathcal{L}(X)^g$ (resp. $\mathcal{L}^{1/d}(\tilde{X})^g$) the complement of $\mathcal{L}(\Delta)$ (resp. $\mathcal{L}^{1/d}(\tilde{\Delta})$) in $\mathcal{L}(X)$ (resp. $\mathcal{L}^{1/d}(\tilde{X})$), and define similarly $\mathcal{L}(X)^g_0$ (resp. $\mathcal{L}^{1/d}(\tilde{X})^g_0$) for the corresponding subsets of arcs with origin 0.
Let φ be a geometric point of $\mathcal{L}(X)^g_0$. So φ is given by a morphism $\varphi : \mathrm{Spec}\, K[[t]] \to X$ with K an algebraically closed overfield of k. The generic point of the image of φ is in $X \setminus \Delta$ and the special point is 0. We can lift φ to a morphism $\tilde{\varphi}$ making the following diagram commutative:

(5.1.1)
$$\begin{array}{ccc} \mathrm{Spec}\, K[[t^{1/d}]] & \xrightarrow{\tilde{\varphi}} & \tilde{X} \\ \downarrow & & \downarrow h \\ \mathrm{Spec}\, K[[t]] & \xrightarrow{\varphi} & X. \end{array}$$

There is a unique element γ in G such that

(5.1.2)
$$\tilde{\varphi}(\xi t^{1/d}) = \gamma\, \tilde{\varphi}(t^{1/d}).$$

If we change $\tilde{\varphi}$ in the diagram (5.1.1), γ will be replaced by a conjugate. If we denote by $\mathcal{L}(X)^g_{0,\gamma}$ the set of φ's in $\mathcal{L}(X)^g_0$ such that there exists $\tilde{\varphi}$ satisfying (5.1.2), we have $\mathcal{L}(X)^g_{0,\gamma} = \mathcal{L}(X)^g_{0,\gamma'}$ for γ and γ' in the same conjugacy class, so we may set $\mathcal{L}(X)^g_{0,[\gamma]} = \mathcal{L}(X)^g_{0,\gamma}$, with $[\gamma]$ the conjugacy class of γ, and and we have a partition

(5.1.3)
$$\mathcal{L}(X)^g_0 = \coprod_{[\gamma] \in \mathrm{Conj}(G)} \mathcal{L}(X)^g_{0,[\gamma]}$$

parametrized by the set of conjugacy classes.
Note that the spaces $\mathcal{L}(X)^g_{0,[\gamma]}$ already appeared long ago in the physic literature as "twisted sectors".
For each γ in G, choose a basis b_γ in which the matrix of γ is diagonal, and denote by $\xi^{e_{\gamma,i}}$, the diagonal coefficients, with $1 \leq e_{\gamma,i} \leq d$, $1 \leq i \leq n$. We define the weight $w(\gamma)$ of γ by $w(\gamma) := \sum_{1 \leq i \leq n} e_{\gamma,i}/d$. Note that $w(\gamma) \in \mathbf{N} \setminus \{0\}$, when $G \subset \mathrm{SL}_n(k)$. In general, when G is a subgroup of $\mathrm{GL}_n(k)$, $w(\gamma)$ may still be defined as above but it becomes a non zero rational number. These numbers were first introduced (up to shift) in [27] where there are called ages.
In the following, we shall have to consider the class of $\mathbf{A}^n_k/G$ in $\mathcal{M}_k$ when G is a finite group acting linearly on $\mathbf{A}^n_k$. When G is abelian one can easily show (cf. Lemma 5.1 of [21]) that $[\mathbf{A}^n_k/G] = \mathbf{L}^n$ in $\mathcal{M}_k$ and it seems reasonable it could also hold in general, but apparently this issue is not settled yet. So we need to consider the quotient $\mathcal{M}_{k/}$ of $\mathcal{M}_k$ obtained by adding all the relations $[\mathbf{A}^n_k/G] = \mathbf{L}^n$. One defines similarly $\bar{\mathcal{M}}_{k/}$, $\widehat{\mathcal{M}}_{k/}$, etc. This is not a serious problem since usual invariants like Eu and H still factor through $\mathcal{M}_{k/}$ and $\bar{\mathcal{M}}_{k/}$ (cf. [9]).
Now, we can state the main result of [9]:

5.1.1. Theorem. *Let G be a finite subgroup of* $\mathrm{SL}_n(k)$. *Under the previous assumptions, for every γ in G,*

$$\mu^{Gor}(\mathcal{L}(X)^g_{0,[\gamma]}) = \mathbf{L}^{-w(\gamma)}$$

in $\widehat{\mathcal{M}}_{k/}$.

Using the partition (5.1.3) and the fact that $\mu^{Gor}(\mathcal{L}(X)_0 \setminus \mathcal{L}(X)^g_0) = 0$, we deduce the following:

5.1.2. Theorem. *Let G be a finite subgroup of* $\mathrm{SL}_n(k)$. *Under the previous assumptions, the relation*

$$\mu^{Gor}(\mathcal{L}(X)_0) = \sum_{[\gamma]\in \mathrm{Conj}(G)} \mathbf{L}^{-w(\gamma)},$$

holds in the ring $\widehat{\mathcal{M}}_{/}$, where $\mathrm{Conj}(G)$ *denotes the set of conjugacy classes in G.*

5.1.3. *Remark.* Theorems 5.1.1 and 5.1.2 still hold, with the same proof, for G a finite subroup of order d of $\mathrm{GL}_n(k)$, replacing $\widehat{\mathcal{M}}_{/}$ by $\widehat{\mathcal{M}}_{/}[\mathbf{L}^{1/d}]$], cf. Remark 4.4.2.

Theorem 5.1.2 was first proved by Batyrev [5] at the level of Hodge numbers (note that $H(\mu^{Gor}(\mathcal{L}(X)) = E_{st}(X, \Delta_x, u, v)$ with the notations of [5]).
The proof of Theorem 5.1.1 has two parts we shall now describe.

5.2. Let γ be in G. A point $\tilde{\varphi}$ in $\mathcal{L}^{1/d}(\tilde{X})^g$ projects to a point in $\mathcal{L}(X)^g_{0,\gamma}$ if and only if it lies in the G-orbit of a point in $\mathcal{L}^{1/d}(\tilde{X})^g$ of the form

$$\tilde{\varphi}(t^{1/d}) = (t^{e_{\gamma,1}/d}\varphi_1(t), \dots, t^{e_{\gamma,n}/d}\varphi_n(t)) \tag{5.2.1}$$

in the basis b_γ. Indeed, it follows from (5.1.2) that a point of $\mathcal{L}^{1/d}(\tilde{X})^g$ which projects to a point in $\mathcal{L}(X)^g_{0,\gamma}$ is in the G-orbit of a point of the form (5.2.1). To conclude observe that, in the basis b_γ, G-invariant polynomials are sums of monomials of the form $x_1^{m_1} \dots x_n^{m_n}$, with d dividing $\sum_{1\leq i\leq n} e_{\gamma,i} m_i$.
We consider the morphism of $k[t]$-schemes

$$\tilde{\lambda} : \mathbf{A}^n_{k[t]} \longrightarrow X \otimes k[t] \qquad (x_1, \dots, x_n) \longmapsto h(t^{e_{\gamma,1}/d}x_1, \dots, t^{e_{\gamma,n}/d}x_n),$$

where $x_1, \dots, x_n$ are the affine coordinates corresponding to the basis b_γ.
The morphism $\tilde{\lambda}$ induces a morphism $\tilde{\lambda}_* : \mathcal{L}(\mathbf{A}^n_k) \to \mathcal{L}(X)_0$. Note that by the remark above

$$\mathcal{L}(X)^g_{0,\gamma} = \tilde{\lambda}_*(\mathcal{L}(\mathbf{A}^n_k)) \cap \mathcal{L}(X)^g. \tag{5.2.2}$$

For γ in G we denote by G_γ the centralizer of γ in G.
One can prove quite easily (cf. Lemma 2.6 of [9]) that the morphism $\tilde{\lambda}$ is invariant under the action of G_γ on $\mathbf{A}^n_{k[t]}$ and that the fibers of $\tilde{\lambda}_*$ above $\mathcal{L}(X)^g_{0,\gamma}$ are G_γ-orbits.
It follows that $\tilde{\lambda}$ induces a morphism of $k[t]$-schemes

$$\lambda : (\mathbf{A}^n_k/G_\gamma) \otimes k[t] \longrightarrow X \otimes k[t]$$

which induces a morphism

$$\lambda_* : \mathcal{L}(\mathbf{A}_k^n/G_\gamma) \longrightarrow \mathcal{L}(X).$$

Furthermore it follows from the previous considerations that λ_* induces a bijection between $(\mathcal{L}(\mathbf{A}_k^n)/G_\gamma) \cap \lambda_*^{-1}(\mathcal{L}(X)^g)$ and $\mathcal{L}(X)^g_{0,\gamma}$.

5.2.1. Proposition. *For any γ in G, we have*

$$\mu^{Gor}(\mathcal{L}(X)^g_{0,\gamma}) = \mathbf{L}^{-w(\gamma)}\,\mu^{Gor}_{\mathcal{L}(\mathbf{A}_k^n/G_\gamma)}(\mathcal{L}(\mathbf{A}_k^n)/G_\gamma)$$

in $\widehat{\mathcal{M}}_k$. We view here $\mathcal{L}(\mathbf{A}_k^n)/G_\gamma$ as embedded in $\mathcal{L}(\mathbf{A}_k^n/G_\gamma)$.

Sketch of proof. Consider the form α_X in $\Omega^n_X \otimes k(X)$ such that $h^*(\alpha_X) = dx_1 \wedge \cdots \wedge dx_n$. We have $\lambda^*(\alpha_X) = t^{w(\gamma)}\alpha_{\mathbf{A}_k^n/G_\gamma}$. This allows to interpret $w(\gamma)$ as the order of the jacobian for the bijection λ. The present situation is not covered by the change of variables formula in the form stated in 3.4.1, since our morphism between varieties is not induced by a birational morphism between k-varieties but only by a morphism of $k[t]$-varieties, but more general forms of Theorem 3.4.1 proven in [9] allow to deduce the result. □

5.3. In view of Proposition 5.2.1, it is now enough to prove the following:

5.3.1. Proposition. *Under the previous assumptions,*

$$\mu^{Gor}_{\mathcal{L}(X)}(\mathcal{L}(\tilde{X}/G)) = 1$$

in $\widehat{\mathcal{M}}_{k/}$, viewing $\mathcal{L}(\tilde{X}/G)$ as embedded in $\mathcal{L}(X)$.

Sketch of proof. Let M be a large integer. For e in $\mathbf{N}$, we consider the subset $\Delta_{e,M}$ of $\mathcal{L}(\mathbf{A}_k^n)$ consisting of all points φ in $\mathcal{L}(\mathbf{A}_k^n)$ such that $\mathrm{ord}h^*(\Omega^n_X)(\varphi) = e$ and $h(\varphi) \in \mathcal{L}^{(M)}(X)$. Note that $\mathrm{ord}\omega_X \circ h = -\mathrm{ord}h^*(\Omega^n_X)$. Indeed,

$$\mathrm{ord}h^*(\omega_X) = \mathrm{ord}\omega_X) \circ h + \mathrm{ord}h^*(\Omega^n_X),$$

and $\mathrm{ord}h^*(\omega_X) = \mathrm{ord}(dx_1 \wedge \cdots \wedge dx_n) = 0$. Thus

$$\mu^{Gor}_{\mathcal{L}(X)}(\mathcal{L}(\mathbf{A}_k^n)/G) = \sum_{e=0}^{M} \mathbf{L}^e \mu_{\mathcal{L}(X)}(h(\Delta_{e,M})) + R_M,$$

with $\lim_{M\to\infty} R_M = 0$, since $\mathbf{L}^{-\mathrm{ord}\omega_X}$ is integrable on $\mathcal{L}(X)$. It follows from Lemma 5.3.2 below, that, for m in $\mathbf{N}$ large enough with respect to M, we have for all $e \leq M$ that $h(\Delta_{e,M})$ is stable at level m and that $[\pi_m(h(\Delta_{e,M}))] = \mathbf{L}^{-e}[\pi_m(\Delta_{e,M})/G]$. It also follows from easy dimension considerations (cf. Lemma 4.4 of [8]), that

$$\begin{aligned}
\mu^{Gor}_{\mathcal{L}(X)}(\mathcal{L}(\mathbf{A}_k^n)/G) &= \sum_{e=0}^{M} [\pi_m(\Delta_{e,M})/G]\,\mathbf{L}^{-(m+1)n} + R_M \\
&= [\pi_m(\cup_{e=0,\dots,M}\Delta_{e,M})/G]\,\mathbf{L}^{-(m+1)n} + R_M \\
&= [\pi_m(\mathcal{L}(\mathbf{A}_k^n))/G]\,\mathbf{L}^{-(m+1)n} + R'_M,
\end{aligned}$$

with $\lim_{M\to\infty} R'_M = 0$. The statement follows now, since $\pi_m(\mathcal{L}(\mathbf{A}_k^n))/G$ is isomorphic to $\mathbf{A}_k^{(m+1)n}/G$, the G-action on $\mathbf{A}_k^{(m+1)n}$ being the diagonal one, and the class of $\mathbf{A}_k^{(m+1)n}/G$ in $\mathcal{M}_{k/}$ is equal to $\mathbf{L}^{(m+1)n}$ (this is the place where we use the fact that we work in $\widehat{\mathcal{M}}_/$ instead of $\widehat{\mathcal{M}}$). □

5.3.2. Lemma. *Let $Y = \mathbf{A}_k^d$ and $X = \mathbf{A}_k^d/G$, with G a finite subgroup of $\mathrm{GL}_d(k)$. Denote by $h : \mathcal{L}(Y) \to \mathcal{L}(X)$ the natural projection. Let $B \subset \mathcal{L}(Y)$ be a cylinder and which is stable under the G-action. Set $A = h(B)$. Assume that $\mathrm{ord} h^*(\Omega_X^n)(\varphi)$ has constant value $e < \infty$ for all $\varphi \in B$, and that $A \subset \mathcal{L}^{(e')}(X)$ for some e' in $\mathbf{N}$. Then, for $n \in \mathbf{N}$ large enough, we have the following:*

(a) *If $\varphi \in B$, $\varphi' \in \mathcal{L}(Y)$ and $\pi_n(h(\varphi)) = \pi_n(h(\varphi'))$, then $\pi_{n-e}(\varphi)$ and $\pi_{n-e}(\varphi')$ have the same image in $\mathcal{L}_{n-e}(Y)/G$.*
(b) *The morphism $h_{n*} : \pi_n(B)/G \to \pi_n(A)$ induced by h may be endowed with the structure of a piecewise vector bundle of rank e.*
(c) $[\pi_n(B)/G] = \mathbf{L}^e\,[\pi_n(A)]$.

Proof. This is Lemma 3.5 of [9], to which we refer for a proof. □

5.4. **Reformulation in terms of resolutions.** From Theorem 5.1.2, Theorem 4.4.1 and Remark 5.1.3 one deduces immediately the following:

5.4.1. Theorem. *Let G be a finite subroup of order d of $\mathrm{SL}_n(k)$, $X = \mathbf{A}_k^n/G$. Let $g : Y \to X$ be a proper birational morphism with Y smooth. Assume that the exceptional locus of h has normal crossings, and keep the notations of Theorem 4.4.1. The following equality holds in $\widehat{\mathcal{M}}_{k/}$:*

$$\sum_{[\gamma]\in \mathrm{Conj}(G)} \mathbf{L}^{-w(\gamma)} = \mathbf{L}^{-n} \sum_{I\subset J} [E_I^\circ \cap g^{-1}(0)] \prod_{i\in I} \frac{\mathbf{L}-1}{\mathbf{L}^{\nu_i^*}-1}. \tag{5.4.1}$$

When G is a finite subroup of order d of $\mathrm{GL}_n(k)$ the same result holds in $\widehat{\mathcal{M}}_{k/}[[\mathbf{L}^{1/d}]]$ (but $w(\gamma)$ and ν_i^* may now lie in $\mathbf{Q}$).

In particular, when the resolution $g : Y \to X$ is crepant we deduce:

5.4.2. Corollary. *Keep the assumptions of the previous Theorem and assume furthermore the resolution $g : Y \to X$ is crepant. Then*

$$[h^{-1}(0)] = \sum_{[\gamma]\in \mathrm{Conj}(G)} \mathbf{L}^{n-w(\gamma)},$$

$$H(h^{-1}(0) = \sum_{[\gamma]\in \mathrm{Conj}(G)} (uv)^{n-w(\gamma)},$$

and

$$\mathrm{Eu}(h^{-1}(0)) = \mathrm{card}\,\mathrm{Conj}(G).$$

5.5. **The global case.** Let now G be a finite group of order d acting effectively on a smooth irreducible k-variety U. We assume k contains all d-th roots of unity and we fix a primitive d-root of unity ξ. We set $X = U/G$.
For γ in G, we denote by U^γ the set of points fixed by γ. The normal bundle $N_{U|U^\gamma}$ of U^γ in U splits as a direct sum of subbundles

$$N_{U|U^\gamma} = \bigoplus_{k=1}^{d} N_\gamma^k$$

with γ acting with eigenvalue ξ^{-k} on N_γ^k. We set $w(\gamma) = \sum_k \frac{k}{d}\mathrm{rk}(N_\gamma^k)$. It is a locally constant function on U^γ, so it takes a constant value w_i on each connected component U_i^γ, $i \in C_\gamma$ of U^γ, where C_γ denotes the set of connected components of U^γ.
Recall we introduced in §2.4 relative Grothendieck rings $K_0(\mathrm{Var}_X)$ and $\mathcal{M}_X$ for X a k-variety. Similarly we may define $\bar{\mathcal{M}}_X$, etc. We have natural forgetful morphisms $\mathcal{M}_X \to \mathcal{M}_k$, etc., so we may extend Eu and H by composing with these forgetful morphisms.
Now we may define the global analogue of the left-hand side of (5.4.1):

$$W(X) := \sum_{[\gamma]\in \mathrm{Conj}(G)} \sum_{i\in C_\gamma} [(U_i^\gamma/G_\gamma)/X]\mathbf{L}^{-w_i}$$

in $\mathcal{M}_X[\mathbf{L}^{1/d}]$.

Theorem 5.1.2 may be generalised as follows to the global case, with a similar proof:

5.5.1. Theorem. *Let G be a finite group acting effectively on a smooth irreducible k-variety U with quotient X. Then the relation*

$$\mu^{Gor}(\mathcal{L}(X)) = W(X)$$

holds in the ring $\widehat{\mathcal{M}}_{X/}[\mathbf{L}^{1/d}]$.

Here we should note that since in the construction $\mu^{Gor}(\mathcal{L}(X)$ everything is done "over" X one may consider $\mu^{Gor}(\mathcal{L}(X)$ as belonging to $\widehat{\mathcal{M}}_{X/}[\mathbf{L}^{1/d}]$

One deduces immediately the following:

5.5.2. Theorem. *Let $g : Y \to X$ be a proper birational morphism with Y smooth. Assume that the exceptional locus of h has normal crossings, and keep the notations of Theorem 4.4.1. The following equality holds in $\widehat{\mathcal{M}}_{X/}[\mathbf{L}^{1/d}]$:*

$$W(X) = \mathbf{L}^{-\dim X} \sum_{I\subset J} [E_I^\circ/X] \prod_{i\in I} \frac{\mathbf{L}-1}{\mathbf{L}^{\nu_i^*}-1}. \tag{5.5.1}$$

Taking H of both sides of (5.5.1) one recovers Theorem 7.5 of [5] which is an equality between stringy Hodge numbers and orbifold Hodge numbers. In fact, the right-hand side of (5.5.1) is of stringy type while the left-hand side is of orbifold type. Let us explain this at the level of Euler numbers. Taking Eu of the right-hand side of (5.5.1), one gets $\mathrm{Eu}_{st}(X)$. Taking Eu of the left-hand side of (5.5.1), one gets $\sum_{[\gamma]\in \mathrm{Conj}(G)} \mathrm{Eu}(U^\gamma/G_\gamma)$. But this can be rewritten as

$$\sum_{[\gamma]\in \mathrm{Conj}(G)} \frac{1}{\mathrm{card}(G_\gamma)} \sum_{h\in G_\gamma} \mathrm{Eu}(U^\gamma \cap U^h) = \frac{1}{\mathrm{card}(G)} \sum_{\gamma h = h\gamma} \mathrm{Eu}(U^\gamma \cap U^h).$$

In the last sum, taken over pairs of commuting elements, we recognize the traditional definition of the orbifold Euler number $\mathrm{Eu}_{orb}(U, G)$.

5.6. **Application to geometric McKay correspondence.** This section is entirely taken from §8 of [5] to which we refer for more details.
Let X be the quotient of $\mathbf{C}^n$ by a finite subgroup G of $\mathrm{SL}(n, \mathbf{C})$. We consider the action of $\mathbf{C}^*$ on X induced by the action of scalar matrices on $\mathbf{C}^n$.
By the McKay correspondence, one usually means relating the geometry of X or some resolution Y to group theoretic invariants of the subgroup G of $\mathrm{SL}(n, \mathbf{C})$.

5.6.1. Theorem (Batyrev). *Let X be the quotient of $\mathbf{C}^n$ by a finite subgroup G of $\mathrm{SL}(n, \mathbf{C})$. Assume X admits a crepant resolution Y. Then $H^{2i+1}(Y, \mathbf{C}) = 0$ and $H^{2i}(Y, \mathbf{C})$ is pure of Hodge type (i, i) and the dimension of $H^{2i}(Y, \mathbf{C})$ is equal to the number $C_i(G)$ of conjugacy classes $[\gamma]$ in G having weight $w(\gamma) = i$.*

Sketch of proof. Batyrev first proves that the $\mathbf{C}^*$-action on X extends to an action on Y. He may then consider the Bialynicki-Birula decomposition $Y = \cup_{1\leq j\leq r} W_j$ for which he proves that each W_j is a vector bundle over a smooth compact variety. By an easy argument using induction and the long exact sequence for cohomology with compact supports he deduces the purity of the cohomology groups $H^i_c(Y, \mathbf{C})$, hence also of $H^i(Y, \mathbf{C})$ by Poincaré duality. To conclude, it is enough to prove that the Hodge polynomial $H(Y)$ of Y is equal to $\sum_i C_i(G)(uv)^{n-i}$ which follows quite easily from Corollary 5.4.2 or from Theorem 5.5.2. □

In particular one deduces the following statement, conjectured by M. Reid [27]:

5.6.2. Corollary (Batyrev). *Let X be the quotient of $\mathbf{C}^n$ by a finite subgroup G of $\mathrm{SL}(n, \mathbf{C})$. Assume X admits a crepant resolution Y. then the Euler number of Y is equal to the number of conjugacy classes in G.*

5.7. **Yasuda's stacky approach.** We shall limit ourselves to a very rough outline here and refer to [29] for more details. Let $\mathcal{X}$ be a Deligne-Mumford stack over k, an algebraically closed field of characteristic zero. Inspired by the work of Abramovich and Vistoli [2], Yasuda [29] introduces for every $n \geq 0$, the stack of twisted n-jets $\widetilde{\mathcal{L}}_n(\mathcal{X})$. The stack $\widetilde{\mathcal{L}}_n(\mathcal{X})$ is the disjoint union $\coprod_{\ell\geq 0} \widetilde{\mathcal{L}}^\ell_n(\mathcal{X})$, where $\widetilde{\mathcal{L}}^\ell_n(\mathcal{X})$ is the substack of twisted n-jets of order ℓ on $\mathcal{X}$. A twisted n-jet of order ℓ on $\mathcal{X}$ is a representable morphism

$$[\mathrm{Spec}(k[[t]]/t^{n+1}k[[t]])/\mu_\ell] \otimes \Omega \longrightarrow \mathcal{X},$$

with Ω an algebraically closed field containg k, and $[\mathrm{Spec}(k[[t]]/t^{n+1}k[[t]])/\mu_\ell]$ the quotient stack of $\mathrm{Spec}(k[[t]]/t^{n+1}k[[t]])$ by the μ_ℓ-action $t \mapsto \zeta t$. In particular, $\widetilde{\mathcal{L}}_0(\mathcal{X})$ is nothing else than the inertia stack $I(\mathcal{X})$.

There are natural morphisms $\widetilde{\mathcal{L}}_{n+1}(\mathcal{X}) \to \widetilde{\mathcal{L}}_n(\mathcal{X})$ that allow to consider the projective limit $\widetilde{\mathcal{L}}(\mathcal{X})$ and truncation morphisms $\tilde{\pi}_n : \widetilde{\mathcal{L}}(\mathcal{X}) \to \widetilde{\mathcal{L}}_n(\mathcal{X})$. Yasuda then constructs a motivic measure $\mu_{\widetilde{\mathcal{L}}(\mathcal{X})}$ on $\widetilde{\mathcal{L}}(\mathcal{X})$ along quite similar lines than our construction of the motivic measure $\mu_{\mathcal{L}(X)}$ on the arc space of a variety X.
Let X be a k-variety with Gorenstein quotient singularities. Yasuda shows there exists a smooth Deligne-Mumford stack $\mathcal{X}$ "without reflections" and with automorphism group of general geometric points trivial such that X is the coarse moduli space of $\mathcal{X}$. He is then able to reformulate and generalize Theorem 5.5.1 as follows[2]:

5.7.1. Theorem. *Let X be a k-variety with Gorenstein quotient singularities. Let $\mathcal{X}$ be a smooth Deligne-Mumford stack having X as coarse moduli space and satisfying the above additional conditions. Then*

$$\mu^{Gor}_{\mathcal{L}(X)}(\mathcal{L}(X)) = \sum_{\mathcal{Y}} \mathbf{L}^{-w(\mathcal{Y})} \mu_{\widetilde{\mathcal{L}}(\mathcal{X})}(\tilde{\pi}_0^{-1}(\mathcal{Y})), \tag{5.7.1}$$

where $\mathcal{Y}$ runs over the connected components of the inertia stack $I(\mathcal{X})$, and w is a weight function defined similarly as in 5.5.

To see the relation with what we did previously, recall that in the case of a global quotient $\mathcal{X} = [U/G]$ of a k-variety U by a finite group G we have an isomorphism

$$I([U/G]) \simeq \coprod_{[\gamma]\in \mathrm{Conj}(G)} [U^{\gamma}/G_{\gamma}].$$

The use of twisted arcs here correponds to our use of ramification in lifting arcs from X to $\tilde{X}$ and the right-hand side of (5.7.1) corresponds to our twisted sectors.

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[2]we make the abuse of keeping the same notation for a stack and its underlying set of points

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École Normale Supérieure, Département de mathématiques et applications, 45 rue d'Ulm, 75230 Paris Cedex 05, France (UMR 8553 du CNRS)
E-mail address: `Francois.Loeser@ens.fr`
URL: `http://www.dma.ens.fr/ loeser/`