ALTERNATIVES FOR PSEUDOFINITE GROUPS

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Abstract. The famous Tits' alternative states that a linear group either contains a nonabelian free group or is soluble-by-(locally finite). We study in this paper similar alternatives in pseudofinite groups. We show for instance that an $\aleph_0$-saturated pseudofinite group either contains a subsemigroup of rank 2 or is nilpotent-by-(uniformly locally finite). We call a class of finite groups G weakly of bounded rank if the radical $rad(G)$ has a bounded Prüfer rank and the index of the sockel of $G/rad(G)$ is bounded. We show that an $\aleph_0$-saturated pseudo-(finite weakly of bounded rank) group either contains a nonabelian free group or is nilpotent-by-abelian-by-(uniformly locally finite). We also obtain some relations between this kind of alternatives and amenability.

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1. Introduction

A group G (respectively a field K) is *pseudofinite* if it is elementarily equivalent to an ultraproduct of finite groups (respectively of finite fields), equivalently if G (respectively K) is a model of the theory of the class of finite groups (respectively of finite fields); that is any sentence true in G (respectively in K) is also true in some finite group (respectively finite field). Note that one usually requires in addition the structure to be infinite, but it is convenient for us to allow a pseudofinite structure to be finite.

Infinite pseudofinite fields have been characterized algebraically by J. Ax [2] and he showed that the theory of all pseudofinite infinite fields is decidable. Natural examples of pseudofinite groups are general linear groups over pseudofinite fields. Pseudofinite simple groups have been investigated first by U. Felgner [17], then by J. Wilson [57] who showed that any pseudofinite simple group is elementarily equivalent to a Chevalley group (of twisted or untwisted type) over a pseudofinite field and it was later observed that it is even isomorphic to such a group [44]. Pseudofinite groups with a theory satisfying various model-theoretic assumptions like stability, supersimplicity or the non independence property (NIP) have been studied [27, 15]; in another direction G. Sabbagh and A. Khélif investigated finitely generated pseudofinite groups.

The Tits alternative [50] states that a linear group, i.e. a subgroup of the general linear group $GL(n, K)$, either contains a free nonabelian group or is soluble-by-(locally finite). It is known that the Tits alternative holds for other classes of groups. For instance a subgroup of a hyperbolic group satisfies a strong form of the Tits alternative, namely it is either virtually cyclic or contains a nonabelian free group. N. Ivanov [23] and J. McCarthy [28] have shown that mapping class groups of compact surfaces satisfy the Tits alternative and M. Bestvina, M. Feighn and M. Handel [3] showed that the alternative holds for $Out(F_n)$ where F_n is the free group of rank n. Note that when Tits alternative holds in a class of groups, then the following dichotomies hold for their finitely generated members: they

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have either polynomial or exponential growth; they are either amenable or contain a free nonabelian group. However, it is well-known that groups which are non-amenable and without nonabelian free subgroups exist [34, 1, 20].

We investigate in this paper, alternatives for pseudofinite groups of the same flavour as the Tits alternative. We show that an $\aleph_0$-saturated pseudofinite group either contains the free subsemigroup of rank 2 or is superamenable. This follows from the following result: an $\aleph_0$-saturated pseudofinite group either contains the free subsemigroup of rank 2 or is nilpotent-by-(uniformly locally finite) (Theorem 4.1). More generally we prove that a pseudofinite group satisfying a finite disjunction of Milnor identities is nilpotent-by-(uniformly locally finite) (Corollary 4.6). This is a straightforward consequence of the analogue proven in the class of finite groups ([51]).

Then we show that whether the following dichotomy holds for $\aleph_0$-saturated pseudofinite groups, namely it either contains a free nonabelian subgroup or it is amenable, is equivalent to whether a finitely generated residually finite group which satisfies a nontrivial identity is amenable (respectively uniformly amenable) (Theorem 5.1).

In the same spirit, we revisit the results of S. Black [4] who considered a "finitary Tits alternative", i.e. an analog of Tits alternative for classes of finite groups. We reformulate Black's results in the context of pseudofinite groups (Theorem 6.1) and we strengthen them to the class of finite groups of weakly bounded rank. A class of finite groups is *weakly of bounded rank* if the class of the radicals has bounded (Prüfer) rank and the index of the sockels are bounded. We obtain the following dichotomies for an $\aleph_0$-saturated pseudo-(finite weakly of bounded rank) group G: either G contains a nonabelian free group or G is nilpotent-by-abelian-by-(uniformly locally finite) (Theorem 6.10). As S. Black, we use results of A. Shalev and D. Segal on classes of finite groups of bounded Prüfer rank ([49], [46]).

We will be also interested in classes of finite groups satisfying some uniformity conditions on centralizer dimension, namely for which there is a bound on the chains of centralizers. A class $\mathcal{C}$ of finite groups has bounded c-dimension, if there is $d \in \mathbb{N}$ such that for each $G \in \mathcal{C}$ the c-dimension of $rad(G)$ and of the index of the sockels of $G/rad(G)$ are bounded by d. We show that an $\aleph_0$-saturated pseudo-(finite of bounded c-dimension) group either contains a nonabelian free group or is soluble-by-(uniformly locally finite) (see Corollary 6.12). We use a result of E. Khukhro [26] on classes of finite soluble groups of finite c-dimension.

In our proofs, we use the following uniformity results which hold in the class of finite groups: the result of J. Wilson [58] who obtained a formula ϕ_R which defines across the class of finite groups the soluble radical, definability results for verbal subgroups of finite groups due to N. Nikolov and D. Segal ([32], [47], [33]) and the positive solution of the restricted Burnside problem due to E. Zelmanov ([61], [52]).

The present paper is organized as follows. In the next section, we relate the notion of being pseudofinite with other approximability properties by a class of groups and we recall some background material. In Section 3, we study some properties of finitely generated pseudofinite groups. Section 4 is devoted to the proof of Theorem 4.1: an $\aleph_0$-saturated pseudofinite group either contains the free subsemigroup of rank 2 or is nilpotent-by-(uniformly locally finite). Then, in Section 5 we study the general problem of the existence of nonabelian free subgroups and its relations with amenability. We end in Section 6 by giving the generalization (in the class of pseudofinite groups) of the above-mentionned Black's results and also some other alternatives under assumptions like bounded c-dimension.

2. Generalities

In this section we will first relate various notions of *approximability* of a group by a class of (finite) groups. The reader interested in a more thorough exposition can consult for instance the survey by T. Ceccherini-Silberstein and M. Coornaert [8]. We point out that Proposition 2.3 (and its Corollaries) seems new and it is important in the proof of Theorem 5.1. At the end of this section we review some basic model-theoretic properties of pseudofinite groups.

In [53], A.Vershik and E. Gordon considered a new version of embedding for groups; they defined LEF-groups, namely groups locally embeddable in a class of finite groups. The definition adapts to any class of groups and it is related to various residual notions that we recall here.

Notation 2.1. Given a class $\mathcal{C}$ of $\mathcal{L}$-structures, we will denote by $Th(\mathcal{C})$ (respectively by $Th_\forall(\mathcal{C})$) the set of sentences (respectively universal sentences) true in all elements of $\mathcal{C}$.

Given a set I, an ultrafilter $\mathscr{U}$ over I and a set of $\mathcal{L}$-structures $(C_i)_{i\in I}$, we denote by $\prod_I^{\mathscr{U}} C_i$ the ultraproduct of the family $(C_i)_{i\in I}$ relative to $\mathscr{U}$. We denote by $\mathcal{P}_{fin}(I)$ the set of all finite subsets of I.

Definition 2.1. Let $\mathcal{C}$ be a class of groups.

• A group G is called *approximable* by $\mathcal{C}$ (or locally $\mathcal{C}$ or locally embeddable into $\mathcal{C}$) if for any finite subset $F \subseteq G$, there exists a group $G_F \in \mathcal{C}$ and an *injective* map $\xi_F : F \to G_F$ such that $\forall g, h \in F$, if $gh \in F$, then $\xi_F(gh) = \xi_F(g)\xi_F(h)$. When $\mathcal{C}$ is a class of finite groups, then G is called LEF.

• A group G is called *residually-$\mathcal{C}$*, if for any nontrivial element $g \in G$, there exists a homomorphism $\varphi : G \to C \in \mathcal{C}$ such that $\varphi(g) \neq 1$.

• A group G is called *fully residually-$\mathcal{C}$*, if for any finite subset S of nontrivial elements of G, there exists a homomorphism $\varphi : G \to C \in \mathcal{C}$ such that $1 \notin \varphi(S)$.

• A group G is called *pseudo-$\mathcal{C}$* if G satisfies $Th(\mathcal{C}) = \bigcap_{C\in\mathcal{C}} Th(C)$.

In particular, when $\mathcal{C}$ is the class of finite groups, a pseudo-$\mathcal{C}$ group is a pseudofinite group. In this case, we will abreviate pseudo-$\mathcal{C}$ group by pseudofinite group. We note that if $\mathcal{C}$ is closed under finite direct products then a group G is residually-$\mathcal{C}$ if and only if it is fully residually-$\mathcal{C}$.

We will use the following variation of a theorem of Frayne ([10] 4.3.13), which can be stated as follows. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $\mathcal{C}$ a class of $\mathcal{L}$-structures. Assume that $\mathcal{A}$ satisfies $Th(\mathcal{C})$. Then there exists I and an ultrafilter $\mathscr{U}$ on I such that $\mathcal{A}$ elementarily embeds into an ultraproduct of elements of $\mathcal{C}$. It follows for instance that a group G is pseudofinite if and only if it is elementarily embeddable in some ultraproduct of finite groups; a property that will be used throughout the paper without explicit reference.

Proposition 2.1. *Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $\mathcal{C}$ a class of $\mathcal{L}$-structures. Assume that $\mathcal{A}$ satisfies $Th_\forall(\mathcal{C})$. Then there exists I and an ultrafilter $\mathscr{U}$ on I such that $\mathcal{A}$ embeds into an ultraproduct of elements of $\mathcal{C}$.*

Proof: We enumerate the elements of A as $(a_\alpha)_{\alpha<\delta}$ and we denote by $\mathcal{L}_A := \mathcal{L} \cup \{c_\alpha : \alpha < \delta\}$. We will consider $\mathcal{A}$ as an $\mathcal{L}_A$-structure interpreting c_α by a_α. Let $\mathcal{F}_A$ be the set of all $\mathcal{L}_A$-quantifier-free sentences $\phi(c_{\alpha_1}, \cdots, c_{\alpha_n})$, where $\alpha_1, \cdots, \alpha_n < \delta$. Let $I := \{\phi \in \mathcal{F}_A : \mathcal{A} \models \phi\}$. Note that if $\mathcal{A} \models \phi(c_{\alpha_1}, \cdots, c_{\alpha_n})$, then there exists $\mathcal{B} \in \mathcal{C}$ such that $\mathcal{B} \models \exists x_1 \cdots \exists x_n\, \phi(x_1, \cdots, x_n)$. Denote $\mathcal{B}_\phi$ such element of $\mathcal{C}$ and the corresponding tuple of elements $b_\phi := (b_{\phi,\alpha_1}, \cdots, b_{\phi,\alpha_n})$ such that $\mathcal{B}_\phi \models \phi(b_\phi)$. For any $\phi(c_{\alpha_1}, \cdots, c_{\alpha_n}) \in I$, we set $J_\phi := \{\psi(c_{\alpha_1}, \cdots, c_{\alpha_n}) \in I : \mathcal{B}_\psi \models \phi(b_\psi)\}$. These subsets J_ϕ have the finite intersection property and so there exists an ultrafilter $\mathscr{U}$ on I containing these J_ϕ.

Finally we define a map f from $\mathcal{A}$ to $\prod_I^{\mathscr{U}} \mathcal{B}_\phi$ by sending a_α to $[b_{\phi\alpha}]_{\mathscr{U}}$ and check this is an embedding. Assume that for $\phi \in \mathcal{F}$, $\mathcal{A} \models \phi(a_{\alpha_1}, \cdots, a_{\alpha_n})$, so $J_{\phi(c_{\alpha_1}, \cdots, c_{\alpha_n})} \in \mathscr{U}$, so $\{\psi(c_{\alpha_1}, \cdots, c_{\alpha_n}) \in I : \mathcal{B}_\psi \models \phi(b_\psi)\} \in \mathscr{U}$. □

Proposition 2.2. *Let G be a group and $\mathcal{C}$ a class of groups. The following properties are equivalent.*

1. *The group G is approximable by $\mathcal{C}$.*
2. *G embeds in an ultraproduct of elements of $\mathcal{C}$.*
3. *G satisfies $Th_\forall(\mathcal{C})$.*
4. *Every finitely generated subgroup of G is approximable by $\mathcal{C}$.*

Proof: (1) ⇒ (2). Let $I = \mathcal{P}_{fin}(G)$ and let $\mathscr{U}$ be an ultrafilter containing all subsets of the form $J_F := \{e \in \mathcal{P}_{fin}(G) : F \subset e\}$, with $F \in \mathcal{P}_{fin}(G)$. Choose $\xi_F : F \to G_F \in \mathcal{C}$ as in Definition 2.1. Then consider the ultraproduct $\prod_I^{\mathscr{U}} G_F$ and let for $g \in G$, $\xi(g) := [\xi_F(g))]_{\mathscr{U}}$. Then ξ is a monomorphism.

(2) ⇒ (1). Assume that G embeds in a ultraproduct of elements of $\mathcal{C}$. Let $F \subset G$ a finite subset. We can describe the partial multiplication table of F by a conjunction of basic formulas. Denote by σ the existential sentence obtained by quantifying over the elements of F. This sentence is true on an infinite family of elements of $\mathcal{C}$. Let $H \in \mathcal{C}$ satisfying this sentence and define a map from G to H accordingly.

(2) ⇒ (3). Let $\sigma \in Th_\forall(\mathcal{C})$. Then since G embeds in an ultraproduct of elements of $\mathcal{C}$, $G \models \sigma$.

The implication (3) ⇒ (2) is the statement of Proposition 2.1 and we see also that the equivalence (1) ⇔ (4) is clear. □

One can derive from the above proposition the following result of Malcev, namely that a group G embeds in an ultraproduct of its finitely generated subgroups, by letting $\mathcal{C}$ to be the class of finitely generated subgroups of G.

Proposition 2.3. *Let G be a group and $\mathcal{C}$ a class of groups. The following properties are equivalent.*

1. *The group G is approximable by $\mathcal{C}$.*
2. *For every finitely generated subgroup L of G, there exists a sequence of finitely generated residually-$\mathcal{C}$ groups $(L_n)_{n\in\mathbb{N}}$ and a sequence of homomorphisms $(\varphi_n : L_n \to L_{n+1})_{n\in\mathbb{N}}$ such the following properties holds:*

   (i) L is the direct limit, $L = \varinjlim L_n$, of the system $\varphi_{n,m} : L_n \to L_m$, $m \geq n$, where $\varphi_{n,m} = \varphi_m \circ \varphi_{m-1} \cdots \circ \varphi_n$.

   (ii) For any $n \geq 0$, for any finite subset S of L_n, if $1 \notin \psi_n(S)$, where $\psi_n : L_n \to L$ is the natural map, there exists a homomorphism $\varphi : L_n \to C \in \mathcal{C}$ such that $1 \notin \varphi(S)$.

Proof: (1) ⇒ (2). Let L be a finitely generated subgroup of G. Let

$$L = \langle a_1, \ldots, a_p \,|\, r_0, \ldots, r_n, \ldots \rangle$$

be a presentation of L and set $D_n = \langle a_1, \ldots, a_p \,|\, r_0, \ldots, r_n \rangle$ for $n \geq 0$. Let $\langle x_1, \cdots, x_p | \rangle$ be the free group generated by $x_1, \cdots, x_p$ and N_n the normal subgroup generated by $r_0(\bar{x}), \ldots, r_n(\bar{x})$, where $\bar{x} := (x_1, \cdots, x_p)$. Then $D_n \cong \langle x_1, \ldots, x_p \rangle / N_n$. We have a direct system of homomorphisms $f_{n,m}$ from D_n to D_m, $n \leq m$, defined by $f_{n,m}(x.N_n) = x.N_m$. It follows that L is the direct limit of the previous system, $L = \varinjlim D_n$.

Let us define L_n to be the group D_n/K_n, where K_n is the intersection of all normal subgroups M for which D_n/M is a subgroup of some $C \in \mathcal{C}$. We see that each L_n is residually-$\mathcal{C}$. Let $\pi_n : D_n \to L_n$ be the natural homomorphism.

Clearly, we have a natural homomorphism $\varphi_{n,m} : L_n \to L_m$ such that $\pi_m \circ f_{n,m} = \varphi_{n,m} \circ \pi_n$. We let $\mathcal{C}L$ to be the direct limit of the given system, $\mathcal{C}L = \varinjlim L_n$. We note also that, we have a natural homomorphism $\pi : L \to \mathcal{C}L$ and we get the following diagram.

$$\begin{array}{ccccccccccc}
D_0 & \xrightarrow{f_{0,1}} & D_1 & \longrightarrow \cdots \longrightarrow & D_n & \xrightarrow{f_{n,n+1}} & D_{n+1} & \longrightarrow \cdots & & L \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} & & & \downarrow{\scriptstyle \pi} \\
L_0 & \xrightarrow[\varphi_{0,1}]{} & L_1 & \longrightarrow \cdots \longrightarrow & L_n & \xrightarrow[\varphi_{n,n+1}]{} & L_{n+1} & \longrightarrow \cdots & & \mathcal{C}L
\end{array}$$

We claim that π is an isomorphism. By definition π is surjective and it is sufficient to show that it is injective. We note that for any word $w(\bar{a})$, $\pi(w(\bar{a})) = 1$ if and only if there exists $n \in \mathbb{N}$ such that $\pi_n(w(\bar{a})) = 1$.

Let $a \in L \setminus \{1\}$. Then there is a word w in $\bar{a}$ such that $a = w(\bar{a})$ and so for all $m \in \mathbb{N}$, $L \models \exists \bar{x}\,(w(\bar{x}) \neq 1\ \&\ \bigwedge_{0 \leq n \leq m} r_n(\bar{x}) = 1)$.

By hypothesis, L is approximable by $\mathcal{C}$, so for all $m \in \mathbb{N}$, there exists $C_m \in \mathcal{C}$ such that $C_m \models \exists \bar{x}\,(w(\bar{x}) \neq 1\ \&\ \bigwedge_{0 \leq n \leq m} r_n(\bar{x}) = 1)$. Let $\bar{b}_m \in C_m$ such that $w(\bar{b}_m) \neq 1$ and $\bigwedge_{0 \leq n \leq m} r_n(\bar{b}_m) = 1$. Hence, there is a homomorphism from L_m to the subgroup of C_m generated by $\bar{b}_m$, so for some normal subgroup M_m of L_m, we get $L_m/M_m \cong \langle \bar{b}_m \rangle \leq C_m$. By definition, we have $K_m \leq M_m$ and thus $\pi_m(w(\bar{a})) \neq 1$ (for all $m \in \mathbb{N}$). Hence $\pi(a) \neq 1$ and thus π is injective as required.

Let $n \geq 0$ and $S = \{g_1, \ldots, g_q\} \subseteq L_n$ be a finite subset such that $1 \notin \psi_n(S)$, where $\psi_n : L_n \to \mathcal{C}L$ is the natural map. Proceeding as above, there exists a finite sequence of words $(w_j(\bar{x}))_{1 \leq j \leq q}$ such that $g_j = w_j(\bar{x})$ and $L \models \exists \bar{x}\,(\bigwedge_{1 \leq j \leq q} w_j(\bar{x}) \neq 1\ \&\ \bigwedge_{0 \leq n \leq m} r_n(\bar{x}) = 1)$. Proceeding as above, we find a normal subgroup $M_n \leq L_n$ such that $K_n \leq M_n$, $\bigwedge_{1 \leq j \leq q} g_j \notin M_n$, L_n/M_n isomorphic to a subgroup of some element $C \in \mathcal{C}$; which gives the required result.

(2) $\Rightarrow$ (1). Let L be a finitely generated subgroup of G. Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of residually-$\mathcal{C}$ groups whose direct limit is L and satisfying the property (ii). Denote the maps in the direct system between L_n and L_m, $n \leq m$, by $f_{n,m}$. Let $L = \varinjlim L_n = \bigsqcup_n L_n/\sim$, and for $x \in L$ we let $x_n \in L_n$ to be a representative of x with respect to the equivalence relation $\sim$, where $x_n \sim x_m$ if and only if there exists $k \geq max\{n, m\}$ such that $f_{n,k}(x_n) = f_{m,k}(x_m)$.

Then we see that if L satisfies a formula of the form, for some tuple $\bar{x}$,

$$\bigwedge_{1 \leq j \leq q} w_j(\bar{x}) \neq 1\ \&\ \bigwedge_{1 \leq i \leq p} r_i(\bar{x}) = 1,$$

then there exists $n \in \mathbb{N}$ such that

$$L_n \models \bigwedge_{1 \leq j \leq q} w_j(\bar{x}_n) \neq 1\ \&\ \bigwedge_{1 \leq i \leq p} r_i(\bar{x}_n) = 1,$$

where $\bar{x}_n$ is a representative of $\bar{x}$. By (ii), we conclude that there exists $C_n \in \mathcal{C}$ such that

$$C_n \models \exists \bar{x}\ (\bigwedge_{1 \leq j \leq q} w_j(\bar{x}) \neq 1\ \&\ \bigwedge_{1 \leq i \leq p} r_i(\bar{x}) = 1).$$

We conclude that L satisfies $Th_\forall(\mathcal{C})$. By the result of Malcev recalled above, G embeds in an ultraproduct of its finitely generated subgroups and so $G \models Th_\forall(\mathcal{C})$. □

Corollary 2.4. *Let G be a group and $\mathcal{C}$ a class of groups closed under finite direct products. Then G is approximable by $\mathcal{C}$ if and only if any finitely generated subgroup of G is a direct limit of finitely generated fully residually-$\mathcal{C}$ groups.* □

As consequence, taking $\mathcal{C}$ to be the class of finite groups, we have the following corollary which seems new and not observed in the literature. A. Vershik and E. Gordon showed that a finitely presented LEF-group is residually finite [53].

Corollary 2.5. *Let G be a finitely generated group. The following properties are equivalent.*
(1) *G is LEF.*
(2) *G is a direct limit of residually finite groups.* □

Remark 2.1.
(1) We note also that a finitely generated group is approximable by $\mathcal{C}$ if and only if G is a limit in an adequate topological space of marked groups (see [8], [9]).
(2) It follows from Proposition 2.2 that the class of pseudofinite groups is included into the class of LEF-groups since any pseudofinite group embeds into an ultraproduct of finite groups. It is easy to see that the class of pseudofinite groups is strictly smaller than the class of LEF-groups (see below).

Examples.
(1) Let $\mathcal{C}$ be the class of finite groups. A locally residually finite group is locally $\mathcal{C}$ [53]. There are groups which are not residually finite and which are approximable by $\mathcal{C}$, for instance, in [8] an example of a finitely generated amenable LEF group which is not residually finite is given. There are residually finite groups which are not pseudofinite, for instance the free group F_2 (see Corollary 3.3).
(2) Let $\mathcal{C}$ be the class of free groups. If G is fully residually-$\mathcal{C}$ (or equivalently ω-residually free or a limit group), then G is approximable by $\mathcal{C}$ [11]. Conversely if G is approximable by $\mathcal{C}$, then G is locally fully residually-$\mathcal{C}$. The same property holds also in hyperbolic groups [48, 41] and more generally in equationally noetherian groups [37].
(3) Let V be a possibly infinite-dimensional vector space over a field K. Denote by $GL(V,K)$ the group of automorphisms of V. Let $g \in GL(V,K)$, then g has finite residue if the subspace $C_V(g) := \{v \in V : g.v = v\}$ has finite-co-dimension. A subgroup G of $GL(V,K)$ is called a *finitary* (infinite-dimensional) linear group, if all its elements have finite residue. A subgroup G of $\prod_{i\in I}^{\mathscr{U}} GL(n_i, K_i)$, where K_i is a field, is *of bounded residue* if for all $g \in G$, where $g := [g_i]_{\mathscr{U}}$, $res(g) := inf\{n \in \mathbb{N} : \{i \in I : res(g_i) \leq n\} \in \mathscr{U}\}$ is finite.
E. Zakhryamin has shown that any finitary (infinite-dimensional) linear group G is isomorphic to a subgroup of bounded residue of some ultraproduct of finite linear groups ([60] Theorem 3). In particular letting $\mathcal{C} := \{GL(n,k)$, where k is a finite field and $n \in \mathbb{N}\ \}$, any finitary (infinite-dimensional) linear group G is approximable by $\mathcal{C}$.

Recall that a group G is said to be a *CSA-group* [30] if every maximal abelian subgroup A of G is malnormal; that is $A^g \cap A = 1$ for any $g \in G \setminus A$. In particular, a nonabelian CSA-group has no nontrivial normal proper abelian subgroup. Let us observe that if G is

CSA, then all the centralizers are abelian. Indeed, let $a \in G \setminus \{1\}$ and let A be a maximal abelian subgroup of G containing a and suppose that there exists $b \in C_G(a) \setminus A$. Consider $A^b \cap A$. This intersection contains a, which is a contradiction. In particular the maximal abelian subgroups of G are centralizers.

Lemma 2.6. [30, 36] *The property of being CSA can be expressed by a universal sentence.*

Proof: Let G be a group. Let us express that $\forall x \neq 1$ $C_G(x)$ is abelian and $\forall y \forall z$ $y \notin C_G(x)$ and $z \in C_G(x) \cap C_G(x)^y$ implies that $z = 1$. Then G is CSA iff G satisfies that sentence. □

Corollary 2.7. [36] *A finite CSA group is abelian.*

Proof: Since the property of being CSA is universal, it is inherited by subgroups. So, a minimal nonabelian CSA finite group has all its proper subgroups abelian and so this group is soluble by a result of O.J.Smidt ([42] 9.1.9) and thus it has a nontrivial proper normal abelian subgroup; a contradiction. □

Proposition 2.8. *A pseudofinite CSA-group is abelian.*

Proof: Indeed, $G \preceq \prod_I^{\mathscr{U}} F_i$, where each F_i is finite, and since the class of CSA-groups is axiomatizable by a single universal sentence (Lemma 2.6), for almost i, F_i is a CSA-group. But a finite CSA-group is abelian and thus G is abelian. □

Corollary 2.9. *The classes of pseudofinite groups and of nonabelian groups approximable by nonabelian free groups have a trivial intersection.* □

Proof: A nonabelian free group is a CSA-group and we apply Lemma 2.8 and Proposition 2.2. □

There are other kinds of approximation by classes of groups related to the previous notions. Gromov [21, Sect.6.E] introduced groups whose Cayley graphs are *initially subamenable* which are afterward called *sofic* by B. Weiss [56]. These groups can be seen as a simultaneous generalization of amenable groups and residually finite groups. We give a definition which is a slight generalization of already known notions by using *invariant metric* and which follows the definition given in [18] (see also [18] for the proof of the fact that this definition is equivalent to the classical one for sofic groups). A group G is an *invariant-metric group*, if there is a distance d on G which is bi-invariant; namely for any $x, y, z \in G$, $d(zx, zy) = d(xz, yz) = d(x, y)$.

Definition 2.2. Let $\mathcal{C}$ be a class of invariant-metric groups. A group G is *$\mathcal{C}$-sofic or sofic relative to $\mathcal{C}$*, if for any finite subset F of G, there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}^*$, there exists $(C, d_C) \in \mathcal{C}$ and an injective map $\xi_F : F \to C$ such that for any $g, h \in F$, if $gh \in F$, then $d_C(\xi_F(gh), \xi_F(g)\xi_F(h)) \leq \frac{1}{n}$ and for all $g \in F \setminus \{1\}$, $d_C(1, \xi_F(g)) \geq \epsilon$.

For $n \in \mathbb{N}^*$ let S_n be the symmetric group on n elements and d_n be the distance on S_n, called the *normalized Hamming distance*, defined by $d_n(\sigma, \tau) = \frac{1}{n}|\{i \in n : \sigma(i) \neq \tau(i)\}|$ with σ, $\tau \in S_n$ (identifying n with the subset $\{1, \cdots, n\}$ of natural numbers). Then a sofic group relative to $\mathcal{C} = \{(S_n, d_n) : n \in \mathbb{N}\}$ is called *sofic.*

We are interested in a characterization of sofic groups relative to $\mathcal{C}$ in terms of embeddings in adequate ultraproducts analogous to that of Proposition 2.2. Elek-Szabó [13] gave such characterization for sofic groups that we generalize here to the general framework of invariant-metric groups (see also [38, 56, 8]).

Definition 2.3. Let $\mathcal{C}$ be a class of invariant-metric groups, I a set and $\mathscr{U}$ a nonprincipal ultrafilter on I. For a sequence $(C_i)_{i\in I}$ from $\mathcal{C}$ we let $\mathcal{G} = \prod_I^{\mathscr{U}} C_i$. Then $\mathcal{G}$ is group endowed with a natural bi-invariant metric $d_{\mathscr{U}}$ with values in $\prod_I^{\mathscr{U}} \mathbb{R}$ by defining $d_{\mathscr{U}}([a_i]_{\mathscr{U}}, [b_i]_{\mathscr{U}}) = [d_{C_i}(a_i, b_i)]_{\mathscr{U}}$. We will say that a distance is *infinitesimal* if it is smaller than any strictly positive rational number. Consider the subset of $\mathcal{G}$ defined by $\mathcal{N} = \{g \in \mathcal{G} | d_{\mathscr{U}}(1, g) \text{ is infinitesimal}\}$. Then $\mathcal{N}$ is a normal subgroup and the quotient group $\mathcal{G}/\mathcal{N}$ will be called an *universal* $\mathcal{C}$*-sofic group.*

Proposition 2.10. *Let $\mathcal{C}$ be a class of invariant-metric groups and G a group. Then the following properties are equivalent.*

(1) *G is $\mathcal{C}$-sofic.*
(2) *G is embeddable in some universal $\mathcal{C}$-sofic group.*

Proof. (1) $\Rightarrow$ (2). Let $J(G) = \mathcal{P}_{fin}(G) \times \mathbb{N}$ and let $\mathscr{U}$ be a nonprincipal ultrafilter over $J(G)$ containing all subsets of the form $J_{F,n_0} = \{(e, n) \in J : \; F \subset e \;\&\; n \geq n_0\}$. For each $(e, n) \in J(G)$ let $C_{(e,n)} \in \mathcal{C}$ and $\xi_{e,n} : e \to C_{(e,n)}$ for which the properties given in Definition 2.2 are fulfilled. Consider the ultraproduct $\mathcal{G}(G) = \prod_{J(G)}^{\mathscr{U}} C_{(e,n)}$ and $\mathcal{N}(G)$ the corresponding normal subgroup as defined in Definition 2.3. Define $\xi : G \to \mathcal{G}(G)$ by $\xi(g) = [\xi_{(e,n)}(g))_{(e,n)\in J(G)}]_{\mathscr{U}}$. We note that $d_{\mathscr{U}}(\xi(g_1 g_2), \xi(g_1)\xi(g_2))$ is infinitesimal and $d_{\mathscr{U}}(\xi(g), 1) > 0$ for every $g \in G \setminus \{1\}$. Hence $\xi : G \to \mathcal{G}(G)/\mathcal{N}(G)$ is an embedding.

(2) $\Rightarrow$ (1). Let $\xi : G \to \mathcal{G}/\mathcal{N}$ be an embedding and set $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{N}$ the natural map. For every finite set F of G, let F' be a subset of $\mathcal{G}$ such that the restriction of π to F' is a bijection from F' to $\xi(F)$ and set $\pi_F^{-1} : \xi(F) \to F'$ the inverse of the restriction of π. Then for any $g, h \in F$, if $gh \in F$, then $d_{\mathscr{U}}(\pi_F^{-1} \circ \xi(gh), \pi_F^{-1} \circ \xi(g) \cdot \pi_F^{-1} \circ \xi(h))$ is infinitesimal and $d_{\mathscr{U}}(1, \pi_F^{-1} \circ \xi(f)) \geq \epsilon$ for any $f \in F$ and some $\epsilon > 0$. By considering an adequate subset of $\mathscr{U}$, we get the required conclusion. □

It is an open problem whether or not any group is sofic. However, it is known that many groups are sofic: residually finite groups, LEF-groups, amenable groups, residually amenable groups (see for instance [8]). More generally any pseudosofic group is sofic [8, Proposition 7.5.10]. We see in particular that any pseudofinite group is sofic.

The next lemma is well-known and holds for any pseudofinite structure, but we give a proof for the reader convenience.

Lemma 2.11. *Let G be a pseudofinite group. Any definable subgroup or any quotient by a definable normal subgroup is pseudofinite.*

Proof: Since G is pseudofinite, there is a family $(F_i)_{i\in I}$ of finite groups and an ultrafilter $\mathscr{U}$ such that $G \preceq \prod_I^{\mathscr{U}} F_i$. Let $\phi(x, \bar{y})$ be a formula and $\bar{b} \in G$ such that $\phi(x, \bar{b})$ defines a subgroup of G. Let $[\bar{b}_i]_{i\in I}$ be a sequence representing $\bar{b}$. Then there exists $U \in \mathscr{U}$ such that for every $i \in U$, $\phi(x, \bar{b}_i)$ defines a subgroup of F_i.

Given a formula $\theta(\bar{x})$ whose prenex form is: $Qz_1 \cdots Qz_n \; \chi(\bar{z}, \bar{x})$, where χ is a quantifier-free formula and Q denotes either $\exists$ or $\forall$, we let $\theta^{\phi}(\bar{x}; \bar{y}) = Qz_1 \cdots Qz_n \; \chi(\bar{z}, \bar{x}) \;\&\; \bigwedge_{i=1}^n \phi(z_i, \bar{y})$.

Assume now that σ is a sentence. Then $\sigma^{\phi}(\bar{y})$ expresses that the subgroup defined by $\phi(x; \bar{y})$ satisfies σ. We conclude that $\prod_I^{\mathscr{U}} F_i \models \sigma^{\phi}(\bar{b})$ for any sentence σ true in any finite group. Hence $\phi(G; \bar{b})$ satisfies any sentence true in any finite group, and thus it is pseudofinite. The same method works for quotients by using relativization to quotients. This time instead of relativizing the quantifiers, we have to replace equality by belonging to the same coset of $\phi(G; \bar{b})$. □

There are many definitions of semi-simple groups in the literature, which differ from a context to another. We adopt here the following. We will say that a group is *semi-simple* if it has no nontrivial normal abelian subgroups. We note in particular that a semi-simple group has no nontrivial soluble normal subgroups.

Let G be a finite group and let $rad(G)$ be the soluble radical, that is the largest normal soluble subgroup of G. In [58], J. Wilson proved the existence of a formula, that will be denoted in the rest of this paper by $\phi_R(x)$, such that in any finite group G, $rad(G)$ is definable by ϕ_R.

Lemma 2.12. *If G is a pseudofinite group then $G/\phi_R(G)$ is a pseudofinite semi-simple group.*

Proof: By the preceding Lemma 2.11 and the above result of Wilson, $G/\phi_R(G)$ is pseudofinite. Let us show it is semi-simple. Suppose there exists $a \in G \setminus \phi_R(G)$ such that for all $h,\ g \in G$, $\phi_R([a^h, a^g])$. By hypothesis $G \equiv \prod_I^{\mathscr{U}} G_i$, where each G_i is finite. So on an element of $\mathscr{U}$, $G_i \models \exists x \forall y \forall z\ (\phi_R([x^y, x^z])\ \&\ \neg\phi_R(x))$; a contradiction. □

We end the section by recalling another well-known result, namely the equivalence in an $\aleph_0$-saturated group of not containing the free group and of satisfying a nontrivial identity.

Notation 2.2. Let F_2 be the free nonabelian group on two generators and M_2 be the free subsemigroup on two generators.

Lemma 2.13. *Let G be an $\aleph_0$-saturated group. Then either G contains F_2, or G satisfies a nontrivial identity (in two variables). In the last case, either G contains M_2, or G satisfies a finite disjunction of positive nontrivial identities in two variables.*

Proof: We enumerate the set $W_{x,y}$ (respectively $M_{x,y}$) of nontrivial reduced words in $\{x, y, x^{-1}, y^{-1}\}$ (respectively in $\{x, y\}$) and we consider the set of atomic formulas $p(x, y) := \{t(x, y) \neq 1 : t(x, y) \in W_{x,y}\}$ (respectively $q(x, y) := \{t_1(x, y) \neq t_2(x, y) : t_1(x, y), t_2(x, y) \in M_{x,y} \cup \{1\}, t_1 \neq t_2\}$).

Either there is a finite subset I of $p(x, y)$ (respectively of $q(x, y)$) which is not satisfiable in G and so $G \models \forall x\ \forall y\ \bigvee_{\theta \in I} \neg\theta(x, y)$, otherwise since G is $\aleph_0$-saturated, $G \supset F_2$ (respectively $G \supset M_2$).

Observe that if a group G satisfies a finite disjunction of nontrivial identities in two variables, then it satisfies one nontrivial identity. For sake of completeness, let us recall here the argument. Suppose $G \models (t_1(x, y) = 1 \vee t_2(x, y) = 1)$. Either $t_1(x, y)$ and $t_2(x, y)$ do not commute in the free group generated by $x,\ y$ and so the commutator $[t_1, t_2] \neq 1$ in the free group and so the corresponding reduced word is nontrivial and $G \models [t_1, t_2] = 1$. Or $t_1,\ t_2$ do commute in the free group and so there exists a nontrivial reduced word t in $x,\ y$ and $z_1,\ z_2 \in \mathbb{Z}$ such that $t_1 = t^{z_1}$ and $t_2 = t^{z_2}$. In that last case $G \models t(x, y)^{z_1 \cdot z_2} = 1$. □

3. Finitely generated pseudofinite groups.

We study in this section some properties of finitely generated pseudofinite groups, led by a question of G. Sabbagh who asked whether all such groups were finite. There will be two main ingredients. First, a definability result due to N. Nikolov and D. Segal that we will recall below (Theorem 3.1), and the following observation.

Recall that a group G is said to be *Hopfian* if any surjective endomorphism of G is bijective. Any finitely generated residually finite group is Hopfian (Malcev) (see for instance [29] page 415). Since in a finite structure, any injective map is surjective and vice-versa, any definable map (with parameters) in a pseudofinite group is injective iff it is surjective. In particular a pseudofinite group is *definably* hopfian; that is any definable surjective homomorphism is injective.

Notation 3.1. Let G^n be the verbal subgroup of G generated by the set of all g^n with $g \in G$, $n \in \mathbb{N}$. The width of this subgroup is the maximal number (if finite) of n^{th}-powers necessary to write an element of G^n.

Theorem 3.1. ([32], [33] Theorem 1) *There exists a function $d \to c(d)$ such that if G is a d-generated finite group and H is a normal subgroup of G, then every element of $[G, H]$ is a product of at most $c(d)$ commutators of the form $[h, g]$, $h \in H$ and $g \in G$.*

Moreover, there exists a function $d \to b(d, n)$ such that in a finite group generated by d elements, the verbal subgroup generated by the n^{th}-powers, is of finite width bounded by $b(d, n)$. □

Finally let us recall the positive solution of the restricted Burnside problem, a long standing problem that was completely solved by E. Zelmanov ([52], [61]). Given k, d, there are only finitely many finite groups generated by k elements of exponent d.

Proposition 3.2. (Sabbagh) *Any abelian finitely generated pseudofinite group is finite.*

Proof: A finitely generated abelian group is a direct sum of a finite group and finitely many copies of $\mathbb{Z}$. So there exists a natural number n such that G^n is a 0-definable subgroup of G which is isomorphic to $\mathbb{Z}^k$ for some k. But $\mathbb{Z}^k$ cannot be pseudofinite since the map $x \to x^2$ is injective but not surjective. □

Corollary 3.3. *There are no nontrivial torsion-free hyperbolic pseudofinite groups.*

Proof: A torsion-free hyperbolic group is a CSA-group and thus if it were pseudofinite then it would be abelian by Proposition 2.8. We conclude by the above proposition. □

Recall that a group is said to be *uniformly locally finite* if for any $n \geq 0$, there exists $\alpha(n)$ such that any n-generated subgroup of G has cardinality bounded by $\alpha(n)$. In particular a uniformly locally finite group is of finite exponent. Examples of uniformly locally finite groups include $\aleph_0$-categorical groups.

Lemma 3.4. *A pseudofinite group of finite exponent is uniformly locally finite.*

Proof: Let $\langle g_1, \cdots, g_k \rangle$ be a k-generated subgroup of G. By definition $G \preceq \prod_J^{\mathscr{U}} G_j$, where G_j is a finite group. If G is of exponent e, on an element of $\mathscr{U}$, G_j is of exponent e. Let $[g_{mj}]_{j \in J}$, $1 \leq m \leq k$, be a representative for g_m and consider the subgroup $\langle g_{1j}, \cdots, g_{kj} \rangle$ on that element of $\mathscr{U}$. Then by the positive solution of the restricted Burnside problem, there is a bound $N(k, e)$ on the cardinality of that subgroup. So the subgroup $\langle g_1, \cdots, g_k \rangle$ embeds into an ultraproduct of groups of cardinality bounded by $N(k, e)$ and so has cardinality bounded by $N(k, e)$. □

Lemma 3.5. *A group G approximable by a class $\mathcal{C}$ of finite groups of bounded exponent is uniformly locally finite.*

Proof: By Proposition 2.2, such group embeds in an ultraproduct of elements of $\mathcal{C}$. So by the same reasoning as in the above lemma, any subgroup of G generated by k elements embeds into an ultraproduct of groups of cardinality bounded by a natural number $N(k, e)$ where e is a bound on the exponent of the elements of $\mathcal{C}$ and so it is finite. □

Proposition 3.6. *Let L be a pseudo-(d-generated finite groups). Then for any definable subgroup H of L, the subgroup $[H, L]$ is definable. In particular the terms of the descending central series of L are of finite width. Moreover, the verbal subgroups L^n, $n \in \mathbb{N}^*$, are 0-definable, of finite width and of finite index.*

Proof: Let $L \preceq \prod_I^{\mathcal{U}} F_i$, where F_i is a finite group generated by d elements.

Let $\phi(x;\bar{y})$ be a formula and $\bar{b} = [\bar{b}_i]$ such that $\phi(x;\bar{b})$ defines a subgroup H. On an element of the ultrafilter, $\phi(x;\bar{b}_i)$ defines a subgroup H_i and the subgroup $[H_i, F_i]$ is of width $\leq c(d)$ (Theorem 3.1). This property can be expressed by a sentence

$$\bigwedge_{1\leq j\leq c(d)+1} \forall u_j \forall v_j \bigwedge_{1\leq j\leq c(d)} \exists x_j \exists y_j (\bigwedge_{1\leq j\leq c(d)} \phi(x_j;\bar{b}_i) \& \bigwedge_{1\leq j\leq c(d)+1} \phi(u_j;\bar{b}_i) \Rightarrow \prod_{j=1}^{c(d)+1} [u_j, v_j] = \prod_{i=1}^{c(d)} [x_j, y_j]),$$

and so, $[H_i, F_i]$ is definable as well as the subgroup $[H, L]$ of L. A similar argument shows that the terms of the descending central series are of finite width.

Moreover by Theorem 3.1, the sentence $\forall u \forall u_1 \cdots \forall u_{b(d,n)} \exists x_1 \cdots \exists x_{b(d,n)}\ u^n. \prod_{i=1}^{b(d,n)} u_i^n = \prod_{i=1}^{b(d,n)} x_i^n$ holds in $\prod_I^{\mathcal{U}} F_j$. Since it holds in $\prod_I^{\mathcal{U}} F_j$, it holds in L and so L^n is 0-definable and of finite width.

By the solution of the restricted Burnside problem, the index of F_j^n in F_j is bounded in terms of d and n only. Then one can express that property by a $\exists\forall\exists$-sentence which transfers in the ultraproduct of the F_j's and therefore in L. □

Remark 3.1. Note that a definable subgroup of a pseudo-(d-generated finite groups) is not in general a pseudo-(d'-generated finite groups), for some d'. Indeed, this would imply that one could apply the preceeding proposition to the derived subgroup. However, there exists a family of finite 2-generated p-groups where the word $[[x_1, x_2], [x_3, x_4]]$ has infinite width ([45] Theorem 4.5.1). So if we take a nonprincipal ultraproduct of the elements of that family, we obtain a pseudo-(2-generated finite p-groups) whose second derived subgroup would be definable but since the class of pseudo-(2-generated finite p-groups) is closed under ultraproducts, this would imply that the second derived subgroup has finite width, a contradiction.

Notation 3.2. Let G be a group and $a, b \in G$, let $n \in \mathbb{N}$. We denote by $B^G_{\{a,b,a^{-1},b^{-1}\}}(n)$ (respectively $B^G_{\{a,b\}}(n)$) the set of elements of G which can be written as a word in a, b, a^{-1}, b^{-1} (respectively in a, b) of length less than or equal to n. By convention the identity of the group is represented by a word of length 0.

Definition 3.1. [4] A (finite) group G contains an approximation of degree n to F_2 (respectively M_2), the free nonabelian group (respectively subsemigroup) on two generators x, y if there exists $a, b \in G$ such that $|B^G_{\{a,b,a^{-1},b^{-1}\}}(n)| = |B^{F_2}_{\{x,y,x^{-1},y^{-1}\}}(n)|$ (respectively $|B^G_{\{a,b\}}(n)| = |B^{M_2}_{\{x,y\}}(n)|$.

Notation 3.3. Let G, L be two groups. Then $G \preceq_\exists L$ if G is a subgroup of L and every existential formula with parameters in G which holds in L, holds in G.

Proposition 3.7. *Let G be a finitely generated pseudofinite group. Then the terms of the derived series are 0-definable of finite width and of finite index.*

Moreover, the sugbroups G^m are 0-definable of finite width and of finite index, $m \in \mathbb{N}^*$.

Proof. Since G is pseudofinite, $G \preceq L = \prod_I^{\mathcal{U}} G_i$, where each G_i is finite. Let $\bar{a}$ be a finite generating tuple of G and set $\bar{a} = [\bar{a}_i]$, $F_i = \langle \bar{a}_i \rangle$ the subgroup of G_i generated by $\bar{a}_i$.

We see that $G \preceq_\exists \prod_I^{\mathscr{U}} F_i$. Since $\prod_I^{\mathscr{U}} F_i$ is pseudo-(d-generated finite groups), as in the proof of Proposition 3.6 any element of the derived subgroup is a product of at most $c(d)$ commutators. Since this can be expressed by $\forall\exists$-sentence and as $G \preceq_\exists \prod_I^{\mathscr{U}} F_i$, we conclude that the same property holds in G, and thus $[G,G]$ is 0-definable and of finite width.

By Lemma 2.11, $G/[G,G]$ is a finitely generated pseudofinite abelian group, and so by Proposition 3.2, it is finite. Hence $[G,G]$ is finitely generated and since it is 0-definable, it is again pseudofinite (Lemma 2.11). Thus the conclusion on the terms of the derived series follows by induction.

We may apply a similar method for the verbal subgroups G^n and we conclude that G^m is 0-definable of finite width. Since G/G^n is pseudofinite of finite exponent it is locally finite by Lemma 3.4 and since it is finitely generated it must be finite. □

Question 1. Is a d-generated pseudofinite group pseudo-(d-generated finite groups)?

We will use the following notation throughout the rest of this section. Let G be an infinite finitely generated pseudofinite group. Assume that G is generated by $g_1, \cdots, g_d$. By Frayne's theorem, there is an ultrapower $\prod_I^{\mathscr{U}} F_i$, where each F_i, $i \in I$, is a finite group, into which G elementarily embeds. Using this elementary embedding, we identify g_k with $[f_{ki}]_{\mathscr{U}}$ with $f_{ki} \in F_j$, $1 \leq k \leq d$. So, G is isomorphic to the subgroup $\langle [f_{1i}]_{\mathscr{U}}, \cdots, [f_{di}]_{\mathscr{U}} \rangle$ of $\prod_I^{\mathscr{U}} F_j$.

Proposition 3.8. *Let G be a finitely generated pseudofinite group and suppose that G satisfies one of the following conditions.*

1. *G is of finite exponent, or*
2. (Khélif) *G is soluble, or*
3. *G is soluble-by-(finite exponent), or*
4. *G is pseudo-(finite linear of degree n in characteristic zero), or*
5. *G is simple, or*
6. *G is hyperbolic.*

Then such a group G is finite.

Proof. (1) G is locally finite by Lemma 3.4 and thus finite as it is finitely generated.

(2) Since G is soluble, $G^{(n)} = 1$ for some n and thus G is finite by Proposition 3.7.

(3) Assume that G is soluble-by-exponent n. By Proposition 3.7 G^n is 0-definable and soluble. Since G/G^n is a finitely generated pseudofinite group of exponent n, by (1), it is finite, so G^n is again a finitely generated soluble pseudofinite group and so it is finite by (2). Thus G is finite as required.

(4) Let $G \preceq L = \prod_I^{\mathscr{U}} F_i$, where each F_i is finite and linear of degree n over $\mathbb{C}$. By a result of C. Jordan [55, Theorem 9.2], there exists a function $d(n)$ depending only on n such that each F_i has an abelian subgroup of index at most $d(n)$. Hence L is abelian-by-finite and since G is a subgroup of L, G is also abelian-by-finite. By (3), G is finite.

(5) In this case one uses Wilson's classification of the simple pseudofinite groups [58] and in particular the fact that they are all linear. Since G is finitely generated and linear it is residually finite by a result of Mal'cev. Since G is simple, it must be finite.

(6) If G is not cyclic-by-finite then the commutator subgroup has an infinite width ([20]). Thus G is cyclic-by-finite and thus finite by (3). One can also argue as follows. Suppose G has an element g of infinite order. Then $C_G(g)$ is (infinite cyclic)-by-finite ([20]) and since it is a definable subgroup of G, it is pseudofinite as so finite by (3). □

Question 2. Is a pseudofinite linear group of degree n, pseudo-(finite and linear of degree n) ?

Question 3. Are there finitely generated infinite residually finite groups G which are pseudofinite?

Question 4. *(Sabbagh)* Are there finitely generated infinite groups G which are pseudofinite?

4. Free subsemigroup, superamenability

We study in this section the existence of free subsemigroups of rank two in pseudofinite groups and its link with superamenability. Recall that a group is *superamenable* if for any nonempty subset A of G, there exists a left-invariant finitely additive mesure $\mu : P(G) \to [0, \infty]$ such that $\mu(A) = 1$. It is known that a group containing a free subsemigroup of rank two is not superamenable [54, Proposition 12.3]. Superamenability is a strong form of amenability which was introduced by Rosenblatt in [43] who also conjectured that a group is superamenable if and only if it is amenable and does not contain a free subsemigroup of rank two. This question was settled negatively by R. Grigorshuck in [19]. In this section, we show in particular, that for $\aleph_0$-saturated pseudofinite groups, superamenability is equivalent to the absence of free subsemigroups of rank two.

Theorem 4.1. *Let G be an $\aleph_0$-saturated pseudofinite group. Then either G contains a free subsemigroup of rank 2 or G is nilpotent-by-(uniformly locally finite).*

Before proving Theorem 4.1, we will state two corollaries.

Definition 4.1. [54, Definition 12.7]

• Let G be a group and S a finite generating set of G. We let $\gamma_S(n)$ to be the cardinal of the ball of radius n in G (for the word distance with respect to $S \cup S^{-1}$), namely $|B^G_{S\cup S^{-1}}(n)|$ (see Notation 3.2).

• A group G is said to be *exponentially bounded* if for any finite subset $S \subseteq G$, and any $b > 1$, there is some $n_0 \in \mathbb{N}$ such that $\gamma_S(n) < b^n$ whenever $n > n_0$.

Corollary 4.2. *Let G be an $\aleph_0$-saturated pseudofinite group. Then the following properties are equivalent.*

(1) *G is superamenable.*
(2) *G has no free subsemigroup of rank 2.*
(3) *G is nilpotent-by-(uniformly locally finite).*
(4) *G is nilpotent-by-(locally finite).*
(5) *Every finitely generated subgroup of G is nilpotent-by-finite.*
(6) *G is exponentially bounded.*

Proof. (1) $\Rightarrow$ (2). This is exactly the statement of [54, Proposition 12.3].
(2) $\Rightarrow$ (3). This is exactly the statement of Theorem 4.1.
(3) $\Rightarrow$ (4) $\Rightarrow$ (5). Clear.
(5) $\Rightarrow$ (6) $\Rightarrow$ (1). This follows from [54] (see page 198 for more details). □

Corollary 4.3. *An infinite finitely generated pseudofinite group has approximation of degree n to M_2 for every $n \in \mathbb{N}$.*

Proof. Suppose not. Let $G \preceq L = \prod_I^{\mathcal{U}} G_i$, where each G_i is finite and assume that for some $n \in \mathbb{N}$, G has not an approximation of degree n to M_2. Then L doesn't contain a free subsemigroup of rank 2 and since L is $\aleph_0$-saturated, it is nilpotent-by-(uniformly locally finite). Therefore if in addition G is finitely generated, G is nilpotent-by-finite and thus finite by Proposition 3.8; a contradiction. □

The rest of the section is devoted to the proof of Theorem 4.1. For $a, b \in G$, we let $H_{a,b} = \langle a^{b^n} | n \in \mathbb{Z} \rangle$ and $H'_{a,b}$ its derived subgroup. Let us recall the following definition (see [39], [40], [51]).

Definition 4.2. A nontrivial word $t(x, y)$ in x, y is a *N-Milnor word* of *degree* $\leq \ell$ if it can be put in the form $yx^{m_1}y^{-1}...y^\ell x^{m_\ell}y^{-\ell}.u = 1$, where $u \in H'_{x,y}$, $\ell \geq 1$, $gcd(m_1, ..., m_\ell) = 1$ (some of the m_i's are allowed to take the value 0 and $\sum_{i=1}^{\ell} |m_i| \leq N$, $N \in \mathbb{N} - \{0\}$.

A group G is *locally N-Milnor* (*of degree* $\leq \ell$) if for all a, b in G there is a nontrivial N-Milnor word $t(x, y)$ (of degree $\leq \ell$) such that $t(a, b) = 1$.

It is straightforward that a group G which contains the free group F_2, cannot be locally N-Milnor.

Any nilpotent-by-finite group is locally 1-Milnor. More generally one has the following property.

Lemma 4.4. ([43] Lemma 4.8.) *Let G be a group without free subsemigroup of rank 2. Then for any $a, b \in G$, the subgroup $H_{a,b}$ is finitely generated, and G is locally 1-Milnor.* □

A finitely generated linear group which is locally N-Milnor is nilpotent-by-finite (see [40] Corollary 2.3).

Example: Let p be a prime number and C_p (respectively C_{p^n}) be the cyclic group of order p (respectively p^n). Then the finite metabelian groups $C_p wr C_{p^n}$, $n \in \omega - \{0\}$ do not satisfy an identity of the form $t(x, y) = 1$, where $t(x, y)$ is a Milnor word of degree $< p^n$ (see [39] Lemma 7).

On Milnor words, we will use the following theorem stated to G. Traustason ([51]). The key fact on these words is that the varieties of groups they define have the property that any finitely generated metabelian group in the variety is nilpotent-by-finite ([7] Theorem A).

To a Milnor word $t(x, y) := yx^{m_1}y^{-1}...y^\ell x^{m_\ell}y^{-\ell}.u$, $u \in H'_{x,y}$, one associates a polynomial q_t of $\mathbb{Z}[X]$ as follows: $q_t[X] = \sum_{i=1}^{\ell} m_i.X^i$ (see [39], [51]).

Theorem 4.5. (See Theorem 3.19 in [51]) *Given a finite number of Milnor words t_i, $i \in I$ and their associated polynomials q_{t_i}, $i \in I$, there exist positive integers $c(q)$ and $e(q)$ only depending on $q := \prod_{i \in I} q_{t_i}$, such that a finite group G satisfying $\bigvee_{i \in I} t_i = 1$, is nilpotent of class $\leq c(q)$-by-exponent dividing $e(q)$.* □

Note that we can express by a universal sentence the property that a group G is nilpotent of class $\leq c(q)$-by-exponent dividing $e(q)$. So we can deduce the following.

Corollary 4.6. *Let G be a group approximable by a class of finite groups which are locally N-Milnor of degree $\leq \ell$. Then G is nilpotent-by-(uniformly locally finite).*

Proof: By Proposition 2.2, $G \leq L := \prod_{i \in I}^{\mathcal{U}} F_i$, where each F_i is a finite group which is locally N-Milnor of degree $\leq \ell$. So there is a finite disjunction $\bigvee_{j \in J} t_j(x, y) = 1$, J finite, where t_j is a N-Milnor word of degree $\leq \ell$, such that each F_i satisfies $\bigvee_{j \in J} t_j(x, y) = 1$. Let $q := \prod_{j \in J} q_{t_j}$. By the theorem above, there exist positive integers $c(q)$ and $e(q)$, such that F_i is nilpotent of class $\leq c(q)$-by-exponent dividing $e(q)$. Since the degree of each q_{t_j} is bounded by ℓ and their coefficients are bounded in absolute value by N, there is a finite number of such polynomials. Let Q denote the set of such polynomials. Let $c_{max} := max\{c(q) : q \in Q\}$ and $e_{max} := \prod_{q \in Q} e(q)$. So for each $i \in I$, we have that $F_i^{e_{max}}$ is nilpotent of class $\leq c_{max}$. So $\prod_I^{\mathcal{U}} F_i$ satisfies that property and it transfers to G since it can be expressed by a universal sentence. So, $G^{e_{max}}$ is nilpotent of class $\leq c_{max}$.

Set $N := \prod_I^{\mathcal{U}} F_i^{e_{max}}$, then N is a definable normal subgroup of L and so L/N is a pseudofinite group by Lemma 2.11. Since L/N is of finite exponent, it is locally finite by Lemma 3.4. Thus $G/G^{e_{max}}$ is also locally finite. □

Proof of Theorem 4.1
Proof: Let G be an $\aleph_0$-saturated pseudofinite group not containing the free subsemigroup of rank 2. Then, by Lemma 2.13, it satisfies a finite disjonction of positive identities. In particular there exists ℓ such that it is approximable by a class of finite groups locally 1-Milnor of degree $\leq \ell$ and so we may apply the preceding corollary. □

Corollary 4.7. *An $\aleph_0$-saturated locally N-Milnor pseudofinite group is nilpotent-by-(uniformly locally finite).*

Proof: It is proven in the same way as the above theorem, using a similar argument as in Lemma 2.13 to show that such group satisfies a finite disjonction of identities of the form $t_i(x, y) = 1$, where $t_i(x, y)$ is a N-Milnor word. And so again we can find a bound on the degrees of the corresponding Milnor words. □

Example: Y. de Cornulier and A. Mann have shown that if one takes the non Milnor word $[[x, y], [z, t]]^q$, then there is a residually finite 2-generated group satisfying the identity $[[x, y], [z, t]]^q = 1$, which is not soluble-by-finite ([12]). They exhibit a family of finite soluble groups R_n generated by two elements, of solubility length n and satisfying the identity $[[x, y], [z, t]]^q = 1$.

Let us recall their construction. On one hand they use an embedding theorem due to B.H. Neumann and H. Neumann in wreath products ([31]) and on the other hand a result of Razmyslov ([52] chapter 4) that for each prime power $q \geq 4$ there exist a finite group B_r generated by r elements, of exponent q and solubility length $n := \lfloor log_2(r) \rfloor$. By [31], B_r embeds in a two generated subgroup R_n of $(B_r WrC_{p^k})WrC_{p^k}$, for some k sufficiently large. (The number k is chosen such that $p^k \geq 4r - 1$.) So R_n is a 2-generated p-group satisfying the identity $[[x, y], [z, t]]^q = 1$.

In particular, we have an example of an $\aleph_0$-saturated pseudofinite group L not containing F_2 and not soluble-by-finite (with $\phi_R(L) = L$) (see Proposition 6.7). Take $L = \prod_{\mathbb{N}}^{\mathcal{U}} R_n$.

5. Free subgroups, amenability

As we have seen in the previous section, the absence of free subsemigroups of rank 2 in pseudofinite ($\aleph_0$-saturated) groups implies superamenability. In this section, we are interested in the similar problem with free subgroups of rank 2. However, as the next proposition shows, the problem is connected to some strong properties that residually finite groups must satisfy. Consequently, we will be interested in this section, more particularly, in the problem of amenability of pseudofinite groups. Then in the next section, we shall give some alternatives under stronger hypotheses.

Recall that a group is said to be *amenable* if there exists a finitely additive left-invariant measure $\mu : \mathcal{P}(G) \to [0,1]$ such that $\mu(G) = 1$. Note that there are many definitions of amenable groups in the literature (see for instance Theorem 10.11 [54]).

M. Bozejko [6] and G. Keller [25] called a group G *uniformly amenable* if there exists a function $\alpha : [0,1] \times \mathbb{N} \to \mathbb{N}$ such that for any finite subset A of G and every $\epsilon \in [0,1]$ there is a finite subset V of G such that $|V| \leq \alpha(\epsilon, |A|)$ and $|AV| < (1+\epsilon)|V|$. By using the equivalent definition of amenability with Følner sequences, we have that a uniformly amenable group is amenable.

Theorem 5.1. *The following properties are equivalent.*

(1) *Every $\aleph_0$-saturated pseudofinite group either contains a free nonabelian group or it is amenable.*
(2) *Every ultraproduct of finite groups either contains a free nonabelian group or it is amenable.*
(3) *Every finitely generated residually finite group satisfying a nontrivial identity is amenable.*
(4) *Every finitely generated residually finite group satisfying a nontrivial identity is uniformly amenable.*

G. Keller showed that a group G is uniformly amenable if and only if all its ultrapowers are amenable. Later J. Wysoczanski [59] gives a more simple combinatorial proof. However, the notion which is behind this, is the saturation property.

Remark 5.1. Let $\sigma_{p,n,f}$ be the following sentence with $(p,n) \in \mathbb{N}^2$ and $f : \mathbb{N}^2 \to \mathbb{N}$:

$$\forall a_1 \cdots \forall a_n \exists y_1 \cdots \exists y_{f(p,n)}\ p.|\{a_i.y_j : 1 \leq i \leq n; 1 \leq j \leq f(p,n)\}| < (p+1).f(p,n).$$

Then we see that G is uniformly amenable iff there exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for any $(p,n) \in \mathbb{N}^2$, $G \models \sigma_{p,n,f}$. In particular, being uniformly amenable is elementary, that is a property preserved by elementary equivalence.

Proposition 5.2. *An $\aleph_0$-saturated group is amenable if and only if it is uniformly amenable.*

Proof. Suppose that G is $\aleph_0$-saturated and amenable. Then for any finite subset A of G and every $\epsilon \in [0,1]$ there is a finite subset V of G such that $|AV| < (1+\epsilon)|V|$.

Let $A = \{a_1, \ldots, a_n\}$ and $\epsilon \in [0,1]$. We may assume without loss of generality that $\epsilon = 1/p$ for some $p \in \mathbb{N}$. Then

$$G \models \bigvee_{m \in \mathbb{N}} \exists x_1 \ldots \exists x_m (p|A.\{x_1, \ldots, x_m\}| < (p+1).m),$$

and by $\aleph_0$-saturation

$$G \models \bigvee_{1 \leq m \leq r} \exists x_1 \ldots \exists x_m (p|A.\{x_1, \ldots, x_m\}| < (p+1).m.$$

By setting $\alpha(\epsilon, n) = r$, we get the uniform bound. Thus G is uniformly amenable. □

Corollary 5.3. ([25], [59]) *A group is uniformly amenable if and only if all its nonprincipal ultrapowers are amenable iff one of its nonprincipal ultrapowers is amenable.* □

In the proof of Theorem 5.1, we will use the fact that the class of amenable groups is closed under various operations (see [54] Theorem 10.4) and in particular a group is amenable iff its finitely generated subgroups are. Recall that no amenable group contains the free subgroup of rank 2 ([54] Corollary 1.11).

We will need the following simple lemma ([25] Theorem 4.5).

Lemma 5.4. *A subgroup of a uniformly amenable group is uniformly amenable.* □

Proof of Theorem 5.1.

(1) ⇒ (2). An ultraproduct of finite groups is $\aleph_0$-saturated and pseudofinite, the conclusion follows.

(2) ⇒ (3). Let G be a finitely generated residually finite group satisfying a nontrivial identity $t = 1$. Then G embeds into an ultraproduct K of finite groups which satisfies a nontrivial identity (see Proposition 2.2 and note that G is residually $\mathcal{C}$ with $\mathcal{C}$ the class of finite groups satisfying $t = 1$). By (2), K is amenable and thus G is amenable.

(3) ⇒ (1). Let G be an $\aleph_0$-saturated pseudofinite group and suppose that G has no free nonabelian subgroup. Let K be an ultraproduct of finite groups such that $G \preceq K$. Since G is $\aleph_0$-saturated, G satisfies a nontrivial identity by Lemma 2.13, as well as K. It is sufficient to show that every finitely generated subgroup of K is amenable. Let L be a finitely generated subgroup of K. Let $\mathcal{C}$ be the class of finite groups satisfying the identity satisfied by K. Then L is approximable by $\mathcal{C}$, and since $\mathcal{C}$ is closed under finite direct products, by Proposition 2.2 L is a direct limit of fully residually-$\mathcal{C}$ groups. Hence L is a direct limit of residually finite groups satisfying a nontrivial identity. By our hypothesis such groups are amenable and so their direct limit is and since L is a quotient of this direct limit, L is amenable as well.

Clearly (4) ⇒ (3) and it remains to show (3) ⇒ (4). Let L be a finitely generated residually finite group satisfying a nontrivial identity, so L is residually $\mathcal{C}$, where $\mathcal{C}$ is a class of finite groups satisfying a nontrivial identity. By Proposition 2.2, L is approximable by $\mathcal{C}$, namely embeds in an ultraproduct K of elements of $\mathcal{C}$. By (1), K is amenable. Since K is $\aleph_0$-saturated, it is uniformly amenable by Proposition 5.2 as well as L by Lemma 5.4. □

As recalled above, a group containing a nonabelian free group cannot be amenable. Von Neumann and Day asked for the converse, namely whether every non-amenable group contains a nonabelian free group. This was answered negatively by Ol'shanskii [34], Adyan [1] and Gromov [20]. However a positive answer can be provided for some classes of groups as the class of linear groups. One may ask if the question has a positive answer in the class of residually finite groups. This was answered negatively by Ershov [16]. Other examples of non-amenable residually finite groups without nonabelian free subgroups were constructed by Osin in [35].

Question 5. [12, Question 14] Does there exist a non-amenable finitely generated residually finite group satisfying a nontrivial identity?

Definition 5.1. Let $\mathcal{G} = (G_i)_{i\in I}$ be a family of groups and $\mathscr{U}$ an ultrafilter over I. We say that $\mathcal{G}$ is *uniformly amenable* relative to $\mathscr{U}$ if the following condition holds. There exists a function $\alpha : [0, 1] \times \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$ and every $\epsilon \in [0, 1]$, there exists $U \in \mathscr{U}$ such that for any $i \in U$, for any finite subset A in G_i with $|A| = n$, there is a finite subset V of G_i such that $|V| \leq \alpha(\epsilon, |A|)$ and $|AV| < (1 + \epsilon)|V|$.

A proof similar to that of Proposition 5.2 yields.

Proposition 5.5. *Let $\mathcal{G} = (G_i)_{i\in I}$ be a family of groups and $\mathscr{U}$ an ultrafilter over I. Then $\prod_I^{\mathscr{U}} G_i$ is amenable if and only if $\mathcal{G}$ is uniformly amenable relative to $\mathscr{U}$.* □

Question 6. Is a pseudofinite amenable group uniformly amenable?

In [22], the notion of *definably amenable* groups was introduced. A group is said to be *definably amenable* if there exists a finitely additive left-invariant measure $\mu : \mathcal{D}(G) \to [0,1]$ with $\mu(G) = 1$; where $\mathcal{D}(G)$ is the boolean algebra of definable subsets of G. They pointed out that there are definably amenable groups which are not amenable as $SO_3(\mathbb{R})$ and also groups that are not definable amenable as $SL_2(\mathbb{R})$.

The following proposition gives natural examples of definably amenable groups (and again shows that there are definably amenable but non amenable groups).

Proposition 5.6. *A pseudofinite group is definably amenable.*

We will show in fact the next more general proposition. Let us first give a definition which is borrowed from non standard analysis.

Definition 5.2. Let I be a set and $(G_i)_{i\in I}$ a family of groups, $\mathscr{U}$ and ultrafilter on I. A subset $A \subseteq \prod_I^{\mathscr{U}} G_i$ is said to be *internal* if there exists $(A_i)_{i\in I}$, $A_i \subseteq G_i$, such that $A = \prod_I^{\mathscr{U}} A_i$.

We see that every definable subset is internal and that the set of internal subsets forms a left-invariant boolean algebra. Recall that a measure on a boolean algebra $\mathcal{B}$, is said *σ-additive* if $\mu(\bigcup_{n\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i)$ whenever $A_i \cap A_j = \emptyset$, $i \neq j \in \mathbb{N}$, and $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{B}$. Given a boolean algebra $\mathcal{B}$, we denote by $\bar{\mathcal{B}}$ the σ-algebra generated by $\mathcal{B}$.

Proposition 5.7. *Let $(G_i)_{i\in I}$ be a family of amenable groups, $\mathscr{U}$ an ultrafilter on I. Let $\mathcal{B}$ be the boolean algebra of internal subsets of $L = \prod_I^{\mathscr{U}} G_i$. Then there exists a finitely additive measure $\mu : \mathcal{P}(L) \to [0,1]$, $\mu(L) = 1$, whose restriction to $\bar{\mathcal{B}}$ is σ-additive and left-invariant.*

Proof: If $A \in \mathcal{B}$ and $A = \prod_I^{\mathscr{U}} A_i$, we define the measure of A by

$$\mu(A) = \lim_{\mathcal{U}} \mu_i(A_i),$$

where each μ_i is a left-invariant probability measure on G_i. It is not difficult to see that μ is a left-invariant finitely additive measure defined on $\mathcal{B}$.

Let us show that μ is σ-additive. It is sufficient to show that if $(B_i | i \in \mathbb{N})$ is a sequence of $\mathcal{B}$, such that $B_{i+1} \subseteq B_i$ and $\bigcap_{i\in\mathbb{N}} B_i = \emptyset$, then $\lim_{i\to\infty} \mu(B_i) = 0$.

But by the saturation of the ultraproduct, we get $B_1 \cap \cdots \cap B_n = \emptyset$, for some $n \geq 1$. Hence $\mu(B_i) = 0$ for $i \geq n$ and so $\lim_{i\to\infty} \mu(B_i) = 0$.

By Carathéodory's theorem, μ can be extended to a σ-additive $\bar{\mu}$ defined over $\bar{\mathcal{B}}$. It is not difficult to see that $\bar{\mu}$ is still left-invariant on $\bar{\mathcal{B}}$. By a theorem of Horn and Tarski [54], $\bar{\mu}$ can be extended to a finitely additive measure defined on $\mathcal{P}(L)$. □

Proof of Proposition 5.6. Let G be a pseudofinite group. Then $G \preceq L := \prod_I^{\mathscr{U}} G_i$, where each G_i is a finite group. By Proposition 5.7, there exists a left-invariant probability measure μ defined on definable subsets of L. For every definable subset X of G, definable by a formula $\phi(x)$, we take $\mu(X) = \mu(\phi(L))$. Then this defines a left-invariant probability measure μ on definable subsets of G. □

6. Free subgroups, alternatives

We study in this section existence of free subgroups in pseudofinite groups under strong hypotheses. Recall that the *Prüfer* rank of a group G is the least integer n such that every finitely generated subgroup of G can be generated by n elements. S. Black [4] has considered famillies $\mathcal{C}$ of finite groups of bounded Prüfer rank and showed that a finitely generated residually-$\mathcal{C}$ group G either contains a free nonabelian group or it is nilpotent-by-abelian-by-finite.

S. Black (Theorem A in [5]) also showed the following *finitary Tits' alternative*: there exists a function $d(n,r)$ such that if G is a finite group of *Prüfer* rank r then either G contains an approximation of degree n to F_2 or G has a soluble subgroup whose derived length and index in G are at most $d(n,r)$. Moreover, in this case, there exists $c = c(n,r)$, $\ell = \ell(n,r)$ such that G is nilpotent of class at most c-by-abelian-by-index-at-most-ℓ.

First, we reformulate the above result in the context of pseudofinite groups.

Theorem 6.1. *Let G be an $\aleph_0$-saturated pseudo-(finite of bounded Prüfer rank) group. Then either G contains a nonabelian free group or G is nilpotent-by-abelian-by-finite.*

Corollary 6.2. *An $\aleph_0$-saturated pseudo-(finite of bounded Prüfer rank) group either contains a nonabelian free group or is uniformly amenable.* □

Remark 6.1. The above theorem is equivalent to the previous mentioned result of Black. One direction is clear: the finitary Tits' alternative implies that if $L = \prod_I^{\mathscr{U}} F_i$, where L satisfies a nontrivial identity then for almost all i, F_i is nilpotent-by-abelian-by-finite with an uniform bound on the nilpotency classes and the indices; hence L is nilpotent-by-abelian-by-finite. For the converse let $(G_n | n \in \mathbb{N})$ be the sequence of all finite groups of Prüfer rank r and without an approximation of degree n to F_2. Suppose that for any $d \in \mathbb{N}$ there exists G_{n_d} such that G_{n_d} is not nilpotent of class at most d-by-abelian-by-index-at-most-d. Hence we get an infinite sequence $H_d = G_{n_d}$. Let $\mathscr{U}$ be any nonprincipal ultrafilter on $\mathbb{N}$ and set $L = \prod_{\mathbb{N}}^{\mathscr{U}} H_d$. Then L satisfies a nontrivial identity and thus L is nilpotent of class at most c-by-abelian-by-index-at-most-c for some $c \in \mathbb{N}$. Since this last property can be expressed by a first order sentence, there exists $U \in \mathscr{U}$ such that for any $d \in U$, H_d is nilpotent of class at most c-by-abelian-by-index-at-most-c. Hence for $d \geq c$ and $d \in U$, H_d is nilpotent of class at most c-by-abelian-by-index-at-most-c; a contradiction.

We give here the proof of Theorem 6.1 from the pseudofinite groups viewpoint. As in [5], one reduces first the problem to finite soluble groups, using a result of A. Shalev ([49]) and then one uses a result of D. Segal on residually (finite soluble groups) ([46]).

We first note that the following alternative holds for simple pseudofinite groups.

Lemma 6.3. *Let G be a nonprincipal ultraproduct of finite nonabelian simple groups. Then either G contains a nonabelian free group, or G is finite.*

Proof: Assume that G doesn't contain a nonabelian free group. Then, since G is $\aleph_0$-saturated, by Proposition 2.13, G satisfies a nontrivial identity. By a theorem of Jones that a proper variety of groups only contains finitely many finite nonabelian simple groups ([24]), so only a finite number of them appear in that ultraproduct. Therefore G is finite. □

Corollary 6.4. *Let G be a nonabelian simple pseudofinite group. Then either G contains a nonabelian free group, or G is finite.*

Proof: By a result of J. Wilson ([57]) and its strenghtening ([44]), G is isomorphic to a nonprincipal ultraproduct of finite nonabelian simple groups (of fixed Lie type). Then we apply the above lemma. □

Recall that a group is said *quasi-linear* if it is embeddable in a finite direct product of linear groups. We say that a function is *r-bounded* if it is bounded in terms of r only.

Proposition 6.5. *Let G be a semi-simple pseudo-(finite of bounded Prüfer rank) group. Then G has a quasi-linear subgroup of finite index.*

Proof. Let $G \prec L$, where $L = \prod_I^{\mathscr{U}} G_i$, $\mathscr{U}$ a nonprincipal ultrafilter on I and each G_i is finite of bounded *Prüfer* rank, $i \in I$. Since $\phi_R(G) = \{1\}$, we have that $\phi_R(L) = \{1\}$ and so on an element U of $\mathscr{U}$, each G_i is semi-simple. Using Proposition 3.6 in [49], for $i \in U$, there exists a characteristic subgroup G_{1i} of G_i such that $|G_i/G_{1i}|$ is r-bounded, say of cardinality $\leq f(r)$ and $G_{1i} \cong S_{1i} \times \cdots \times S_{ki}$ where $1 \leq k \leq g(r)$ and each S_{1i} is a simple pseudofinite group of Lie type of r-bounded Lie rank n_j over the finite field $F_{p_j^{e_j}}$ where e_j is r-bounded and $1 \leq j \leq r$, for each $i \in U$.

We have $1 \lhd \prod_I^{\mathscr{U}} G_{1i} \trianglelefteq \prod_I^{\mathscr{U}} G_i = L$ and the subgroup $L_0 := \prod_I^{\mathscr{U}} G_{1i}$ is of finite index in L since $|\prod_I^{\mathscr{U}} G_i / \prod_I^{\mathscr{U}} G_{1i}| \leq f(r)$. Moreover, $\prod_I^{\mathscr{U}} G_{1i} \cong \prod_I^{\mathscr{U}} (S_{1i} \times \cdots \times S_{ki}) \cong (\prod_I^{\mathscr{U}} S_{1i}) \times \cdots \times (\prod_I^{\mathscr{U}} S_{ki})$ and each factor is a simple linear group.

Since G embeds in L and $G \cap L_0$ is a subgroup of finite index in G which embeds in a quasi-linear group, G has a quasi-linear group of finite index. □

Corollary 6.6. *Let G be a pseudo-(finite of bounded Prüfer rank) group. Then $G/\phi_R(G)$ has a quasi-linear subgroup of finite index.*

Proof: This follows from the preceding proposition and Lemma 2.12. □

Proposition 6.7. *Let G be a pseudo-(finite of bounded Prüfer rank) group satisfying a non trivial identity and such that $G = \phi_R(G)$. Then G is nilpotent-by-abelian-by-finite.*

Proof. Let $G \preceq L = \prod_I^{\mathscr{U}} G_i$, where each G_i is finite of Prüfer rank $\leq r$. Since $G = \phi_R(G)$ and $G \preceq L$, without loss of generality we may assume that each G_i is soluble. Similarly since G satisfies a nontrivial identity, L satisfies a nontrivial identity, say $t = 1$. Hence, w.l.o.g. each G_i satisfies the same nontrivial identity $t = 1$.

Let $F = \langle x_1, \ldots, x_r | \rangle$ be the free group on $\{x_1, \ldots, x_r\}$ and for each $i \in U$ let $S_i = \{s_{1i}, \ldots, s_{ri}\}$ be a finite generating set of G_i. Let $\varphi_i : F \to G_i$ be the natural homomorphism which sends x_j to s_{ji}. Let $H = F/\bigcap_{i \in U} \ker(\varphi_i)$. Then H is a residually (finite soluble of bounded rank) group, satisfying a nontrivial identity $t = 1$. By a result of Segal (see Theorem page 2 in [46]), H has a nilpotent normal subgroup N such that H/N is quasi-linear. Since H/N doesn't contain F_2, by Tits alternative for linear groups, H/N is soluble-by-finite.

Hence H has a soluble normal subgroup K of finite index. Again by the same theorem of Segal ([46]), K is nilpotent-by-abelian-by finite, so is H. Hence there exists c and f such that each $F/\ker(\varphi_i) \cong G_i$ is (nilpotent of class at most c)-by-abelian-by-finite index f. So, $[L^f, L^f]$ is nilpotent of class at most c.

Since L does not contain F_2, by Proposition 3.6, L^f is of finite index in L, L^f is 0-definable and so we can express in a first-order way that $[L^f, L^f]$ is nilpotent of class at most c. These (first-order) properties transfer in G. □

Proof of Theorem 6.1. Let $G \preceq L = \prod_I^{\mathscr{U}} G_i$, where each G_i is finite of rank $\leq r$. Suppose that G contains no free nonabelian subgroup. Then G satisfies a nontrivial identity by Lemma 2.13, as well as L and $L/\phi_R(L)$. By the proof of Proposition 6.5 and Lemma 6.3, $L/\phi_R(L)$ is finite, say of cardinality $\leq f(r)$ and so is $G/\phi_R(G)$. By Proposition 3.6, $G^{f(r)}$ is 0-definable and of finite index in G. Applying Proposition 6.7 to $G^{f(r)}$, we get that $G^{f(r)}$ is nilpotent-by-abelian-by-finite. So the conclusion also applies to G. □

We place ourselves now in a slightly more general context than Theorem 6.1.

Definition 6.1. Let us say that a class $\mathcal{C}$ of finite groups is *weakly of r-bounded rank* if for each element $G \in \mathcal{C}$, the index of the sockel of $G/rad(G)$ is r-bounded and $rad(G)$ has r-bounded rank.

By the above result of A. Shalev ([49]), a class of finite groups of r-bounded Prüfer rank is weakly of r-bounded rank.

Definition 6.2. ([26]) A group G has *finite c-dimension* if there is a bound on the chains of centralizers. We will say that a class $\mathcal{C}$ of finite groups has *bounded c-dimension* if there is $d \in \mathbb{N}$ such that for each element $G \in \mathcal{C}$, the c-dimensions of $rad(G)$ and of the sockel of $G/rad(G)$ are d-bounded. (Note that a class of finite groups of bounded Prüfer rank is of bounded c-dimension.)

Lemma 6.8. *Let $\mathcal{C}$ be a class of finite groups satisfying a nontrivial identity. Suppose that for any $G \in \mathcal{C}$, either $Soc(G/rad(G))$ is of r-bounded rank, or of r-bounded index (in $G/rad(G)$) or of r-bounded c-dimension. Then $G/rad(G)$ is of bounded exponent depending only on r and on the identity.*

Proof: Recall that the sockel $Soc(G)$ of a group G is the union of its minimal normal nontrivial subgroups. In case G is a finite group, then $Soc(G)$ is a direct sum of simple groups and is completely reducible (see [45] 7.4.12).

Let $S := Soc(G/rad(G))$. Since a nontrivial identity can only be satisfied by finitely many finite simple groups (see [24]), by hypothesis on the class $\mathcal{C}$, we have a bound on the cardinality of the simple groups appearing in $Soc(G/rad(G))$, for $G \in \mathcal{C}$. So if the index of $Soc(G/rad(G))$ in $G/rad(G)$ is r-bounded, then the exponent of $G/rad(G)$ is bounded in terms on r and the identity only.

In the other cases, we note the following. The centralizer of S is trivial and so in order to show that some power of an element of $G/rad(G)$ is equal to 1, it suffices to show that the corresponding inner automorphism on S is the identity.

Let $\bar{g} \in G/rad(G)$ and let $\alpha_{\bar{g}}$ be the conjugation by $\bar{g}$ in $G/rad(G)$. It induces a permutation of the copies of a given finite simple group appearing in S. So if the subgroups of S generated by $\alpha_{\bar{g}^z}(\bar{h})$, $z \in \mathbb{Z}$, are r-generated, or if the c-dimension of S is r-bounded, we get the result. □

Corollary 6.9. *Let G be a pseudo-(finite weakly of r-bounded rank) group satisfying a nontrivial identity. Then $G/\phi_R(G)$ is uniformly locally finite.*

Proof. Let $G \preceq L = \prod_I^{\mathcal{U}} G_i$, where each G_i is finite. Let $\mathcal{C} := \{G_i \,:\, i \in I\}$, then it satisfies the hypothesis of the lemma above. So, $L/\phi_R(L)$ is r-bounded exponent. It transfers to $G/\phi_R(G)$. We conclude by applying Lemma 3.4. □

Theorem 6.10. *Let G be an $\aleph_0$-saturated pseudo-(finite weakly of bounded rank) group. Then either G contains a nonabelian free group or G is nilpotent-by-abelian-by-(uniformly locally finite).*

Proof: By applying Proposition 6.7 to $\phi_R(G)$, we get that $\phi_R(G)$ is nilpotent-by-abelian-by-finite. By the above corollary, $G/\phi_R(G)$ is uniformly locally finite. So, G is nilpotent-by-abelian-by-uniformly locally finite. □

Lemma 6.11. *Let $\mathcal{C}$ be a class of finite groups of bounded c-dimension and suppose G is an ultraproduct of elements of $\mathcal{C}$. Then $\phi_R(G)$ is soluble and $Soc(G/\phi_R(G))$ is a finite direct product of simple pseudofinite groups.*

Proof: Let $G = \prod_I^{\mathscr{U}} G_i$, where each G_i is finite. Then by hypothesis there is $d \in \mathbb{N}$ such that the c-dimension of each $\phi_R(G_i)$ is d-bounded (as well as the c-dimensions of $Soc(G_i/rad(G_i)))$ and so by a result of E. Khukhro (see Theorem 2 in [26]), the derived length of $\phi_R(G_i)$ is d-bounded. So, $\phi_R(G)$ is soluble.

Since the c-dimension of $Soc(G_i/rad(G_i))$ is bounded, we can bound the number of copies a each simple finite group occurring in $Soc(G_i/rad(G_i))$, the ranks of the groups of (twisted) Lie type (see [27] Proposition 3.1) and the size of the alternating groups occurring as simple factors. So, there exist finitely many groups of (twisted) Lie type $L_j(K_{ji})$, $1 \leq j \leq n$, where K_{ji} is a finite field and a finite product F of finitely many finite simple groups such that on an element of $\mathscr{U}$, $Soc(G_i/rad(G_i))$ is isomorphic to $\prod_{1 \leq j \leq n} L_j(K_{ji}) \times F$. Therefore, $Soc(G/\phi_R(G)) \cong \prod_{1 \leq j \leq n} L_j(\prod_I^{\mathscr{U}} K_{ji}) \times F$. □

Proposition 6.12. *Let $\mathcal{C}$ be a class of finite groups of bounded c-dimension and suppose G is a pseudo-$\mathcal{C}$ group satisfying a nontrivial identity. Then G is soluble-by-(uniformly locally finite).*

Proof. Let $G \preceq L = \prod_I^{\mathscr{U}} G_i$, where each G_i is finite. By the preceding lemma, $\phi_R(L)$ is soluble and so it is inherited by $\phi_R(G)$.

By Lemma 6.8, the $G_i/\phi_R(G_i)$ are of d-bounded exponent and so $L/\phi_R(L)$ is of finite exponent as well as $G/\phi_R(G)$. By Lemma 3.4, $G/\phi_R(G)$ is uniformly locally finite. □

Corollary 6.13. *Let G be an $\aleph_0$-saturated pseudo-(finite of bounded c-dimension) group. Then either G contains a nonabelian free group or G is soluble-by-(uniformly locally finite).* □

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