Seattle lectures on motivic integration

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These notes are the written-up version of a series of 4 lectures given at the Summer Institute. Though I tried as much as possible to keep the basic structure of the lectures as well as their rather informal style, some flesh has also been added to the bones. Motivic integration being born exactly ten years ago, nothing could be more timely than the proposition by the organizers of the Institute to review the achievements of the past decade in a series of lectures. I would like to thank them for providing me such a unique opportunity.

Lecture 1: Before Motivic Integration

1.1. Modifications. One may start the whole story of motivic integration with a somewhat intriguing result obtained by Jan Denef and myself in 1987 and only published in 1992 [**24**]. At the time we certainly would never have guessed the fantastic developments that would arise later.

Let us consider a smooth complex algebraic variety X and a closed nowhere dense subscheme F. By a log-resolution

$$h : Y \to X$$

of (X, F) we mean a proper morphism $h : Y \to X$ with Y smooth such that the restriction of h:

$$Y \setminus h^{-1}(F_{\mathrm{red}}) \to X \setminus F_{\mathrm{red}}$$

is an isomorphism, and $h^{-1}(F_{\mathrm{red}})$ is a divisor with simple normal crossings. We denote by E_i, i in A, the set of irreducible components of the divisor $h^{-1}(F_{\mathrm{red}})$. Hence, by definition the E_i's are smooth and intersect transversally. If $h : Y \to X$ is a log-resolution of (X, F) for some F, we call h a DNC-modification.

For $I \subset A$, we set

$$E_I := \bigcap_{i \in I} E_i$$

and

$$E_I^\circ := E_I \setminus \bigcup_{j \notin I} E_j.$$

If $\mathcal{I}$ is an ideal sheaf defining the closed subscheme F of X and $h^{-1}(\mathcal{I})\mathcal{O}_Y$ is locally principal, we define $N_i(\mathcal{I})$, the multiplicity of $\mathcal{I}$ along E_i, by

$$h^*(\mathcal{I})\mathcal{O}_Y \simeq \mathcal{O}_Y\Big(-\sum_{i\in A} N_i(\mathcal{I})E_i\Big).$$

If $\mathcal{I}$ is principal, generated by a function g, we write $N_i(g)$ for $N_i(\mathcal{I})$.

Similarly, one defines integers ν_i, called log discrepancies, by the equality of divisors

$$K_Y = h^*K_X + \sum_{i\in A}(\nu_i - 1)E_i.$$

Let X be a complex algebraic variety (not necessarily smooth). If X is proper, $X(\mathbb{C})$ is compact and we may define its Euler Characteristic as

$$\mathrm{Eu}(X) := \sum_i (-1)^i \mathrm{rk}\, H^i(X(\mathbb{C}), \mathbb{C}).$$

There is a unique way to extend Eu additively to the category of all complex algebraic varieties, by requiring that

$$\mathrm{Eu}(X) = \mathrm{Eu}(X') + \mathrm{Eu}(X \setminus X')$$

for X' closed in X. Indeed, just set

$$\mathrm{Eu}(X) := \sum_i (-1)^i \mathrm{rk}\, H^i_c(X(\mathbb{C}), \mathbb{C}),$$

where $H^i_c(\_, \mathbb{C})$ stands for cohomology with compact supports.

The following result was obtained in 1987 and published in 1992:

1.1.1. Theorem (Denef and Loeser [**24**]). (1) *Let $h : Y \to X$ be a DNC modification between smooth complex algebraic varieties. We have*

$$(\dagger) \qquad \mathrm{Eu}\,(X) = \sum_{I\subset A} \frac{\mathrm{Eu}\,(E_I^\circ)}{\prod_{i\in I}\nu_i}.$$

(2) *Let F be a nowhere dense subscheme of X defined by an ideal $\mathcal{I}$ and let $h : Y \to X$ be a log-resolution of (X, F). Then the rational function*

$$(\ddagger) \qquad Z_{top,F}(s) := \sum_{I\subset A} \frac{\mathrm{Eu}\,(E_I^\circ)}{\prod_{i\in I} N_i(\mathcal{I})s + \nu_i},$$

does not depend on the log-resolution h.

1.1.2. Remarks. The result also holds in the complex analytic setting. Initially 2) was stated only when $\mathcal{I}$ is principal, but the proof is the same in general, cf. [**97**].

The original proof of Theorem 1.1.1 was quite surprizing at the time, since it used integration over p-adic numbers to prove a purely complex statement. That proof, we shall explain now, used also the change of variables formula for p-adic integrals, expression of p-adic integrals in term of number of points on varieties over finite fields, and computing Euler characteristics as limits of number of points on varieties over finite fields.

1.2. Quick review of p-adic integration. Most of the material in this subsection is detailed in the book [**56**].

Let p be a prime number. We endow $\mathbb{Q}$ with the p-adic valuation $\mathrm{ord}_p : \mathbb{Q}^\times \to \mathbb{Z}$ and the p-adic norm $|x|_p := p^{-\mathrm{ord}_p(x)}$, $|0|_p = 0$, and consider its completion $\mathbb{Q}_p$ with ring of integers

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p; |x|_p \leq 1\}.$$

More generally one can consider a field K with a valuation $\mathrm{ord} : K^\times \to \mathbb{Z}$, extended to K by $\mathrm{ord}(0) = \infty$. We denote by $\mathcal{O}_K$ the valuation ring $\mathcal{O}_K = \{x \in K | \mathrm{ord}(x) \geq 0\}$ and we fix an uniformizing parameter ϖ, i.e. an element of valuation 1 in $\mathcal{O}_K$. The ring $\mathcal{O}_K$ is a local ring with maximal ideal $\mathcal{M}_K$ of $\mathcal{O}_K$ generated by ϖ. We shall assume the residue field $k := \mathcal{O}_K/\mathcal{M}_K$ is finite with $q = p^e$ elements. One endows K with a norm by setting $|x| := q^{-\mathrm{ord}(x)}$ for x in K. We shall furthermore assume K is complete for $|\_|$.

It follows in particular that the abelian groups $(K^n, +)$ are locally compact, hence they have a canonical Haar measure μ_n, unique up to multiplication by a non zero constant, so we may assume $\mu_n(\mathcal{O}_K^n) = 1$.

The measure μ_n is the unique $\mathbb{R}$-valued Borel measure on K^n which is invariant by translation and such that $\mu_n(\mathcal{O}_K^n) = 1$. For instance

$$\mu_n(a + \varpi^m \mathcal{O}_K^n) = q^{-mn}$$

and for any measurable subset A of K^n and any λ in K,

$$\mu_n(\lambda A) = |\lambda|^n \mu_n(A).$$

More generally, for every g in $\mathrm{GL}_n(K)$,

$$\mu_n(gA) = |\det g| \mu_n(A).$$

If f is, say, a K-analytic function on A, we set

$$\int_A |f| \mu_n := \int_A |f||dx| := \sum_{m \in \mathbb{Z}} \mu_n(\mathrm{ord}(f) = m) q^{-m},$$

assuming the series $\sum_{m \in \mathbb{Z}} \mu_n(\mathrm{ord}(f) = m) q^{-m}$ is convergent in $\mathbb{R}$.

More generally, one can define similarly $\int_A |f|^s |dx|$ by

$$\sum_{m \in \mathbb{Z}} \mu_n(\mathrm{ord}(f) = m) q^{-ms}$$

whenever it makes sense. For instance, when $n = 1$, we have, for $s > 0$ in $\mathbb{R}$,

$$\begin{aligned}\int_{x \in \mathcal{O}_K, \mathrm{ord}(x) \geq m} |x|^s |dx| &= \sum_{j \geq m} q^{-sj} \int_{\mathrm{ord}(x) = j} |dx| \\ &= \sum_{j \geq m} q^{-sj}(q^{-j} - q^{-j-1}) \\ &= (1 - q^{-1}) q^{-(s+1)m} / (1 - q^{-(s+1)}).\end{aligned}$$

The formula for change of volumes under $\mathrm{GL}_n(K)$-action is a very special form of the following fundamental change of variables formula:

1.2.1. PROPOSITION (p-adic change of variables formula). *Let $f = (f_1, \ldots, f_n)$ be a K-analytic isomorphism between open subsets U and V of K^n. Then*

$$\mu_{n|V} = |\operatorname{Jac} f| f_*(\mu_{n|U}),$$

where $\operatorname{Jac} f$ *is the determinant of the jacobian matrix* $(\partial f_i / \partial x_j)$ *of* f. *In other terms:*

$$\int_V \varphi \mu_n = \int_U (\varphi \circ f) |\operatorname{Jac} f| \mu_n,$$

for every integrable function φ on V.

Let X be an d-dimensional smooth K-analytic manifold. One assigns to any K-analytic d-differential form ω on X a measure $\mu_\omega := |\omega|$ as follows.

Take an atlas $\{(U, \phi_U)\}$ of X. Write

$$(\phi_U^{-1})^* \omega_{|U} = f_U dx_1 \wedge \cdots \wedge dx_d.$$

If A is small enough to be contained in some U, we set

$$\mu_\omega(A) := \int_{\phi_U(A)} |f_U| |dx|.$$

It follows from the change of variables formula that the measure may be extended uniquely by additivity to any A in a way which is independent of the choice of the atlas.

Assume now X is a (smooth) closed d-dimensional submanifold of $\mathcal{O}_K^N$. There is a canonical measure μ_X on X defined as follows. For any subset $I = \{i_1 < \cdots < i_d\}$ of cardinality d of $\{1, \ldots, N\}$, we consider the measure $\mu_{X,I}$ on X induced by $dx_{i_1} \wedge \cdots \wedge dx_{i_d}$ on X and we set $\mu_X := \sup_I \mu_{X,I}$.

The canonical volume of X is $\operatorname{vol}(X) := \mu_X(X)$. The relation between the volume $\operatorname{vol}(X)$ and counting points is the following. Denote by X_n the image of X in the finite set $(\mathcal{O}_K / \varpi^n \mathcal{O}_K)^N$.

If X is smooth over $\mathcal{O}_K$, then

$$\operatorname{vol}(X) = |X_1| q^{-d}.$$

In general, if X is smooth over K,

$$\operatorname{vol}(X) = |X_n| q^{-nd} \text{ for } n \gg 0.$$

In particular, $\operatorname{vol}(X)$ lies in $\mathbb{Z}[1/q]$.

For singular X, one may define $\operatorname{vol}(X)$ as the limit in $\mathbb{R}$ of the volume of the complement in X of a tubular neighborhood of small radius around the singular locus, and by a result of Oesterlé [**81**]:

$$\operatorname{vol}(X) = \lim_{n \mapsto \infty} |X_n| q^{-nd}$$

in $\mathbb{R}$.

1.3. Sketch of proof of Theorem 1.1.1. For simplicity, we shall assume that $X = \mathbb{A}^d$ and $F = f^{-1}(0)$, with f a polynomial in $\mathbb{C}[x_1, \ldots, x_d]$ but the proof in general works just the same.

Let us first prove 2). We shall write $Z_{\mathrm{top},f}(s)$ for $Z_{\mathrm{top},F}(s)$. We shall make the assumption that the coefficients of f all lie in the same number field K, i.e. f is in $K[x_1, \ldots, x_d]$ (in general, we can only assume they lie in a field of finite type over $\mathbb{Q}$, but the basic idea of the proof still remains the same). Now for every prime ideal $\mathfrak{P}$ in the ring of integers $\mathcal{O}_K$, we denote by $K_\mathfrak{P}$ the corresponding local field, with ring of integers $\mathcal{O}_\mathfrak{P}$ and residue field $k_\mathfrak{P}$.

We consider Igusa's local zeta function

$$Z_{f,K_\mathfrak{P}}(s) := \int_{\mathcal{O}_\mathfrak{P}^d} |f|_\mathfrak{P}^s |dx|_\mathfrak{P},$$

where $|\_|_\mathfrak{P}$ stands for the $\mathfrak{P}$-adic norm on $K_\mathfrak{P}$.

Consider now a log-resolution $h : Y \to X$ defined over K. It follows from a result of Denef proved in [**23**], that, for almost all $\mathfrak{P}$,

$$(*) \qquad Z_{f,K_\mathfrak{P}}(s) = q^{-d} \sum_{I \subset A} \operatorname{card}((E_I^\circ)_{k_\mathfrak{P}}(k_\mathfrak{P})) \prod_{i \in I} \frac{(q-1)q^{-(N_i s + \nu_i)}}{1 - q^{-(N_i s + \nu_i)}},$$

with $q = \operatorname{card} k_\mathfrak{P}$.

Here we should explain what we mean by $(E_I^\circ)_{k_\mathfrak{P}}$. For Z a variety over K we choose a model $\mathcal{Z}$ over $\mathcal{O}_K$, and we denote by $(E_I^\circ)_{k_\mathfrak{P}}$ its reduction mod $\mathfrak{P}$. Of course, this may depend on the choice of the model $\mathcal{X}$, but if one takes another model $\mathcal{X}'$, the reductions will differ only for a finite number of prime ideals $\mathfrak{P}$.

Denef's proof of (*) is based on the change of variables formula and the fact that $h(K_\mathfrak{P}) : Y(K_\mathfrak{P}) \to X(K_\mathfrak{P})$ is an isomorphism outside closed analytic subsets of dimension $< d$, which are of measure 0 for d-dimensional measures. Also note that any model over $\mathcal{O}_K$ of a smooth K-variety will be smooth over $\mathcal{O}_\mathfrak{P}$ for almost all $\mathfrak{P}$.

For $e \geq 1$, let us write $K_\mathfrak{P}^{(e)}$ for the unramified extension of $K_\mathfrak{P}$ of degree e. Its residue field $k_\mathfrak{P}^{(e)}$ has q^e elements. Also, for almost all $\mathfrak{P}$, equation (*) still holds when replacing $K_\mathfrak{P}$ by $K_\mathfrak{P}^{(e)}$, yielding

$$(**) \qquad Z_{f,K_\mathfrak{P}^{(e)}}(s) = q^{-ed} \sum_{I \subset A} \operatorname{card}((E_I^\circ)_{k_\mathfrak{P}}(k_\mathfrak{P}^{(e)})) \prod_{i \in I} \frac{(q^e-1)q^{-e(N_i s + \nu_i)}}{1 - q^{-e(N_i s + \nu_i)}}.$$

Now, taking formally the limit as $e \mapsto 0$ in (**) would give us the right-hand side of (‡), if only we could make sense of the following statement:

$$\lim_{e \mapsto 0} \operatorname{card} W_{k_\mathfrak{P}}(k_\mathfrak{P}^{(e)}) = \operatorname{Eu} W,$$

for almost all $\mathfrak{P}$, when W is a variety over K.

Indeed, it follows from Grothendieck's trace formula for the Frobenius acting on ℓ-adic cohomology together with standard comparison theorems between ℓ-adic and classical Betti cohomology, that, when W is a variety over K, for almost all $\mathfrak{P}$, there exist complex numbers α_i, $i \in B$ and β_j, $j \in C$, such that

$$\operatorname{card} W_{k_\mathfrak{P}}(k_\mathfrak{P}^{(e)}) = \sum_B \alpha_i^e - \sum_C \beta_j^e$$

and

$$\operatorname{Eu} W = |B| - |C|.$$

One can actually take as α_i and β_j the eigenvalues, respectively in even and odd degree, of the Frobenius acting on ℓ-adic cohomology groups with compact supports of $W_{k_\mathfrak{P}} \otimes \overline{k_\mathfrak{P}}$.

Of course, this is just a rough sketch of the proof of 2) and further work is required in order to show this process of taking limits as $e \mapsto 0$ really makes sense.

To prove 1), one proceeds similarly, setting $s = 0$ in 2) and noticing that $Z_{f,K_{\mathfrak{P}}}(0) = 1 = \mathrm{Eu}(\mathbb{A}^d)$. □

1.4. A reformulation. One can reformulate (†) in terms of constructible functions. Note that constructible functions will occur again in Lecture 4. Recall that the algebra $\mathcal{C}(X, \mathbb{C})$ of complex-valued constructible functions on X is generated by characteristic functions $\mathbf{1}_W$ of constructible subsets W of X. If $h : Y \to X$ is a morphism one defines the pushforward $h_! : \mathcal{C}(Y, \mathbb{C}) \to \mathcal{C}(X, \mathbb{C})$ by

$$h_!(\mathbf{1}_{W'})(x) := \mathrm{Eu}(h^{-1}(x) \cap W'),$$

for W' constructible in Y.

Now if h is a DNC modification, one can reformulate (†) by saying that

$$\mathbf{1}_W \longmapsto \sum_{I \subset A} \frac{\mathbf{1}_{E_I^\circ \cap h^{-1}(W)}}{\prod_{i \in I} \nu_i},$$

for W constructible in X, is an inverse to $h_!$. This formulation has been recently rediscovered in [**4**].

1.5. Other proofs of Theorem 1.1.1. Nowadays, there are two other proofs of Theorem 1.1.1. The one using Motivic Integration will be explained in Lecture 2. The other one relies on the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [**2**] that was not available at the time.

The weak factorization theorem is a very strong structure result for birational morphisms:

1.5.1. THEOREM (Weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk). *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper smooth irreducible varieties. Let $U \subset X_1$ be the largest open subset on which ϕ is an isomorphism. Then ϕ can be factored into a sequence of blowing ups and blowing down with smooth centers disjoint from U: $\phi_i : V_{i-1} \dashrightarrow V_i$, $i = 1, \dots, \ell$, with $V_0 = X_1$, $V_\ell = X_2$, with ϕ_i or ϕ_i^{-1} blowing ups with smooth centers away from U. Moreover there exists i_0 such that $V_i \dashrightarrow X_1$ is defined everywhere and projective for $i \leq i_0$ and $V_i \dashrightarrow X_2$ is defined everywhere an projective for $i \geq i_0$.*

Indeed, by Theorem 1.5.1, it is enough to prove Theorem 1.1.1 when $h : Y \to X$ is a blowing-up of a smooth center, in which case it can be checked directly by an easy computation! Note however that the proof of Theorem 1.5.1 is long and difficult.

Finding a direct topological proof of (†) still seems to be a very challenging problem.

1.6. Birational Calabi-Yau varieties. The next occurrence of p-adic integration as a tool to prove new results in birational geometry took place in 1995, with Batyrev's striking proof that birational Calabi-Yau varieties have the same Betti numbers [**6**].

Let X be a smooth complex projective variety of dimension n. We shall say X is Calabi-Yau if X admits a nowhere vanishing degree n algebraic differential form ω, that is, the sheaf Ω_X^n is trivial. Recall the Betti numbers $b_i(X)$ are the ranks of the cohomology groups $H^i(X(\mathbb{C}), \mathbb{C})$. Considerations from Mirror Symmetry led to the guess that birational Calabi-Yau varieties should have the same Betti numbers (in fact the same Hodge numbers).

This was proved by Batyrev in 1995 using p-adic integration and the Weil conjectures:

1.6.1. THEOREM (Batyrev [**6**]). *Let X and X' be complex Calabi-Yau varieties of dimension n. Assume X and X' are birationally equivalent. Then they have the same Betti numbers.*

Let us sketch the proof. For simplicity, we assume, as in the proof of the Theorem 1.1.1 that X, X' and all the data are defined over some number field K (in general they are defined only over some field of finite type, but the basic idea of the proof is the same). By Hironaka there exists a smooth projective Y defined over K, and birational proper morphisms (also defined over K) $h : Y \to X$ and $h' : Y \to X'$. Furthermore we may assume there exists a divisor with normal crossings $E = \cup_{i \in J} E_i$ such that the exceptional locus of h and h' respectively, is a finite union of E_i's. We may write $K_Y = h^* K_X + \sum_{i \in J} (\nu_i - 1) E_i$ and $K_Y = h'^* K_{X'} + \sum_{i \in J} (\nu'_i - 1) E_i$. Since $h^* K_X$ and $h'^* K_{X'}$ are both trivial, it follows - check it as an exercise! - that $\nu_i = \nu'_i$ for every i in J.

One then deduces from the change of variables formula, that for almost all $\mathfrak{P}$, with a slight abuse of notation, we have

$$\int_{X(K_{\mathfrak{P}})} |\omega|_{\mathfrak{P}} = \int_{X'(K_{\mathfrak{P}})} |\omega'|_{\mathfrak{P}}$$

and the same holds for all unramified extensions $K_{\mathfrak{P}}^{(e)}$.

Indeed, we may express by the change of variables formula both integrals as the same integral over the rational points of Y.

Since, for almost all $\mathfrak{P}$ and every e,

$$\int_{X(K_{\mathfrak{P}}^{(e)})} |\omega|_{\mathfrak{P}} = q^{-en} \operatorname{card}(X_{K_{\mathfrak{P}}}(k_{\mathfrak{P}}^{(e)})),$$

it follows that for almost all $\mathfrak{P}$, the reductions of (some model of) X and X' modulo the maximal ideal at $\mathfrak{P}$ have same the zeta function.

On the other side, by the part of the Weil conjectures proved by Deligne, for proper smooth varieties over a finite field, the zeta function determines the ℓ-adic Betti numbers, hence the result follows from standard comparison results between ℓ-adic and usual Betti numbers.

1.6.2. REMARK. The above proof gives in fact the following stronger result: if X and X' are two n-dimensional smooth proper complex varieties that are K-equivalent, meaning that there exists birational proper morphisms $h : Y \to X$ and $h' : Y \to X'$ with Y smooth proper such that the invertible sheaves $h^*(\Omega^n_X)$ and $h^*(\Omega^n_{X'})$ are isomorphic, then X and X' have the same Betti numbers.

Shortly after Batyrev's proof, M. Kontsevich found a direct approach to Batyrev's Theorem, avoiding the use of p-adic integrals and involving arc spaces, which he explained in a seminal Orsay talk[1] on December 7, 1995 [**61**].

Motivic integration was born ...

[1]The curious reader will find in [**3**] explanations about the title of Kontsevich's talk.

Lecture 2: Basics of Motivic Integration

2.1. Arc spaces. We shall now work over a field k of characteristic 0. By a variety X over k we shall always mean a separated and reduced scheme, of finite type over k. For $n \geq 0$, we introduce the space of n-arcs on X, denoted by $\mathcal{L}_n(X)$. It is a k-scheme of finite type which represents the functor:

$$k-\text{algebras} \longrightarrow \text{Sets}$$

$$R \longmapsto \text{Hom}_{k-\text{schemes}}(\text{Spec}(R[t]/(t^{n+1}), X) = X(R[t]/(t^{n+1})).$$

For example, when X is an affine variety with equations $f_i(\vec{x}) = 0$, $i = 1, \cdots, m$, $\vec{x} = (x_1, \cdots, x_r)$, then $\mathcal{L}_n(X)$ is given by the equations, in the variables $\vec{a}_0, \cdots, \vec{a}_n$, expressing that

$$f_i(\vec{a}_0 + \vec{a}_1 t + \cdots + \vec{a}_n t^n) \equiv 0 \mod t^{n+1}, i = 1, \cdots, m.$$

We have canonical isomorphisms $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X) = TX$, where TX denotes the tangent space of the variety X.

For $m \geq n$, there are canonical morphisms $\theta^n_m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$. In general, when X is not smooth, they may not be surjective. When X is smooth of dimension d, θ^n_m is a locally trivial fibration for the Zariski topology with fiber $\mathbb{A}_k^{(m-n)d}$ (more precisely it is an affine bundle).

Taking the projective limit of these algebraic varieties $\mathcal{L}_n(X)$, we obtain the arc space $\mathcal{L}(X)$ of X. A priori this is just a pro-scheme, but, the transition maps θ^n_m being affine, it is indeed a k-scheme. In general, $\mathcal{L}(X)$ is not of finite type over k. For K a field extension of k, the K-rational points of $\mathcal{L}(X)$ are the $K[[t]]$-rational points of X and they are called K-arcs on X. For instance, when X is an affine variety with equations $f_i(\vec{x}) = 0, i = 1, \cdots, m, \vec{x} = (x_1, \cdots, x_r)$, then the K-rational points of $\mathcal{L}(X)$ are the sequences $(\vec{a}_0, \vec{a}_1, \vec{a}_2, \cdots) \in (K^r)^{\mathbb{N}}$ satisfying $f_i(\vec{a}_0 + \vec{a}_1 t + \vec{a}_2 t^2 + \cdots) = 0$, for $i = 1, \cdots, m$.

For every n we have natural morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

obtained by truncation. For any arc γ on X (i.e. a K-arc for some field K containing k), we call $\pi_0(\gamma)$ the origin of the arc γ.

The arc space was first introduced by Nash in [**74**] in connection with the study of essential components of the exceptional locus of resolution of singularities. We refer to [**64**] [**58**] for some recent results on the Nash problem and [**47**] [**36**] [**85**] for recent results on the geometry of the arc space when X is singular.

2.2. Additive invariants. Let R and S be rings. An additive invariant λ from the category Var_R of algebraic varieties over R with values in S, assigns to any X in Var_R an element $\lambda(X)$ of S, such that

$$\lambda(X) = \lambda(X')$$

for $X \simeq X'$,

$$\lambda(X) = \lambda(X') + \lambda(X \setminus X'),$$

for X' closed in X, and

$$\lambda(X \times X') = \lambda(X) \cdot \lambda(X')$$

for every X and X'.

One should note that additive invariants λ naturally extend to take their values on constructible subsets of algebraic varieties. Indeed a constructible subset W may be written as a finite disjoint union of locally closed subvarieties Z_i, $i \in I$. One may define $\lambda(W)$ to be $\sum_{i\in I} \lambda(Z_i)$. By the very axioms, this is independent of the decomposition into locally closed subvarieties.

2.3. Some examples.

2.3.1. *Euler characteristic.* Here $R = k$ is a field. When k is a subfield of $\mathbb{C}$, the Euler characteristic

$$\mathrm{Eu}(X) := \sum_i (-1)^i \mathrm{rk} H^i_c(X(\mathbb{C}), \mathbb{C})$$

gives rise to an additive invariant $\mathrm{Eu} : \mathrm{Var}_k \to \mathbb{Z}$.

For general k, replacing Betti cohomology with compact support of $X(\mathbb{C})$ by ℓ-adic cohomology with compact support of $X \otimes \overline{k}$, $\ell \neq \mathrm{char}\, k$, one gets an additive invariant $\mathrm{Eu} : \mathrm{Var}_k \to \mathbb{Z}$, which does not depend on ℓ.

2.3.2. *Hodge polynomial.* Let us assume $R = k$ is a field of characteristic zero. Then it follows from Deligne's Mixed Hodge Theory that there is a unique additive invariant $H : \mathrm{Var}_k \to \mathbb{Z}[u, v]$, which assigns to a smooth projective variety X over k its usual Hodge polynomial

$$H(u, v) := \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q,$$

with $h^{p,q}(X) = \dim H^q(X, \Omega^p_X)$ the (p, q)-Hodge number of X.

2.3.3. *Virtual motives.* More generally, when $R = k$ is a field of characteristic zero, there exists by work of Gillet and Soulé [**42**], and Guillen and Navarro-Aznar [**50**], a unique additive invariant

$$\chi_c : \mathrm{Var}_k \to K_0(\mathrm{CHMot}_k),$$

which assigns to a smooth projective variety X over k the class of its Chow motive, where $K_0(\mathrm{CHMot}_k)$ denotes the Grothendieck ring of the category of Chow motives over k (with rational coefficients).

2.3.4. *Counting points.* Assume $k = \mathbb{F}_q$, then $N_n : X \mapsto |X(\mathbb{F}_{q^n})|$ gives rise to an additive invariant $N_n : \mathrm{Var}_k \to \mathbb{Z}$.

Similarly, if R is (essentially) of finite type over $\mathbb{Z}$, for every maximal ideal $\mathfrak{P}$ of R with finite residue field $k(\mathfrak{P})$, we have an additive invariant $N_{\mathfrak{P}} : \mathrm{Var}_R \to \mathbb{Z}$, which assigns to X the cardinality of $(X \otimes k(\mathfrak{P}))(k(\mathfrak{P}))$.

2.4. Grothendieck rings. There exists a universal additive invariant

$$[\_] : \mathrm{Var}_R \longrightarrow K_0(\mathrm{Var}_R)$$

in the sense that composition with $[\_]$ gives a bijection between ring morphisms $K_0(\mathrm{Var}_R) \to S$ and additive invariants $\mathrm{Var}_R \to S$.

The construction of $K_0(\mathrm{Var}_R)$ is quite easy: take the free abelian group on isomorphism classes $[S]$ of objects of Var_R and mod out by the relations

$$[S] = [S'] + [S \setminus S']$$

for S' closed in S. The product is defined by

$$[S] \cdot [S'] = [S \times S'].$$

We shall denote by $\mathbb{L}$ the class of the affine line $\mathbb{A}^1_R$ in $K_0(\mathrm{Var}_R)$.

An important role will be played by the ring

$$\mathcal{M}_R := K_0(\mathrm{Var}_R)[\mathbb{L}^{-1}]$$

obtained by localization with respect to the multiplicative set generated by $\mathbb{L}$.

This construction is analogous to the construction of the category of Chow motives from the category of effective Chow motives by localization with respect to the Lefschetz motive. (Note that the morphism χ_c induced by the previous morphism χ_c sends $\mathbb{L}$ to the class of the Lefschetz motive.)

Very little is known about the structure of the rings $K_0(\mathrm{Var}_R)$ and $\mathcal{M}_R$ even when R is a field. B. Poonen [**83**] proved that, when k is a field of characteristic zero, the ring $K_0(\mathrm{Var}_k)$ is not a domain. Even for a field k of characteristic zero, it is not known whether the localization morphism $K_0(\mathrm{Var}_k) \to \mathcal{M}_k$ is injective or not (although we guess it is not). We shall denote by $\overline{\mathcal{M}}_k$ the image of $K_0(\mathrm{Var}_k)$ in $\mathcal{M}_k$.

We shall also need relative Grothendieck rings defined as follows. Let X be a variety over R. One defines $K_0(\mathrm{Var}_X)$ similarly as $K_0(\mathrm{Var}_R)$ using the category of varieties over X instead of the category of varieties over R. Recall that objects in this category are arrows $f : Y \to X$ in Var_R and that a morphism between $f : Y \to X$ and $f' : Y' \to X$ is just a morphism $g : Y \to Y'$ such that $f = f' \circ g$.

One also defines $\mathcal{M}_X$ by inverting the class $\mathbb{L}$ of $\mathbb{A}^1_X \to X$ in $K_0(\mathrm{Var}_X)$. We shall write $[f : Y \to X]$ for the class of $f : Y \to X$.

2.5. Using the weak factorization Theorem, F. Bittner gave a very convenient description of $K_0(\mathrm{Var}_k)$. Let us denote by $K_0^{\mathrm{bl}}(\mathrm{Var}_k)$ the quotient of the free abelian group on isomorphism classes of irreducible smooth projective varieties over k by the relations $[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$, for Y and X irreducible smooth projective over k, Y closed in X, $\mathrm{Bl}_Y X$ the blowup of X with center Y and E the exceptional divisor in $\mathrm{Bl}_Y X$. As for $K_0(\mathrm{Var}_k)$, cartesian product induces a product on $K_0^{\mathrm{bl}}(\mathrm{Var}_k)$ which endowes it with a ring structure. There is a canonical ring morphism $\theta : K_0^{\mathrm{bl}}(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$, which sends $[X]$ to $[X]$. Bittner's result [**10**] asserts that, when k is of characteristic zero, the morphism θ is an isomorphism. In particular, using that result, one gets direct alternate constructions of the first three examples in 2.3.

2.6. So, what is motivic integration? Roughly speaking, motivic integration assigns to a reasonable class of subsets A of $\mathcal{L}(X)$, the arc space of a k-variety X, a volume $\mu(A)$[2].

The most naive idea would be to construct a real valued measure on $\mathcal{L}(X)$ similarly as in the p-adic case. Such attempts are doomed to fail immediately since, as soon as k is infinite, $k((t))$ is not locally compact. Kontsevich's real breakthrough was to realize that a reasonable measure on $k((t))$ could in fact be constructed once $\mathbb{R}$ is replaced by $\mathcal{M}_k$ (or its completion). The motivic measure $\mu(A)$ will be an element of $\mathcal{M}_k$, or of some completion or localization of $\mathcal{M}_k$.

2.7. Motivic integration: the original construction. We shall present today the original construction of motivic integration as it has been developed in papers by Batyrev and Denef - Loeser, [**7**], [**26**], [**32**], following insights of Kontsevich [**61**]. In the last lecture, we shall present the more recent approach of Cluckers and Loeser developed in [**12**], [**13**], [**14**]. Amongst the main features of the new

[2]See also [**43**].

approach, let us mention that completion is no longer needed and that it also allows to deal with integrals depending on parameters.

We want to assign a measure to subsets of $\mathcal{L}(X)$ that will take its values into a ring related to $K_0(\mathrm{Var}_k)$. We shall proceed by analogy with p-adic integration: $K_0(\mathrm{Var}_k)$ will be the analogue of $\mathbb{Z}$ and $\mathcal{M}_k$ the analogue of $\mathbb{Z}[p^{-1}]$ (recall the number of rational points of the affine line over $\mathbb{F}_p$ is p). Since in $\mathbb{R}$, p^{-i} has limit 0 as $i \mapsto \infty$, we should complete $\mathcal{M}_k$ is such a way that $\mathbb{L}^{-i}$ has limit 0 as $i \mapsto \infty$. This is achieved in the following way: we define $F^m\mathcal{M}_k$ to be the subgroup of $\mathcal{M}_k$ generated by elements of the form $[S]\mathbb{L}^{-i}$, with $\dim S - i \leq -m$. We have $F^{m+1} \subset F^m$, $\mathbb{L}^{-m} \in F^m$ and $F^nF^m \subset F^{n+m}$. We denote by $\widehat{\mathcal{M}}_k$ the completion of $\mathcal{M}_k$ with respect to that filtration.

A minor technical problem shows up here, namely that it is not known whether the canonical morphism $\mathcal{M}_k \to \widehat{\mathcal{M}}_k$ is injective or not. Nevertheless, this is not much an issue since one can show invariants $\mathrm{Eu} : \mathcal{M}_k \to \mathbb{Z}$ (Euler number) and $H : \mathcal{M}_k \to \mathbb{Z}[u, v, (uv)^{-1}]$ (Hodge polynomial) factor through the image $\overline{\mathcal{M}}_k$ of $\mathcal{M}_k$ in $\widehat{\mathcal{M}}_k$.

Let X be an algebraic variety over k of dimension d, maybe singular.

By a cylinder in $\mathcal{L}(X)$, we mean a subset A of $\mathcal{L}(X)$ of the form $A = \pi_n^{-1}(C)$ with C a constructible subset of $\mathcal{L}_n(X)$, for some n. We say A is stable (at level n) if furthermore the restriction of $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ over $\pi_m(A)$ is a piecewise Zariski fibration over $\pi_m(A)$ with fiber $\mathbb{A}^d_k$ for all $m \geq n$. By being a piecewise Zariski fibration over $\pi_m(A)$ we mean that there exists a finite partition of $\pi_m(A)$ into locally closed subsets of $\mathcal{L}_m(X)$ over which the morphism is a locally trivial fibration for the Zariski topology.

If A is a stable cylinder at level n, we set

$$\tilde{\mu}(A) := [\pi_n(A)]\mathbb{L}^{-(n+1)d}$$

in $\mathcal{M}_k$. The stability condition insures that we would get the same value by viewing A as a stable cylinder at level m, $m \geq n$. When X is smooth, all cylinders are stable. In particular, in this case, $\mathcal{L}(X)$ itself is a stable cylinder and

$$\tilde{\mu}(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}.$$

In general, we can assign to any cylinder A in $\mathcal{L}(X)$ a measure $\mu(A)$ in $\widehat{\mathcal{M}}_k$ by a limit process as follows: for $e \geq 0$, set

$$\mathcal{L}^{(e)}(X) := \mathcal{L}(X) \setminus \pi_e^{-1}(\pi_e(\mathcal{L}(X_{\mathrm{sing}}))),$$

where X_{sing} denote the singular locus of X and we view $\mathcal{L}(X_{\mathrm{sing}})$ as a subset of $\mathcal{L}(X)$. The set $\mathcal{L}^{(e)}(X)$ should be viewed as $\mathcal{L}(X)$ minus some tubular neighborhood around the singular locus.

It can be proved that $A \cap \mathcal{L}^{(e)}(X)$ is a stable cylinder and that $\tilde{\mu}(A \cap \mathcal{L}^{(e)}(X))$ does have a limit in $\widehat{\mathcal{M}}_k$ as e goes to ∞ which we define to be $\mu(A)$. This applies in particular to $A = \mathcal{L}(X)$ when X is not smooth.

We define

$$||\text{-}|| : \widehat{\mathcal{M}}_k :\to \mathbb{R}_{\geq 0}$$

to be given by $||a|| = 2^{-n}$ if $a \in F^n\widehat{\mathcal{M}}_k$ and $a \notin F^{n+1}\widehat{\mathcal{M}}_k$, where $F^{\cdot}\widehat{\mathcal{M}}_k$ denotes the induced filtration on $\widehat{\mathcal{M}}_k$.

We say a subset A of $\mathcal{L}(X)$ is measurable if, for every $\varepsilon > 0$, there exist cylinders $A_i(\varepsilon)$, $i \in \mathbb{N}$, such that $(A \cup A_0(\varepsilon)) \setminus (A \cap A_0(\varepsilon))$ is contained in $\cup_{i\geq 1} A_i(\varepsilon)$, and $||\mu(A_i(\varepsilon))|| \leq \varepsilon$, for every $i \geq 1$.

One shows that

$$\mu(A) := \lim_{\varepsilon \mapsto 0} \mu(A_0(\varepsilon))$$

exists and is independent of the choice of the $A_i(\varepsilon)$'s. We say that A is strongly measurable if moreover we can take $A_0(\varepsilon) \subset A$.

Let A be a measurable subset of $\mathcal{L}(X)$ and $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ be a function such that all its fibers are measurable. We shall say $\mathbb{L}^{-\alpha}$ is integrable if the series

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{n\in\mathbb{Z}} \mu(A \cap \alpha^{-1}(n))\mathbb{L}^{-n}$$

is convergent in $\widehat{\mathcal{M}}_k$.

We have the following motivic analogue of the p-adic change of variables formula:

2.7.1. Theorem (Change of variables formula). *Let X be an algebraic variety over k of dimension d. Let $h : Y \to X$ be a proper birational morphism. We assume Y to be smooth. Let A be a subset of $\mathcal{L}(X)$ such that A and $h^{-1}(A)$ are strongly measurable. Assume $\mathbb{L}^{-\alpha}$ is integrable on A. Then*

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{h^{-1}(A)} \mathbb{L}^{-\alpha\circ h - \mathrm{ord} h^*(\Omega^d_X)} d\mu.$$

Let us explain what is meant by $\mathrm{ord} h^*(\Omega^d_X)$, the order of the jacobian of h, when X is not smooth.

If $\mathcal{I}$ is some ideal sheaf on Y, we denote by $\mathrm{ord}\mathcal{I}$ the function which to a arc φ in $\mathcal{L}(Y)$ assigns $\inf \mathrm{ord} g(\varphi)$ where g runs over local sections of $\mathcal{I}$ at $\pi_0(\varphi)$.

We denote by Ω^d_X the d-th exterior power of Ω^1_X, the Kähler differentials. The image of $h^*(\Omega^d_X)$ in Ω^d_Y is of the form $\mathcal{I}\Omega^d_Y$ and we set

$$\mathrm{ord} h^*(\Omega^d_X) := \mathrm{ord}\mathcal{I}.$$

The key geometrical statement behind the proof of the change of variables formula is the following:

2.7.2. Theorem (Denef-Loeser). *Let X be an algebraic variety over k. Let $h : Y \to X$ be proper birational morphism. We assume Y to be smooth. For e and e' in $\mathbb{N}$, we set*

$$\Delta_{e,e'} := \Big\{\varphi \in \mathcal{L}(Y) \;\Big|\; \mathrm{ord} h^*(\Omega^d_X)(\varphi) = e \quad \text{and} \quad h(\varphi) \in \mathcal{L}^{(e')}(X)\Big\}.$$

Then there exists $c > 0$ such that, for $n \geq \sup(2e, e + ce')$,

1. *The image $\Delta_{e,e',n}$ of $\Delta_{e,e'}$ in $\mathcal{L}_n(Y)$ is a union of fibers of h_n, the morphism induced by h.*
2. *The morphism $h_n : \Delta_{e,e',n} \to h_n(\Delta_{e,e',n})$ is a piecewise Zariski fibration with fiber $\mathbb{A}^e_k$.*

When X is smooth, one can take $e' = 0$.

2.8. Application to DNC modifications. Let $h : Y \to X$ be a DNC modification between smooth varieties, as in 1.1.

Since

$$\mathbb{L}^{-d}[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord} h^*(\Omega_X^d)} d\mu,$$

we have the following:

2.8.1. THEOREM. *Let $h : Y \to X$ be a DNC modification between smooth varieties. Then*

$$[X] = \sum_{I \subset A} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i} - 1}$$

in $\widehat{\mathcal{M}}_k$.

Taking Eu of both sides one recovers the the first statement in Theorem 1.1.1.

2.9. Application to birationally equivalent Calabi-Yau varieties. Let X and X' be two birationally equivalent smooth proper Calabi-Yau varieties. Take birational morphisms of proper smooth varieties $h : Y \to X$ and $h' : Y \to X$'. Since

$$\mathbb{L}^{-d}[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord} h^*(\Omega_X^d)} d\mu$$
$$= \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord} h'^*(\Omega_{X'}^d)} d\mu = \mu(\mathcal{L}(X')) = \mathbb{L}^{-d}[X'],$$

we deduce:

2.9.1. THEOREM. *Let X and X' be two birationally equivalent smooth proper Calabi-Yau varieties. Then*

$$[X] = [X']$$

in $\widehat{\mathcal{M}}_k$.

Taking H of both sides, one deduces that X and X' have same Hodge numbers, hence same Betti numbers.

The same result and proof hold for K-equivalent varieties.

2.10. Application to stringy invariants. Let X be a normal $\mathbb{Q}$-Gorenstein variety. Let $h : Y \to X$ be a log-resolution of X, that is Y is smooth, h is proper, the restriction of $h : Y \setminus h^{-1}(X_{\mathrm{sing}}) \to X \setminus X_{\mathrm{sing}}$ is an isomorphism, and $h^{-1}(X_{\mathrm{sing}})$ is a divisor with simple normal crossings with irreducible components E_i, i in A.

We may define log discrepancies ν_i in $\mathbb{Q}$ by

$$K_Y = h^* K_X + \sum_{i \in A} (\nu_i - 1) E_i.$$

When all ν_i's are > 0 we say X is log terminal.

For X a normal log terminal $\mathbb{Q}$-Gorenstein variety

$$E_{st}(X) := \sum_{I \subset A} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i} - 1}$$

does not depend on the log-resolution h. This is a direct consequence of the change of variables formula, or more exactly of Theorem 2.7.2.

Note that $E_{st}(X)$ belongs to $\widehat{\mathcal{M}}_k[\mathbb{L}^{1/m}]$, with m a multiple of the denominators of the ν_i's. Also, setting $j : X \setminus X_{\mathrm{sing}} \hookrightarrow X$ and denoting by $\mathcal{I}_X \omega_X$ the image of the natural morphism $\Omega_X^d \to \omega_X := j_* j^* \Omega_X^d$, $E_{st}(X)$ may be defined intrisically as

$$E_{st}(X) = \int_{\mathcal{L}(X)} \mathbb{L}^{\mathrm{ord}_t \mathcal{I}_X} d\mu.$$

Applying Eu and H to $E_{st}(X)$ one recovers the stringy Betti numbers and the stringy Hodge numbers of Batyrev [**7**]. It is an open issue to generalize these stringy invariants to general singularities, cf. the work of Veys [**95**].

2.11. Application to the log-canonical threshold. Let X be a smooth variety of dimension d and let F be a closed nowhere dense subscheme of X defined by an ideal sheaf $\mathcal{I}$. Let $h : Y \to X$ be a log-resolution of (X, F) such that $h^{-1}(\mathcal{I})\mathcal{O}_Y$ is locally free.

For $n \geq 0$, consider

$$\mathcal{X}_n(\mathcal{I}) := \Big\{ \varphi \in \mathcal{L}_n(X) \;\Big|\; \inf_{g \in \mathcal{I}_{\pi_0(\varphi)}} \mathrm{ord}_t(g \circ \varphi) = n \Big\}.$$

It follows directly from the geometric form of the change of variables formula, Theorem 2.7.2, that $[\mathcal{X}_n(\mathcal{I})]$ can be computed in $\mathcal{M}_k$ on the log-resolution h as follows, cf. [**30**]:

$$(\natural) \qquad [\mathcal{X}_n(\mathcal{I})] = \mathbb{L}^{nd} \sum_{I \subset A} (\mathbb{L} - 1)^{|I|} [E_I^\circ] \left(\sum_{k_i \geq 1, i \in I, \sum_I k_i N_i(\mathcal{I}) = n} \mathbb{L}^{-\sum k_i \nu_i} \right)$$

in $\mathcal{M}_k$.

Recall the log-canonical threshold of the pair (X, F) is defined by

$$(\natural\natural) \qquad \mathrm{lct}(X, \mathcal{I}) = \min_{i \in A} \frac{\nu_i}{N_i(\mathcal{I})}.$$

From $(\natural)$ and $(\natural\natural)$ one deduces (cf. [**73**], [**39**]):

2.11.1. Theorem (Mustaţă [**73**]).

$$(\sharp) \qquad \mathrm{lct}(X, \mathcal{I}) = \min_n \left(\frac{\mathrm{codim}_{\mathcal{L}_n(X)} \mathcal{L}_n(F)}{n+1} \right).$$

From $(\sharp)$ Mustaţă is able to derive a simple algebraic proof of the semicontinuity in families of the log-canonical threshold [**73**].

2.12. Some other applications. Amongst other applications, let us mention without further details or pretention to exhaustiveness:

- the work of Batyrev [**8**], Denef-Loeser [**32**] and Yasuda [**98**] [**99**], see also [**87**] [**68**] [**66**], related to Miles Reid's homological McKay correspondence [**86**].
- further work by Mustaţă, Ein, Lazarsfeld, Yasuda, de Fernex on log-canonical threshold, inversion of adjunction and application to birational rigidity [**72**] [**38**] [**41**] [**39**].

- the rationality of the series

$$P(T) : \sum_{n \geq 0} [\pi_n(\mathcal{L}(X))] \, T^n$$

in $\mathcal{M}_k[[T]]$ proved by Denef and Loeser in [**26**] (note that the series makes sense because $\pi_n(\mathcal{L}(X))$ is constructible in $\mathcal{L}_n(X)$) and computation of $P(T)$ for some singularities by Lejeune-Jalabert and Reguera [**65**] and Nicaise [**78**] [**79**].

Lecture 3: The motivic Milnor fiber

3.1. The Milnor fiber and the monodromy. Let X be a smooth complex algebraic variety and $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a morphism to the affine line. Let x be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$.

Fix $0 < \eta \ll \varepsilon \ll 1$. By Milnor's local fibration Theorem [**71**] the morphism f restricts to a fibration (the Milnor fibration)[3]

$$B(x, \varepsilon) \cap f^{-1}(B(0, \eta) \setminus \{0\}) \to B(0, \eta) \setminus \{0\}.$$

Here $B(a, r)$ denotes the closed ball of center a and radius r.

The Milnor fiber at x,

$$F_x = f^{-1}(\eta) \cap B(x, \varepsilon),$$

has a diffeomorphism type that does not depend on η and ε and is endowed with an automorphism, defined up to homotopy, the monodromy M_x, induced by the characteristic mapping of the fibration.

In particular the cohomology groups $H^q(F_x, \mathbb{C})$ are endowed with an automorphism M_x.

3.2. Nearby and vanishing cycles. The above description is not very convenient when, for instance, one wants to make the point x vary. In [**20**] Deligne introduced a sheaf theoretic version of Milnor's constructions. Let us change notations slightly, denoting by D a very small open disk around the origin in $\mathbb{C}$, and by $f : X \to D$ the restriction of f above D. We set $D^* := D \setminus \{0\}$ and denote by $\widetilde{D}^*$ the universal covering of D^*. We denote by X^* the preimage of D^* in D and by $\widetilde{X}^*$ the fiber product $X^* \times_{D^*} \widetilde{D}^*$. We consider morphisms $i : X_0 := f^{-1}(0) \hookrightarrow X$ and $j : \widetilde{X}^* \to X$.

If $\mathcal{F}$ is a constructible sheaf on X, for $q \geq 0$, one sets

$$R^q\psi_f \mathcal{F} := i^* R^q j_* j^* \mathcal{F}.$$

It is a constructible sheaf on X_0. Furthermore the deck transformation on $\widetilde{X}^*$ induces the action of a canonical monodromy automorphism on $R^q\psi_f\mathcal{F}$. One may check - do it as an exercise - that, as a vector space with automorphism, the stalk $(R^q\psi_f\mathbb{C}_X)_x$ is canonically isomorphic to $(H^q(F_x, \mathbb{C}), M_x)$.

One could work as well in derived categories D^b_{constr} and $D^b_{\mathrm{constr,aut}}$ of bounded complexes with constructible cohomology, resp. bounded complexes with automorphism and constructible cohomology, defining the nearby cycle functor

$$R\psi_f : D^b_{\mathrm{constr}}(X) \longrightarrow D^b_{\mathrm{constr,aut}}(X_0)$$

by

$$R\psi_f K := i^* R j_* j^* K.$$

Note that, while $R\psi_f K$ depends only on the restriction of K to X^*, this is no longer the case for the vanishing cycle functor $R\phi_f$ defined by the triangle

$$i^* K \longrightarrow R\psi_f K \longrightarrow R\phi_f K.$$

[3]Strictly speaking, in [**71**], only isolated singularities are considered and the fibration is somewhat different. A proof for non isolated singularities can be found in [**63**].

3.3. Arcs and monodromy. A first connection between arcs and monodromy is the following.

Denote by $\mathcal{X}^1_{n,x}(f)$ the set of arcs φ in $\mathcal{L}_n(X)$ with $\varphi(0) = x$ such that

$$f(\varphi(t)) = t^n + (\text{higher order terms}).$$

Consider a log-resolution $h : Y \to X$ of $f^{-1}(0)$ such that $h^{-1}(x)$ is a union of components E_i, $i \in A_0$. Similarly as above, one deduces from Theorem 2.7.2,

$$[\mathcal{X}^1_{n,x}(f)] = \mathbb{L}^{nd} \sum_{I \cap A_0 \neq \emptyset} (\mathbb{L}-1)^{|I|-1} [\widetilde{E}^\circ_I] \left(\sum_{k_i \geq 1, i \in I, \sum_I k_i N_i(f) = n} \mathbb{L}^{-\sum k_i \nu_i} \right)$$

with $\widetilde{E}^\circ_I \to E^\circ_I$ an étale cover of degree $\gcd(N_i(f))_{i \in I}$.

Taking Eu of both sides, all terms with $|I| \geq 2$ cancel out, and one gets

$$(\flat) \qquad \mathrm{Eu}(\mathcal{X}^1_{n,x}(f)) = \sum_{N_i(f)|n, i \in A_0} N_i(f)\ \mathrm{Eu}(E^0_{\{i\}}).$$

By a classical result of A'Campo [**1**], which can be deduced from the Leray-Serre spectral sequence associated to the direct image of nearby cycles, the right hand side of ($\flat$) is equal, for $n \geq 1$, to the n-th Lefschetz number

$$\Lambda^n(M_x) := \sum_j (-1)^j\ \mathrm{tr}\,(M^n_x; H^j(F_x)).$$

Hence we get :

3.3.1. Theorem (Denef-Loeser). *For $n \geq 1$,*

$$\mathrm{Eu}(\mathcal{X}^1_{n,x}(f)) = \Lambda^n(M_x).$$

It would be interesting to find a direct geometric proof not using resolution of singularities.

Recently, Nicaise and Sebag [**80**] have been able to restate and generalize that result within the framework of rigid geometry using the motivic Serre invariant introduced in [**67**].

3.4. The motivic Milnor fiber. We now work more generally over any field k of characteristic 0. Let X be a smooth variety over k of pure dimension d and consider a morphism $f : X \to \mathbb{A}^1_k$. We consider, for $n \geq 1$, the variety

$$\mathcal{X}_n(f) := \Big\{ \varphi \in \mathcal{L}_n(X) \ \Big|\ \mathrm{ord}_t f(\varphi) = n \Big\}.$$

Note that $\mathcal{L}_n(X)$ - and $\mathcal{L}(X)$ - is endowed with a canonical $\mathbf{G}_m$-action

$$a \cdot \varphi(t) := \varphi(at)$$

under which $\mathcal{X}_n(f)$ is invariant.

Furthermore f induces a morphism

$$f_n : \mathcal{X}_n(f) \to \mathbf{G}_m,$$

assigning to a point φ in $\mathcal{X}_n(f)$ the coefficient $\overline{\mathrm{ac}}(f(\varphi))$ of t^n in $f(\varphi)$. Since

$$f_n(a \cdot \varphi) = a^n f_n(\varphi),$$

the fiber

$$\mathcal{X}^1_n(f) := f_n^{-1}(1)$$

is canonically endowed with a μ_n-action, where μ_n is the group scheme

$$\mu_n := \operatorname{Spec} k[T]/(T^n - 1).$$

We consider the projective limit

$$\widehat{\mu} = \varprojlim \mu_n$$

relative to the morphisms $\mu_{nm} \to \mu_n$ given by $\zeta \mapsto \zeta^m$. Denoting by $X_0(f)$ the zero locus of f, we assign to $\mathcal{X}_n^1(f)$ a class $[\mathcal{X}_n^1(f)]$ in the equivariant relative Grothendieck group $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}$ which we now define.

We fix a k-variety S that we endow with the trivial μ_n-action and we denote by $K_0^{\mu_n}(\mathrm{Var}_S)$ the quotient of the free abelian group on isomorphism classes of equivariant μ_n-morphisms $Y \to S$ with Y a variety with good (i.e. every orbit is contained in an affine open subset) μ_n-action by the additivity relation and the following additional relation

$$[A \to Y \to S] = [A' \to Y \to S]$$

if A and A' are two affine bundles of the same rank over $Y \to S$ with affine μ_n-action lifting the same μ_n-action on Y. It is naturally endowed with a ring structure.

We then set

$$\mathcal{M}_S^{\mu_n} := K_0^{\mu_n}(\mathrm{Var}_S)[\mathbb{L}^{-1}]$$

with $\mathbb{L}$ the class of the trivial rank one affine bundle over S and

$$\mathcal{M}_S^{\widehat{\mu}} := \varprojlim \mathcal{M}_S^{\mu_n}$$

under the projective system $\mu_{nm} \to \mu_n$.

We can now consider the following series in $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}[[T]]$

$$Z_f(T) := \sum_{n \geq 1} [\mathcal{X}_n^1(f)] \, \mathbb{L}^{-nd} \, T^n,$$

which is a motivic analogue of Igusa's local zeta function.

Note that $Z_f = 0$ if f is identically 0 on X, so we may assume $X_0(f)$ is nowhere dense in X.

We shall now express $Z_f(T)$ in terms of a log-resolution $h : Y \to X$ of $X_0(f)$.

We denote by ν_{E_i} - resp. ν_{E_I} - the normal bundle to E_i -resp. E_I - in Y, by U_{E_i} the complement of the zero section in ν_{E_i} and by U_I the fiber product of the restrictions of the spaces U_{E_i}, i in I, to E_I°. There is a canonical $\mathbf{G}_m$-action on each U_{E_i} and we consider the diagonal action on U_I.

We fix I such that there exists i in I with $N_i(f) > 0$. Note that the function $f \circ h$ induces a function

$$\bigotimes_{i \in I} \nu_{E_i}^{\otimes N_i(f)}{}_{|E_I} \longrightarrow \mathbf{A}_k^1,$$

vanishing only over the zero section. We define $f_I : \nu_{E_I} \to \mathbf{A}_k^1$ as the composition of this last function with the natural morphism $\nu_{E_I} \to \otimes_{i \in I} \nu_{E_i}^{\otimes N_i(f)}{}_{|E_I}$, sending (y_i) to $\otimes y_i^{\otimes N_i(f)}$. We still denote by f_I the induced morphism from U_I to $\mathbf{G}_m$.

Since $f_I(\lambda \cdot x) = \lambda^n f_I(x)$ with $n = \sum_{i \in I} N_i(f)$, it follows that $U_I^1 := f_I^{-1}(1)$ is endowed with a μ_n-action, so we can consider its class $[U_I^1]$ in $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}$.

Using Theorem 2.7.2 again, one gets

$$Z_f(T) = \sum_{\emptyset \neq I \subset A} [U_I^1] \prod_{i \in I} \frac{1}{T^{-N_i(f)}\mathbb{L}^{\nu_i} - 1}$$

in $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}[[T]]$. In particular, the function $Z_f(T)$ is rational.

Little is known about the poles of $Z_f(T)$.

3.4.1. Monodromy Conjecture. *Let r be a rational number. If $\mathbb{L}^r$ is a pole of $Z_f(T)$, then $\exp(2i\pi r)$ is an eigenvalue of the monodromy on the stalk of the nearby cycles at some point of $X_0(f)$.*

Besides some very specific examples and low dimensional cases, the conjecture is largely open. (Note that the notion of poles is not totally obvious here since the ring could have zero divisors.)

Since

$$\lim_{T \mapsto \infty} \frac{1}{T^{-N_i(f)}\mathbb{L}^{\nu_i} - 1} = -1,$$

we get

3.4.2. Theorem (Denef-Loeser). *The limit*

$$-\mathcal{S}_f := \lim_{T \mapsto \infty} Z_f(T)$$

is well-defined in $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}$ and given a log-resolution h we have

$$(\sharp) \qquad \mathcal{S}_f = - \sum_{\emptyset \neq I \subset A} (-1)^{|I|} [U_I^1].$$

Note that it is a priori non trivial that the right-hand side of ($\sharp$) is independent from the log-resolution h. If x is a closed k-point of $X_0(f)$, by restricting to arcs with origin at x, one defines similarly $\mathcal{S}_{f,x}$ in $\mathcal{M}_{k(x)}^{\widehat{\mu}}$.

We claim that $\mathcal{S}_f$ is the motivic incarnation in $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}$ of the complex of nearby cycles $R\psi_f \mathbb{Q}_X$.

Assume for simplicity $k = \mathbb{C}$. The group $\widehat{\mu}$ is topologically generated by $\zeta := (\exp(2i\pi/n))$. We have a Betti realization morphism

$$\mathrm{Eu} : \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}} \longrightarrow K_0(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{mon}})$$

with $K_0(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{mon}})$ the Grothendieck ring of finite dimensional vector spaces with an automorphism sending the class of variety to its virtual cohomology with compact support, with automorphism the one induced by ζ.

It follows from A'Campo's formula that

$$\mathrm{Eu}(\mathcal{S}_{f,x}) = [(H^{\cdot}(F_x), M_x)]$$

in $K_0(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{mon}})$.

A similar result holds at the Hodge level. There is a Hodge realization morphism

$$H : \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}} \longrightarrow K_0(\mathrm{HS}^{\mathrm{mon}})$$

with $K_0(\mathrm{HS}^{\mathrm{mon}})$ the Grothendieck ring of Hodge structures with an automorphism of finite order.

One can prove, cf. [**25**], that $H(\mathcal{S}_{f,x})$ is equal to the class of the mixed Hodge structure - constructed by Steenbrink [**92**] and Navarro-Aznar [**75**] - on $H^{\cdot}(F_x)$

with the monodromy automorphism, whose semi-simplification is of finite order by the monodromy Theorem.

There is a natural linear map, called the Hodge spectrum,

$$\mathrm{hsp} : K_0(\mathrm{HS}^{\mathrm{mon}}) \longrightarrow \mathbb{Z}[\mathbb{Q}],$$

such that

$$\mathrm{hsp}([(H,T)]) := \sum_{\alpha \in \mathbb{Q}\cap[0,1)} t^{\alpha}\Big(\sum_{p,q\in\mathbb{N}} \dim(H_{\alpha}^{p,q})t^{p}\Big),$$

for any Hodge structure H with an automorphism of finite order T, where $H_{\alpha}^{p,q}$ is the eigenspace of $H^{p,q}$ with respect to the eigenvalue $\exp(2\pi i\alpha)$. Here we identify $\mathbb{Z}[\mathbb{Q}]$ with $\cup_{n\geq 1}\mathbb{Z}[t^{1/n}, t^{-1/n}]$.

The Steenbrink Hodge spectrum of f at x, cf. [**93**], is defined as

$$\mathrm{Sp}(f,x) := (-1)^{d-1}\mathrm{hsp}([(H^{\cdot}(F_x), M_x)]).$$

Now if one defines the motivic vanishing cycles by

$$\mathcal{S}_{f,x}^{\phi} := (-1)^{d-1}(\mathcal{S}_{f,x} - 1),$$

it follows from [**25**] that

$$\mathrm{Sp}(f,x) = (\mathrm{hsp}\circ H)(\mathcal{S}_{f,x}^{\phi}).$$

3.5. Convolution. Applications. If G is a finite group scheme and X and Y are two varieties with good G-action, we denote by $X\times^{G} Y$ the quotient of the product $X\times Y$ by the equivalence relation $(gx,y)\equiv(x,gy)$. The G-action on one factor induces a good G-action on $X\times^{G} Y$.

For $n\geq 1$, we consider the Fermat varieties

$$F_1^n := \Big\{(x,y)\in \mathbf{G}_m^2 \;\Big|\; x^n + y^n = 1\Big\}$$

and

$$F_0^n := \Big\{(x,y)\in \mathbf{G}_m^2 \;\Big|\; x^n + y^n = 0\Big\}.$$

Let X be a variety with good $\mu_n\times\mu_n$-action. We set

$$\Psi_{\Sigma}(X) := -[F_1 \times^{\mu_n\times\mu_n} X] + [F_0\times^{\mu_n\times\mu_n} X]$$

in $\mathcal{M}_{\mathbb{C}}^{\mu_n}$, the μ_n-action being the diagonal one. The construction goes through the projective limit, so we get a group morphism

$$\Psi_{\Sigma} : \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}\times\widehat{\mu}} \longrightarrow \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}}.$$

The convolution product

$$* : \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}} \times \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}} \longrightarrow \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}}$$

is then defined by

$$[X]*[Y] := \psi_{\Sigma}([X\times Y]).$$

It is commutative and associative. It was first constructed at the level of Chow motives in [**27**], and then on $\mathcal{M}_{\mathbb{C}}^{\widehat{\mu}}$ and its relative versions by Looijenga in [**68**].

Let X_1 and X_2 be smooth varieties of pure dimension d_1 and d_2 and consider functions $f_1 : X_1\to\mathbb{A}^1$ and $f_2 : X_2\to\mathbb{A}^1$. Denote by $f_1\oplus f_2$ the function on $X_1\times X_2$ sending (x_1,x_2) to $f_1(x_1)+f_2(x_2)$.

When f_1 and f_2 have isolated singular points x_1 and x_2, Thom and Sebastiani [**89**] proved that the Milnor fiber with monodromy action of $f_1\oplus f_2$ at (x_1,x_2) is the join of those of f_1 and f_2 at x_1 and x_2, respectively.

The corresponding statement for the Hodge spectrum has been proved by Steenbrink, Varchenko and Saito, cf. [**92**], [**94**], [**93**], [**88**]:

3.5.1. Theorem (Steenbrink, Varchenko, Saito).

$$\mathrm{Sp}(f_1 \oplus f_2, (x_1, x_2)) = \mathrm{Sp}(f_1, x_1) \cdot \mathrm{Sp}(f_2, x_2).$$

We can now state the motivic version of the Thom-Sebastiani Theorem [**27**], [**68**], [**29**], [**48**], [**49**]:

3.5.2. Theorem (Denef-Loeser, Looijenga).

$$\mathcal{S}^{\phi}_{f_1 \oplus f_2, (x_1, x_2)} = \mathcal{S}^{\phi}_{f_1, x_1} * \mathcal{S}^{\phi}_{f_2, x_2}.$$

Let us now move to a situation where guessing the motivic analogue of a statement involving the Hodge spectrum can be less immediate.

Consider a function f on a smooth complex variety X and a point x in the zero locus of f. Let us assume that the singular locus of f is a curve Γ, having r local components Γ_ℓ, $1 \leq \ell \leq r$, in a neighborhood of x. We denote by m_ℓ the multiplicity of Γ_ℓ at x.

Let g be a function vanishing at x whose differential at x is a generic linear form. For N large enough, the function $f + g^N$ has an isolated singularity at x. In a neighborhood of the complement Γ°_ℓ to $\{x\}$ in Γ_ℓ, we may view f as a family of isolated hypersurface singularities parametrized by Γ°_ℓ. Such a construction has been first considered by Y. Yomdin in [**57**].

The cohomology of the Milnor fiber of that isolated hypersurface singularity is naturally endowed with the action of two commuting monodromies: the monodromy of the function and the monodromy of a generator of the local fundamental group of Γ°_ℓ.

We denote by $\alpha_{\ell,j}$ the exponents - counted with multiplicity - of the Hodge spectrum of that isolated hypersurface singularity and by $\beta_{\ell,j}$ the corresponding rational numbers in $[0, 1)$ such that the complex numbers $\exp(2\pi i \beta_{\ell,j})$ are the eigenvalues of the monodromy along Γ°_ℓ.

The following statement has been conjectured by J. Steenbrink [**93**] and proved by M. Saito in [**88**]:

3.5.3. Theorem (M. Saito). *For $N \gg 0$,*

$$\mathrm{Sp}(f + g^N, x) - \mathrm{Sp}(f, x) = \sum_{\ell,j} t^{\alpha_{\ell,j} + (\beta_{\ell,j}/m_\ell N)} \frac{1 - t}{1 - t^{1/m_\ell N}}.$$

3.6. Before stating the motivic version of the Steenbrink conjecture let us explain an extension of $\mathcal{S}_f$ to a morphism

$$\mathcal{S}_f : \mathcal{M}_{X_0(f)} \longrightarrow \mathcal{M}^{\widehat{\mu}}_{X_0(f)}$$

that has been constructed by Bittner in [**11**] using weak factorization and by Guibert-Loeser-Merle in [**48**] using motivic integration. This should be viewed as the analogue of considering nearby cycles for complexes of contructible sheaves instead of just the constant sheaf.

Let us start by the construction of $\mathcal{S}_f([U])$ when U is a dense open subset of X.

Denote by F the closed subset $X \setminus U$ and by $\mathcal{I}_F$ the ideal of functions vanishing on F.

Fix $\gamma \geq 1$ a positive integer. For $n \geq 1$, we consider the constructible set

$$\mathcal{X}_n^{\gamma n}(f,U) := \Big\{\varphi \in \mathcal{L}_{\gamma n}(X) \;\Big|\; \mathrm{ord}_t f(\varphi) = n, \mathrm{ord}_t \varphi^*(\mathcal{I}_F) \leq \gamma n\Big\},$$

we set $\mathcal{X}_n^{\gamma n,1}(f,U) = \mathcal{X}_n^{\gamma n}(f,U) \cap f_n^{-1}(1)$ and define the modified zeta function as

$$Z_{f,U}^{\gamma}(T) := \sum_{n \geq 1} [\mathcal{X}_n^{\gamma n,1}(f,U)]\, \mathbb{L}^{-\gamma n d}\, T^n$$

in $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}[[T]]$.

Note that for $U = X$, $Z_{f,U}^{\gamma}(T)$ is equal to $Z_f(T)$ for every γ and that if f vanishes on X it is 0. If $X_0(f)$ is nowhere dense in X, considering a log-resolution of $(X, F \cup X_0(f))$, one proves

3.6.1. Proposition. *There exists γ_0 such that for every $\gamma > \gamma_0$ the series $Z_{f,U}^{\gamma}(T)$ is rational and admits a limit $\lim_{T \mapsto \infty} Z_{f,U}^{\gamma}(T)$ which is independent of $\gamma > \gamma_0$.*

One sets

$$\mathcal{S}_f([U]) := -\lim_{T \mapsto \infty} Z_{f,U}^{\gamma}(T)$$

for $\gamma \gg 0$.

We can then state

3.6.2. Theorem ([**48**]). *Let X be a variety with a function $f : X \to \mathbf{A}_k^1$. There exists a unique $\mathcal{M}_k$-linear group morphism*

$$\mathcal{S}_f : \mathcal{M}_X \longrightarrow \mathcal{M}_{X_0(f)}^{\widehat{\mu}} \tag{3.6.1}$$

such that, for every proper morphism $p : Z \to X$, with Z smooth, and every dense open subset U in Z,

$$\mathcal{S}_f([U \to X]) = p_!(\mathcal{S}_{f \circ p}[U]). \tag{3.6.2}$$

Here $p_!$ denotes the morphism $\mathcal{M}_{Z_0(f \circ p)}^{\widehat{\mu}} \to \mathcal{M}_{X_0(f)}^{\widehat{\mu}}$ induced by composition with p.

The construction in the previous theorem can be carried out equivariantly, cf. [**48**], leading to a morphism

$$\mathcal{S}_f : \mathcal{M}_{X_0(f)}^{\widehat{\mu}} \longrightarrow \mathcal{M}_{X_0(f)}^{\widehat{\mu} \times \widehat{\mu}}.$$

3.7. One can now state the motivic version of Steenbrink's conjecture:

3.7.1. Theorem (Guibert-Loeser-Merle [**48**]). *Let X be a smooth variety and f and g be two functions from X to $\mathbb{A}^1$. Let x be a closed point of $X_0(f) \cap X_0(g)$. For $N \gg 0$, the equality*

$$\mathcal{S}_{f,x}^{\phi} - \mathcal{S}_{f+g^N,x}^{\phi} = \Psi_{\Sigma}(i_x^*(\mathcal{S}_{g^N}(\mathcal{S}_f^{\phi})))$$

holds.

Here $\mathcal{S}_{g^N}(\mathcal{S}_f^{\phi})$ lives in $\mathcal{M}_{X_0(f) \cap X_0(g)}^{\widehat{\mu}}$ and i_x^* stands for taking the fiber over x.

The proof of Theorem 3.7.1 takes place on the arc space $\mathcal{L}(X)$. The basic idea behind it is the following trichotomy:

- If $\mathrm{ord}_t f < N \mathrm{ord}_t g$, then $\mathrm{ord}_t(f + g^N) = \mathrm{ord}_t f$ and $\overline{\mathrm{ac}}(f + g^N) = \overline{\mathrm{ac}}(f)$, so the contributions to $\mathcal{S}_{f+g^N}$ and $\mathcal{S}_f$ are the same.

- Arcs with $\mathrm{ord}_t f > N \mathrm{ord}_t g$ essentially do not contribute to $\mathcal{S}_{f+g^N}$ except for a term $\mathcal{S}_{g^N}([X_0(f)])$.
- The main contribution to $\mathcal{S}_{f+g^N}$ comes from arcs with $\mathrm{ord}_t f = N \mathrm{ord}_t g$. The key geometric fact is that on a log-resolution h, the function $f_I + g_I^N$ is smooth near arcs with $\mathrm{ord}_t(f \circ h) = N \mathrm{ord}_t(g \circ h)$.

For more results on the motivic Milnor Fiber we refer to [**11**] and [**49**].

Lecture 4: A general setting for Motivic Integration

In this final lecture, we shall present a new general setting for motivic integration developed in joint work with Raf Cluckers [**12**] [**13**] [**14**] [**16**][**17**].

4.1. A detour through semialgebraic geometry. Let us start by a quick detour through semialgebraic geometry.

Semialgebraic subsets of $\mathbb{R}^n$ are defined by a finite boolean combination of equations

$$f_i(x_1, \cdots, x_n) = 0$$

and inequalities

$$g_j(x_1, \cdots, x_n) \geq 0$$

with f_i and g_j polynomials in $\mathbb{R}[x_1, \cdots, x_n]$.

Semialgebraic subsets of $\mathbb{R}^n$ for varying n form a category $\mathrm{SA}_{\mathbb{R}}$, morphisms being functions with semialgebraic graph.

Imagine we know no topology at all, except that Eu is an additive invariant and that

$$\mathrm{Eu}(\text{point}) = 1,$$

and we want to define

$$\mathrm{Eu}(X) := \sum_i (-1)^i \operatorname{rk} H_c^i(X, \mathbb{Q})$$

when X is a semialgebraic subset of $\mathbb{R}^n$.

By cutting an open interval into the disjoint union of two open intervals and a point, one deduces

$$\mathrm{Eu}(\text{open interval}) = -1$$

and a natural way to proceed is to use cell decomposition as follows. Fix $n \geq 0$.

A 0-cell in $\mathbb{R}^n$ is a semialgebraic subset Z_A^0 of the form

$$\Big\{(x_1, \cdots, x_n) \in \mathbb{R}^n \ \Big|\ x' := (x_1, \cdots, x_{n-1}) \in A, x_n = c(x')\Big\},$$

with A semialgebraic in $\mathbb{R}^{n-1}$ and $c : \mathbb{R}^{n-1} \to \mathbb{R}$ a semialgebraic morphism.

A 1-cell in $\mathbb{R}^n$ is a semialgebraic subset Z_A^1 of the form

$$\Big\{(x_1, \cdots, x_n) \in \mathbb{R}^n \ \Big|\ x' := (x_1, \cdots, x_{n-1}) \in A, a(x') < x_n < b(x')\Big\},$$

with A semialgebraic in $\mathbb{R}^{n-1}$, a and b semialgebraic morphisms $\mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$. Here we assume $a < b$ on A, and we leave to the reader the task of extending the definition of semialgebraic morphisms to handle morphisms with values in $\mathbb{R} \cup \{\pm\infty\}$.

4.1.1. Theorem (Cell decomposition). *Every semialgebraic subset of $\mathbb{R}^n$ is a finite disjoint union of 0-cells and 1-cells.*

Now, going back to our problem of defining $\mathrm{Eu}(X)$ when X is a semialgebraic subset of $\mathbb{R}^n$, without topology, we remark that, by the above theorem it is enough, by additivity, to define Eu for cells, which we do by induction on n, setting

$$\mathrm{Eu}(Z_A^0) = \mathrm{Eu}(A)$$

and

$$\mathrm{Eu}(Z_A^1) = -\,\mathrm{Eu}(A).$$

4.1.2. THEOREM. (1) *The above definition of* Eu(X)*, for X semialgebraic, makes sense.*
(2) *If X and X' are isomorphic,*

$$\mathrm{Eu}(A) = \mathrm{Eu}(A').$$

For the proof of (2), the key point is to prove that Eu is invariant under permutation of coordinates. The above results still hold in the abstract setting of o-minimal structures (expansions of the theory of densely ordered sets without endpoints for which definable subsets of the line are finite unions of points and intervals) in which case Theorem 4.1.1 is due to Knight, Pillay and Steinhorn [**55**] and Theorem 4.1.2 to van den Dries [**35**]. The reader will find in the marvellous book [**35**] of van den Dries a definitive introduction to o-minimal structures.

For X semialgebraic, we denote by $\mathcal{C}(X)$ the algebra of $\mathbb{Z}$-valued constructible functions on X. It is generated by characteristic functions $\mathbf{1}_Z$ with Z semialgebraic subset of X.

Constructing Eu is equivalent to constructing functorial push-forward morphisms $f_! : \mathcal{C}(X) \to \mathcal{C}(Y)$ for $f : X \to Y$ a morphism in $\mathrm{SA}_\mathbb{R}$. Indeed, by a graph construction one reduces to defining the push-forward for inclusions and for projections. For inclusions, one defines push-forward to be extension by zero. For projections one reduces to projections $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and one defines $p_!$ by

$$p_!(\mathbf{1}_{Z_A^0}) := \mathbf{1}_A$$

and

$$p_!(\mathbf{1}_{Z_A^1}) := -\mathbf{1}_A.$$

The key point amounts to proving the follow Fubini type result

$$p_{i!} \circ p_{j!} = p_{j!} \circ p_{i!},$$

for $i \neq j$, with p_i the projection omitting the x_i-coordinate, which is exactly the key point in (2) of Theorem 4.1.2.

One can reinterpret constructible functions in terms of Grothendieck rings as follows. Consider the Grothendieck ring $K_0(\mathrm{SA}_\mathbb{R})$ defined as the quotient of the free abelian group on isomorphism classes of semialgebraic sets modulo the relation

$$[X] = [X'] + [X \setminus X']$$

for X' contained in X.

Since every semialgebraic set admits a semialgebraic triangulation, Eu induces an isomorphism

$$\mathrm{Eu} : K_0(\mathrm{SA}_\mathbb{R}) \to \mathbb{Z}.$$

More generally, for X in $\mathrm{SA}_\mathbb{R}$, one may consider the category SA_X whose objects are morphisms $p : Y \to X$ and the corresponding Grothendieck rings $K_0(\mathrm{SA}_X)$.

The mapping

$$[p : Y \to X] \longmapsto p_!(\mathbf{1}_Y)$$

induces an isomorphism between $K_0(\mathrm{SA}_X)$ and $\mathcal{C}(X)$.

If $f : X \to Z$ is a morphism in $\mathrm{SA}_\mathbb{R}$, the morphism $K_0(\mathrm{SA}_X) \to K_0(\mathrm{SA}_Z)$ corresponding to $f_!$ under the above isomorphism is given by

$$[p : Y \to X] \longmapsto [f \circ p : Y \to Z].$$

4.2. Setting the picture. Since, by Tarski's Theorem, the projection of a semialgebraic set is still semialgebraic, semialgebraic subsets of $\mathbb{R}^n$ are nothing but subsets that are definable by a formula in the language of rings - or in the language of ordered rings - with coefficients in $\mathbb{R}$ and n free variables.

Let us recall that a first order formula in a language $\mathcal{L}$ is a formula written with symbols in $\mathcal{L}$, logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (negation), quantifiers $\exists$, $\forall$, and variables. The language of rings with coefficients in S is given by symbols 0, $+$, $-$, 1, $\times$ and a symbol for each element of S. For the language of ordered rings one adds a symbol $<$.

Let us fix a field k of characteristic zero.
What are the definable subsets of $k((t))^n$ we would like to consider?

- Certainly subsets definable in the language of rings
- Subsets definable in the language of valued rings, e.g. by conditions like
$$\operatorname{ord}(f(x)) < \operatorname{ord}(g(x))$$
- We also encountered subsets defined by conditions like
$$\overline{\operatorname{ac}}(f(x)) = \text{something}.$$

In fact, it is more convenient to consider definable subsets of $k((t))^m \times k^n \times \mathbb{Z}^r$ in the following language $\mathcal{L}_{\mathrm{DP}}$ of Denef-Pas with three sorts of variables.

- For the valued field sort, the language of rings with coefficients in $k((t))$.
- For the residue field sort, the language of rings with coefficients in k.
- For the value group sort, the language $\{0, 1, +, -, <\}$ of ordered groups.
- two additional symbols $\overline{\operatorname{ac}}$ and ord from the valued field sort to the residue field and value group sort, respectively.

We shall ignore the minor, and easily settled, issue about $\operatorname{ord} 0$.

Let φ be a formula in the language $\mathcal{L}_{\mathrm{DP}}$ having respectively m, n, and r free variables in the various sorts. To such a formula φ we assign, for every field K containing k, the subset $h_\varphi(K)$ of $K((t))^m \times K^n \times \mathbb{Z}^r$ consisting of all points satisfying φ.

We shall call the datum of such subsets for all K definable (sub)assignments. In analogy with algebraic geometry, where the emphasis is not put anymore on equations but on the functors they define, we consider instead of formulas the corresponding subassignments (note $K \mapsto h_\varphi(K)$ is in general not a functor).

Let us make these definitions more precise. Let $F : \mathcal{C} \to \mathrm{Sets}$ be a functor from a category $\mathcal{C}$ to the category of sets. By a subassignment h of F we mean the datum, for every object C of $\mathcal{C}$, of a subset $h(C)$ of $F(C)$. Most of the standard operations of elementary set theory extend trivially to subassignments. For instance, given subassignments h and h' of the same functor, one defines subassignments $h \cup h'$, $h \cap h'$ and the relation $h \subset h'$, etc. When $h \subset h'$ we say h is a subassignment of h'.

A morphism $f : h \to h'$ between subassignments of functors F_1 and F_2 consists of the datum for every object C of a map
$$f(C) : h(C) \to h'(C).$$

The graph of f is the subassignment
$$C \mapsto \operatorname{graph}(f(C))$$
of $F_1 \times F_2$.

Let k be a field and consider the category F_k of fields containing k. We denote by $h[m,n,r]$ the functor $F_k \to \text{Sets}$ given by

$$h[m,n,r](K) = K((t))^m \times K^n \times \mathbb{Z}^r.$$

In particular, $h[0,0,0]$ assigns the one point set to every K. We sometimes write $\mathbb{Z}^r$ for $h[0,0,r]$.

To any formula φ in $\mathcal{L}_{\text{DP}}$ having respectively m, n, and r free variables in the various sorts, corresponds a subassignment h_φ of $h[m,n,r]$, which associates to K in F_k the subset $h_\varphi(K)$ of $h[m,n,r](K)$ consisting of all points satisfying φ. Such subassignments are called definable subassignments.

We denote by Def_k the category whose objects are definable subassignments of some $h[m,n,r]$, morphisms in Def_k being morphisms of subassignments $f : h \to h'$ with h and h' definable subassignments of $h[m,n,r]$ and $h[m',n',r']$ respectively such that the graph of f is a definable subassignment. Note that $h[0,0,0]$ is the final object in this category. When Y is an object in Def_k, we denote by $Y[m,n,r]$ the definable subassignment $K \mapsto Y(K) \times K((t))^m \times K^n \times \mathbb{Z}^r$.

4.3. What should be the general motivic measure? We want to define, for any definable subassignment S, a measure $\mu(S)$ in some Grothendieck ring of objects defined over k.

We consider the subcategory RDef_k of Def_k consisting of definable subassignments of some $h[0,n,0]$. One can prove they are defined by formulas involving residue field variables only, and no other sorts.

One denotes by $SK_0(\text{RDef}_k)$ the free abelian semigroup on isomorphism classes of objects of RDef_k modulo the additivity relation

$$[X] + [X'] = [X \cup X'] + [X \cap X'].$$

It is endowed with a natural semiring structure.

One defines similarly the Grothendieck ring $K_0(\text{RDef}_k)$, which is the ring associated to the semiring $SK_0(\text{RDef}_k)$.

We consider the ring

$$\mathbb{A} := \mathbb{Z}\Big[\mathbb{L}, \mathbb{L}^{-1}, \Big(\frac{1}{1-\mathbb{L}^{-i}}\Big)_{i>0}\Big].$$

Note that this ring has already been considered long ago in topology [**46**] and it also occurred naturally in [**9**].

For q a real number > 1, we denote by ϑ_q the ring morphism

$$\vartheta_q : \mathbb{A} \longrightarrow \mathbb{R}$$

sending $\mathbb{L}$ to q and we consider the semiring

$$\mathbb{A}_+ := \Big\{x \in \mathbb{A} \;\Big|\; \vartheta_q(x) \geq 0, \forall q > 1\Big\}.$$

We denote by $\mathbb{L} - 1$ the class of the subassignment $x \neq 0$ of $h[0,1,0]$ in $SK_0(\text{RDef}_k)$ and in $K_0(\text{RDef}_k)$.

We set

$$\mathcal{C}_+(\text{point}) := SK_0(\text{RDef}_k) \otimes_{\mathbb{N}[\mathbb{L}-1]} \mathbb{A}_+$$

and

$$\mathcal{C}(\text{point}) := K_0(\text{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}.$$

There are canonical morphisms

$$\mathcal{C}_+(\text{point}) \longrightarrow \mathcal{C}(\text{point})$$

and

$$\mathcal{C}(\text{point}) \longrightarrow \widehat{\mathcal{M}}_k.$$

The second one is constructed by assigning to any formula over residue field variables the class of the corresponding constructible[4] set and by developing $(1-\mathbb{L}^{-i})^{-1}$ into series.

To any bounded - that is, contained in $\mathrm{ord} x_i \geq \alpha$ for some small enough α - definable subassignment A of $h[n,0,0]$, we assign a volume $\mu(A)$ in $\mathcal{C}_+(\text{point})$ whose image in $\widehat{\mathcal{M}}_k$ coincides with the one given by the previous construction. Note that completion is not anymore needed !

Let us now sketch the construction of the motivic measure μ. We shall use coordinates $x = (x', z)$ on $h[n+1, m, r]$ with x' running over $h[n, m, r]$ and z over $h[1, 0, 0]$.

A 0-cell in $h[n+1, m, r]$ is a definable subassignment Z^0_A defined by

$$x' \in A \quad \text{and} \quad z = c(x')$$

with A a definable subassignment of $h[n, m, r]$ and c a morphism $A \to h[1, 0, 0]$.

A 1-cell in $h[n+1, m, r]$ is a definable subassignment Z^1_A defined by

$$x' \in A, \quad \overline{\mathrm{ac}}(z - c(x')) = \xi(x') \quad \text{and} \quad \mathrm{ord}(z - c(x')) = \alpha(x')$$

with A a definable subassignment of $h[n, m, r]$, c, ξ and α morphisms from A to $h[1, 0, 0]$, $h[0, 1, 0] \setminus \{0\}$ and $h[0, 0, 1]$, respectively.

Clearly, it cannot be true that every definable subassignment is a finite disjoint union of cells, but this becomes true after adding a finite number of auxiliary parameters in the residue field and value group sorts. More precisely, we have:

4.3.1. Theorem (Denef-Pas Cell Decomposition [**82**]). *Let A be a definable subassignment $h[n+1, m, r]$. After adding a finite number of auxiliary parameters in the residue field and value group sorts, A is a finite disjoint union of cells, that is, there exists an embedding*

$$\lambda : h[n+1, m, r] \longrightarrow h[n+1, m+m', r+r']$$

such that the composition of λ with the projection to $h[n+1, m, r]$ is the identity on A and such that $\lambda(A)$ is a finite disjoint union of cells.

The construction of the motivic measure $\mu(A)$ for a definable subassignment A of $h[n, m, r]$ goes roughly as follows (more details will be given in 4.6).

- Using cell decomposition one may by induction reduce to $n = 0$, at the cost of increasing m and r.
- When $n = 0$, one may consider the counting measure on the $\mathbb{Z}^r$-factor and the tautological measure on the $h[0, m, 0]$-factor, assigning to a definable subassigment of $h[0, m, 0]$ its class in $\mathcal{C}_+(\text{point})$.

That such a construction is invariant under permutations of valued field coordinates is absolutely non trivial! It is in fact a key statement underlying the motivic Fubini Theorem.

[4]By Chevalley constructibility.

4.4. Constructible functions. In fact it is much more convenient to consider from start functions instead of just subsets.

So let us define for X in Def_k the semiring $\mathcal{C}_+(X)$, of non negative constructible motivic functions and the ring $\mathcal{C}(X)$ of constructible motivic functions.

One denotes by Def_X the category of morphisms $Y \to X$ in Def_k. One considers the subcategory RDef_X of Def_X whose objects are definable subassignments of $X \times h[0, n, 0]$, for variable n, and the corresponding semiring, resp. ring, $SK_0(\mathrm{RDef}_X)$ and $K_0(\mathrm{RDef}_X)$.

We denote by $|X|$ the set of points of X and consider the subring $\mathcal{P}(X)$ of the ring of functions $|X| \to \mathbb{A}$ generated by constants in $\mathbb{A}$ and by all functions α and $\mathbb{L}^\alpha$ with $\alpha : X \to \mathbb{Z}$ definable morphisms.

We define $\mathcal{P}_+(X)$ as the semiring of functions in $\mathcal{P}(X)$ taking their values in $\mathbb{A}_+$.

We consider the subring $\mathcal{P}^0(X)$ of $\mathcal{P}(X)$, resp. the subsemiring $\mathcal{P}^0_+(X)$ of $\mathcal{P}_+(X)$, generated by functions of the form $\mathbf{1}_Y$ with Y a definable subassignment of X, and by the constant function $\mathbb{L} - 1$. We have canonical morphisms $\mathcal{P}^0(X) \to K_0(\mathrm{RDef}_X)$ and $\mathcal{P}^0_+(X) \to SK_0(\mathrm{RDef}_X)$.

We may now set

$$\mathcal{C}_+(X) = SK_0(\mathrm{RDef}_X) \otimes_{\mathcal{P}^0_+(X)} \mathcal{P}_+(X)$$

and

$$\mathcal{C}(X) = K_0(\mathrm{RDef}_X) \otimes_{\mathcal{P}^0(X)} \mathcal{P}(X).$$

There are some easy functorialities. For every morphism $f : S \to S'$, there is a natural pullback by $f^* : SK_0(\mathrm{RDef}_{S'}) \to SK_0(\mathrm{RDef}_S)$ induced by the fiber product. If $f : S \to S'$ is a morphism in $\mathrm{RDef}_{S'}$, composition with f induces a morphism $f_! : SK_0(\mathrm{RDef}_S) \to SK_0(\mathrm{RDef}_{S'})$. Similar constructions apply to K_0. If $f : S \to S'$ is a morphism in Def_k, one shows in [**14**] that the morphism f^* may naturally be extended to a morphism

$$f^* : \mathcal{C}_+(S') \longrightarrow \mathcal{C}_+(S). \tag{4.4.1}$$

If, furthermore, f is a morphism in $\mathrm{RDef}_{S'}$, one shows that the morphism $f_!$ may naturally be extended to

$$f_! : \mathcal{C}_+(S) \longrightarrow \mathcal{C}_+(S'). \tag{4.4.2}$$

Similar functorialities exist for $\mathcal{C}$.

4.5. Taking care of dimensions. In fact, we shall need to consider not only functions as we just defined, but functions defined almost everywhere in a given dimension, that we call Functions. (Note the capital in Functions.)

We start by defining a good notion of dimension for objects of Def_k. Heuristically, that dimension corresponds to counting the dimension only in the valued field variables, without taking in account the remaining variables. More precisely, to any algebraic subvariety Z of $\mathbf{A}^m_{k((t))}$ we assign the definable subassignment h_Z of $h[m, 0, 0]$ given by $h_Z(K) = Z(K((t)))$. The Zariski closure of a subassignment S of $h[m, 0, 0]$ is the intersection W of all algebraic subvarieties Z of $\mathbf{A}^m_{k((t))}$ such that $S \subset h_Z$. We define the dimension of S as $\dim S := \dim W$. In the general case, when S is a subassignment of $h[m, n, r]$, we define $\dim S$ as the dimension of the image of S under the projection $h[m, n, r] \to h[m, 0, 0]$. One can prove, using

Theorem 4.3.1 and results of van den Dries [**34**], the following result, which is by no means obvious: that isomorphic objects of Def_k have the same dimension.

For every non negative integer d, we denote by $\mathcal{C}_+^{\leq d}(S)$ the ideal of $\mathcal{C}_+(S)$ generated by functions $\mathbf{1}_Z$ with Z definable subassignments of S with $\dim Z \leq d$. We set $C_+(S) = \oplus_d C_+^d(S)$ with $C_+^d(S) := \mathcal{C}_+^{\leq d}(S)/\mathcal{C}_+^{\leq d-1}(S)$. It is a graded abelian semigroup, and also a $\mathcal{C}_+(S)$-semimodule. Elements of $C_+(S)$ are called positive constructible Functions on S. If φ is a function lying in $\mathcal{C}_+^{\leq d}(S)$ but not in $\mathcal{C}_+^{\leq d-1}(S)$, we denote by $[\varphi]$ its image in $C_+^d(S)$. One defines similarly $C(S)$ from $\mathcal{C}(S)$.

One of the reasons why we consider functions which are defined almost everywhere originates in the differentiation of functions with respect to the valued field variables: one may show that a definable function $c : S \subset h[m,n,r] \to h[1,0,0]$ is differentiable (in fact even analytic) outside a definable subassignment of S of dimension $< \dim S$. In particular, if $f : S \to S'$ is an isomorphism in Def_k, one may define a function $\mathrm{ordjac} f$, the order of the jacobian of f, which is defined almost everywhere and is equal almost everywhere to a definable function, so we may define $\mathbb{L}^{-\mathrm{ordjac} f}$ in $C_+^d(S)$ when S is of dimension d.

4.6. Construction of the general motivic measure. Let k be a field of characteristic zero. Given S in Def_k, we define S-integrable Functions and construct pushforward morphisms for these:

4.6.1. THEOREM (Cluckers-Loeser [**14**]). *Let k be a field of characteristic zero and let S be in Def_k. There exists a unique functor $Z \mapsto \mathrm{I}_S C_+(Z)$ from Def_S to the category of abelian semigroups, the functor of S-integrable Functions, assigning to every morphism $f : Z \to Y$ in Def_S a morphism $f_! : \mathrm{I}_S C_+(Z) \to \mathrm{I}_S C_+(Y)$ such that for every Z in Def_S, $\mathrm{I}_S C_+(Z)$ is a graded subsemigroup of $C_+(Z)$ and $\mathrm{I}_S C_+(S) = C_+(S)$, satisfying the following list of axioms (A1)-(A8).*

(A1a) **(Naturality)**
If $S \to S'$ is a morphism in Def_k and Z is an object in Def_S, then any S'-integrable Function φ in $C_+(Z)$ is S-integrable and $f_!(\varphi)$ is the same, considered in $\mathrm{I}_{S'}$ or in I_S.

(A1b) **(Fubini)**
A positive Function φ on Z is S-integrable if and only if it is Y-integrable and $f_!(\varphi)$ is S-integrable.

(A2) **(Disjoint union)**
If Z is the disjoint union of two definable subassignments Z_1 and Z_2, then the isomorphism $C_+(Z) \simeq C_+(Z_1) \oplus C_+(Z_2)$ induces an isomorphism $\mathrm{I}_S C_+(Z) \simeq \mathrm{I}_S C_+(Z_1) \oplus \mathrm{I}_S C_+(Z_2)$, under which $f_! = f_{|Z_1 !} \oplus f_{|Z_2 !}$.

(A3) **(Projection formula)**
For every α in $\mathcal{C}_+(Y)$ and every β in $\mathrm{I}_S C_+(Z)$, $\alpha f_!(\beta)$ is S-integrable if and only if $f^*(\alpha)\beta$ is, and then $f_!(f^*(\alpha)\beta) = \alpha f_!(\beta)$.

(A4) **(Inclusions)**
If $i : Z \hookrightarrow Z'$ is the inclusion of definable subassignments of the same object of Def_S, $i_!$ is induced by extension by zero outside Z and sends injectively $\mathrm{I}_S C_+(Z)$ to $\mathrm{I}_S C_+(Z')$.

(A5) **(Integration along residue field variables)**
Let Y be an object of Def_S and denote by π the projection $Y[0,n,0] \to Y$. A Function $[\varphi]$ in $C_+(Y[0,n,0])$ is S-integrable if and only if, with notations of 4.4.2, $[\pi_!(\varphi)]$ is S-integrable and then $\pi_!([\varphi]) = [\pi_!(\varphi)]$.

Basically this axiom means that integrating with respect to variables in the residue field just amounts to taking the pushforward induced by composition at the level of Grothendieck semirings.

(A6) **(Integration along $\mathbb{Z}$-variables)** Basically, integration along $\mathbb{Z}$-variables corresponds to summing over the integers, but to state precisely (A6), we need to perform some preliminary constructions.

Let us consider a function φ in $\mathcal{P}(S[0,0,r])$, hence φ is a function $|S| \times \mathbb{Z}^r \to \mathbb{A}$. We shall say φ is S-integrable if for every $q > 1$ and every x in $|S|$, the series $\sum_{i \in \mathbb{Z}^r} \vartheta_q(\varphi(x,i))$ is summable. One proves that if φ is S-integrable there exists a unique function $\mu_S(\varphi)$ in $\mathcal{P}(S)$ such that $\vartheta_q(\mu_S(\varphi)(x))$ is equal to the sum of the previous series for all $q > 1$ and all x in $|S|$. We denote by $\mathrm{I}_S\mathcal{P}_+(S[0,0,r])$ the set of S-integrable functions in $\mathcal{P}_+(S[0,0,r])$ and we set

$$\mathrm{I}_S\mathcal{C}_+(S[0,0,r]) = \mathcal{C}_+(S) \otimes_{\mathcal{P}_+(S)} \mathrm{I}_S\mathcal{P}_+(S[0,0,r]). \tag{4.6.1}$$

Hence $\mathrm{I}_S\mathcal{C}_+(S[0,0,r])$ is a sub-$\mathcal{C}_+(S)$-semimodule of $\mathcal{C}_+(S[0,0,r])$ and μ_S may be extended by tensoring to

$$\mu_S : \mathrm{I}_S\mathcal{C}_+(S[0,0,r]) \to \mathcal{C}_+(S). \tag{4.6.2}$$

Now we can state (A6):
Let Y be an object of Def_S and denote by π the projection $Y[0,0,r] \to Y$. A Function $[\varphi]$ in $C_+(Y[0,0,r])$ is S-integrable if and only if there exists φ' in $\mathcal{C}_+(Y[0,0,r])$ with $[\varphi'] = [\varphi]$ which is Y-integrable in the previous sense and such that $[\mu_Y(\varphi')]$ is S-integrable. We then have $\pi_!([\varphi]) = [\mu_Y(\varphi')]$.

(A7) **(Volume of balls)** It is natural to require (by analogy with the p-adic case) that the volume of a ball $\{z \in h[1,0,0] | \mathrm{ord}(z-c) = \alpha, \overline{\mathrm{ac}}(z-c) = \xi\}$, with α in $\mathbb{Z}$, c in $k((t))$ and ξ non zero in k, should be $\mathbb{L}^{-\alpha-1}$. (A7) is a relative version of that statement:
Let Y be an object in Def_S and let Z be the definable subassignment of $Y[1,0,0]$ defined by $\mathrm{ord}(z - c(y)) = \alpha(y)$ and $\overline{\mathrm{ac}}(z - c(y)) = \xi(y)$, with z the coordinate on the $\mathbf{A}^1_{k((t))}$-factor and α, ξ, c definable functions on Y with values respectively in $\mathbb{Z}$, $h[0,1,0] \setminus \{0\}$, and $h[1,0,0]$. We denote by $f : Z \to Y$ the morphism induced by projection. Then $[\mathbf{1}_Z]$ is S-integrable if and only if $\mathbb{L}^{-\alpha-1}[\mathbf{1}_Y]$ is, and then $f_!([\mathbf{1}_Z]) = \mathbb{L}^{-\alpha-1}[\mathbf{1}_Y]$.

(A8) **(Graphs)** This last axiom expresses the pushforward for graph projections. It relates volume and differentials and is a special case of the change of variables Theorem 4.7.1.
Let Y be in Def_S and let Z be the definable subassignment of $Y[1,0,0]$ defined by $z - c(y) = 0$ with z the coordinate on the $\mathbf{A}^1_{k((t))}$-factor and c a morphism $Y \to h[1,0,0]$. We denote by $f : Z \to Y$ the morphism induced by projection. Then $[\mathbf{1}_Z]$ is S-integrable if and only if $\mathbb{L}^{(\mathrm{ordjac} f)\circ f^{-1}}$ is, and then $f_!([\mathbf{1}_Z]) = \mathbb{L}^{(\mathrm{ordjac} f)\circ f^{-1}}$.

Once Theorem 4.6.1 is proved, one may proceed as follows to extend the constructions from C_+ to C . One defines $\mathrm{I}_S C(Z)$ as the subgroup of $C(Z)$ generated by the image of $\mathrm{I}_S C_+(Z)$. One shows that if $f : Z \to Y$ is a morphism in Def_S, the morphism $f_! : \mathrm{I}_S C_+(Z) \to \mathrm{I}_S C_+(Y)$ has a natural extension $f_! : \mathrm{I}_S C(Z) \to \mathrm{I}_S C(Y)$.

The relation of Theorem 4.6.1 with motivic integration is the following. When S is equal to $h[0,0,0]$, the final object of Def_k, one writes $\mathrm{I}C_+(Z)$ for $\mathrm{I}_S C_+(Z)$ and we shall say integrable for S-integrable, and similarly for C. Note that $\mathrm{I}C_+(h[0,0,0]) = C_+(h[0,0,0]) = SK_0(\mathrm{RDef}_k) \otimes_{\mathbb{N}[\mathbb{L}-1]} \mathbb{A}_+$ and that $\mathrm{I}C(h[0,0,0]) = K_0(\mathrm{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}$. For φ in $\mathrm{I}C_+(Z)$, or in $\mathrm{I}C(Z)$, one defines the motivic integral $\mu(\varphi)$ by $\mu(\varphi) = f_!(\varphi)$ with f the morphism $Z \to h[0,0,0]$. Working in the more general framework of Theorem 4.6.1 to construct μ appears to be very convenient for inductions occurring in the proofs. Also, it is not clear how to characterize μ alone by existence and unicity properties. Note also, that one reason for the statement of Theorem 4.6.1 to look somewhat cumbersone, is that we have to define at once the notion of integrability and the value of the integral.

The proof of Theorem 4.6.1 is quite long and involved. In a nutshell, the basic idea is the following. Integration along residue field variables is controlled by (A5) and integration along $\mathbb{Z}$-variables by (A6). Integration along valued field variables is constructed one variable after the other. To integrate with respect to one valued field variable, one may, using (a variant of) the cell decomposition Theorem 4.3.1 (at the cost of introducing additional new residue field and $\mathbb{Z}$-variables), reduce to the case of cells which is covered by (A7) and (A8). An important step is to show that this is independent of the choice of a cell decomposition. When one integrates with respect to more than one valued field variable (one after the other) it is crucial to show that it is independent of the order of the variables, for which we use a notion of bicells.

4.7. Motivic calculus. Let X be in Def_k of dimension d. Let φ be a function in $\mathcal{C}_+(X)$, or in $\mathcal{C}(X)$. We shall say φ is integrable if its class $[\varphi]_d$ in $C_+^d(X)$, resp. in $C^d(X)$, is integrable, and we shall set

$$\mu(\varphi) = \int_X \varphi \, d\mu = \mu([\varphi]_d).$$

In this setting one can state the following general form of the Change of Variables Theorem.

4.7.1. Theorem (Cluckers-Loeser [**14**]). *Let $f : Y \to X$ be an isomorphism in Def_k. For any integrable function φ in $\mathcal{C}_+(X)$ or $\mathcal{C}(X)$,*

$$\int_X \varphi d\mu = \int_Y \mathbb{L}^{-\mathrm{ord\,jac}(f)} f^*(\varphi) d\mu.$$

Once the Change of Variables Theorem 4.7.1 is available, one may develop the integration on global (non affine) objects endowed with a differental form of top degree (similarly as in the p-adic case), cf. [**14**].

Also, the construction we outlined of the motivic measure carries over almost literally to a relative setting: one can develop a relative theory of motivic integration: integrals depending on parameters of functions in $\mathcal{C}_+$ or $\mathcal{C}$ still belong to $\mathcal{C}_+$ or $\mathcal{C}$ as functions of parameters.

More specifically, if $f : X \to \Lambda$ is a morphism and φ is a function in $\mathcal{C}_+(X)$ or $\mathcal{C}(X)$ that is relatively integrable (a notion defined in [**14**]), one constructs in [**14**] a function

$$\mu_\Lambda(\varphi)$$

in $\mathcal{C}_+(\Lambda)$, resp. $\mathcal{C}(\Lambda)$, whose restriction to every fiber of f coincides with the integral of φ restricted to that fiber.

4.8. Adding exponentials. It is also possible to enlarge $\mathcal{C}(X)$ to a ring $\mathcal{C}(X)^{\exp}$ containing also motivic analogues of exponential functions and to construct a natural extension of the previous theory to $\mathcal{C}^{\exp}$.

This is performed as follows in [**16**] [**17**]. Let X be in Def_k. We consider the category $\mathrm{RDef}_X^{\exp}$ whose objects are triples $(Y \to X, \xi, g)$ with Y in RDef_X and $\xi : Y \to h[0,1,0]$ and $g : Y \to h[1,0,0]$ morphisms in Def_k. In such a triple, ξ should be understood as a formal substitute for a function $\exp(\xi)$ with exp a formal non trivial additive character on the residue field and g as a formal substitute for a function $E(g)$ with E a formal non trivial additive character on the valued field.

A morphism $(Y' \to X, \xi', g') \to (Y \to X, \xi, g)$ in $\mathrm{RDef}_X^{\exp}$ is a morphism $h : Y' \to Y$ in Def_X such that $\xi' = \xi \circ h$ and $g' = g \circ h$. The functor sending Y in RDef_X to $(Y, 0, 0)$, with 0 denoting the constant morphism with value 0 in $h[0,1,0]$, resp. $h[1,0,0]$ being fully faithful, we may consider RDef_X as a full subcategory of $\mathrm{RDef}_X^{\exp}$.

To the category $\mathrm{RDef}_Z^{\exp}$ one assigns a Grothendieck ring $K_0(\mathrm{RDef}_Z^{\exp})$ defined as follows. As an abelian group it is the quotient of the free abelian group over symbols $[Y \to Z, \xi, g]$ with $(Y \to Z, \xi, g)$ in $\mathrm{RDef}_Z^{\exp}$ by the following four relations (R1)-(R4). The first two are natural analogues of the ones occuring in the definition of $K_0(\mathrm{RDef}_Z)$ while the last two are specific to the exponential setting.

Isomorphism: For $(Y \to Z, \xi, g)$ isomorphic to $(Y' \to Z, \xi', g')$,

$$[Y \to Z, \xi, g] = [Y' \to Z, \xi', g']. \tag{R1}$$

Additivity: For Y and Y' definable subassignments of some X in RDef_Z and ξ, g defined on $Y \cup Y'$,

$$\begin{aligned}[(Y \cup Y') \to Z, \xi, g] &+ [(Y \cap Y') \to Z, \xi_{|Y \cap Y'}, g_{|Y \cap Y'}] \\ &= [Y \to Z, \xi_{|Y}, g_{|Y}] + [Y' \to Z, \xi_{|Y'}, g_{|Y'}].\end{aligned} \tag{R2}$$

Compatibility with reduction: For $h : Y \to h[1,0,0]$ a definable morphism with $\mathrm{ord}(h(y)) \geq 0$ for all y in Y and $\bar{h}$ the reduction of h modulo (t),

$$[Y \to Z, \xi, g + h] = [Y \to Z, \xi + \bar{h}, g]. \tag{R3}$$

Sum over the line: When $p : Y[0,1,0] \to h[0,1,0]$ is the projection and when the morphisms $Y[0,1,0] \to Z$, g, and ξ all factorize through the projection $Y[0,1,0] \to Y$,

$$[Y[0,1,0] \to Z, \xi + p, g] = 0. \tag{R4}$$

Relation (R3) expresses compatibility under reduction modulo the uniformizing parameter between the exponential over the valued field and over the residue field. It can be considered as an analogue of the relation $\exp((2\pi i/p)x) = \exp((2\pi i/p)\bar{x})$ for x in $\mathbf{Z}_p$ reducing mod p to $\bar{x}$ in $\mathbf{Z}/p\mathbf{Z}$. Relation (R4) expresses abstractly the familiar fact that the sum of the values of a non trivial character over all points in a finite field is zero. Fiber product endows $K_0(\mathrm{RDef}_X^{\exp})$ with a ring structure.

Finally, one defines the ring $\mathcal{C}(X)^{\exp}$ of exponential constructible functions as $\mathcal{C}(X)^{\exp} := \mathcal{C}(X) \otimes_{K_0(\mathrm{RDef}_X)} K_0(\mathrm{RDef}_X^{\exp})$. One defines similarly $C(X)^{\exp}$ and $\mathrm{I}_S C(X)^{\exp}$.

In [**16**] [**17**], given S in Def_k, we contruct for a morphism $f : X \to Y$ in Def_S a push-forward $f_! : \mathrm{I}_S C(X)^{\mathrm{exp}} \to \mathrm{I}_S C(Y)^{\mathrm{exp}}$ extending $f_! : \mathrm{I}_S C(X) \to \mathrm{I}_S C(Y)$ and characterized by certain natural axioms. In particular, the construction of the measure μ and its relative version μ_Λ extend to the exponential setting.

At this stage, one is able to develop a motivic calculus as flexible and easy to use as the usual calculus over reals or the p-adics.

For instance, we are able to construct in loc. cit. natural motivic versions of Fourier transformation, convolution, Schwartz-Bruhat spaces, for which we prove various forms of Fourier inversion.

4.9. The transfer principle. In the first talk, we showed how p-adic integration has been used as a substitute for motivic integration before it was invented. It is now time to loop the loop by explaining how motivic integrals specialize to p-adic integrals and may be used to obtain a general transfer principle allowing to transfer relations between integrals from $\mathbb{Q}_p$ to $\mathbb{F}_p((t))$ and vice-versa.

We shall assume from now on that k is a number field with ring of integers $\mathcal{O}$. We denote by $\mathcal{A}_\mathcal{O}$ the set of p-adic completions of all finite extensions of k and by $\mathcal{B}_\mathcal{O}$ the set of all local fields of characteristic > 0 which are $\mathcal{O}$-algebras.

Notation: For K in $\mathcal{C}_\mathcal{O} := \mathcal{A}_\mathcal{O} \cup \mathcal{B}_\mathcal{O}$, we denote by

- R_K the valuation ring
- M_K the maximal ideal
- k_K the residue field
- $q(K)$ the cardinal of k_K
- ϖ_K a uniformizing parameter of R_K.

There exists a unique morphism $\overline{\mathrm{ac}} : K^\times \to k_K^\times$ extending $R_K^\times \to k_K^\times$ and sending ϖ_K to 1. We set $\overline{\mathrm{ac}}(0) = 0$. We denote by $\mathcal{D}_K$ the set of additive characters $\psi : K \to \mathbb{C}^\times$ such that $\psi(x) = \exp((2\pi i/p)\mathrm{Tr}_{k_K}(\bar{x}))$ for $x \in R_K$, with p the characteristic of k_K, Tr_{k_K} the trace of k_K relatively to its prime field and $\bar{x}$ the class of x in k_K.

For $N > 0$, we denote by $\mathcal{A}_{\mathcal{O},N}$ the set of fields K in $\mathcal{A}_\mathcal{O}$ such that k_K has characteristic $> N$, and similarly for $\mathcal{B}_{\mathcal{O},N}$ and $\mathcal{C}_{\mathcal{O},N}$.

To be able to interpret our formulas to fields in $\mathcal{C}_\mathcal{O}$, we restrict the language $\mathcal{L}_{\mathrm{DP}}$ to the sub-language $\mathcal{L}_\mathcal{O}$ for which coefficients in the valued field sort are assumed to belong to the subring $\mathcal{O}[[t]]$ of $k((t))$.

We denote by $\mathrm{Def}(\mathcal{L}_\mathcal{O})$ the sub-category of Def_k of objects definable in $\mathcal{L}_\mathcal{O}$, and similarly for functions, etc. For instance, for X in $\mathrm{Def}(\mathcal{L}_\mathcal{O})$, we denote by $\mathcal{C}(S, \mathcal{L}_\mathcal{O})$ the ring of contructible functions on X definable in $\mathcal{L}_\mathcal{O}$ and by $\mathcal{C}(S, \mathcal{L}_\mathcal{O})^{\mathrm{exp}}$ its exponential version.

We consider K as a $\mathcal{O}[[t]]$-algebra via

$$\lambda_{\mathcal{O},K} : \sum_{i \in \mathbb{N}} a_i t^i \longmapsto \sum_{i \in \mathbb{N}} a_i \varpi_K^i.$$

Hence, if we interpret a in $\mathcal{O}[[t]]$ by $\lambda_{\mathcal{O},K}(a)$, every $\mathcal{L}_\mathcal{O}$-formula ψ defines for K in $\mathcal{C}_\mathcal{O}$ a subset φ_K of some $K^m \times k_K^n \times \mathbb{Z}^r$. One proves that if two $\mathcal{L}_\mathcal{O}$-formulas ψ and ψ' define the same subassignment X of $h[m, n, r]$, then $\psi_K = \psi'_K$ for K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$. This allows us to denote by X_K the subset defined by φ_K, for K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$. Similarly, every $\mathcal{L}_\mathcal{O}$-definable morphism $f : X \to Y$ specializes to $f_K : X_K \to Y_K$ for K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$.

We now explain how φ in $\mathcal{C}(X, \mathcal{L}_{\mathcal{O}})$ can be specialized to $\varphi_K : X_K \to \mathbb{Q}$ for K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$. Let us consider φ in $K_0(\mathrm{RDef}_X(\mathcal{L}_{\mathcal{O}}))$ of the form $[\pi : W \to X]$ with W in $\mathrm{RDef}_X(\mathcal{L}_{\mathcal{O}})$. For K in $\mathcal{C}_{\mathcal{O},N}$ with $N \gg 0$, we have $\pi_K : W_K \to X_K$, so we may define $\varphi_K : X_K \to \mathbb{Q}$ by

$$x \longmapsto \mathrm{card}\left(\pi_K^{-1}(x)\right).$$

For φ in $\mathcal{P}(X)$, we specialize $\mathbb{L}$ into q_K and $\alpha : X \to \mathbb{Z}$ into $\alpha_K : X_K \to \mathbb{Z}$. By tensor product we get $\varphi \mapsto \varphi_K$ for φ in $\mathcal{C}(X, \mathcal{L}_{\mathcal{O}})$. Note that, under that construction, functions in $\mathcal{C}_+(X, \mathcal{L}_{\mathcal{O}})$ specialize into non negative functions. This construction extends to the exponential case as follows. Let φ be in $K_0(\mathrm{RDef}_X(\mathcal{L}_{\mathcal{O}}))^{\mathrm{exp}}$ of the form $[W, g, \xi]$. For ψ_K in $\mathcal{D}_K$, one specializes φ in $\varphi_{K,\psi_K} : X_K \to \mathbb{C}$ given by $x \mapsto \sum_{y \in \pi_K^{-1}(x)} \psi_K(g_K(y)) \exp((2\pi i/p)\mathrm{Tr}_{k_K}(\xi_K(y)))$ for K in $\mathcal{C}_{\mathcal{O},N}$ with $N \gg 0$. One defines the specialization $\varphi \mapsto \varphi_{K,\psi_K}$ for φ in $\mathcal{C}(X, \mathcal{L}_{\mathcal{O}})^{\mathrm{exp}}$ by tensor product.

Let K be in $\mathcal{C}_{\mathcal{O}}$ and A be a subset of $K^m \times k_K^n \times \mathbb{Z}^r$. We consider the Zariski closure $\bar{A}$ of the projection of A into $\mathbb{A}_K^m$. One defines a measure μ on A by restriction of the product of the canonical measure on $\bar{A}(K)$ with the counting measure on $k_K^n \times \mathbb{Z}^r$.

Fix a morphism $f : X \to \Lambda$ in $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$ and consider φ in $\mathcal{C}(X, \mathcal{L}_{\mathcal{O}})^{\mathrm{exp}}$. One can show that if φ is relatively integrable, then, for $N \gg 0$ and every K in $\mathcal{C}_{\mathcal{O},N}$, for every λ in Λ_K and every ψ_K in $\mathcal{D}_K$, the restriction $\varphi_{K,\psi_K,\lambda}$ of φ_{K,ψ_K} to $f_K^{-1}(\lambda)$ is integrable.

We denote by $\mu_{\Lambda_K}(\varphi_{K,\psi_K})$ the function on Λ_K defined by

$$\lambda \longmapsto \mu(\varphi_{K,,\psi_K,\lambda}).$$

The following theorem says that motivic integrals specialize to the corresponding integrals over local fields of high enough residue field characteristic.

4.9.1. Theorem (Specialization, Cluckers-Loeser [**16**] [**17**]). *Let $f : S \to \Lambda$ be a morphism in* $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$. *Let φ be in $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})^{\mathrm{exp}}$ relatively integrable with respect to f. For $N \gg 0$, for every K in $\mathcal{C}_{\mathcal{O},N}$ and every ψ_K in $\mathcal{D}_K$, we have*

$$(\mu_\Lambda(\varphi))_{K,\psi_K} = \mu_{\Lambda_K}(\varphi_{K,\psi_K}).$$

We are now ready to state the following abstract transfer principle:

4.9.2. Theorem (Abstract transfer principle, Cluckers-Loeser [**16**] [**17**]). *Let φ be in $\mathcal{C}(\Lambda, \mathcal{L}_{\mathcal{O}})^{\mathrm{exp}}$. There exists N such that for every K_1, K_2 in $\mathcal{C}_{\mathcal{O},N}$ with $k_{K_1} \simeq k_{K_2}$,*

$$\varphi_{K_1,\psi_{K_1}} = 0 \quad \textit{for all} \quad \psi_{K_1} \in \mathcal{D}_{K_1} \quad \textit{if and only if} \quad \varphi_{K_2,\psi_{K_2}} = 0 \quad \textit{for all} \quad \psi_{K_2} \in \mathcal{D}_{K_2}.$$

Putting together the two previous theorems, one immediatly gets:

4.9.3. Theorem (Transfer principle for integrals with parameters, Cluckers-Loeser [**16**] [**17**]). *Let $S \to \Lambda$ and $S' \to \Lambda$ be morphisms in* $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$. *Let φ and φ' be relatively integrable functions in $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})^{\mathrm{exp}}$ and $\mathcal{C}(S', \mathcal{L}_{\mathcal{O}})^{\mathrm{exp}}$, respectively. There exists N such that for every K_1, K_2 in $\mathcal{C}_{\mathcal{O},N}$ with $k_{K_1} \simeq k_{K_2}$,*

$$\mu_{\Lambda_{K_1}}(\varphi_{K_1,\psi_{K_1}}) = \mu_{\Lambda_{K_1}}(\varphi'_{K_1,\psi_{K_1}}) \quad \textit{for all} \quad \psi_{K_1} \in \mathcal{D}_{K_1}$$

if and only if

$$\mu_{\Lambda_{K_2}}(\varphi_{K_2,\psi_{K_2}}) = \mu_{\Lambda_{K_2}}(\varphi'_{K_2,\psi_{K_2}}). \quad \textit{for all} \quad \psi_{K_2} \in \mathcal{D}_{K_2}.$$

In the special case where $\Lambda = h[0,0,0]$ and φ and φ' are in $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$ and $\mathcal{C}(S', \mathcal{L}_{\mathcal{O}})$, respectively, this follows from previous results of Denef-Loeser [**28**].

Note that when $S = S' = \Lambda = h[0,0,0]$, one recovers the celebrated

4.9.4. Theorem (Ax-Kochen-Eršov [**5**] [**40**]). *Let φ be a first order sentence (= formula with no free variables) in the language of rings. For almost all prime number p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.*

As shown in [**18**], Theorem 4.9.3 applies in particular to the integrals occuring in the Fundamental Lemma, which is of special interest in view of the recent advances by Laumon and Ngô [**62**] and Ngô [**77**].

For other recent work relating motivic integration with the Langlands program, see [**44**][**45**] [**19**] [**53**].

4.10. Let us end by giving a specific example where Theorem 4.9.3 applies. It is a relative version of the Fundamental Lemma.

Let E/F be a degree two unramified extension of non archimedean local fields of residue characteristic $\neq 2$ and let ψ be a non trivial additive character of F of conductor $\mathcal{O}_F$.

Let N_n be the group of upper triangular matrices with 1's on the diagonal and consider the character $\theta : N_n(F) \to \mathbb{C}^\times$ given by

$$\theta(u) := \psi(\sum_i u_{i,i+1}).$$

For a the diagonal matrix $(a_1, \cdots, a_n)$ with a_i in $F^\times$, we consider the integral

$$I(a) := \int_{N_n(F)\times N_n(F)} \mathbf{1}_{M_n(\mathcal{O}_F)}({}^t u_1 a u_2)\, \theta(u_1 u_2)\, du_1 du_2.$$

Here du denote the Haar measure on $N_n(F)$ with the normalisation $\int_{N_n(\mathcal{O}_F)} du = 1$.

Similarly, one defines

$$J(a) := \int_{N_n(E)} \mathbf{1}_{M_n(\mathcal{O}_E)\cap H_n}({}^t \bar{u} a u)\, \theta(u\bar{u})\, du,$$

with H_n the set of Hermitian matrices.

The Jacquet-Ye Conjecture [**59**], proved by Ngô [**76**] over function fields and by Jacquet [**60**] in general, asserts that

$$(\lozenge) \qquad I(a) = \gamma(a)\, J(a)$$

with

$$\gamma(a) := \prod_{1\leq i\leq n-1} \eta(a_1 \cdots a_i),$$

and η the multiplicative character of order 2 on $F^\times$.

Theorem 4.9.3 for exponential integrals applies to integrals of Jacquet-Ye type, yielding a general principle why $(\lozenge)$ holds over functions fields of large characteristic if and only if it holds over unequal characteristic local fields of large residual characteristic.

It is natural to expect that relations between non archimedean integrals holding over all local fields of large residual characteristic already hold at the motivic level, as equalities between constructible motivic functions.

What will be next?

While it is certainly impossible to anticipate what will be the main topics of the next 2015 Summer Institute, we should mention two recent new directions of research that look particularly promising for future developments, and that were not touched upon in these lectures. The first one is the work of Drinfeld [**37**] that provides the first steps for replacing virtual objects living in Grothendieck rings by complexes of ℓ-adic or motivic sheaves. The second one is the recent work of Hrushovski and Kazhdan [**54**] that sets the basis for integration in general valued fields, with possibly non discrete valuation.

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