Lectures on Motivic Integration

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Outline I

Outline II

- Constructing the general motivic measure
- Motivic calculus
- The transfer principle

Let X be a smooth complex algebraic variety and F be a closed nowhere dense subscheme. By a log-resolution $h : Y \to X$ of (X, F) we mean a proper morphism $h : Y \to X$ with Y smooth such that the restriction of $h : Y \setminus h^{-1}(F_{\mathrm{red}}) \to X \setminus F_{\mathrm{red}}$ is an isomorphism, and $h^{-1}(F_{\mathrm{red}})$ is a divisor with simple normal crossings.

We denote by E_i, i in A, the set of irreducible components of the divisor $h^{-1}(F_{\mathrm{red}})$. By definition the E_i's are smooth and intersect transversally.

If $h : Y \to X$ is log-resolution of (X, F) for some F, we call h a DNC modification.

For $I \subset A$, we set

$$E_I := \bigcap_{i \in I} E_i$$

and

$$E_I^\circ := E_I \setminus \bigcup_{j \notin I} E_j.$$

If $\mathcal{I}$ is an ideal sheaf defining the closed subscheme F of X and $h^{-1}(\mathcal{I})\mathcal{O}_Y$ is locally principal, we define $N_i(\mathcal{I})$, the multiplicity of $\mathcal{I}$ along E_i, by

$$h^*(\mathcal{I})\mathcal{O}_Y \simeq \mathcal{O}_Y\Big(-\sum_{i\in A} N_i(\mathcal{I})E_i\Big).$$

If $\mathcal{I}$ is principal, generated by a function g, we write $N_i(g)$ for $N_i(\mathcal{I})$.

Similarly, we define integers ν_i (log discrepancies) by the equality of divisors

$$K_Y = h^*K_X + \sum_{i\in A}(\nu_i - 1)E_i.$$

Let X be a complex algebraic variety (not necessarily smooth). If X is proper, $X(\mathbb{C})$ is compact and we may define its Euler Characteristic as

$$\mathrm{Eu}(X) := \sum_i (-1)^i \operatorname{rk} H^i(X(\mathbb{C}), \mathbb{C}).$$

There is a unique way to extend Eu additively to the category of all complex algebraic varieties, ie by requiring that

$$\mathrm{Eu}(X) = \mathrm{Eu}(X') + \mathrm{Eu}(X \setminus X')$$

for X' closed in X. Indeed, just set

$$\mathrm{Eu}(X) := \sum_i (-1)^i \operatorname{rk} H^i_c(X(\mathbb{C}), \mathbb{C}),$$

where $H^i_c(\_, \mathbb{C})$ stands for cohomology with compact supports.

We can now state the following result, obtained in 1987 and published in 1992:

Theorem (Denef and Loeser)

1) *Let $h : Y \to X$ be a DNC modification between smooth complex algebraic varieties. We have*

$$\mathrm{Eu}(X) = \sum_{I \subset A} \frac{\mathrm{Eu}(E_I^\circ)}{\prod_{i \in I} \nu_i}. \tag{$\dagger$}$$

2) *Let F be a nowhere dense subscheme of X defined by an ideal $\mathcal{I}$ and let $h : Y \to X$ be a log-resolution of (X, F). Then the rational function*

$$Z_{top,F}(s) := \sum_{I \subset A} \frac{\mathrm{Eu}(E_I^\circ)}{\prod_{i \in I} N_i(\mathcal{I})s + \nu_i}, \tag{$\ddagger$}$$

does not depend on the log-resolution h.

Remarks

The result also holds in the complex analytic setting. Initially 2) was stated only when $\mathcal{I}$ is principal, but the proof is the same in general.

The proof was by no means direct. Main steps are

- use of p-adic integrals
- change of variable formula for p-adic integrals
- expression of p-adic integrals in term of number of points on varieties over finite fields
- computing Euler characteristics as limits of number of points on varieties over finite fields.

Let p be a prime number. We endow $\mathbb{Q}$ with the p-adic valuation $\operatorname{ord}_p : \mathbb{Q}^\times$ and the p-adic norm $|x|_p := p^{-\operatorname{ord}_p(x)}$, $|0|_p = 0$, and consider its completion $\mathbb{Q}_p$ with ring of integers

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p; |x|_p \leq 1\}.$$

More generally we consider a field K with a valuation $\operatorname{ord} : K^\times \to \mathbb{Z}$, extended to K by $\operatorname{ord}(0) = \infty$. We denote by $\mathcal{O}_K$ the valuation ring $\mathcal{O}_K = \{x \in K | \operatorname{ord}(x) \geq 0\}$ and we fix an uniformizing parameter ϖ, i.e. an element of valuation 1 in $\mathcal{O}_K$.

The ring $\mathcal{O}_K$ is a local ring with maximal ideal $\mathcal{M}_K$ of $\mathcal{O}_K$ generated by ϖ. We shall assume the residue field $k := \mathcal{O}_K/\mathcal{M}_K$ is finite with $q = p^e$ elements. We endow K with a norm by setting $|x| := q^{-\operatorname{ord}(x)}$ for x in K. We shall furthermore assume K is complete for $|\_|$.

It follows in particular that the abelian groups $(K^n, +)$ are locally compact, hence they have a canonical Haar measure μ_n, unique up to multiplication by a non zero constant, so we may assume $\mu_n(\mathcal{O}_K^n) = 1$.

The measure μ_n is the unique $\mathbb{R}$-valued Borel measure on K^n which is invariant by translation and such that $\mu_n(\mathcal{O}_K^n) = 1$. For instance

$$\mu_n(a + \varpi^m + \mathcal{O}_K^n) = q^{-mn}.$$

For any measurable subset A of K^n and any λ in K,

$$\mu_n(\lambda A) = |\lambda|^n \mu_n(A).$$

More generally, for every g in $\mathrm{GL}_n(K)$,

$$\mu_n(gA) = |\det g| \mu_n(A).$$

If f is, say, a K-analytic function on A, we set

$$\int_A |f|\mu_n := \int_A |f||dx| := \sum_{m\in\mathbb{Z}} \mu_n(\mathrm{ord}(f) = m)q^{-m},$$

assuming the series $\sum_{m\in\mathbb{Z}} \mu_n(\mathrm{ord}(f) = m)q^{-m}$ is convergent in $\mathbb{R}$.

More generally, we define similarly $\int_A |f|^s|dx|$ by

$$\sum_{m\in\mathbb{Z}} \mu_n(\mathrm{ord}(f) = m)q^{-ms}$$

whenever it makes sense.

For instance, when $n = 1$, we have, for $s > 0$ in $\mathbb{R}$,

$$
\begin{aligned}
\int_{x \in \mathcal{O}_K, \mathrm{ord}(x) \geq m} |x|^s |dx| &= \sum_{j \geq m} q^{-sj} \int_{\mathrm{ord}(x) = j} |dx| \\
&= \sum_{j \geq m} q^{-sj} (q^{-j} - q^{-j-1}) \\
&= (1 - q^{-1}) q^{-(s+1)m} / (1 - q^{-(s+1)}).
\end{aligned}
$$

The formula for change of volumes under $\mathrm{GL}_n(K)$-action is a very special form of the following fundamental change of variables formula

Proposition (The p-adic change of variables formula)

Let $f = (f_1, \dots, f_n)$ be a K-analytic isomorphism between open subsets U and V of K^n. Then

$$\mu_{n|V} = |\operatorname{Jac} f| f_*(\mu_{n|U}),$$

where $\operatorname{Jac} f$ is the determinant of the jacobian matrix $(\partial f_i/\partial x_j)$ of f. In other terms:

$$\int_V \varphi \mu_n = \int_U (\varphi \circ f) |\operatorname{Jac} f| \mu_n,$$

for every integrable function φ on V.

Let X be an d-dimensional smooth K-analytic manifold. One assigns to any K-analytic d-differential form ω on X a measure $\mu_\omega := |\omega|$ as follows.

Take an atlas $\{(U, \phi_U)\}$ of X. Write

$$(\phi_U^{-1})^*\omega_{|U} = f_U dx_1 \wedge \cdots \wedge dx_d.$$

If A is small enough to be contained in some U, we set

$$\mu_\omega(A) := \int_{\phi_U(A)} |f_U||dx|.$$

It follows from the change of variables formula that the measure may be extended uniquely by additivity to any A in a way which is independent of the choice of the atlas.

Assume now X is a (smooth) closed d-dimensional submanifold of $\mathcal{O}_K^N$.

There is a canonical measure μ_X on X defined as follows.

For any subset $I = \{i_1 < \cdots < i_d\}$ of cardinality d of $\{1, \ldots, N\}$, we consider the measure $\mu_{X,I}$ on X induced by $dx_{i_1} \wedge \cdots \wedge dx_{i_d}$ on X and we set $\mu_X := \sup_I \mu_{X,I}$.

The canonical volume of X is $\mathrm{vol}(X) := \mu_X(X)$.

The relation between the volume $\mathrm{vol}(X)$ and counting points is the following. Denote by X_n the image of X in the finite set $(\mathcal{O}_K/\varpi^n\mathcal{O}_K)^N$.

If X is smooth over $\mathcal{O}_K$, then

$$\mathrm{vol}(X) = |X_1| q^d.$$

In general, if X is smooth over K,

$$\mathrm{vol}(X) = |X_n| q^{nd} \text{ for } n \gg 0.$$

For singular X, we can define $\mathrm{vol}(X)$ as the limit of the volume of the complement in X of a tubular neighbothood of small radius around the singular locus. Then

$$\mathrm{vol}(X) = \lim_{n \mapsto \infty} |X_n| q^{nd}.$$

Let us explain the idea of the proof of the result of Denef and Loeser. For simplicity, we shall assume that $X = \mathbb{A}^d$ and $F = f^{-1}(0)$, with f a polynomial in $\mathbb{C}[x_1, \dots, x_d]$ but the proof in general works just the same.

Let us first prove 2). We shall write $Z_{\mathrm{top},f}(s)$ for $Z_{\mathrm{top},F}(s)$. We shall make the assumption that the coefficients of f all lie in the same number field K, i.e. f is in $K[x_1, \dots, x_d]$ (in general, we can only assume they lie in a field of finite type over $\mathbb{Q}$, but the basic idea of the proof still remains the same). Now for every prime ideal $\mathfrak{P}$ in the ring of integers $\mathcal{O}_K$, we denote by $K_{\mathfrak{P}}$ the corresponding local field, with ring of integers $\mathcal{O}_{\mathfrak{P}}$ and residue field $k_{\mathfrak{P}}$.

We consider the Igusa local zeta function

$$Z_{f,K_{\mathfrak{P}}}(s) := \int_{\mathcal{O}_{\mathfrak{P}}^d} |f|_{\mathfrak{P}}^s |dx|_{\mathfrak{P}},$$

where $|\_|_{\mathfrak{P}}$ stands for the $\mathfrak{P}$-adic norm on $K_{\mathfrak{P}}$.

Consider now a log-resolution $h : Y \to X$ defined over K. It follows from a result of Denef, that, for almost all $\mathfrak{P}$,

$$Z_{f,K_{\mathfrak{P}}}(s) = q^{-d} \sum_{I \subset J} \mathrm{card}((E_I^\circ)_{k_{\mathfrak{P}}}(k_{\mathfrak{P}})) \prod_{i \in I} \frac{(q-1)q^{-(N_i s + \nu_i)}}{1 - q^{-(N_i s + \nu_i)}}, \quad (*)$$

with $q = \mathrm{card}\, k_{\mathfrak{P}}$.

Here we should explain what we mean by $(E_I^\circ)_{k_{\mathfrak{P}}}$.

For Z a variety over K we choose a model $\mathcal{Z}$ over $\mathcal{O}_K$, and we denote by $(E_I^\circ)_{k_{\mathfrak{P}}}$ its reduction mod $\mathfrak{P}$. Of course, this may depend on the choice of the model $\mathcal{X}$, but if one takes another model $\mathcal{X}'$, the reductions will differ only for a finite number of prime ideals $\mathfrak{P}$.

Denef's proof is based on the change of variable formula and the fact that $h(K_{\mathfrak{P}}) : Y(K_{\mathfrak{P}}) \to X(K_{\mathfrak{P}})$ is an isomorphism outside closed analytic subsets of dimension $< d$, which are of measure 0 for d-dimensional measures.
Also note that any model over $\mathcal{O}_K$ of a smooth K-variety will be smooth over $\mathcal{O}_{\mathfrak{P}}$ for almost all $\mathfrak{P}$.

For $e \geq 1$, let us write $K_{\mathfrak{P}}^{(e)}$ for the unramified extension of $K_{\mathfrak{P}}$ of degree e. Its residue field $k_{\mathfrak{P}}^{(e)}$ has q^e elements. Also, for almost all $\mathfrak{P}$, equation (*) still holds when replacing $K_{\mathfrak{P}}$ by $K_{\mathfrak{P}}^{(e)}$, yielding

$$Z_{f,K_{\mathfrak{P}}^{(e)}}(s) = q^{-ed} \sum_{I \subset J} \mathrm{card}((E_I^{\circ})_{k_{\mathfrak{P}}}(k_{\mathfrak{P}}^{(e)})) \prod_{i \in I} \frac{(q^e - 1)q^{-e(N_i s + \nu_i)}}{1 - q^{-e(N_i s + \nu_i)}}. \tag{**}$$

Now, taking formally the limit as $e \mapsto 0$ in (**) would give us the right-hand side of (‡), if only we could make sense of the following statement:

$$\lim_{e \mapsto 0} \mathrm{card}\, W_{k_{\mathfrak{P}}}(k_{\mathfrak{P}}^{(e)}) = \mathrm{Eu}\, W,$$

for almost all $\mathfrak{P}$, when W is a variety over K.

Indeed, it follows from Grothendieck's trace formula for the Frobenius acting on ℓ-adic cohomology together with standard comparison theorems between ℓ-adic and classical Betti cohomology, that, when W is a variety over K, for almost all $\mathfrak{P}$, there exists complex numbers α_i, $i \in B$ and β_j, $j \in C$ such that

$$\mathrm{card}\, W_{k_{\mathfrak{P}}}(k_{\mathfrak{P}}^{(e)}) = \sum_A \alpha_i^e - \sum_B \beta_j^e$$

and

$$\mathrm{Eu} W = |B| - |C|.$$

One can actually take as α_i and β_j the eigenvalues, respectively in even and odd degree, of the Frobenius acting on ℓ-adic cohomology groups with compact supports of $W_{k_{\mathfrak{P}}} \otimes \overline{k_{\mathfrak{P}}}$.

Of course, this is just a rough sketch of the proof of 2) and further work is required in order to show this process of taking limits as $e \mapsto 0$ really makes sense.

To prove 1), one just sets $s = 0$ in 2). $\square$

Comments

Denote by $\mathcal{C}(X, \mathbb{C})$ the algebra of complex-valued constructible functions on X, generated by characteristic functions $\mathbf{1}_W$ of constructible subsets W of X. If $h : Y \to X$ is a morphism one defines $h_! : \mathcal{C}(Y, \mathbb{C}) \to \mathcal{C}(X, \mathbb{C})$ by

$$h_!(\mathbf{1}_{W'})(x) := \mathrm{Eu}(h^{-1}(x) \cap W'),$$

for W' constructible in Y.

Now if h is a DNC modification, one can reformulate (†) by saying that

$$\mathbf{1}_W \longmapsto \sum_{I \subset A} \frac{\mathbf{1}_{E_I^\circ \cap h^{-1}(W)}}{\prod_{i \in I} \nu_i},$$

for W constructible in X, is an inverse to $h_!$.

Other known proofs of the theorem of Denef and Loeser:

- via Motivic Integration, following Kontsevich (1995)
- via the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk (2002)

Challenging problem: Find a direct topological proof of (†).

Let us recall the weak factorization theorem:

Theorem (Weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk)

Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper smooth irreducible varieties. Let $U \subset X_1$ be the largest open subset on which ϕ is an isomorphism. Then ϕ can be factored into a sequence of blowing ups and blowing down with smooth centers disjoint from U: $\phi_i : V_{i-1} \dashrightarrow V_i$, $i = 1, \dots, \ell$, with $V_0 = X_1$, $V_\ell = X_2$, with ϕ_i or ϕ_i^{-1} blowing ups with smooth centers away from U. Moreover there exists i_0 such that $V_i \dashrightarrow X_1$ is defined everywhere and projective for $i \leq i_0$ and $V_i \dashrightarrow X_2$ is defined everywhere an projective for $i \geq i_0$.

Let X be a smooth complex projective variety of dimension n. We say X is Calabi-Yau if X admits a nowhere vanishing degree n algebraic differential form ω. This is equivalent to the sheaf Ω_X^n being trivial. Recall the Betti numbers $b_i(X)$ are the ranks of the cohomology groups $H^i(X(\mathbb{C}), \mathbb{C})$.

Considerations from Mirror Symmetry led to the guess that birational Calabi-Yau varieties should have the same Betti numbers (in fact the same Hodge numbers).

This was proved by Batyrev in 1995 using p-adic integration and the Weil conjectures.

Theorem (Batyrev)

Let X and X' be complex Calabi-Yau varieties of dimension n. Assume X and X' are birationally equivalent. Then they have the same Betti numbers.

Let us sketch the proof. For simplicity, we assume, as in the proof of the theorem of Denef and Loeser that X, X' and all the data are defined over some number field K (in general they are defined only over some field of finite type, but the basic idea of the proof is the same).

By Hironaka there exists a smooth projective Y defined over K, and birational proper morphisms (also defined over K) $h : Y \to X$ and $h' : Y \to X'$. Furthermore we may assume there exists a divisor with normal crossings $E = \cup_{i \in J} E_i$ such that the exceptional locus of h and h' respectively, is a finite union of E_i's. We may write $K_Y = h^* K_X + \sum_{i \in J} (\nu_i - 1) E_i$ and $K_Y = h'^* K_{X'} + \sum_{i \in J} (\nu'_i - 1) E_i$.

Since $h^* K_X$ and $h'^* K_{X'}$ are both trivial, it follows that $\nu_i = \nu'_i$ for every i in J [Exercise: check it!].

One then deduces follows from the change of variables formula, that for almost all $\mathfrak{P}$, with a slight abuse of notation, we have

$$\int_{X(K_{\mathfrak{P}})} |\omega|_{\mathfrak{P}} = \int_{X'(K_{\mathfrak{P}})} |\omega'|_{\mathfrak{P}}$$

and the same holds for all unramified extensions $K_{\mathfrak{P}}^{(e)}$. Indeed, we may express by the change of variables formula both integrals as the same integral over the rational points of Y. Since, for almost all $\mathfrak{P}$ and every e,

$$\int_{X(K_{\mathfrak{P}}^{(e)})} |\omega|_{\mathfrak{P}} = q^{-en} \operatorname{card}(X(k_{\mathfrak{P}}^{(e)})),$$

it follows that for almost all $\mathfrak{P}$, the reductions of (some model of) X and X' mod $\mathcal{M}_{\mathfrak{P}}$ have same the zeta function.

On the other side, by the part of the Weil conjectures proved by Deligne, for proper smooth varieties over a finite field, the zeta function determines the ℓ-adic Betti numbers, hence the result follows from standard comparison results between ℓ-adic and usual Betti numbers.

Remark

The above proof gives in fact the following stronger result: if X and X' are two n-dimensional smooth proper complex varieties that are K-equivalent, meaning that there exists birational proper morphisms $h : Y \to X$ and $h' : Y \to X'$ with Y smooth proper such that the invertible sheaves $h^*(\Omega^n_X)$ and $h^*(\Omega^n_{X'})$ are isomorphic, then X and X' have the same Betti numbers.

Shortly after Batyrev's proof, M. Kontsevich found a direct approach to Batyrev's Theorem, avoiding the use of p-adic integrals and involving arc spaces, which he explained in a seminal Orsay talk of December 7, 1995, entitled "String cohomology".

Motivic integration was born . . .

We shall now work over a field k of characteristic 0.

By a variety X over k will shall always mean a separated and reduced scheme, of finite type over k.

For $n \geq 0$, we introduce the space of n-arcs on X, denoted by $\mathcal{L}_n(X)$. It is a k-scheme of finite type which represents the functor:

$$k-\text{algebras} \longrightarrow \text{Sets}$$
$$R \longmapsto \mathrm{Hom}_{k-\text{schemes}}(\mathrm{Spec}(R[t]/(t^{n+1}), X) := X(R[t]/(t^{n+1})).$$

For example, when X is an affine variety with equations $f_i(\vec{x}) = 0$, $i = 1, \cdots, m$, $\vec{x} = (x_1, \cdots, x_r)$, then $\mathcal{L}_n(X)$ is given by the equations, in the variables $\vec{a}_0, \cdots, \vec{a}_n$, expressing that

$$f_i(\vec{a}_0 + \vec{a}_1 t + \cdots + \vec{a}_n t^n) \equiv 0 \mod t^{n+1}, i = 1, \cdots, m.$$

We have canonical isomorphisms $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X) = TX$, where TX denotes the tangent space of the variety X.

For $m \geq n$, there are canonical morphisms $\theta_m^n : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$. In general, when X is not smooth, they may not be surjective.

When X is smooth of dimension d, θ_m^n is a locally trivial fibration for the Zariski topology with fiber $\mathbb{A}_k^{(m-n)d}$ (more precisely it is an affine bundle).

Taking the projective limit of these algebraic varieties $\mathcal{L}_n(X)$, we obtain the arc space $\mathcal{L}(X)$ of X. A priori this is just a pro-scheme, but, the transition maps θ_m^n being affine, it is indeed a k-scheme.

In general, $\mathcal{L}(X)$ is not of finite type over k. For K a field extension of k, the K-rational points of $\mathcal{L}(X)$ are the $K[[t]]$-rational points of X and they are called K-arcs on X.

For instance, when X is an affine variety with equations $f_i(\vec{x}) = 0, i = 1, \cdots, m, \vec{x} = (x_1, \cdots, x_r)$, then the K-rational points of $\mathcal{L}(X)$ are the sequences $(\vec{a}_0, \vec{a}_1, \vec{a}_2, \cdots) \in (K^n)^{\mathbb{N}}$ satisfying $f_i(\vec{a}_0 + \vec{a}_1 t + \vec{a}_2 t^2 + \cdots) = 0$, for $i = 1, \cdots, m$.

For every n we have natural morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

obtained by truncation. For any arc γ on X (i.e. a K-arc for some field K containing k), we call $\pi_0(\gamma)$ the origin of the arc γ.

Let R and S be rings. An additive invariant λ from the category Var_R of algebraic varieties over R with values in S, assigns to any X in Var_R an element $\lambda(X)$ of S, such that

$$\lambda(X) = \lambda(X')$$

for $X \simeq X'$,

$$\lambda(X) = \lambda(X') + \lambda(X \setminus X'),$$

for X' closed in X, and

$$\lambda(X \times X') = \lambda(X) \cdot \lambda(X')$$

for every X and X'.

Let us remark that additive invariants λ naturally extend to take their values on constructible subsets of algebraic varieties.

Indeed a constructible subset W may be written as a finite disjoint union of locally closed subvarieties Z_i, $i \in I$. One may define $\lambda(W)$ to be $\sum_{i \in I} \lambda(Z_i)$. By the very axioms, this is independent of the decomposition into locally closed subvarieties.

Some examples:

Euler characteristic Here $R = k$ is a field. When k is a subfield of $\mathbb{C}$, the Euler characteristic

$$\mathrm{Eu}(X) := \sum_i (-1)^i \mathrm{rk} H_c^i(X(\mathbb{C}), \mathbb{C})$$

give rise to an additive invariant $\mathrm{Eu} : \mathrm{Var}_k \to \mathbb{Z}$.

For general k, replacing Betti cohomology with compact support by ℓ-adic cohomology with compact support, $\ell \neq \mathrm{char}\, k$, one gets an additive invariant $\mathrm{Eu}_\ell : \mathrm{Var}_k \to \mathbb{Z}$, which does not depend on ℓ when k is of characteristic zero.

Hodge polynomial Let us assume $R = k$ is a field of characteristic zero. Then it follows from Deligne's Mixed Hodge Theory that there is a unique additive invariant $H : \mathrm{Var}_k \to \mathbb{Z}[u, v]$, which assigns to a smooth projective variety X over k its usual Hodge polynomial

$$H(u, v) := \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q,$$

with $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$ the (p, q)-Hodge number of X.

Virtual motives More generally, when $R = k$ is a field of characteristic zero, there exists by work of Gillet and Soulé, and Guillen and Navarro-Aznar, a unique additive invariant

$$\chi_c : \mathrm{Var}_k \longrightarrow K_0(\mathrm{CHMot}_k),$$

which assigns to a smooth projective variety X over k the class of its Chow motive, where $K_0(\mathrm{CHMot}_k)$ denotes the Grothendieck ring of the category of Chow motives over k (with rational coefficients).

Counting points Assume $k = \mathbb{F}_q$, then $N_n : X \mapsto |X(\mathbb{F}_{q^n})|$ gives rise to an additive invariant $N_n : \mathrm{Var}_k \to \mathbb{Z}$.

Similarly, if R is (essentially) of finite type over $\mathbb{Z}$, for every maximal ideal $\mathfrak{P}$ of R with finite residue field $k(\mathfrak{P})$, we have an additive invariant $N_{\mathfrak{P}} : \mathrm{Var}_R \to \mathbb{Z}$, which assigns to X the cardinality of $(X \otimes k(\mathfrak{P}))(k(\mathfrak{P}))$.

There exists a universal additive invariant

$$[\_] : \mathrm{Var}_R \longrightarrow K_0(\mathrm{Var}_R)$$

in the sense that composition with $[\_]$ gives a bijection between ring morphisms $K_0(\mathrm{Var}_R) \to S$ and additive invariants $\mathrm{Var}_R \to S$.

The construction of $K_0(\mathrm{Var}_R)$ is quite easy: take the free abelian group on isomorphism classes $[S]$ of objects of Var_R and mod out by the relations

$$[S] = [S'] + [S \setminus S']$$

for S' closed in S.

The product is defined by

$$[S] \cdot [S'] = [S \times S'].$$

We shall denote by $\mathbb{L}$ the class of the affine line $\mathbb{A}^1_R$ in $K_0(\mathrm{Var}_R)$.

An important role will be played by the ring

$$\mathcal{M}_R := K_0(\mathrm{Var}_R)[\mathbb{L}^{-1}]$$

obtained by localization with respect to the multiplicative set generated by $\mathbb{L}$.

This construction is analogous to the construction of the category of Chow motives from the category of effective Chow motives by localization with respect to the Lefschetz motive. (Note that the morphism χ_c sends $\mathbb{L}$ to the class of the Lefschetz motive.)

Very little is known about the structure of the rings $K_0(\mathrm{Var}_R)$ and $\mathcal{M}_R$ even when R is a field. B. Poonen proved that, when k is a field of characteristic zero, the ring $K_0(\mathrm{Var}_k)$ is not a domain.

Even for a field k of characteristic zero, it is not known whether the localization morphism $K_0(\mathrm{Var}_k) \to \mathcal{M}_k$ is injective or not (although we guess it is not).

We shall denote by $\overline{\mathcal{M}}_k$ the image of $K_0(\mathrm{Var}_k)$ in $\mathcal{M}_k$.

We shall also need relative Grothendieck rings defined as follows.

Let X be a variety over R. One defines $K_0(\mathrm{Var}_X)$ similarly as $K_0(\mathrm{Var}_R)$ using the category of varieties over X instead of the category of varieties over R.

Recall that objects in this category are arrows $f : Y \to X$ in Var_R and that a morphism between $f : Y \to X$ and $f' : Y' \to X$ is just a morphism $g : Y \to Y'$ such that $f = f' \circ g$.

One also defines $\mathcal{M}_X$ by inverting the class $\mathbb{L}$ of $\mathbb{A}^1_X \to X$ in $K_0(\mathrm{Var}_X)$. We shall write $[f : Y \to X]$ for the class of $f : Y \to X$.

So, what is motivic integration?

Roughly speaking, motivic integration assigns to a reasonable class of subsets A of $\mathcal{L}(X)$, the arc space of a k-variety X, a volume $\mu(A)$.
The most naive idea would be to construct a real valued measure on $\mathcal{L}(X)$ similarly as in the p-adic case. Such attempts are doomed to fail immediately since, as soon as k is infinite, $k((t))$ is not locally compact.
Kontsevich's real breakthrough was to realize that a reasonable measure on $k((t))$ could in fact be constructed once $\mathbb{R}$ is replaced by $\mathcal{M}_k$ (or its completion).
The motivic measure $\mu(A)$ will be an element of $\mathcal{M}_k$, or of some completion or localization of $\mathcal{M}_k$.

We shall present today the original construction of motivic integration as it has been developed by Denef and Loeser, following insights of Kontsevich. It is nowadays superseded by the more recent approach of Cluckers and Loeser we shall present in the last lecture.

Amongst the main features of the new approach, let us mention that completion is no more needed and that it also allows to deal with integrals depending on parameters.

We want to assign a measure to subsets of $\mathcal{L}(X)$.

That measure will take its values into a ring related to $K_0(\mathrm{Var}_k)$. In the analogy with p-adic integration, $K_0(\mathrm{Var}_k)$ is the analogue of $\mathbb{Z}$ and $\mathcal{M}_k$ is the analogue of $\mathbb{Z}[p^{-1}]$ (the number of rational points of the affine line over $\mathbb{F}_p$ is p). Since in $\mathbb{R}$, p^{-i} has limit 0 as $i \mapsto \infty$, we should complete $\mathcal{M}_k$ is such a way that $\mathbb{L}^{-i}$ has limit 0 as $i \mapsto \infty$.

This is achieved in the following way: we define $F^m\mathcal{M}_k$ to be the subgroup of $\mathcal{M}_k$ generated by elements of the form $[S]\mathbb{L}^{-i}$, with $\dim S - i \leq -m$. We have $F^{m+1} \subset F^m$, $\mathbb{L}^{-m} \in F^m$ and $F^nF^m \subset F^{n+m}$. We denote by $\widehat{\mathcal{M}}_k$ the completion of $\mathcal{M}_k$ with respect to that filtration.

A minor technical problem shows up here, since it is not known whether the canonical morphism $\mathcal{M}_k \to \widehat{\mathcal{M}}_k$ is injective or not.

Nevertheless, this is not much an issue since invariants $\mathrm{Eu} : \mathcal{M}_k \to \mathbb{Z}$ (Euler number) and $H : \mathcal{M}_k \to \mathbb{Z}[u, v, (uv)^{-1}]$ (Hodge polynomial) factor through the image $\overline{\mathcal{M}}_k$ of $\mathcal{M}_k$ in $\widehat{\mathcal{M}}_k$.

Let X be an algebraic variety over k of dimension d, maybe singular.

By a cylinder in $\mathcal{L}(X)$, we mean a subset A of $\mathcal{L}(X)$ of the form $A = \pi_n^{-1}(C)$ with C a constructible subset of $\mathcal{L}_n(X)$, for some n.

We say A is stable (at level n) if furthermore $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ is a piecewise Zariski fibration over $\pi_m(A)$ with fiber $\mathbb{A}_k^d$ for all $m \geq n$.

By being a piecewise Zariski fibration over $\pi_m(A)$ we mean that there exists a finite partition of $\pi_m(A)$ into locally closed subsets of $\mathcal{L}_m(X)$ over which the morphism is a locally trivial fibration for the Zariski topology.

If A is a stable cylinder at level n, we set

$$\tilde{\mu}(A) := [\pi_n(A)]\mathbb{L}^{-(n+1)d}$$

in $\mathcal{M}_k$.

Remark that the stability condition insures that we would get the same value by viewing A as a stable cylinder at level m, $m \geq n$.

When X is smooth, all cylinders are stable. In particular, in this case, $\mathcal{L}(X)$ itself is a stable cylinder and

$$\tilde{\mu}(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}.$$

In general, we can assign to any cylinder A in $\mathcal{L}(X)$ a measure $\mu(A)$ in $\widehat{\mathcal{M}}_k$ by a limit process as follows: for $e \geq 0$, set

$$\mathcal{L}^{(e)}(X) := \mathcal{L}(X) \setminus \pi_e^{-1}(\pi_e(\mathcal{L}(X_{\mathrm{sing}}))),$$

where X_{sing} denote the singular locus of X and we view $\mathcal{L}(X_{\mathrm{sing}})$ as a subset of $\mathcal{L}(X)$. The set $\mathcal{L}^{(e)}(X)$ should be viewed as $\mathcal{L}(X)$ minus some tubular neighborhood around the singular locus.

It can be proved that $A \cap \mathcal{L}^{(e)}(X)$ is a stable cylinder and that $\tilde{\mu}(A \cap \mathcal{L}^{(e)}(X))$ does have a limit in $\widehat{\mathcal{M}}_k$ as e goes to ∞ which we define to be $\mu(A)$. This apply in particular to $A = \mathcal{L}(X)$ when X is not smooth.

We define

$$||\_|| : \widehat{\mathcal{M}}_k :\longrightarrow \mathbb{R}_{\geq 0}$$

to be given by $||a|| = 2^{-n}$ if $a \in F^n\widehat{\mathcal{M}}_k$ and $a \notin F^{n+1}\widehat{\mathcal{M}}_k$, where $F^{\cdot}\widehat{\mathcal{M}}_k$ denotes the induced filtration on $\widehat{\mathcal{M}}_k$.

We say a subset A of $\mathcal{L}(X)$ is measurable if, for every $\varepsilon > 0$, there exists cylinders $A_i(\varepsilon)$, $i \in \mathbb{N}$, such that $(A \cup A_0(\varepsilon)) \setminus (A \cap A_0(\varepsilon))$ is contained in $\cup_{i\geq 1} A_i(\varepsilon)$, and $||\mu(A_i(\varepsilon))|| \leq \varepsilon$, for every $i \geq 1$.

One shows that

$$\mu(A) := \lim_{\varepsilon \mapsto 0} \mu(A_0(\varepsilon))$$

exists and is independent of the choice of the $A_i(\varepsilon)$'s. We say that A is strongly measurable if moreover we can take $A_0(\varepsilon) \subset A$.

Let A be a measurable subset of $\mathcal{L}(X)$ and $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ be a function such that all its fibers are measurable. We shall say $\mathbb{L}^{\alpha}$ is integrable if the series

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{n \in \mathbb{Z}} \mu(A \cap \alpha^{-1}(n)) \mathbb{L}^{-n}$$

is convergent in $\widehat{\mathcal{M}}_k$.

We have the following motivic analogue of the p-adic change of variables formula:

Theorem (Change of variables formula)

Let X be an algebraic variety over k of dimension d. Let $h : Y \to X$ be proper birational morphism. We assume Y to be smooth. Let A be a subset of $\mathcal{L}(X)$ such that A and $h^{-1}(A)$ are strongly measurable. Assume $\mathbb{L}^{-\alpha}$ is integrable on A. Then

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{h^{-1}(A)} \mathbb{L}^{-\alpha \circ h - \operatorname{ord} h^*(\Omega^d_X)} d\mu.$$

Let us explain what is meant by $\operatorname{ord} h^*(\Omega^d_X)$, the order of the jacobian of h, when X is not smooth.

If $\mathcal{I}$ is some ideal sheaf on Y, we denote by $\operatorname{ord}\mathcal{I}$ the function which to a arc φ in $\mathcal{L}(Y)$ assigns $\inf \operatorname{ord} g(\varphi)$ where g runs over local sections of $\mathcal{I}$ at $\pi_0(\varphi)$.

We denote by Ω^d_X to be the d-th exterior power of Ω^1_X, the Kähler differentials. The image of $h^*(\Omega^d_X)$ in Ω^d_Y is of the form $\mathcal{I}\Omega^d_Y$ and we set

$$\operatorname{ord} h^*(\Omega^d_X) := \operatorname{ord}\mathcal{I}.$$

The key geometrical statement behind the proof of the change of variables formula is the following:

Theorem (Denef-Loeser)

Let X be an algebraic variety over k. Let $h : Y \to X$ be proper birational morphism. We assume Y to be smooth. For e and e' in $\mathbb{N}$, we set

$$\Delta_{e,e'} := \left\{ \varphi \in \mathcal{L}(Y) \,\middle|\, \operatorname{ord} h^*(\Omega_X^d)(\varphi) = e \quad \textit{and} \quad h(\varphi) \in \mathcal{L}^{(e')}(X) \right\}.$$

Then there exists $c > 0$ such that, for $n \geq \sup(2e, e + ce')$,

1. *The image $\Delta_{e,e',n}$ of $\Delta_{e,e'}$ in $\mathcal{L}_n(Y)$ is a union of fibers of h_n, the morphism induced by h.*
2. *The morphism $h_n : \Delta_{e,e',n} \to h_n(\Delta_{e,e',n})$ is a piecewise Zariski fibration with fiber $\mathbb{A}_k^e$.*

When X is smooth, one can take $e' = 0$.

Let us give some direct applications.

DNC modifications

Let $h : Y \to X$ be a DNC modification between smooth varieties. Then
$$[X] = \sum_{I \subset A} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i} - 1}$$
in $\widehat{\mathcal{M}}_k$.

Taking Eu of both sides one recovers the 1987 Denef-Loeser formula.

Proof:

$$\mathbb{L}^{-d}[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord} h^*(\Omega_X^d)} d\mu. \quad \square$$

Birationally equivalent Calabi-Yau varieties

Let X and X' be two birationally equivalent smooth proper Calabi-Yau varieties. Then

$$[X] = [X']$$

in $\widehat{\mathcal{M}}_k$.

Taking H of both sides, one deduces that X and X' have same Hodge numbers, hence same Betti numbers.

Proof: Take birational morphisms of proper smooth varieties $h : Y \to X$ and $h' : Y \to X$'. We have

$$\mathbb{L}^{-d}[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord} h^*(\Omega^d_X)} d\mu$$
$$= \int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord} h'^*(\Omega^d_{X'})} d\mu = \mu(\mathcal{L}(X')) = \mathbb{L}^{-d}[X']. \quad \square$$

The same result and proof hold for K-equivalent varieties.

Stringy invariants Let X be a normal $\mathbb{Q}$-Gorenstein variety. Let $h : Y \to X$ be a log-resolution of X, that is Y is smooth, h is proper, the restriction of $h : Y \setminus h^{-1}(X_{\mathrm{sing}}) \to X \setminus X_{\mathrm{sing}}$ is an isomorphism, and $h^{-1}(X_{\mathrm{sing}})$ is a divisor with simple normal crossings with irreducible components E_i, i in A.

We may define log discrepancies ν_i in $\mathbb{Q}$ by

$$K_Y = h^* K_X + \sum_{i \in A} (\nu_i - 1) E_i.$$

When all ν_i's are > 0 we say X is log terminal.

For X a normal log terminal $\mathbb{Q}$-Gorenstein variety

$$E_{st}(X) := \sum_{I \subset A} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i} - 1}$$

does not depend on the log-resolution h.

This is a direct consequence of the change of variables formula.

Note that $E_{st}(X)$ belongs to $\widehat{\mathcal{M}}_k[\mathbb{L}^{1/m}]$, with m a multiple of the denominators of the ν_i's.

Also, setting $j : X \setminus X_{\text{sing}} \hookrightarrow X$ and denoting by $\mathcal{I}_X \omega_X$ the image of the natural morphism $\Omega^d_X \to \omega_X := j_* j^* \Omega^d_X$, $E_{st}(X)$ may be defined intrisically as

$$E_{st}(X) = \int_{\mathcal{L}(X)} \mathbb{L}^{\operatorname{ord}_t \mathcal{I}_X} d\mu.$$

Applying Eu and H to $E_{st}(X)$ one gets the stringy Betti numbers and the stringy Hodge numbers of Batyrev.

It is an open issue to generalize these stringy invariants to general singularities (cf. work of Veys).

Let X be a smooth variety of dimension d and F be a closed nowhere dense subscheme of X defined by an ideal sheaf $\mathcal{I}$. Let $h : Y \to X$ be a log-resolution of (X, F) such that $h^{-1}(\mathcal{I})\mathcal{O}_Y$ is locally free.

For $n \geq 0$, consider

$$\mathcal{X}_n(\mathcal{I}) := \left\{ \varphi \in \mathcal{L}_n(X) \;\middle|\; \inf_{g \in \mathcal{I}_{\pi_0(\varphi)}} \mathrm{ord}_t(g \circ \varphi) = n \right\}.$$

It follows directly from the geometric form of the change of variables formula that $[\mathcal{X}_n(\mathcal{I})]$ can be computed in $\mathcal{M}_k$ on the log-resolution h as follows:

$$[\mathcal{X}_n(\mathcal{I})] = \mathbb{L}^{nd} \sum_{I \in A} (\mathbb{L}-1)^{|I|} [E_I^\circ] \left(\sum_{k_i \geq 1, i \in I, \sum_I k_i N_i(\mathcal{I}) = n} \mathbb{L}^{-\sum k_i \nu_i} \right) \qquad (\natural)$$

in $\mathcal{M}_k$.

Recall the log-canonical threshold of the pair is defined by

$$\operatorname{lct}(X, \mathcal{I}) = \min_{i \in I} \frac{\nu_i}{N_i(\mathcal{I})}.$$

Theorem (Mustață)

$$\operatorname{lct}(X, \mathcal{I}) = \min_n \left(\frac{\operatorname{codim}_{\mathcal{L}_n(X)} \mathcal{L}_n(F)}{n+1} \right). \qquad (\natural\natural)$$

Proof: Follows from the previous formula ($\natural$). □

Mustață deduces from ($\natural\natural$) the semicontinuity in families of the log-canonical threshold.

Some other applications

- McKay correspondence (Batyrev, Denef-Loeser, Yasuda, ...)
- rationality of the series

$$\sum_{n\geq 0}[\pi_n(\mathcal{L}(X))]\, T^n$$

  in $\mathcal{M}_k[[T]]$ (Denef-Loeser). Note that $\pi_n(\mathcal{L}(X))$ is constructible in $\mathcal{L}_n(X)$
- further work on log-canonical threshold and inversion of adjunction (Mustaţă, Ein, Lazarsfeld, Yasuda, de Fernex, ...).

Let X be a smooth complex algebraic variety and $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a morphism to the affine line.

Let x be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$.

Fix $0 < \eta \ll \varepsilon \ll 1$.

The morphism f restricts to a fibration (the Milnor fibration)

$$B(x, \varepsilon) \cap f^{-1}(B(0, \eta) \setminus \{0\}) \to B(0, \eta) \setminus \{0\}.$$

Here $B(a, r)$ denotes the closed ball of center a and radius r.

The Milnor fiber at x,

$$F_x = f^{-1}(\eta) \cap B(x, \varepsilon),$$

has a diffeomorphism type that does not depend on η and ε and is endowed with an automorphism, the monodromy M_x, induced by the characteristic mapping of the fibration.

In particular the cohomology groups $H^q(F_x, \mathbb{C})$ are endowed with an automorphism M_x.

Let us now explain how to make these invariants vary with x.

We change notations slightly, denoting by D a very small open disk around the origin in $\mathbb{C}$, and by $f : X \to D$ the restriction of f above D. We set $D^* := D \setminus \{0\}$ and denote by $\widetilde{D}^*$ the universal covering of D^*. We denote by X^* the preimage of D^* in D and by $\widetilde{X}^*$ the fiber product $X^* \times_{D^*} \widetilde{D}^*$. We consider morphisms $i : X_0 := f^{-1}(0) \hookrightarrow X$ and $j : \widetilde{X}^* \to X$.

If $\mathcal{F}$ is a constructible sheaf on X, for $q \geq 0$, one sets

$$R^q\psi_f\mathcal{F} := i^* R^q j_* j^* \mathcal{F}.$$

It is a constructible sheaf on X_0. Furthermore the deck transformation on $\widetilde{X}^*$ induces the action of a canonical monodromy automorphism on $R^q\psi_f\mathcal{F}$.

One may check [exercise] that, as a vector space with automorphism, $R^q\psi_f\mathbb{C}_X$ is canonically isomorphic to $(H^q(F_x, \mathbb{C}), M_x)$.

One could work as well in derived categories D^b_{constr} and $D^b_{\mathrm{constr,aut}}$ of bounded complexes with constructible cohomology, resp. bounded complexes with automorphism and constructible cohomology, defining the nearby cycle functor

$$R\psi_f : D^b_{\mathrm{constr}}(X) \longrightarrow D^b_{\mathrm{constr,aut}}(X_0)$$

by

$$R\psi_f K := i^* Rj_* j^* K.$$

Note that, while $R\psi_f K$ depends only on the restriction of K to X^*, this is not the case for the vanishing cycle functor $R\phi_f$ defined by the triangle

$$i^* K \longrightarrow R\psi_f K \longrightarrow R\phi_f K.$$

Denote by $\mathcal{X}^1_{n,x}$ the set of arcs φ in $\mathcal{L}_n(X)$ with $\varphi(0) = x$ such that

$$f(\varphi(t)) = t^n + (\text{higher order terms}).$$

Consider a log-resolution $h : Y \to X$ of $f^{-1}(0)$ such that $h^{-1}(x)$ in a union of components E_i, $i \in A_0$. Similarly as above,

$$[\mathcal{X}^1_{n,x}] = \mathbb{L}^{nd} \sum_{I \cap A_0 \neq \emptyset} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}^\circ_I] \left(\sum_{k_i \geq 1, i \in I, \sum_I k_i N_i(f) = n} \mathbb{L}^{-\sum k_i \nu_i} \right)$$

with $\widetilde{E}^\circ_I \to E^\circ_I$ an étale cover of degree $\gcd(N_i(f))_{i \in I}$.

Taking Eu of both sides, all terms with $|I| \geq 2$ cancel out, and one gets

$$\mathrm{Eu}(\mathcal{X}^1_{n,x}) = \sum_{N_i(f)|n, i \in A_0} N_i(f)\, \mathrm{Eu}(E^0_{\{i\}}). \qquad (\flat)$$

By a classical result of A'Campo, the right hand side of $(\flat)$ is equal, for $n \geq 1$, to the n-th Lefschetz number

$$\Lambda^n(M_x) := \sum_j (-1)^j \,\mathrm{tr}\,(M_x^n; H^j(F_x)).$$

Hence we get :

Theorem (Denef-Loeser)

For $n \geq 1$,

$$\mathrm{Eu}(\mathcal{X}^1_{n,x}) = \Lambda^n(M_x).$$

Challenging Problem: Find a direct, geometric proof.

Recently, Nicaise and Sebag have been able to restate and generalize that result within the framework of rigid geometry.

We now work over a field k of characteristic 0. Let X be a smooth variety over k of pure dimension d and consider a morphism $f : X \to \mathbb{A}^1_k$. We consider, for $n \geq 1$, the variety

$$\mathcal{X}_n(f) := \left\{ \varphi \in \mathcal{L}_n(X) \;\middle|\; \operatorname{ord}_t f(\varphi) = n \right\}.$$

Note that $\mathcal{L}_n(X)$ - and $\mathcal{L}(X)$ - is endowed with a canonical $\mathbb{G}_m$-action

$$a \cdot \varphi(t) := \varphi(at)$$

under which $\mathcal{X}_n(g)$ is invariant.

Furthermore f induces a morphism

$$f_n : \mathcal{X}_n(f) \to \mathbb{G}_m,$$

assigning to a point φ in $\mathcal{X}_n(f)$ the coefficient $\mathrm{ac}(f(\varphi))$ of t^n in $f(\varphi)$. Since

$$f_n(a \cdot \varphi) = a^n f_n(\varphi),$$

the fiber

$$\mathcal{X}_n^1 := f_n^{-1}(1)$$

is canonically endowed with a μ_n-action, with μ_n the group scheme

$$\mu_n := \operatorname{Spec} k[T]/T^n - 1.$$

We set

$$\widehat{\mu} = \varprojlim \mu_n$$

under $\mu_{nm} \to \mu_n$ given by $\zeta \mapsto \zeta^m$.

Denoting by $X_0(f)$ the zero locus of f, we assign to $\mathcal{X}_n^1$ a class $[\mathcal{X}_n^1]$ in the equivariant relative Grothendieck groupe $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}$ which we now define.

We fix a k-variety S that we endow with the trivial μ_n-action and we denote by $K_0^{\mu_n}(\mathrm{Var}_S)$ the quotient of the free abelian group on isomorphism classes of equivariant μ_n morphisms $Y \to S$ with Y a variety with (good) μ_n-action by the additivity relation and the following additional relation

$$[A \to Y \to S] = [A' \to Y \to S]$$

if A and A' are two affine bundles of the same rank over $Y \to S$ with affine μ_n-action lifting the same μ_n-action on Y.

It is naturally endowed with a ring structure.

We then set

$$\mathcal{M}_S^{\mu_n} := K_0^{\mu_n}(\mathrm{Var}_S)[\mathbb{L}^{-1}]$$

with $\mathbb{L}$ the class of the trivial rank one affine bundle over S and

$$\mathcal{M}_S^{\widehat{\mu}} := \varprojlim \mathcal{M}_S^{\mu_n}$$

under the the projective system $\mu_{nm} \longrightarrow \mu_n$.

We can now consider the following series in $\mathcal{M}^{\widehat{\mu}}_{X_0(f)}[[T]]$

$$Z_f(T) := \sum_{n \geq 1} [\mathcal{X}^1_n(f)]\,\mathbb{L}^{-nd}\,T^n,$$

which is a motivic analogue of Igusa's local zeta function.

Note that $Z_f = 0$ if f is identically 0 on X, so we may assume $X_0(f)$ is nowhere dense in X.

We shall now express $Z_f(T)$ in terms of a log-resolution $h : Y \to X$ of $X_0(f)$.

We denote by ν_{E_i} the normal bundle to E_i in Y, by U_{E_i} the complement of the zero section in ν_{E_i} and by U_I the fiber product of the restrictions of the spaces U_{E_i}, i in I, to E_I°.

There is a canonical $\mathbb{G}_m$-action on each U_{E_i} and we consider the diagonal action on U_I.
After taking the quotient of $f \circ h$ by a suitable monomial we get a morphism

$$f_I : U_I \to \mathbb{G}_m$$

such that

$$f_I(\lambda \cdot x) = \lambda^n f_I(x)$$

with

$$n = \sum_{i \in I} N_i(f).$$

It follows that $U_I^1 := f_I^{-1}(1)$ is endowed with a μ_n-action, so we can consider its class $[U_I^1]$ in $\mathcal{M}^{\widehat{\mu}}_{X_0(f)}$.

Using the change of variables formula, one gets

$$Z_f(T) = \sum_{\emptyset \neq I \subset A} [U_I^1] \prod_{i \in I} \frac{1}{T^{-N_i(f)} \mathbb{L}^{\nu_i} - 1}$$

in $\mathcal{M}^{\widehat{\mu}}_{X_0(f)}[[T]]$.

In particular, the function $Z_f(T)$ is rational.

Little is known about the poles of $Z_f(T)$.

Monodromy Conjecture

Let r be a rational number. If $\mathbb{L}^r$ is a pole of $Z_f(T)$, then $\exp(2i\pi r)$ is an eigenvalue of the monodromy on the stalk of the nearby cycles at some point of $X_0(f)$.

Besides some very specific examples and low dimensional cases, the conjecture is largely open.

Since

$$\lim_{T \mapsto \infty} \frac{1}{T^{-N_i(f)}\mathbb{L}^{\nu_i} - 1} = -1,$$

we get

Theorem (Denef-Loeser)

The limit

$$-\mathcal{S}_f := \lim_{T \mapsto \infty} Z_f(T)$$

is well-defined in $\mathcal{M}^{\widehat{\mu}}_{X_0(f)}$ *and given a log-resolution h we have*

$$\mathcal{S}_f = - \sum_{\emptyset \neq I \subset A} (-1)^{|I|} [U_I^1]. \tag{$\sharp$}$$

Note it is a priori non trivial that the right-hand side of ($\sharp$) is independent from the log-resolution h.

If x is a closed k-point of $X_0(f)$, by restricting to arcs with origin at x, one defines similarly $\mathcal{S}_{f,x}$ in $\mathcal{M}_{k(x)}^{\widehat{\mu}}$.

We claim that $\mathcal{S}_f$ is the motivic incarnation in $\mathcal{M}_{X_0(f)}^{\widehat{\mu}}$ of the complex of nearby cycles $R\psi_f\mathbb{Q}_X$.

Assume for simplicity $k = \mathbb{C}$. The group $\widehat{\mu}$ is topologically generated by $\zeta := (\exp(2i\pi/n))$. We have a Betti realization morphism

$$\mathrm{Eu} : \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}} \longrightarrow K_0(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{mon}})$$

with $K_0(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{mon}})$ the Grothendieck ring of finite dimensional vector spaces with an automorphism sending the class of variety to its virtual cohomology with compact support, with automorphism the one induced by ζ.

It follows from A'Campo formula that

$$\mathrm{Eu}(\mathcal{S}_{f,x}) = [(H^{\cdot}(F_x), M_x)]$$

in $K_0(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{mon}})$.

A similar result holds at the Hodge level. There is a Hodge realization morphism

$$H : \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}} \longrightarrow K_0(\mathrm{HS}^{\mathrm{mon}})$$

with $K_0(\mathrm{HS}^{\mathrm{mon}})$ the Grothendieck ring of Hodge structures with an automorphism of finite order.

One can prove that $H(\mathcal{S}_{f,x})$ is equal to the class of the mixed Hodge structure - constructed by Steenbrink and Navarro-Aznar - on $H^{\cdot}(F_x)$ with the monodromy automorphism, whose semi-simplification is of finite order by the monodromy Theorem.

There is a natural linear map, called the Hodge spectrum,

$$\mathrm{hsp} : K_0(\mathrm{HS}^{\mathrm{mon}}) \longrightarrow \mathbb{Z}[\mathbb{Q}],$$

such that

$$\mathrm{hsp}([(H, T)]) := \sum_{\alpha \in \mathbb{Q} \cap [0,1)} t^{\alpha} \Big(\sum_{p,q \in \mathbb{N}} \dim(H_{\alpha}^{p,q}) t^{p} \Big),$$

for any Hodge structure H with an automorphism of finite order T, where $H_{\alpha}^{p,q}$ is the eigenspace of $H^{p,q}$ with respect to the eigenvalue $\exp(2\pi i \alpha)$.

Here we identify $\mathbb{Z}[\mathbb{Q}]$ with $\cup_{n \geq 1} \mathbb{Z}[t^{1/n}, t^{-1/n}]$.

The Steenbrink Hodge spectrum of f at x is defined as

$$\mathrm{Sp}(f,x) := (-1)^{d-1}\,\mathrm{hsp}([(H^{\cdot}(F_x), M_x)]).$$

If one defines the motivic vanishing cycles by

$$\mathcal{S}^{\phi}_{f,x} := (-1)^{d-1}(\mathcal{S}_{f,x} - 1),$$

then

$$\mathrm{Sp}(f,x) = (\mathrm{hsp} \circ \mathrm{Eu})(\mathcal{S}^{\phi}_{f,x}).$$

If G is a finite group scheme and X and Y are two varieties with (good) G-action, we denote by

$$X \times^G Y$$

the quotient of the product $X \times Y$ by the equivalence relation $(gx, y) \equiv (x, gy)$.

The G-action on one factor induces a good G-action on $X \times^G Y$.

For $n \geq 1$, we consider the Fermat varieties

$$F_1^n := \left\{(x, y) \in \mathbb{G}_m^2 \,\middle|\, x^n + y^n = 1\right\}$$

and

$$F_0^n := \left\{(x, y) \in \mathbb{G}_m^2 \,\middle|\, x^n + y^n = 0\right\}.$$

Let X be a variety with $\mu_n \times \mu_n$-action.
We set

$$\Psi_\Sigma(X) := -[F_1 \times^{\mu_n \times \mu_n} X] + [F_0 \times^{\mu_n \times \mu_n} X]$$

in $\mathcal{M}_{\mathbb{C}}^{\mu_n}$,

the μ_n-action being the diagonal one.

The construction goes through the projective limit, so we get a group morphism

$$\Psi_\Sigma : \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}\times\widehat{\mu}} \longrightarrow \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}}.$$

The convolution product

$$* : \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}} \times \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}} \longrightarrow \mathcal{M}_{\mathbb{C}}^{\widehat{\mu}}$$

is then defined by

$$[X] * [Y] := \psi_\Sigma([X \times Y]).$$

It is commutative and associative.

Let X_1 and X_2 be smooth varieties of pure dimension d_1 and d_2 and consider functions $f_1 : X_1 \to \mathbb{A}^1$ and $f_2 : X_2 \to \mathbb{A}^1$. Denote by $f_1 \oplus f_2$ the function on $X_1 \times X_2$ sending (x_1, x_2) to $f_1(x_1) + f_2(x_2)$.

When f_1 and f_2 have isolated singular points x_1 and x_2, Thom and Sebastiani proved that the Milnor fiber with monodromy action of $f_1 \oplus f_2$ at (x_1, x_2) is the join of those of f_1 and f_2 at x_1 and x_2 respectively.

Theorem (Steenbrink, Varchenko, Saito)

$$\mathrm{Sp}(f_1 \oplus f_2, (x_1, x_2)) = \mathrm{Sp}(f_1, x_1) \cdot \mathrm{Sp}(f_2, x_2).$$

We can now state the motivic version of the Thom-Sebastiani Theorem:

Theorem (Denef-Loeser, Looijenga)

$$\mathcal{S}^{\phi}_{f_1 \oplus f_2,(x_1,x_2)} = \mathcal{S}^{\phi}_{f_1,x_1} * \mathcal{S}^{\phi}_{f_2,x_2}.$$

Sometimes guessing the motivic analogue of a statement involving the Hodge spectrum can be less immediate.

Consider a function f on a smooth complex variety X and a point x in the zero locus of f. Let us assume that the singular locus of f is a curve Γ, having r local components Γ_ℓ, $1 \leq \ell \leq r$, in a neighborhood of x. We denote by m_ℓ the multiplicity of Γ_ℓ.

Let g be a function vanishing at x whose differential at x is a generic linear form).

For N large enough, the function $f + g^N$ has an isolated singularity at x. In a neighborhood of the complement Γ_ℓ° to $\{x\}$ in Γ_ℓ, we may view f as a family of isolated hypersurface singularities parametrized by Γ_ℓ°.

The cohomology of the Milnor fiber of that isolated hypersurface singularity is naturally endowed with the action of two commuting monodromies: the monodromy of the function and the monodromy of a generator of the local fundamental group of Γ_ℓ°.

We denote by $\alpha_{\ell,j}$ the exponents - counted with multiplicity - of the Hodge spectrum of that isolated hypersurface singularity and by $\beta_{\ell,j}$ the corresponding rational numbers in $[0,1)$ such that the complex numbers $\exp(2\pi i \beta_{\ell,j})$ are the eigenvalues of the monodromy along Γ_ℓ°.

The following statement has been conjectured by J. Steenbrink:

Theorem (M. Saito)

For $N \gg 0$,

$$\mathrm{Sp}(f + g^N, x) - \mathrm{Sp}(f, x) = \sum_{\ell,j} t^{\alpha_{\ell,j} + (\beta_{\ell,j}/m_\ell N)} \frac{1-t}{1-t^{1/m_\ell N}}.$$

The motivic analogue of the Steenbrink conjecture is the following:

Theorem (Guibert -Loeser -Merle)

Let X be a smooth variety and f and g be two functions from X to $\mathbb{A}^1$. Let x be a closed point of $X_0(f) \cap X_0(g)$. For $N \gg 0$, the equality

$$\mathcal{S}^{\phi}_{f,x} - \mathcal{S}^{\phi}_{f+g^N,x} = \Psi_{\Sigma}(i_x^*(\mathcal{S}_{g^N}(\mathcal{S}^{\phi}_f)))$$

holds.

We should explain the meaning of the term

$$\Psi_{\Sigma}(i_x^*(\mathcal{S}_{g^N}(\mathcal{S}^{\phi}_f))).$$

An extension of $\mathcal{S}_f$ to a morphism

$$\mathcal{S}_f : \mathcal{M}_{X_0(f)} \longrightarrow \mathcal{M}^{\widehat{\mu}}_{X_0(f)}$$

has been constructed by Bittner (using weak factorization) and by Guibert-Loeser-Merle (using motivic integration). This should be viewed as the analogue of considering nearby cycles for complexes of contructible sheaves instead of just the constant sheaf.

The construction can be carried out equivariantly leading to a morphism

$$\mathcal{S}_f : \mathcal{M}^{\widehat{\mu}}_{X_0(f)} \longrightarrow \mathcal{M}^{\widehat{\mu}\times\widehat{\mu}}_{X_0(f)}.$$

Thus $\mathcal{S}_{g^N}(\mathcal{S}^{\phi}_f)$ lives in $\mathcal{M}^{\widehat{\mu}}_{X_0(f)\cap X_0(g)}$ and i_x^* stands for taking the fiber over x.

To give a flavour of the way the extension of $\mathcal{S}_f$ to the whole Grothendieck group is performed by Guibert, Loeser and Merle, let us explain a key example, the construction of $\mathcal{S}_f([U])$ when U is a dense open subset of X.

Denote by F the closed subset $X \setminus U$ and by $\mathcal{I}_F$ the ideal of functions vanishing on F.

Fix $\gamma \geq 1$ a positive integer. For $n \geq 1$, we consider the constructible set

$$\mathcal{X}_n^{\gamma n}(f, U) := \Big\{ \varphi \in \mathcal{L}_{\gamma n}(X) \;\Big|\; \operatorname{ord}_t f(\varphi) = n, \operatorname{ord}_t \varphi^*(\mathcal{I}_F) \leq \gamma n \Big\}.$$

We consider the modified zeta function

$$Z^{\gamma}_{f,U}(T) := \sum_{n\geq 1} [\mathcal{X}^{\gamma n}_{n}(f,U)]\, \mathbb{L}^{-\gamma nd}\, T^{n}$$

in $\mathcal{M}^{\widehat{\mu}}_{X_0(f)}[[T]]$.

Note that for $U = X$, $Z^{\gamma}_{f,U}(T)$ is equal to $Z_f(T)$ for every γ and that if f vanishes on X it is 0.

If $X_0(f)$ is nowhere dense in X, considering a log-resolution of $(X, F \cup X_0(f))$, one proves

Proposition

There exists γ_0 such that for every $\gamma > \gamma_0$ the series $Z^{\gamma}_{f,U}(T)$ lies is rational and admits a limit $\lim_{T \mapsto \infty} Z^{\gamma}_{f,U}(T)$ which is independent of $\gamma > \gamma_0$.

One sets

$$\mathcal{S}_f([U]) := - \lim_{T \mapsto \infty} Z^{\gamma}_{\gamma,U}(T)$$

for $\gamma \gg 0$.

The proof of the motivic analogue of Steenbrink's conjecture takes place on the arc space $\mathcal{L}(X)$. The basic idea behind is the following:

- If $\operatorname{ord}_t f < N \operatorname{ord}_t g$, then $\operatorname{ord}_t(f + g^N) = \operatorname{ord}_t f$ and $\operatorname{ac}(f + g^N) = \operatorname{ac}(f)$, so the contributions to $\mathcal{S}_{f+g^N}$ and $\mathcal{S}_f$ are the same.
- Arcs with $\operatorname{ord}_t f > N \operatorname{ord}_t g$ essentially do not contribute to $\mathcal{S}_{f+g^N}$ except for a term $\mathcal{S}_{g^N}([X_0(f)])$.
- The main contribution to $\mathcal{S}_{f+g^N}$ comes from arcs with $\operatorname{ord}_t f = N \operatorname{ord}_t g$. The key geometric fact is that on a log-resolution h, the function $f_I + g_I^N$ is smooth near arcs with $\operatorname{ord}_t(f \circ h) = N \operatorname{ord}_t(g \circ h)$.

In this final lecture, we shall present a new general setting for motivic integration developed in joint work with Raf Cluckers.

Let us start by a quick detour through semialgebraic geometry.

Semialgebraic subsets of $\mathbb{R}^n$ are defined by finite boolean combination of equations

$$f_i(x_1, \cdots, x_n) = 0$$

and inequations

$$g_j(x_1, \cdots, x_n) \geq 0$$

with f_i and g_j polynomials in $\mathbb{R}[x_1, \cdots, x_n]$.

Semialgebraic subsets of $\mathbb{R}^n$ for varying n form a category $\mathrm{SA}_{\mathbb{R}}$, morphisms being functions with semialgebraic graph.

Imagine we know no topology at all and we want to define

$$\mathrm{Eu}(X) := \sum_i (-1)^i \operatorname{rk} H^i_c(X, \mathbb{Q})$$

when X is a semialgebraic subset of $\mathbb{R}^n$.

We know

$$\mathrm{Eu}(\text{point}) = 1$$

and

$$\mathrm{Eu}(\text{open interval}) = -1.$$

Fix $n \geq 0$. A 0-cell in $\mathbb{R}^n$ is a semialgebraic subset Z_A^0 of the form

$$\left\{(x_1, \cdots, x_n) \in \mathbb{R}^n \;\middle|\; x' := (x_1, \cdots, x_{n-1}) \in A, x_n = c(x')\right\},$$

with A semialgebraic in $\mathbb{R}^{n-1}$ and $c : \mathbb{R}^{n-1} \to \mathbb{R}$ a semialgebraic morphism.

A 1-cell in $\mathbb{R}^n$ is a semialgebraic subset Z_A^1 of the form

$$\Big\{(x_1, \cdots, x_n) \in \mathbb{R}^n \;\Big|\; x' := (x_1, \cdots, x_{n-1}) \in A, a(x') < x_n < b(x')\Big\},$$

with A semialgebraic in $\mathbb{R}^{n-1}$, a and b semialgebraic morphisms $\mathbb{R}^{n-1} \to \mathbb{R} \cup \{\pm\infty\}$.

Basic cell decomposition Theorem

Every semialgebraic subset of $\mathbb{R}^n$ is a finite disjoint union of 0-cells and 1-cells.

By additivity it enough to define Eu for cells. By induction on n, one sets

$$\mathrm{Eu}(Z_A^0) = \mathrm{Eu}(A)$$

and

$$\mathrm{Eu}(Z_A^1) = -\,\mathrm{Eu}(A).$$

Theorem

1. *The above definition of* $\mathrm{Eu}(A)$, *for* A *semialgebraic, makes sense.*
2. *If* A *and* A' *are isomorphic,*
$$\mathrm{Eu}(A) = \mathrm{Eu}(A').$$

For the proof of 2), the key point is to prove that Eu is invariant under permutation of coordinates.

The above results still hold in the abstract setting of *o*-minimal structures (expansions of the theory of ordered abelian groups for which definable subsets of the line are finite unions of intervals). The Theorem is then due to Knight, Pillay and Steinhorn.

For X semialgebraic, we denote by $\mathcal{C}(X)$ the algebra of $\mathbb{Z}$-valued constructible functions on X. It is generated by characteristic functions $\mathbf{1}_Z$ with Z semialgebraic subset of X.

Constructing Eu is equivalent to constructing functorial push-forward morphisms $f_! : \mathcal{C}(X) \to \mathcal{C}(Y)$ for $f : X \to Y$ a morphism in $\mathrm{SA}_{\mathbb{R}}$.

By a graph construction one reduces to defining the push-forward for inclusions and for projections.

For inclusions, push-forward is just extension by zero.
For projections one reduces to projections $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and one defines $p_!$ by

$$p_!(\mathbf{1}_{Z_A^0}) := \mathbf{1}_A$$

and

$$p_!(\mathbf{1}_{Z_A^1}) := -\mathbf{1}_A.$$

The key point amounts to proving the follow Fubini type result

$$p_{i!} \circ p_{j!} = p_{j!} \circ p_{i!},$$

for $i \neq j$, with p_i the projection omitting the x_i-coordinate.

Consider the Grothendieck ring $K_0(\mathrm{SA}_{\mathbb{R}})$ defined as the quotient of the free abelian group on isomorphism classes of semialgebraic sets modulo the relation

$$[X] = [X'] + [X \setminus X']$$

for X' contined in X.

Since every semialgebraic sets m admits a semialgebraic triangulation, Eu induces an isomorphism

$$\mathrm{Eu} : K_0(\mathrm{SA}_{\mathbb{R}}) \to \mathbb{Z}.$$

More generally, for X in $\mathrm{SA}_{\mathbb{R}}$, one may consider the category SA_X whose objects are morphisms $p : Y \to X$ and the corresponding Grothendieck rings $K_0(\mathrm{SA}_X)$.

The mapping

$$[p : Y \to X] \longmapsto p_!(\mathbf{1}_Y)$$

induces an isomorphism between $K_0(\mathrm{SA}_X)$ and $\mathcal{C}(X)$.

If $f : X \to Z$ is a morphism in $\mathrm{SA}_{\mathbb{R}}$, the morphism $K_0(\mathrm{SA}_X) \to K_0(\mathrm{SA}_Z)$ corresponding to $f_!$ under the above isomorphism is given by

$$[p : Y \to X] \longmapsto [f \circ p : Y \to Z].$$

By Tarski's Theorem, the projection of a semialgebraic set is still semialgebraic.

It follows that semialgebraic subsets of $\mathbb{R}^n$ are just the subsets definable by a formula in the language of rings - or in the language of ordered rings - with coefficients in $\mathbb{R}$ and n free variables.

A first order formula in a language $\mathcal{L}$ is a formula written with symbols in $\mathcal{L}$, logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (negation), quantifiers $\exists$, $\forall$, and variables.

The language of rings with coefficients in S is given by symbols 0, $+$, $-$, 1, $\times$ and a symbol for each element of S.

For the language of ordered rings one adds a symbol $<$.

Let us fix a field k of characteristic zero.

What are the definable subsets of $k((t))^n$ we would like to consider?

- Certainly subsets definable in the language of rings
- Subsets definable in the language of valued rings, e.g. by conditions like

$$\operatorname{ord}(f(x)) < \operatorname{ord}(g(x))$$

- We also encountered subsets defined by conditions like

$$\operatorname{ac}(f(x)) = \text{something}.$$

In fact, it is more convenient to consider definable subsets of $k((t))^n \times k^m \times \mathbb{Z}^m$ in the following language $\mathcal{L}_{\mathrm{DP}}$ of Denef-Pas with three sorts of variables.

- For the valued field sort, the language of rings with coefficients in $k((t))$.
- For the residue field sort, the language of rings with coefficients in k.
- For the value group sort, the language $\{0, 1, +, -, <\}$ of ordered groups.
- two additional symbols ac and ord from the valued field sort to the residue field and valued groups sort, respectively.

We shall ignore the minor, and easily settled, issue about ord 0.

Let φ be a formula in the language $\mathcal{L}_{\mathrm{DP}}$ having respectively m, n, and r free variables in the various sorts. To such a formula φ we assign, for every field K containing k, the subset $h_\varphi(K)$ of $K((t))^m \times K^n \times \mathbb{Z}^r$ consisting of all points satisfying φ.

We shall call the datum of such subsets for all K definable (sub)assignments. In analogy with algebraic geometry, where the emphasis is not put anymore on equations but on the functors they define, we consider instead of formulas the corresponding subassignments (note $K \mapsto h_\varphi(K)$ is in general not a functor).

Let $F : \mathcal{C} \to \mathrm{Ens}$ be a functor from a category $\mathcal{C}$ to the category of sets. By a subassignment h of F we mean the datum, for every object C of $\mathcal{C}$, of a subset $h(C)$ of $F(C)$.

Most of the standard operations of elementary set theory extend trivially to subassignments. For instance, given subassignments h and h' of the same functor, one defines subassignments $h \cup h'$, $h \cap h'$ and the relation $h \subset h'$, etc.

When $h \subset h'$ we say h is a subassignment of h'.

A morphism $f : h \to h'$ between subassignments of functors F_1 and F_2 consists of the datum for every object C of a map

$$f(C) : h(C) \to h'(C).$$

The graph of f is the subassignment

$$C \mapsto \mathrm{graph}(f(C))$$

of $F_1 \times F_2$.

Let k be a field and consider the category F_k of fields containing k. We denote by $h[m,n,r]$ the functor $F_k \to \mathrm{Ens}$ given by

$$h[m,n,r](K) = K((t))^m \times K^n \times \mathbb{Z}^r.$$

In particular, $h[0,0,0]$ assigns the one point set to every K.

To any formula φ in $\mathcal{L}_{\mathrm{DP}}$ having respectively m, n, and r free variables in the various sorts, we assign a subassignment h_φ of $h[m,n,r]$, which associates to K in F_k the subset $h_\varphi(K)$ of $h[m,n,r](K)$ consisting of all points satisfying φ. Such subassignments are called definable subassignements.

We denote by Def_k the category whose objects are definable subassignments of some $h[m, n, r]$, morphisms in Def_k being morphisms of subassignments $f : h \to h'$ with h and h' definable subassignments of $h[m, n, r]$ and $h[m', n', r']$ respectively such that the graph of f is a definable subassignment.

Note that $h[0, 0, 0]$ is the final object in this category.

We want to assign to any definable subassignment S a measure $\mu(S)$ in some Grothendieck ring of objects defined over k.

We consider the subcategory RDef_k of Def_k consisting of definable subassignments of some $h[0, n, 0]$. One can prove they are defined by formulas involving residue field variables only, and no other sorts.

One denotes by $SK_0(\mathrm{RDef}_k)$ the free abelian semigroup on isomorphism classes of objects of RDef_k modulo the additivity relation

$$[X] + [X'] = [X \cup X'] + [X \cap X'].$$

It is endowed with a natural semiring structure. One defines similarly the Grothendieck ring $K_0(\mathrm{RDef}_k)$, which is the ring associated to the semiring $SK_0(\mathrm{RDef}_k)$.

We consider the ring

$$\mathbb{A} := \mathbb{Z}\Big[\mathbb{L}, \mathbb{L}^{-1}, \Big(\frac{1}{1-\mathbb{L}^{-i}}\Big)_{i>0}\Big].$$

For q a real number > 1, we denote by ϑ_q the ring morphism

$$\vartheta_q : \mathbb{A} \longrightarrow \mathbb{R}$$

sending $\mathbb{L}$ to q and we consider the semiring

$$\mathbb{A}_+ := \Big\{x \in \mathbb{A} \;\Big|\; \vartheta_q(x) \geq 0, \forall q > 1\Big\}.$$

We denote by $\mathbb{L} - 1$ the class of the subassignment $x \neq 0$ of $h[0, 1, 0]$ in $SK_0(\mathrm{RDef}_k)$, resp $K_0(\mathrm{RDef}_k)$.

We set

$$\mathcal{C}_+(\text{point}) := SK_0(\mathrm{RDef}_k) \otimes_{\mathbb{N}[\mathbb{L}-1]} \mathbb{A}_+$$

and

$$\mathcal{C}(\text{point}) := K_0(\mathrm{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}.$$

There are canonical morphisms

$$\mathcal{C}_+(\text{point}) \longrightarrow \mathcal{C}(\text{point})$$

and

$$\mathcal{C}(\text{point}) \longrightarrow \widehat{\mathcal{M}}_k.$$

The second one is constructed by assigning to any formula over residue field variables the class of the corresponding constructible set and by developing $(1 - \mathbb{L}^{-i})^{-1}$ into series.

To any bounded - that is, contained in $\operatorname{ord} x_i \geq \alpha$ for α small enough - definable subassignment A of $h[n, 0, 0]$, we assign a volume $\mu(A)$ in $\mathcal{C}_+(\text{point})$ whose image in $\widehat{\mathcal{M}}_k$ coincides with the one given by the previous construction.

Completion is not needed anymore!

We shall use coordinates $x = (x', z)$ on $h[n+1, m, r]$ with x' running over $h[n, m, r]$ and z over $h[1, 0, 0]$.

A 0-cell in $h[n+1, m, r]$ is a definable subassignment Z_A^0 defined by

$$x' \in A \quad \text{and} \quad z = c(x')$$

with A a definable subassignment of $h[n, m, r]$ and c a morphism $A \to h[1, 0, 0]$.

A 1-cell in $h[n+1, m, r]$ is a definable subassignment Z_A^1 defined by

$$x' \in A, \quad \mathrm{ac}(z - c(x')) = \xi(x') \quad \text{and} \quad \mathrm{ord}(z - c(x')) = \alpha(x')$$

with A a definable subassignment of $h[n, m, r]$, c, ξ and α morphisms from A to $h[1, 0, 0]$, $h[0, 1, 0] \setminus \{0\}$ and $h[0, 0, 1]$, respectively.

Clearly , it cannot be true that every definable subassignment is a finite disjoint union of cells, but . . .

Theorem (Denef-Pas Cell Decomposition)

Let A be a definable subassignment $h[n+1, m, r]$. After adding a finite number of auxiliary parameters in the residue field and value group sorts, A is a finite disjoint union of cells, that is, there exists an embedding

$$\lambda : h[n+1, m, r] \longrightarrow h[n+1, m+m', r+r']$$

such that the projection of λ with the projection to $h[n+1, m, r]$ is the identity on A and such that $\lambda(A)$ is a finite disjoint union of cells.

The construction of the motivic measure $\mu(A)$ for a definable subassignment A of $h[n, m, r]$ goes roughly as follows:

- Using cell decomposition one may by induction reduce to $n = 0$, at the cost of increasing m and r.
- When $n = 0$, one may consider the counting measure on the $\mathbb{Z}^r$-factor and the tautological measure on the $h[0, m, 0]$-factor, assigning to a definable subassigment of $h[0, m, 0]$ its class in $\mathcal{C}_+(\text{point})$.

That such a construction is invariant under permutations of valued field coordinates is absolutely non trivial!

It is in fact a key statement underlying the motivic Fubini Theorem.

In fact it is much more convenient to consider from start functions instead of just subsets.

So let us define for X in Def_k the semiring $\mathcal{C}_+(X)$, resp ring $\mathcal{C}(X)$, of non negative constructible motivic functions, resp. constructible motivic functions:

One considers the subcategory RDef_X of Def_x whose objects are definable subassignments of $X \times h[0, n, 0]$, for variable n, and the corresponding semiring, resp. ring, $SK_0(\mathrm{RDef}_X)$ and $K_0(\mathrm{RDef}_X)$.

We denote by $|X|$ the set of points of X and consider the subring $\mathcal{P}(X)$ of the ring of functions $|X| \to \mathbb{A}$ generated by constants in $\mathbb{A}$ and by all functions α and $\mathbb{L}^{\alpha}$ with $\alpha : X \to \mathbb{Z}$ definable morphisms.

We define $\mathcal{P}_+(X)$ as the semiring of functions in $\mathcal{P}(X)$ taking their values in $\mathbb{A}_+$.

We consider the subring $\mathcal{P}^0(X)$ of $\mathcal{P}(X)$, resp. the subsemiring $\mathcal{P}^0_+(X)$ of $\mathcal{P}_+(X)$, generated by functions of the form $\mathbf{1}_Y$ with Y a definable subassignment of X, and by the constant function $\mathbb{L} - 1$.

We have canonical morphisms $\mathcal{P}^0(X) \to K_0(\mathrm{RDef}_X)$ and $\mathcal{P}^0_+(X) \to SK_0(\mathrm{RDef}_X)$.

We may now set

$$\mathcal{C}_+(X) = SK_0(\mathrm{RDef}_X) \otimes_{\mathcal{P}^0_+(X)} \mathcal{P}_+(X)$$

and

$$\mathcal{C}(X) = K_0(\mathrm{RDef}_X) \otimes_{\mathcal{P}^0(X)} \mathcal{P}(X).$$

Using Denef-Pas Cell Decomposition, Cluckers and Loeser are able to give an inductive definition of integrability, for a function φ in $\mathcal{C}_+(X)$, resp . $\mathcal{C}(X)$, and when it is integrable to define its integral

$$\mu(\varphi) = \int_X \varphi d\mu$$

in $\mathcal{C}_+(X)$, resp. in $\mathcal{C}(X)$.

At the same time, one proves a very general form of Fubini's Theorem.

For the constructions it is useful to work first with non negative functions.

In this setting one can state the following general form of the Change of Variables Theorem.

Theorem (Cluckers-Loeser)

Let $f : Y \to X$ be an isomorphism in Def_k. For any integrable function φ in $\mathcal{C}_+(X)$ or $\mathcal{C}(X)$,

$$\int_X \varphi d\mu = \int_Y \mathbb{L}^{-\operatorname{ord}\operatorname{jac}(f)} f^*(\varphi) d\mu.$$

Here $\operatorname{ord}\operatorname{jac}(f)$ stands for the order of the jacobian of f (which is defined only almost everywhere) and $f^*(\varphi)$ for the natural pullback of φ under f.

Some of the main features of the Theory

- Once the Change of Variables Theorem is available, one may develop the integration on global (non affine) objects endowed with a differental form of to degree (similarly as in the p-adic case).
- One can develop a relative theory of motivic integration: integrals depending on parameters of functions in $\mathcal{C}_+$ or $\mathcal{C}$ still belong to $\mathcal{C}_+$ or $\mathcal{C}$ as functions of parameters.

More specifically, if $f : X \to \Lambda$ is a morphism and φ is a function in $\mathcal{C}_+(X)$ or $\mathcal{C}(X)$ that is relatively integrable, one constructs a function

$$\mu_\Lambda(\varphi)$$

in $\mathcal{C}_+(\Lambda)$, resp. $\mathcal{C}(\Lambda)$, whose restriction to every fiber of f coincides with the integral of φ restricted to that fiber.

It is also possible to enlarge $\mathcal{C}(X)$ to a ring $\mathcal{C}(X)^{\mathrm{exp}}$ containing also motivic analogues of exponential functions and to construct a natural extension of the previous theory to $\mathcal{C}^{\mathrm{exp}}$.

At this stage, one is able to develop a motivic calculus as flexible and easy to use as the usual calculus over reals or the p-adics. For instance, we are able to construct

- Fourier transformation
- Convolution
- Schwartz-Bruhat spaces

and we are able to prove

- various forms of Fourier inversion.

In the first talk, we showed how p-adic integration has been used as a substitute for motivic integration before it was invented.

It is now time to loop the loop by explaining how motivic integrals specialize to p-adic integrals and may be used to obtain a general transfert principle allowing to transfer relations between integrals from $\mathbb{Q}_p$ to $\mathbb{F}_p((t))$ and vice-versa.

We shall assume from now on that k is a number field with ring of integers $\mathcal{O}$.

We denote by $\mathcal{A}_{\mathcal{O}}$ the set of p-adic completions of all finite extensions of k and by $\mathcal{B}_{\mathcal{O}}$ the set of all local fields of characteristic > 0 which are $\mathcal{O}$-modules.

Notation: For K in $\mathcal{C}_{\mathcal{O}} := \mathcal{A}_{\mathcal{O}} \cup \mathcal{B}_{\mathcal{O}}$, we denote by

- R_K the valuation ring
- M_K the maximal ideal
- k_K the residue field
- $q(K)$ the cardinal of k_K
- ϖ_K a uniformizing parameter of R_K.

There exists a unique morphism $\mathrm{ac} : K^\times \to k_K^\times$ extending $R_K^\times \to k_K^\times$ and sending ϖ_K to 1. We set $\mathrm{ac}(0) = 0$.

For $N > 0$, we denote by $\mathcal{A}_{\mathcal{O},N}$ the set of fields K in $\mathcal{A}_{\mathcal{O}}$ such that k_K has characteritic $> N$, and similarly for $\mathcal{B}_{\mathcal{O},N}$ and $\mathcal{C}_{\mathcal{O},N}$.

To be able to interpret our formulas to fields in $\mathcal{C}_{\mathcal{O}}$, we restrict the language $\mathcal{L}_{\mathrm{DP}}$ to the sub-language $\mathcal{L}_{\mathcal{O}}$ for which coefficients in the valued field sort are assumed to belong to the subring $\mathcal{O}[[t]]$ of $k((t))$.

We denote by $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$ the sub-category of Def_k of objects definable in $\mathcal{L}_{\mathcal{O}}$, and similarly for functions, etc.

For instance, for X in $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$, we denote by $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$ the ring of contructible functions on X definable in $\mathcal{L}_{\mathcal{O}}$.

We consider K as a $\mathcal{O}[[t]]$-algebra via

$$\lambda_{\mathcal{O},K} : \sum_{i \in \mathbf{N}} a_i t^i \longmapsto \sum_{i \in \mathbf{N}} a_i \varpi_K^i.$$

Hence, if we interpret a in $\mathcal{O}[[t]]$ by $\lambda_{\mathcal{O},K}(a)$, every $\mathcal{L}_{\mathcal{O}}$-formula φ defines for K in $\mathcal{C}_{\mathcal{O}}$ a subset φ_K of some $K^m \times k_K^n \times \mathbb{Z}^r$.

One proves that if two $\mathcal{L}_{\mathcal{O}}$-formulas φ and φ' define the same subassignment X of $h[m, n, r]$, then $\varphi_K = \varphi'_K$ for K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$.

This allows us to denote by X_K the subset defined by φ_K, for K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$.

Similarly, every $\mathcal{L}_{\mathcal{O}}$-definable morphism $f : X \to Y$ specializes to $f_K : X_K \to Y_K$ for K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$.

We now explain how φ in $\mathcal{C}(X, \mathcal{L}_{\mathcal{O}})$ can be specialized to $\varphi_K : X_K \to \mathbb{Q}$ for K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$.

Let us consider φ in $K_0(\mathrm{RDef}_X(\mathcal{L}_{\mathcal{O}}))$ of the form $[\pi : W \to X]$ when W in $\mathrm{RDef}_X(\mathcal{L}_{\mathcal{O}})$. For K in $\mathcal{C}_{\mathcal{O},N}$ when $N \gg 0$, we have $\pi_K : W_K \to X_K$, so we may define $\varphi_K : X_K \to \mathbb{Q}$ by

$$x \longmapsto \mathrm{card}\left(\pi_K^{-1}(x)\right).$$

For φ in $\mathcal{P}(X)$, we specialize $\mathbb{L}$ into q_K and $\alpha : X \to \mathbb{Z}$ into $\alpha_K : X_K \to \mathbb{Z}$.

By tensor product we get $\varphi \mapsto \varphi_K$ for φ in $\mathcal{C}(X, \mathcal{L}_{\mathcal{O}})$.

Note that, under that construction, functions in $\mathcal{C}_+(X, \mathcal{L}_{\mathcal{O}})$ specialize into non negative functions.

Let K be in $\mathcal{C}_{\mathcal{O}}$ and A be a subset of $K^m \times k_K^n \times \mathbb{Z}^r$. We consider the Zariski closure $\bar{A}$ of the projection of A into $\mathbb{A}_K^m$. One defines a measure μ on A by restriction of the product of the canonical measure on $\bar{A}(K)$ with the counting measure on $k_K^n \times \mathbb{Z}^r$.

Fix a morphism $f : X \to \Lambda$ in $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$ and consider φ in $\mathcal{C}(X, \mathcal{L}_{\mathcal{O}})$. One can show that if φ is relatively integrable, for $N \gg 0$ and every K in $\mathcal{C}_{\mathcal{O},N}$, then, for every λ in Λ_K, the restriction $\varphi_{K,\lambda}$ of φ_K to $f_K^{-1}(\lambda)$ is integrable.

We denote by $\mu_{\Lambda_K}(\varphi_K)$ the function on Λ_K defined by

$$\lambda \longmapsto \mu(\varphi_{K,\lambda}).$$

The following theorem says that motivic integrals specialize to the corresponding integrals over local fields of high enough residue field characteristic.

Theorem (Specialization, Cluckers-Loeser)

Let $f : S \to \Lambda$ be a morphism in Def$(\mathcal{L}_{\mathcal{O}})$. Let φ be in $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$ relatively integrable with respect to f. For $N \gg 0$, for every K in $\mathcal{C}_{\mathcal{O},N}$, we have

$$(\mu_{\Lambda}(\varphi))_K = \mu_{\Lambda_K}(\varphi_K).$$

We are now ready to state the following abstract transfer principle:

Theorem (Abstract transfer principle, Cluckers-Loeser)

Let φ be in $\mathcal{C}(\Lambda, \mathcal{L}_{\mathcal{O}})$. There exists N such that for every K_1, K_2 in $\mathcal{C}_{\mathcal{O},N}$ with $k_{K_1} \simeq k_{K_2}$,

$$\varphi_{K_1} = 0 \quad \textit{if and only if} \quad \varphi_{K_2} = 0.$$

Putting together the two previous theorems, one immediatly gets:

Theorem (Transfert principle for integrals with parameters, Cluckers-Loeser)

Let $S \to \Lambda$ and $S' \to \Lambda$ be morphisms in Def($\mathcal{L}_{\mathcal{O}}$). Let φ and φ' be relatively integrable functions in $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$ and $\mathcal{C}(S', \mathcal{L}_{\mathcal{O}})$, respectively. There exists N such that for every K_1, K_2 in $\mathcal{C}_{\mathcal{O},N}$ with $k_{K_1} \simeq k_{K_2}$,

$$\mu_{\Lambda_{K_1}}(\varphi_{K_1}) = \mu_{\Lambda_{K_1}}(\varphi'_{K_1}) \quad \textit{if and only if} \quad \mu_{\Lambda_{K_2}}(\varphi_{K_2}) = \mu_{\Lambda_{K_2}}(\varphi'_{K_2}).$$

In the special case where $\Lambda = h[0, 0, 0]$, this follows from previous results of Denef-Loeser.

Note that when $S = S' = \Lambda = h[0,0,0]$, one recovers the celebrated

Theorem (Ax-Kochen-Eršov)

Let φ be a first order sentence ($=$ formula with no free variables) in the language of rings. For almost all prime number p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.

As shown by Cunningham and Hales, this applies in particular to the integrals occuring in the Fundamental Lemma.

Let us recall that the Fundamental Lemma for GL_n over function fields has been recently proved by Laumon and Ngô.

Using specific group theoretical methods, Waldspurger proved that one can then deduce it for local fields of inequal characteristic.

One can extend the previous results to integrals of functions involving exponentials (Cluckers-Loeser).

In particular one can prove a transfert principle for integrals with parameters in the exponential case.

Let us end by giving a specific example where this applies.

Let E/F be a degree two extension of non archimedean local fields of residue characteristic $\neq 2$.

Let ψ be a non trivial additive character of F of conductor $\mathcal{O}_F$.

Let N_n be the group of upper triangular matrices with 1's on the diagonal and consider the character $\theta : N_n(F) \to \mathbb{C}^\times$ given by

$$\theta(u) := \psi(\sum_i u_{i,i+1}).$$

For a the diagonal matrix $(a_1, \cdots, a_n)$ with a_i in $F^\times$, we consider the integral

$$I(a) := \int_{N_n(F) \times N_n(F)} \mathbf{1}_{M_n(\mathcal{O}_F)}({}^t u_1 a u_2)\, \theta(u_1 u_2)\, du_1 du_2,$$

with the normalisation $\int_{N_n(\mathcal{O}_F)} du = 1$.

Similarly, one defines

$$J(a) := \int_{N_n(E)} \mathbf{1}_{M_n(\mathcal{O}_E) \cap H_n}({}^t \bar{u} a u)\, \theta(u \bar{u})\, du,$$

with H_n the set of Hermitian matrices.

The Jacquet-Ye Conjecture, proved by Ngô over function fields and by Jacquet in general, asserts that

$$I(a) = \gamma(a)\, J(a) \qquad (\Diamond)$$

with

$$\gamma(a) := \prod_{1 \leq i \leq n-1} \eta(a_1 \cdots a_i),$$

and η the multiplicative character of order 2 on $K^\times$.

The Transfert Theorem for exponential integrals applies to integrals of Jacquet-Ye type, yielding that $(\Diamond)$ holds over functions fields of large characteristic if and only if it holds over unequal characteristic local fields of large residual characteristic.

It is natural to expect that relations between non archimedean integrals holding over all local fields of large residual characteristic already hold at the motivic level, as equalities between constructible motivic functions.