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# Lefschetz numbers of iterates of the monodromy and truncated arcs

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## Abstract

We express the Lefschetz number of iterates of the monodromy of a function on a smooth complex algebraic variety in terms of the Euler characteristic of a space of truncated arcs. We also construct a canonical representative of the Milnor fibre in a suitable monodromic Grothendieck group. © 2002 Elsevier Science Ltd. All rights reserved.

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## 1. Introduction

Let  $X$  be a smooth complex algebraic variety and let  $f : X \rightarrow \mathbf{C}$  be a non constant morphism of complex algebraic varieties. We fix a smooth metric on  $X$ . Let  $x$  be a point of  $f^{-1}(0)$ . We set  $X_{\varepsilon, \eta}^{\times} := B(x, \varepsilon) \cap f^{-1}(D_{\eta}^{\times})$ , with  $B(x, \varepsilon)$  the open ball of radius  $\varepsilon$  centered at  $x$  and  $D_{\eta}^{\times} = D_{\eta} \setminus \{0\}$ , with  $D_{\eta}$  the open disk of radius  $\eta$  centered at 0. For  $0 < \eta \ll \varepsilon \ll 1$ , the restriction of  $f$  to  $X_{\varepsilon, \eta}^{\times}$  is a locally trivial fibration – called the Milnor fibration – onto  $D_{\eta}^{\times}$  with fiber  $F_x$ , the Milnor

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fibre at  $x$ . The action of a characteristic homeomorphism of this fibration on cohomology gives rise to the monodromy operator

$$M : H^*(F_x, \mathbf{Q}) \rightarrow H^*(F_x, \mathbf{Q}).$$

For any natural number  $n$ , we consider the Lefschetz number

$$\Lambda(M^n) := \sum_{q \geq 0} (-1)^q \text{Trace} [M^n, H^q(F_x, \mathbf{Q})],$$

of the  $n$ -th iterate of  $M$ . These numbers are related to the monodromy zeta function

$$Z(t) := \prod_{q \geq 0} [\det(\text{Id} - tM, H^q(F_x, \mathbf{Q}))]^{(-1)^q}$$

as follows: if one writes

$$\Lambda(M^n) = \sum_{i|n} s_i,$$

for  $n \geq 1$ , then

$$Z(t) = \prod_{i \geq 1} (1 - t^i)^{s_i/i}.$$

In the paper [1], A'Campo gives a formula for the Lefschetz numbers  $\Lambda(M^n)$  in terms of a resolution of the singularities of  $f=0$ . The aim of the present note is to give a formula for  $\Lambda(M^n)$  directly in terms of geometric objects associated to the function  $f$ . This formula will involve truncated arcs on  $X$ . Let us recall from [3,7], that there is a  $\mathbf{C}$ -scheme  $\mathcal{L}(X)$ , the space of formal arcs on  $X$ , whose set of  $\mathbf{C}$ -rational points  $\mathcal{L}(X)(\mathbf{C})$  is naturally in bijection with  $X(\mathbf{C}[[t]])$ . Similarly, for  $n \geq 0$ , we can consider the space  $\mathcal{L}_n(X)$  of arcs modulo  $t^{n+1}$ : a  $\mathbf{C}$ -rational point of  $\mathcal{L}_n(X)$  corresponds to a  $\mathbf{C}[t]/t^{n+1}\mathbf{C}[t]$ -rational point on  $X$ . The space  $\mathcal{L}_n(X)$  is canonically endowed with the structure of a complex algebraic variety. For instance, when  $X$  is the affine space  $\mathbf{A}_{\mathbf{C}}^m$ , a  $\mathbf{C}$ -rational point of  $\mathcal{L}(X)$  is just an  $m$ -tuple of power series in the variable  $t$  with coefficients in  $\mathbf{C}$ , while  $\mathcal{L}_n(X)(\mathbf{C})$  is the set of  $m$ -tuples of complex polynomials of degree  $\leq n$  in the variable  $t$ . Furthermore, there is a natural morphism

$$\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X),$$

which corresponds to truncation on  $\mathbf{C}$ -rational points. Since  $X$  is assumed to be smooth, the morphism  $\pi_n$  is surjective (also for  $\mathbf{C}$ -rational points). In what follows, we identify  $\mathcal{L}(X)$  with its set of  $\mathbf{C}$ -rational points, and similarly for  $\mathcal{L}_n(X)$ . With a ‘‘complex algebraic variety’’, we mean the set of  $\mathbf{C}$ -rational points of a separated reduced scheme of finite type over  $\mathbf{C}$ , not necessarily irreducible. Now, we consider the set  $\mathcal{X}_n$  of points  $\varphi$  in  $\mathcal{L}_n(X)$  such that  $\pi_0(\varphi) = x$  and such that the  $t$ -valuation of  $f(\varphi)$  is exactly  $n$ . This subset  $\mathcal{X}_n$  is a locally closed subvariety of  $\mathcal{L}_n(X)$  and we may consider the morphism

$$\bar{f}_n : \mathcal{X}_n \rightarrow \mathbf{C}^\times,$$

sending a point  $\varphi$  in  $\mathcal{X}_n$  to the coefficient of  $t^n$  in  $f(\varphi)$ . There is a natural action of  $\mathbf{C}^\times$  on  $\mathcal{X}_n$  given by  $a \cdot \varphi(t) = \varphi(at)$ . Since  $\bar{f}_n(a \cdot \varphi) = a^n \bar{f}_n(\varphi)$ , it follows that  $\bar{f}_n$  is a locally trivial

fibration and has a geometric monodromy of order  $n$ . We denote by  $\mathcal{X}_{n,1}$  the fiber  $\bar{f}_n^{-1}(1)$ . The group  $\mu_n$  of  $n$ -th roots of unity in  $\mathbf{C}$  acts naturally on  $\mathcal{X}_{n,1}$ , by restriction of the  $\mathbf{C}^\times$ -action on  $\mathcal{X}_n$ .

We can now state the main result of this note.

**1.1. Theorem.** *For every integer  $n \geq 1$ , the Lefschetz number  $\Lambda(M^n)$  is equal to  $\chi(\mathcal{X}_{n,1})$ , the Euler characteristic of  $\mathcal{X}_{n,1}$ .*

In fact, we shall deduce Theorem 1.1 from a more general result, Theorem 2.4, where we give a formula for the class of  $\mathcal{X}_{n,1}$  in the ring  $\mathcal{M}_{\text{loc}}$  obtained by localisation of the class of the affine line in the Grothendieck ring of complex algebraic varieties, whose definition is recalled at the beginning of Section 2. Actually, it would be also possible to prove Theorem 1.1 as an easy consequence of Theorem 2.2.1 in [2], by a reasoning involving motives and Hodge polynomials. In fact, Theorem 2.4 may be further generalized to take in account the monodromy action. This is done in Theorem 2.10, once we introduced the “monodromic” Grothendieck ring  $\mathcal{M}_{\text{loc}}^{\text{mon}}$ .

We denote by  $T_n$  the monodromy operator

$$T_n : H^*(\mathcal{X}_{n,1}, \mathbf{Q}) \rightarrow H^*(\mathcal{X}_{n,1}, \mathbf{Q}),$$

of the locally trivial fibration  $\bar{f}_n : \mathcal{X}_n \rightarrow \mathbf{C}^\times$ . Note that  $T_n$  is induced by the automorphism  $\varphi(t) \mapsto \varphi(e^{2\pi i/n}t)$  of  $\mathcal{X}_{n,1}$ . For any  $d$  in  $\mathbf{N}$ , we denote the Lefschetz number of  $T_n^d$  by

$$\Lambda(T_n^d) := \sum_{q \geq 0} (-1)^q \text{Trace} [T_n^d, H^q(\mathcal{X}_{n,1}, \mathbf{Q})].$$

Since  $T_n^n$  is the identity, we have  $\Lambda(T_n^d) = \Lambda(T_n^{\text{gcd}(d,n)})$ . Hence to know the  $\Lambda(T_n^d)$ ’s for every  $d$  in  $\mathbf{N}$ , it is enough to know them for every  $d$  dividing  $n$ . This information is provided by the following result:

**1.2. Theorem.** *If  $n \geq 1$  and  $d$  divides  $n$ , then  $\Lambda(T_n^d) = \Lambda(M^d)$ .*

Finally, in Section 3, we extend Theorem 2.10 to the case of quasi-projective varieties over a field of characteristic zero. This enables us to construct what we believe to be the “virtual motivic incarnation” of the Milnor fibre at  $x$  in  $\mathcal{M}_{k,\text{loc}}^{\text{mon}}$ , the analogue over  $k$  of the ring  $\mathcal{M}_{\text{loc}}^{\text{mon}}$ . Taking Euler characteristic with values into virtual Chow motives, we apply this to settle an issue that remained open in [2].

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## 2. Calculation of a motivic volume and proof of the main results

To state Theorem 2.4, we shall start with some reminders from motivic integration as developed in [3,5].

**2.1.** We denote by  $\mathcal{M}$  the abelian group generated by symbols  $[S]$ , for  $S$  a complex algebraic variety, with the relations  $[S] = [S']$  if  $S$  and  $S'$  are isomorphic and  $[S] = [S'] + [S \setminus S']$  if  $S'$  is Zariski closed in  $S$ . There is a natural ring structure on  $\mathcal{M}$ , the product being induced by the cartesian product of varieties, and to any constructible set  $S$  in some complex algebraic variety one naturally associates a class  $[S]$  in  $\mathcal{M}$ . We denote by  $\mathbf{L}$  the class of the affine line in  $\mathcal{M}$  and we denote by  $\mathcal{M}_{\text{loc}}$  the localisation  $\mathcal{M}_{\text{loc}} := \mathcal{M}[\mathbf{L}^{-1}]$ .

Let  $X$  be a smooth complex algebraic variety of pure dimension  $m$ . We call a subset  $A$  of  $\mathcal{L}(X)$  cylindrical at level  $n$  if  $A = \pi_n^{-1}(C)$ , with  $C$  a constructible subset of  $\mathcal{L}_n(X)$ . We say that  $A$  is cylindrical if it is cylindrical at some level  $n$ .

Let  $X, Y$  and  $F$  be complex algebraic varieties, and let  $A$ , resp.  $B$ , be a constructible subset of  $X$ , resp.  $Y$ . We say that a map  $\pi : A \rightarrow B$  is a piecewise morphism if there exists a finite partition of the domain of  $\pi$  into locally closed subvarieties of  $X$  such that the restriction of  $\pi$  to any of these subvarieties is a morphism of varieties. We say that a map  $\pi : A \rightarrow B$  is a piecewise trivial fibration with fiber  $F$ , if there exists a finite partition of  $B$  in subsets  $S$  which are locally closed subvarieties of  $Y$  such that  $\pi^{-1}(S)$  is a locally closed subvariety of  $X$  and isomorphic, as a complex algebraic variety, to  $S \times F$ , with  $\pi$  corresponding under the isomorphism to the projection  $S \times F \rightarrow S$ . We say that the map  $\pi$  is a piecewise trivial fibration over some constructible subset  $C$  of  $B$ , if the restriction of  $\pi$  to  $\pi^{-1}(C)$  is a piecewise trivial fibration onto  $C$ .

We call a cylindrical subset  $A$  of  $\mathcal{L}(X)$  stable at level  $n \in \mathbf{N}$  if  $A$  is cylindrical at level  $n$  and  $\pi_{k+1}(\mathcal{L}(X)) \rightarrow \pi_k(\mathcal{L}(X))$  is a piecewise trivial fibration over  $\pi_k(A)$  with fiber  $\mathbf{A}_{\mathbb{C}}^m$ , for all  $k \geq n$ . We call  $A$  stable, if it is stable at some level  $n$ .

Since  $X$  is smooth, any cylindrical subset  $A$  of  $\mathcal{L}(X)$  is stable (at the same level), by Lemma 4.1 of [3]. Denote by  $\mathbf{C}_0$  the family of stable cylindrical subsets of  $\mathcal{L}(X)$ . Clearly there exists a unique additive measure

$$\tilde{\mu} : \mathbf{C}_0 \rightarrow \mathcal{M}_{\text{loc}},$$

satisfying

$$\tilde{\mu}(A) = [\pi_n(A)]\mathbf{L}^{-(n+1)m},$$

when  $A \in \mathbf{C}_0$  is stable at level  $n$ .

In particular, the relation

$$[\mathcal{X}_{n,1}] = \tilde{\mu}(\mathcal{Z}_{n,1})\mathbf{L}^{(n+1)m}$$

holds in  $\mathcal{M}_{\text{loc}}$ , where  $\mathcal{Z}_{n,1}$  is the set of points  $\varphi$  in  $\mathcal{L}(X)$  such that  $\pi_0(\varphi) = x$ , such that the  $t$ -valuation of  $f(\varphi)$  is exactly  $n$ , and such that the coefficient of  $t^n$  in  $f(\varphi)$  is equal to 1.

The following geometric lemma, which is a special case of Lemma 3.4 in [3], will play a crucial role in the proof of Theorem 2.4.

**2.2. Lemma.** *Let  $X$  and  $Y$  be connected smooth complex algebraic varieties and let  $h : Y \rightarrow X$  be a birational morphism. For  $e$  in  $\mathbf{N}$ , let  $\Delta_e$  be the subset of  $\mathcal{L}(Y)$  defined by*

$$\Delta_e := \{ \varphi \in Y(\mathbf{C}[[t]]) \mid \text{ord}_t \det \text{Jac}_h(\varphi) = e \},$$

where  $\text{Jac}_h(\varphi)$  is the jacobian of  $h$  at  $\varphi$ . For  $k$  in  $\mathbf{N}$ , let  $h_{k*} : \mathcal{L}_k(Y) \rightarrow \mathcal{L}_k(X)$  be the morphism induced by  $h$ , and let  $\Delta_{e,k}$  be the image of  $\Delta_e$  in  $\mathcal{L}_k(Y)$ . If  $k \geq 2e$ , the following holds.

- (a) *The constructible subset  $\Delta_{e,k}$  of  $\mathcal{L}_k(Y)$  is a union of fibers of  $h_{k^*}$ .*
- (b) *The restriction of  $h_{k^*}$  to  $\Delta_{e,k}$  is a piecewise trivial fibration with fiber  $\mathbf{A}_{\mathbb{C}}^e$  onto its image.*

**2.3.** Now we shall use Lemma 2.2 to compute  $\tilde{\mu}(\mathcal{L}_{n,1})$  on a resolution of  $f$ . Let  $D$  be the divisor defined by  $f=0$  in  $X$ . Let  $(Y, h)$  be a resolution of  $f$ . By this, we mean that  $Y$  is a smooth and connected complex algebraic variety,  $h : Y \rightarrow X$  is proper, that the restriction  $h : Y \setminus h^{-1}(D) \rightarrow X \setminus D$  is an isomorphism, and that  $(h^{-1}(D))_{\text{red}}$  has only normal crossings as a subvariety of  $Y$ . Furthermore, we choose  $h$  in such a way that  $(h^{-1}(x))_{\text{red}}$  is a union of irreducible (smooth) components of  $(h^{-1}(D))_{\text{red}}$ . We shall denote the irreducible (smooth) components of  $(h^{-1}(D))_{\text{red}}$  by  $E_i, i \in J$ . We set  $J_0 = \{i \in J \mid E_i \subset h^{-1}(x)\}$ . For each  $i \in J$ , denote by  $N_i$  the multiplicity of  $E_i$  in the divisor of  $f \circ h$  on  $Y$ , and by  $v_i - 1$  the multiplicity of  $E_i$  in the divisor of  $h^*dx$ , where  $dx$  is a local non-vanishing volume form at  $x$ , i.e. a local generator of the sheaf of differential forms of maximal degree at  $x$ . For  $i \in J$  and  $I \subset J$ , we consider the varieties  $E_i^\circ := E_i \setminus \bigcup_{j \neq i} E_j$ ,  $E_I := \bigcap_{i \in I} E_i$ , and  $E_I^\circ := E_I \setminus \bigcup_{j \in J \setminus I} E_j$ . We shall also set  $m_I = \text{gcd}(N_i)_{i \in I}$ . We introduce an unramified Galois cover  $\tilde{E}_I^\circ$  of  $E_I^\circ$ , with Galois group  $\mu_{m_I}$ , as follows. Let  $U$  be an affine Zariski open subset of  $Y$ , such that, on  $U, f \circ h = uv^{m_I}$ , with  $u$  a unit on  $U$  and  $v$  a morphism from  $U$  to  $\mathbf{A}_{\mathbb{C}}^1$ . Then the restriction of  $\tilde{E}_I^\circ$  above  $E_I^\circ \cap U$ , denoted by  $\tilde{E}_I^\circ \cap U$ , is defined as

$$\{(z, y) \in \mathbf{A}_{\mathbb{C}}^1 \times (E_I^\circ \cap U) \mid z^{m_I} = u^{-1}\}.$$

Note that  $E_I^\circ$  can be covered by such affine open subsets  $U$  of  $Y$ . Gluing together the covers  $\tilde{E}_I^\circ \cap U$ , in the obvious way (cf. the proof of Lemma 3.2.2 in [2]), we obtain the cover  $\tilde{E}_I^\circ$  of  $E_I^\circ$  which has a natural  $\mu_{m_I}$ -action (obtained by multiplying the  $z$ -coordinate with the elements of  $\mu_{m_I}$ ).

**2.4. Theorem.** *With the previous notations, the following relation holds in  $\tilde{\mathcal{M}}_{\text{loc}}$ :*

$$[\mathcal{X}_{n,1}] = \mathbf{L}^{nm} \sum_{\substack{I \subset J \\ I \cap J_0 \neq \emptyset}} (\mathbf{L} - 1)^{|I|-1} [\tilde{E}_I^\circ] \left( \sum_{\substack{k_i \geq 1, i \in I \\ \sum k_i N_i = n}} \mathbf{L}^{-\sum_{i \in I} k_i v_i} \right). \tag{2.1}$$

**Proof.** Let  $\tilde{\mathcal{Z}}_{n,1}$  be the preimage of  $\mathcal{Z}_{n,1}$  in  $\mathcal{L}(Y)$ . Remark that, the morphism  $h$  being proper, the induced function  $h_* : \tilde{\mathcal{Z}}_{n,1} \rightarrow \mathcal{Z}_{n,1}$  is bijective. Now, for  $e \geq 0$ , define  $\tilde{\mathcal{Z}}_{n,1,e}$  as the set of points  $\varphi$  in  $\tilde{\mathcal{Z}}_{n,1}$  such that  $\text{ord}_t \det \text{Jac}_h(\varphi) = e$ .

**2.5. Lemma.** *Let  $I$  be a subset of  $J$  such that  $I \cap J_0 \neq \emptyset$ . Let  $U$  be an affine Zariski open subset of  $Y$ , such that, on  $U, f \circ h = u \prod_{i \in I} y_i^{N_i}$  and  $\det \text{Jac}_h = v \prod_{i \in I} y_i^{v_i - 1}$ , with  $u$  and  $v$  units on  $U$  and  $y_i$  a regular function on  $U$  with divisor  $E_i \cap U$ . Let  $k_i, i \in I$ , be natural numbers with  $\sum_{i \in I} k_i N_i = n, k_i \geq 1$ . Let  $\mathcal{U}_{(k_i)}$  be the set of points  $\varphi$  in  $\mathcal{L}_n(U)$  such that  $\text{ord}_t y_i(\varphi) = k_i$ , for  $i \in I$ , and  $\tilde{f}_n(h_n^*(\varphi)) = 1$ , where  $h_n^* : \mathcal{L}_n(Y) \rightarrow \mathcal{L}_n(X)$  is the morphism induced by  $h$ .*

Then

$$[\mathcal{U}_{(k_i)}] = (\mathbf{L} - 1)^{|I|-1} [\tilde{E}_I^\circ \cap U] \mathbf{L}^{mn - \sum_{i \in I} k_i}$$

in  $\mathcal{M}_{\text{loc}}$ .

**Proof.** Note that the projection  $\mathcal{L}_n(U) \rightarrow U$  maps  $\mathcal{U}_{(k_i)}$  into  $E_I^\circ \cap U$ . Thus we may assume that  $E_I^\circ \cap U$  is not empty, and – using additivity – that there exist  $m - |I|$  functions on  $U$  which, together with the functions  $(y_i)_{i \in I}$ , induce an étale map  $U \rightarrow \mathbf{A}_{\mathbf{C}}^m$ . By Lemma 4.2 of [3] (with  $n = e = 0$ ), this map induces an isomorphism  $\mathcal{L}_n(U) \simeq U \times_{\mathbf{A}_{\mathbf{C}}^m} \mathcal{L}_n(\mathbf{A}_{\mathbf{C}}^m)$ . It follows now from the very definitions that

$$[\mathcal{U}_{(k_i)}] = [W_I] \mathbf{L}^{mn - \sum_{i \in I} k_i},$$

with

$$W_I := \left\{ (z_i, y) \in (\mathbf{C}^\times)^{|I|} \times (E_I^\circ \cap U) \mid \prod_{i \in I} z_i^{N_i} u = 1 \right\},$$

and we have

$$[W_I] = (\mathbf{L} - 1)^{|I|-1} [\tilde{E}_I^\circ \cap U].$$

To verify the last equality, consider an automorphism  $z \mapsto (z^{a_i})_{i \in I}$  of  $(\mathbf{C}^\times)^{|I|}$ , with  $(a_i)_{i \in I}$  a basis for  $\mathbf{Z}^{|I|}$  and  $a_1 = (N_i/m_I)_{i \in I}$ .  $\square$

**End of proof of Theorem 2.4.** By covering  $Y$  with affine Zariski open subsets  $U$  verifying the assumptions in Lemma 2.5, one sees that all the subsets  $\tilde{\mathcal{Z}}_{n,1,e}$  of  $\mathcal{L}(Y)$  are cylindrical and that there exists  $e_0$  such that  $\tilde{\mathcal{Z}}_{n,1,e}$  is empty for  $e > e_0$ . Now set  $\mathcal{Z}_{n,1,e} = h_*(\tilde{\mathcal{Z}}_{n,1,e})$ . Since  $\pi_k \circ h_* = h_{k*} \circ \pi_k$ , with the notation of Lemma 2.2, it follows from assertion (a) of Lemma 2.2 that the subsets  $\mathcal{Z}_{n,1,e}$  of  $\mathcal{L}(X)$  are cylindrical. Since  $\mathcal{Z}_{n,1}$  is equal to the disjoint union of the subsets  $\mathcal{Z}_{n,1,e}$  for  $e \leq e_0$ , we have

$$\tilde{\mu}(\mathcal{Z}_{n,1}) = \sum_{e \leq e_0} \tilde{\mu}(\mathcal{Z}_{n,1,e}).$$

Now, it follows from Lemma 2.2 that

$$\tilde{\mu}(\mathcal{Z}_{n,1,e}) = \mathbf{L}^{-e} \tilde{\mu}(\tilde{\mathcal{Z}}_{n,1,e}).$$

By using Lemma 2.5, one gets, for every  $U$  as in 2.5,

$$\tilde{\mu}(\tilde{\mathcal{Z}}_{n,1,e} \cap \mathcal{L}(U)) = \mathbf{L}^{-m} \sum_{\substack{I \subset J \\ I \cap J_0 \neq \emptyset}} (\mathbf{L} - 1)^{|J|-1} [\tilde{E}_J^\circ \cap U] \sum_{\substack{k_i \geq 1, i \in I \\ \sum k_i N_i = n, \sum k_i (v_i - 1) = e}} \mathbf{L}^{-\sum_{i \in I} k_i},$$

and the result follows by additivity of  $\tilde{\mu}$ .  $\square$

**2.6. Proof of Theorem 1.1.** We use a resolution  $(Y, h)$  of  $f$  satisfying the conditions in 2.3. We shall view the Euler characteristic of complex constructible sets as a ring morphism  $\chi: \mathcal{M} \rightarrow$

**Z.** Since  $\chi(\mathbf{L}) = 1$ , this morphism extends uniquely to a ring morphism  $\chi: \mathcal{M}_{\text{loc}} \rightarrow \mathbf{Z}$ . Since  $\chi((\mathbf{L} - 1)^{|I|-1}) = 0$  when  $|I| > 1$ , it follows from Theorem 2.4 that

$$\chi(\mathcal{X}_{n,1}) = \sum_{\substack{N_i|n \\ i \in J_0}} N_i \chi(E_i^\circ).$$

The result follows since  $\Lambda(M^n)$  is equal to the right hand side of the previous formula by A' Campo's formula for  $\Lambda(M^n)$  given in [1].  $\square$

**2.7. Remark.** As observed by Paul Seidel, Theorem 1.1 bears some similarity with properties of Floer homology for a symplectic automorphism (see, e.g. [6]).

**2.8. Remark.** Of course, it is also possible to prove Theorem 1.1 directly, without considering the ring  $\mathcal{M}_{\text{loc}}$ , by using Lemma 2.2 and working with Euler characteristics all the way.

**2.9.** To take in account the monodromy action we introduce a ring  $\mathcal{M}^{\text{mon}}$ . As an abelian group  $\mathcal{M}^{\text{mon}}$  is generated by symbols  $[S, \tau]$ , with  $S$  a complex algebraic variety and  $\tau$  an automorphism of  $S$ . The relations are  $[S, \tau] = [S', \tau']$  if the pairs  $(S, \tau)$  and  $(S', \tau')$  are isomorphic,  $[S, \tau] = [S', \tau|_{S'}] + [S \setminus S', \tau|_{S \setminus S'}]$  when  $S'$  is Zariski closed in  $S$  and stable under  $\tau$ , and  $[S \times \mathbf{A}_{\mathbf{C}}^n, \sigma] = [S \times \mathbf{A}_{\mathbf{C}}^n, \sigma']$  whenever  $\sigma$  and  $\sigma'$  are liftings<sup>3</sup> of a same automorphism  $\tau$  of  $S$ . There is a natural ring structure on  $\mathcal{M}^{\text{mon}}$ , induced by the cartesian product of varieties. We denote by  $\mathbf{L}$  the class of  $(\mathbf{A}_{\mathbf{C}}^1, \text{id})$  in  $\mathcal{M}^{\text{mon}}$  and we denote by  $\mathcal{M}_{\text{loc}}^{\text{mon}}$  the localisation  $\mathcal{M}_{\text{loc}}^{\text{mon}} := \mathcal{M}^{\text{mon}}[\mathbf{L}^{-1}]$ .

We will often write  $[S]$  instead of  $[S, \tau]$  when it is clear from the context what  $\tau$  is. For example we write  $[\mathcal{X}_{n,1}]$  and  $[\tilde{E}_I]$ , the automorphism being the obvious one induced by multiplication by  $e^{2\pi i/n}$ , resp.  $e^{2\pi i/m_I}$ .

Let  $T$  be an automorphism of the scheme  $\mathcal{L}(X)$  which permutes the fibers of  $\pi_n$  for every  $n$ . Denote by  $\mathbf{C}_{0,T}$  the family of stable cylindrical subsets  $A$  of  $\mathcal{L}(X)$  with  $T(A) = A$ . Clearly, there exists a unique additive measure

$$\tilde{\mu}^{\text{mon}} : \mathbf{C}_{0,T} \rightarrow \mathcal{M}_{\text{loc}}^{\text{mon}},$$

satisfying

$$\tilde{\mu}^{\text{mon}}(A) = [\pi_n(A)] \mathbf{L}^{-(n+1)m},$$

when  $A \in \mathbf{C}_{0,T}$  is stable at level  $n$ .

**2.10. Theorem.** *Relation (2.1) holds in  $\mathcal{M}_{\text{loc}}^{\text{mon}}$ .*

**Proof.** The proof of Theorem 2.4 carries over literally to the monodromic situation.  $\square$

**2.11.** For any complex algebraic variety with an action of a finite abelian group  $G$ , and for any character  $\alpha$  of  $G$ , we denote by  $H^*(X, \mathbf{C})_\alpha$  the part of  $H^*(X, \mathbf{C})$  on which  $G$  acts by multiplication

<sup>3</sup> Meaning that  $\tau \circ p = p \circ \sigma = p \circ \sigma'$ , where  $p$  is the projection of  $S \times \mathbf{A}_{\mathbf{C}}^n$  onto  $S$ .

by  $\alpha$ , and we set

$$\chi(X, \alpha) := \sum_{q \geq 0} (-1)^q \dim H^q(X, \mathbf{C})_\alpha.$$

**2.12. Proof of Theorem 1.2.** The monodromy zeta function  $Z(t)$  of the Milnor fibration being equal to  $\prod_{i \in J_0} (1 - t^{N_i})^{\chi(E_i^\circ)}$  by [1], Theorem 1.2 is equivalent to the assertion that the monodromy zeta function of the fibration  $\tilde{f}_n$  is equal to

$$\prod_{\substack{i \in J_0 \\ N_i | n}} (1 - t^{N_i})^{\chi(E_i^\circ)}.$$

But this assertion is equivalent to the validity of the equality

$$\chi(\mathcal{X}_{n,1}, \alpha) = \sum_{\substack{i \in J_0 \\ \text{ord}(\alpha) | N_i | n}} \chi(E_i^\circ),$$

for every character  $\alpha$  of  $\mu_n$ , which is a direct consequence of Theorem 2.10.  $\square$

**2.13. Remark.** In fact, Theorem 2.10 still remains valid if one changes the definition of  $\mathcal{M}_{\text{loc}}^{\text{mon}}$  in 2.9 by imposing the relation  $[S \times \mathbf{A}_{\mathbf{C}}^n, \sigma] = [S \times \mathbf{A}_{\mathbf{C}}^n, \sigma']$  only when  $\sigma$  and  $\sigma'$  are cartesian products that coincide on  $S$  and that are linear on  $\mathbf{A}_{\mathbf{C}}^n$ . To verify this claim one needs a straightforward refinement of Lemma 4.1 of [3] and of Lemma 2.2.

### 3. Some further results

**3.1.** There is an algebraic analogue of Theorem 2.10 over an arbitrary field  $k$  of characteristic zero. To state it, we first define the ring  $\mathcal{M}_{k, \text{loc}}^{\text{mon}}$ , which is the algebraic analogue of the ring  $\mathcal{M}_{\text{loc}}^{\text{mon}}$ . For  $n \geq 1$  an integer, we denote by  $\mu_n$  the group scheme over  $k$  of  $n$ -th roots of unity. Note that it is not assumed that all geometric points of  $\mu_n$  are rational over  $k$ . By an action of  $\mu_n$  on a quasi-projective scheme over  $k$ , we mean an action in the sense of group schemes and schemes over  $k$ . Set  $\hat{\mu} := \varprojlim_n \mu_n$ . By an action of  $\hat{\mu}$  on a quasi-projective scheme over  $k$ , we mean an action which factors through a suitable  $\mu_n$ . We define  $\mathcal{M}_{k, \text{loc}}^{\text{mon}}$  in the same way as  $\mathcal{M}_{\text{loc}}^{\text{mon}}$ , working now with pairs consisting of a quasi-projective scheme over  $k$  and a  $\hat{\mu}$ -action on it.

Let  $X$  be a smooth algebraic variety over  $k$  and let  $f : X \rightarrow \mathbf{A}_k^1$  be a non constant morphism. Similarly as in the complex case one defines the arc space  $\mathcal{L}(X)$  and the quasi-projective variety  $\mathcal{X}_{n,1}$  with a natural  $\hat{\mu}$ -action. One defines also similarly as before resolutions  $(Y, h)$  of  $f$  and the varieties  $\tilde{E}_I^\circ$  with  $\hat{\mu}$ -action.

The proof of Theorem 2.10 carries over to the algebraic case to give the following:

**3.2. Theorem.** *Let  $X$  be a smooth algebraic variety over  $k$  and let  $f : X \rightarrow \mathbf{A}_k^1$  be a non constant morphism. Let  $(Y, h)$  be a resolution of  $f$ . With the previous notations, relation (2.1) holds in  $\mathcal{M}_{k, \text{loc}}^{\text{mon}}$ .*

**3.3.** Consider the power series

$$P(T) := \sum_{n \geq 1} [\mathcal{X}_{n,1}] \mathbf{L}^{-nm} T^n$$

in the variable  $T$  over the ring  $\mathcal{M}_{k, \text{loc}}^{\text{mon}}$ . It follows directly from Theorem 3.2 that  $P(T)$  is a rational power series and that its “limit for  $T \rightarrow \infty$ ” (cf. Section 4 of [2]) is equal to  $-\mathcal{S}$ , where

$$\mathcal{S} := \sum_{\substack{I \subset J \\ I \cap J_0 \neq \emptyset}} (1 - \mathbf{L})^{|I|-1} [\tilde{E}_I^\circ]. \tag{3.1}$$

In particular it follows that the right hand side of (3.1) is independent of the resolution  $(Y, h)$ , as an element of  $\mathcal{M}_{k, \text{loc}}^{\text{mon}}$ . As we shall explain in 3.5, we believe that  $\mathcal{S}$  is the “virtual motivic incarnation” of the Milnor fibre at  $x$ .

**3.4.** We denote by  $E$  the smallest subfield of  $\mathbf{C}$  containing all roots of unity. Assume  $E$  is contained in  $k$ . To any quasi-projective variety  $X$  over  $k$  with a  $\hat{\mu}$ -action, and to any character  $\alpha$  of  $\hat{\mu}$  of finite order, we associate, as in Theorem 1.3.1 of [2] (see also [4]), an element  $\chi_{\text{mot},c}(X, \alpha)$  of the Grothendieck group  $K_0(\text{Mot}_{k,E})$  of the category  $\text{Mot}_{k,E}$  of Chow motives over  $k$  with coefficients in  $E$ . One can check that  $\chi_{\text{mot},c}(-, \alpha)$  factorizes through a map  $\mathcal{M}_{k, \text{loc}}^{\text{mon}} \rightarrow K_0(\text{Mot}_{k,E})$ . One verifies that  $\chi_{\text{mot},c}(-, \alpha)$  respects the last relation in the definition of  $\mathcal{M}_{k, \text{loc}}^{\text{mon}}$  by an argument similar to the proof of Proposition 2.6 in [5].

We still write  $\mathbf{L}$  for  $\chi_{\text{mot},c}(\mathbf{A}_k^1, 1)$ . When  $k = \mathbf{C}$ , the topological Euler characteristic of  $\chi_{\text{mot},c}(X, \alpha)$  is equal to  $\chi(X, \alpha)$ , with the notation of 2.11.

**3.5.** Let  $\alpha$  be a character of  $\hat{\mu}$  of order  $d$  and set  $S_\alpha := \chi_{\text{mot},c}(\mathcal{S}, \alpha^{-1})$ . It follows from the definition that

$$S_\alpha = \sum_{\substack{I \subset J, I \cap J_0 \neq \emptyset \\ d | m_I}} (1 - \mathbf{L})^{|I|-1} \chi_{\text{mot},c}(\tilde{E}_I^\circ, \alpha^{-1}).$$

This element  $S_\alpha$  plays a key role in Section 4 of [2], where it was proved that modulo  $(\mathbf{L} - 1)$ -torsion  $S_\alpha$  does not depend on the chosen resolution  $(Y, h)$  of  $f$ . However, it follows now from the above considerations that  $S_\alpha$ , as an element of  $K_0(\text{Mot}_{k,E})$ , does not depend on the chosen resolution. As explained in [2], we believe that  $S_\alpha$  is the “virtual motivic incarnation” of the  $\alpha$ -isotypic part of the Milnor fiber. It was shown in [2] that this is indeed true at the level of  $\mathbf{C}$ -Hodge realizations.

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