FINITELY PRESENTED ABELIAN LATTICE-ORDERED GROUPS.

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To Daniele Mundici on his 60^{th} birthday.

Abstract. We give necessary and sufficient conditions for the first-order theory of a finitely presented abelian lattice-ordered group to be decidable. We also show that if the number of generators is at most 3, then elementary equivalence implies isomorphism. We deduce from our methods that the theory of the free MV-algebra on at least 2 generators is undecidable.

1. Introduction.

Throughout, let n be a fixed positive integer and $FA\ell(n)$ be the free abelian lattice-ordered group on n generators. Let $\mathcal{L}$ be the language $\{+, \wedge, \vee, 0\}$ for this structure.

The additive group C of all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ is a lattice-ordered group under the pointwise ordering. The sublattice subgroup of C generated by the standard n projections $\pi_i : \mathbb{R}^n \to \mathbb{R}$ mapping $(x_1, \cdots, x_n)$ to x_i $(i = 1, \ldots, n)$ is (isomorphic to) the free abelian lattice-ordered group $FA\ell(n)$ on n generators ([1]).

For $f \in FA\ell(n)$, let $Z(f)$ be the zero set of f and $\langle f \rangle_{cl}$ be the principal ℓ-ideal of $FA\ell(n)$ generated by f. There is a one-to-one correspondence between the closed integral simplicial cones (see Definition 2.1) and the zero-sets of elements of $FA\ell(n)$. One can use this to interpret the lattice of zero-sets of the elements of $FA\ell(n)$ by first-order formulae. This played a central role in the proofs of the results obtained in [6]. We first showed that $FA\ell(2) \not\equiv FA\ell(n)$ for any $n > 2$. Then, using induction and a duality result due to Beynon [3], we proved that $FA\ell(m) \not\equiv FA\ell(n)$ if $m \neq n$. As a consequence of our proof and an undecidability result due to A. Grzegorczyk for some topological theories [8], we derived that the theory of $FA\ell(n)$ is undecidable if $n > 2$. In contrast, the first-order theory of the free abelian group on any finite number of generators is decidable.

W. M. Beynon generalised K. Baker's characterisation of projective vector lattices on a finite number of generators ([1]) and showed that finitely generated projective abelian lattice-ordered groups are precisely the finitely presented abelian lattice-ordered groups ([3] Theorem 3.1). These in turn are the quotients of $FA\ell(n)$ by principal ℓ-ideals. D. Mundici asked whether one could generalise the results in [6] and also classify the theories of finitely generated projective abelian lattice-ordered groups. This seems intractable. However, we will prove:

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Theorem 1.1. *Let $m, n \leq 3$, $f \in FA\ell(m)$ and $g \in FA\ell(n)$. Then $FA\ell(m)/\langle f\rangle_{cl}$ and $FA\ell(n)/\langle g\rangle_{cl}$ are elementarily equivalent iff they are isomorphic.*

Theorem 1.2. *The first-order theory of $FA\ell(n)/\langle f\rangle_{cl}$ is decidable iff the dimension of $Z(f)$ is at most 2.*

Note that we are not concerned here with decidability questions concerning isomorphisms. Undecidability results in group theory can be converted into algorithmic insolubility statements in topology. For instance, in dimension 5, it is known that there is no algorithm for deciding whether a compact piecewise linear manifold is piecewise linearly equivalent to a standard piecewise linear sphere ([10] page 22). A.M.W. Glass and J. J. Madden used similar facts to show that the isomorphism problem for projective abelian lattice-ordered groups on 10 generators is undecidable (see [7]).

2. Projective finitely generated abelian ℓ-groups

We recall some notation and basic results from [1], [2], [3], and [5] Chapter 5.

First, $FA\ell(n) := \{f = \bigwedge_i \bigvee_j f_{ij} : f_{ij} \in Hom(\mathbb{Z}^n, \mathbb{Z})\}$, and any $g \in Hom(\mathbb{Z}^n, \mathbb{Z})$ is equal to $\sum_{i=1}^n m_i.\pi_i$, where $m_i := g(e_i) \in \mathbb{Z}$ for all $i \in \{1, \ldots, n\}$.

Definition 2.1. A subspace $\sum_{i=1}^n m_i x_i = 0$ (with all $m_i \in \mathbb{Z}$) will be called an *integral hyperspace*, and the corresponding n-dimensional subsets $\sum_{i=1}^n m_i x_i > 0$, $\sum_{i=1}^n m_i x_i < 0$, $\sum_{i=1}^n m_i x_i \geq 0$ and $\sum_{i=1}^n m_i x_i \leq 0$ (with all $m_i \in \mathbb{Z}$) will be called *integral half spaces*. A *cone* in $\mathbb{R}^n$ is a subset which is invariant under multiplication by elements of $\mathbb{R}^+$. A *closed cone* is a cone which is closed in the standard topology of $\mathbb{R}^n$; the vertex is the origin. We will always confine ourselves to such cones defined by integral half spaces. A closed (or open) *integral simplicial cone* is a cone obtainable by finite unions and intersections from closed (or open) integral half spaces. It is convex if it is obtained using only intersections. Note that on each ray contained in such a cone and containing a point with rational coordinates, there is a unique non-zero point p with integral coordinates such that the open line segment $(0,\ p)$ contains no point with integral coordinates. Following [3] Section 2, we will call such a point *the initial integer lattice point* on this ray.

Definition 2.2. For $f \in FA\ell(n)$, let $Z(f)$ be the zero set of f; *i.e.*,

$$Z(f) = \{x \in \mathbb{R}^n : f(x) = 0\}.$$

Let $S(f)$ be the support of f; *i.e.*,

$$S(f) = \{x \in \mathbb{R}^n : f(x) \neq 0\}.$$

Let $\mathcal{K}$ be a subset of $\mathbb{R}^n$; then $S_{\mathcal{K}}(f)$ is the support of f on $\mathcal{K}$ (=$\{x \in \mathcal{K} : f(x) \neq 0\}$). In the special case that $\mathcal{K}$ is the $(n-1)$-sphere $S^{(n-1)}$, we write $\mathcal{S}(f)$ for $S_{\mathcal{K}}(f)$ and

$\mathcal{Z}(f)$ for $Z(f) \cap S^{(n-1)}$. Note that $f(rx) = rf(x)$ for all $x \in S^{(n-1)}$ and $r \in \mathbb{R}_+$. Hence $\mathcal{Z}(f)$ completely determines $Z(f)$ and $\mathcal{S}(f)$ completely determines $S(f)$.

As mentioned in the introduction, there is a one-to-one correspondence between the closed integral simplicial cones and the zero-sets of the elements of $FA\ell(n)$. Let $f \in FA\ell(n)$. We define the *dimension* of a zero set $Z(f)$ to be k if it contains the positive span of k $\mathbb{R}$-linearly independent vectors in $\mathbb{R}^n$ (but not $(k+1)$ such).

As is standard, we will write *ℓ-group* as a shorthand for lattice-ordered group.

Given an element $f \in FA\ell(n)$, let $|f| = f \vee -f$; then $|f| \in FA\ell(n)^+ := \{g \in FA\ell(n) : g(x) \geq 0 \text{ for all } x \in \mathbb{R}^n\}$. If $f \neq 0$, then $|f| \in FA\ell(n)_+ := FA\ell(n)^+ \setminus \{0\}$. Note that $f_1 = 0$ & ... & $f_m = 0$ iff $|f_1| \vee \cdots \vee |f_m| = 0$. Hence every finitely presented abelian ℓ-group can be written in the form $FA\ell(n)/\langle f\rangle_{cl}$ for some $n \in \mathbb{Z}_+$ and $f \in FA\ell(n)_+$, where $\langle f\rangle_{cl}$ is the ℓ-ideal of $FA\ell(n)$ generated by f; *i.e.*, $\langle f\rangle_{cl}$ is the subgroup of $FA\ell(n)$ generated by all elements g with $|g| \leq m.f$, for some $m \in \mathbb{N}$. Then $\mathcal{S}(f) = \mathcal{S}(g)$ iff $\langle f\rangle_{cl} = \langle g\rangle_{cl}$ (see [1]).

Since $FA\ell(m)/\langle f\rangle_{cl} \cong FA\ell(m+k)/\langle f'\rangle_{cl}$ where $f' = f \vee |\pi_{m+1}| \vee \cdots \vee |\pi_{m+k}|$, we may assume that $m = n$ in Theorem 1.1 and that $f, g \in FA\ell(n)^+$. We will consider the two cases $m = n = 2$ and $m = n = 3$ separately. These are proved in Sections 5 and 9, respectively.

Let $\mathcal{K}$ be a closed simplicial cone in $\mathbb{R}^n$. A map $h : \mathcal{K} \to \mathbb{R}^k$ is *piecewise homogeneous linear* if h is continuous and there is a finite subdivision $\{\mathcal{K}_s : s = 1, \ldots, m\}$ of $\mathcal{K}$ and a finite set of homogeneous linear functions $h_1, \cdots, h_m : \mathbb{R}^n \to \mathbb{R}^k$ such that $h(x) = h_s(x)$ for all $x \in \mathcal{K}_s$ $(s = 1, \ldots, m)$. If there is a retract r from $\mathbb{R}^n$ to $\mathcal{K}$ such that the composite $h \circ r$ is a piecewise homogeneous linear map from $\mathbb{R}^n$ to $\mathbb{R}^k$, then h can be expressed as $h = \bigwedge_i \bigvee_j g_{ij}$ where each g_{ij} is some h_s, $s = 1, \ldots, m$ (see Theorem 3.1 in [2]). We call such a piecewise homogeneous linear map an *ℓ-map*. If we restrict ourselves to rational closed simplicial cones $\mathcal{K} \subseteq \mathbb{R}^n$, any ℓ-map from $\mathcal{K}$ to $\mathbb{R}^k$ with integer coefficients is called an *integral ℓ-map*; it has the form $(u_1(x), \cdots, u_k(x))$ with $u_1, \cdots, u_k \in FA\ell(n)$ (see Corollary 1 to Theorem 3.1 in [3]).

Let $f \in FA\ell(m)^+$ and $g \in FA\ell(n)^+$. We say that $Z(f)$ and $Z(g)$ are *ℓ-equivalent* if there is an ℓ-map $\theta : Z(f) \to Z(g)$ with inverse $\tau : Z(g) \to Z(f)$ which is also an ℓ-map. If the ℓ-maps are integral, we also say that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ are *integrally ℓ-equivalent* and write

$$\mathcal{Z}(f) \sim_\ell \mathcal{Z}(g).$$

Let $\theta : \mathbb{R}^n \to \mathbb{R}^m$ be an integral ℓ-map mapping $Z(g)$ onto $Z(f)$; say, $\theta(x) = (u_1(x), \cdots, u_m(x))$ with $u_1, \cdots, u_m \in FA\ell(n)$. Let $T(\theta)$ be the induced map from $FA\ell(m)$ to $FA\ell(n)$; *i.e.*, $T(\theta) : h \mapsto h \circ \theta$. We identify $h \circ \theta$ with $h|_{Z(f)}$. The kernel of $T(\theta)$ is the ℓ-ideal $\langle f\rangle_{cl}$. For convenience we will denote the image $T(\theta)(h)$ by h^θ and write $T^*(\theta)$ for the induced ℓ-isomorphism between $FA\ell(m)/\langle f\rangle_{cl}$ and $FA\ell(n)/\langle g\rangle_{cl}$ (see Corollary 5.2.2 in [5]).

The Baker-Beynon Duality (see Corollary 2 of Theorem 3.1 in [2]) is

Theorem 2.1. *Let $f \in FA\ell(m)_+$ and $g \in FA\ell(n)_+$. Then $FA\ell(m)/\langle f\rangle_{cl} \cong FA\ell(n)/\langle g\rangle_{cl}$ (as $\mathcal{L}$-structures) iff $\mathcal{Z}(f) \sim_\ell \mathcal{Z}(g)$.*

As a consequence, one obtains ([3], Theorem 3.1)

Theorem 2.2. *The class of finitely generated projective abelian ℓ-groups is precisely the class of finitely presented abelian ℓ-groups; each has the form $FA\ell(n)/\langle f\rangle_{cl}$ for some $n \in \mathbb{Z}_+$ and $f \in FA\ell(n)_+$.*

Theorems 2.1 and 2.2 give a correspondence between equivalence classes of zero-sets modulo the relation $\sim_\ell$ and equivalence classes of finitely presented abelian ℓ-groups under isomorphism.

Remarks:

Let $f, g \in FA\ell(n)^+$.

(1) Suppose that $\mathcal{Z}(f) \sim_\ell \mathcal{Z}(g)$. Then the integral ℓ-map $\theta : \mathbb{R}^n \to \mathbb{R}^n$ realising this equivalence and mapping $Z(f)$ to $Z(g)$ need not be a homeomorphism of $\mathbb{R}^n$. (See for instance [10], Annex C.1).

(2) If $Z(f)$ and $Z(g)$ are *simplicially equivalent*, then $\mathcal{Z}(f) \sim_\ell \mathcal{Z}(g)$.

We elaborate on (2). Recall that a (rational) simplicial cone $Z(f)$ can be presented as a union of (rational) convex simplicial cones belonging to a complex $\mathcal{K}$ (Lemma 0.1 in [3]). Subdivide this presentation into a primitive rational simplicial one and denote this simplicial presentation of $\mathcal{K}$ by $\mathcal{S}$ — if two rational closed simplicial cones have isomorphic subdivisions into closed simplicial convex cones, then they have isomorphic subdivisions into primitive rational convex simplicial cones (Corollary 3 in Section 2 of [3]).

We will occasionally pass without mention from a simplicial complex in $\mathbb{R}^n$ to its domain and view it as a subset of $\mathbb{R}^n$.

Let $0x_1, \cdots, 0x_r$ be the 1-dimensional simplicial cones in $\mathcal{S}$ emanating from the origin. Choose initial integer lattice points $p_1, \cdots, p_r$ on these rays. Let $P(f)$ be the (rational) simplicial cone associated with $\mathcal{S}$ and let ${}_SP(f)$ be the image of this simplicial cone on $S^{(n-1)}$ obtained by taking the intersection with $S^{(n-1)}$ of all rays from 0 to the simplicial cone $P(f)$; we will call such a simplicial complex an *$\mathcal{S}$-simplicial complex*. Let $\{u_i : i = 1, \ldots, r\}$ be the Schauder hats associated with $\mathcal{S}$. That is, they are the continuous functions that are linear on each cone of $\mathcal{S}$ with $u_i(p_j) = \delta_{ij}$ $(i, j = 1, \ldots, r)$. Let I be the ℓ-ideal generated by $\pi_1 \vee 0, \cdots, \pi_r \vee 0$, $\bigvee_{j=1}^{t}(\bigwedge_{k\in X_j} \pi_k)$, where $X_1, \cdots, X_t$ are the subsets of $\{1, \cdots, r\}$ for which the corresponding subsets of $\{p_1, \cdots, p_r\}$ do not span a simplex of $\mathcal{S}$. Then $FA\ell(n)/\langle f\rangle_{cl}$ is isomorphic to $FA\ell(r)/I$ by the map $T(\theta)$ where $\theta : \mathbb{R}^r \to \mathbb{R}^n$ is the integral ℓ-map sending $Z(f)$ to $Z(I)$ (see Corollary 2, Section 2 in [3]).

Let $f,\ g \in FA\ell(n)$. Then $Z(f)$ and $Z(g)$ are rational closed simplicial cones in $\mathbb{R}^n$. If $P(f)$ and $P(g)$ are simplicially equivalent (by integral ℓ-maps), then $\mathcal{Z}(f) \sim_\ell \mathcal{Z}(g)$ by Corollary 3, Section 3 in [3] and Corollary 2 to Theorem 4.1 in [2]. Hence $FA\ell(n)/\langle f\rangle_{cl} \cong FA\ell(n)/\langle g\rangle_{cl}$.

For further background and more details, see the survey article [9].

We next consider first-order theories. In our proof of the undecidability result of $FA\ell(n)$ for $n \geq 3$ (see Theorem 4.8 in [6]), we showed how to express in $\mathcal{L}$ that two elements $f,\ g \in FA\ell(n)$ had the same zero-sets (or equivalently the same supports). (The formula depended on n.) We first showed how to express in $\mathcal{L}$ the notion of "dimension" of a zero-set by induction. Let $\psi_{n,k}(x)$ be such formulae ($k = -1, \ldots, n-1$). That is, for each $k \in \{-1, \ldots, n-1\}$,

$$FA\ell(n) \models \psi_{n,k}(f) \quad \text{iff} \quad dim(\mathcal{Z}(f)) = k.$$

This allowed us to express that a zero-set is empty, and then to interpret the lattice of zero-sets of the elements of $FA\ell(n)$. This last result implies that for any $f \in FA\ell(n)$, the structure $FA\ell(n)/\langle f\rangle_{cl}$ is first-order interpretable in $FA\ell(n)$.

We will frequently implicitly use

Lemma 2.3. *Let $g \in FA\ell(n)$. Then $FA\ell(n) \models \psi_{n,k}(|f| \vee |g|)$ iff the zero-set of the restriction of g to $\mathcal{Z}(f)$ has dimension k.*

3. Components.

In this section we reduce determining elementary equivalence to $FA\ell(n)/\langle f\rangle_{cl}$ to the special case that $\mathcal{Z}(f)$ has a single connected component. We will show

Proposition 3.1. *Let $f,\ g \in FA\ell(n)^+$. If $FA\ell(n)/\langle f\rangle_{cl} \equiv FA\ell(n)/\langle g\rangle_{cl}$, then $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ have the same number of connected components.*

We first consider connectedness for supports and zero-sets of elements of a finitely presented abelian ℓ-group. Since we can only use definable open (respectively closed) subsets and their restrictions to the zero-set of a distinguished element, we will use the term definably connected.

Example A: Let $g, h \in FA\ell(3)_+$ with $\mathcal{S}(g)$ the northern hemisphere, and $\mathcal{S}(h) = \mathcal{S}(g)\backslash(\{X\}\cup A)$, where X is the north pole and A is an arc in the northern hemisphere. Now $k \perp g$ iff $k \perp h$ (in either case, $\mathcal{S}(k)$ is contained in the southern hemisphere). Then $\mathcal{S}(g)$ and $\mathcal{S}(h)$ are connected and differ by a set with empty interior; and $\mathcal{Z}(g)$ is connected, but $\mathcal{Z}(h)$ is not.

Recall the formulae we used in [6] for $FA\ell(n)$.

We expressed that the support $\mathcal{S}(h)$, of an element $h > 0$, is *definably connected* by the formula $\theta_S(h)$ given by

$$(h > 0)\ \&\ \neg((\exists h_1, h_2 > 0)(h_1 \perp h_2\ \&\ h_1 \vee h_2 = h)),$$

where $h \perp g$ is a shorthand for $h \wedge g = 0$ (equivalently that $\mathcal{S}(h) \subseteq \mathcal{Z}(g)$).

[We have used the subscript S on θ to make clear that we are dealing with support.]

That is, $FA\ell(n) \models \theta_S(h)$ iff $\mathcal{S}(h)$ is connected.

More generally, if $g, h \in FA\ell(n)^+$, write $g \sim h$ as a shorthand for

$$(\forall k > 0)(k \perp g \leftrightarrow k \perp h).$$

So

Lemma 3.2. *$FA\ell(n) \models g \sim h$ iff the interior of the symmetric difference of $\mathcal{Z}(g)$ and $\mathcal{Z}(h)$ is empty.*

Caution: In Example A, if $\mathcal{Z}(f) = \{X\} \cup B$ where B is any closed disc in the northern hemisphere disjoint from A with $X \notin B$, then $g \sim h$ but $FA\ell(3)/\langle f\rangle_{cl} \models \theta_S(h + \langle f\rangle_{cl})$ & $\neg\theta_S(g + \langle f\rangle_{cl})$.

Let $h \in FA\ell(n)_+$ and $\mathcal{Z}(h)$ be such that for each $x \in \mathcal{Z}(h)$, there is a neighbourhood $N(x)$ of x such that $N(x) \cap \mathcal{Z}(h)$ contains the support of a non-zero element. We expressed that $\mathcal{Z}(h)$ is connected by the formula $\theta_Z(h)$ given by

$$(\exists k > 0)(k \perp h) \ \ \& \ \ (\forall k > 0)\ [(k \perp h) \rightarrow (\exists g \geq k)(\theta_S(g) \ \& \ g \perp h)].$$

[We have used the subscript Z on θ to make clear that we are dealing with zero sets. The first conjunct has been included for when we relativise to $\mathcal{Z}(f)$ later.]

Lemma 3.3. *Let $h \in FA\ell(n)_+$ satisfy the above hypothesis. Then $FA\ell(n) \models \theta_Z(h)$ iff $\mathcal{Z}(h)$ is connected.*

More generally, if $g \in FA\ell(n)_+$, we say that $\mathcal{Z}(g)$ is *definably connected* if $g \sim h$ for some $h \in FA\ell(n)_+$ with h as above $\mathcal{Z}(h)$ connected.

Caution. Let $f \in FA\ell(n)_+$ and consider the above formula in $FA\ell(n)/\langle f\rangle_{cl}$. It is possible to have that both $FA\ell(n)/\langle f\rangle_{cl} \models \theta_Z(0+\langle f\rangle_{cl})$ and $FA\ell(n) \models \neg\theta_Z(f)$. Such an example is provided by letting $\mathcal{Z}(f)$ be two closed discs on $S^{(2)}$ whose intersection is a single point.)

Let $\mathcal{S}$ be a simplicial complex in $S^{(n-1)}$. If the rays from the origin to the vertices of $\mathcal{S}$ all contain initial integral lattice points, then we say that the simplicial complex is *rationally determined*. If P_1, P_2 are non-empty disjoint open simplicial complexes whose union contains $\mathcal{Z}(f)$ ($f \in FA\ell(n)_+$), then by the density of $\mathbb{Q}^n$ in $\mathbb{R}^n$, we may choose open rationally determined simplicial complexes $P_j' \subseteq P_j$ ($j = 1, 2$) so that $\mathcal{Z}(f) \subseteq P_1' \cup P_2'$. We will always do this.

Lemma 3.4. *$FA\ell(n)/\langle f\rangle_{cl} \models \theta_Z(0 + \langle f\rangle_{cl})$ iff whenever there are two disjoint open simplicial complexes P_1, P_2 in $S^{(n-1)}$ with $P_1 \cup P_2 \supseteq \mathcal{Z}(f)$, the intersection of one of them with $\mathcal{Z}(f)$ is trivial.*

Proof: Suppose that P_1, P_2 are non-empty disjoint open simplicial complexes in $S^{(n-1)}$ with $P_1 \cup P_2 \supseteq \mathcal{Z}(f)$. As just remarked, we may assume that P_j is rationally determined $(j = 1, 2)$. Hence there are $h_j \in FA\ell(n)_+$ with $\mathcal{S}(h_j) = P_j$ $(j = 1, 2)$. Let $g \geq h_1 \vee h_2$. If $P_j \cap \mathcal{Z}(f) \neq \emptyset$ $(j = 1, 2)$, then g witnesses that $FA\ell(n)/\langle f\rangle_{cl} \models \neg\theta_Z(0 + \langle f\rangle_{cl})$.

Conversely, suppose that $FA\ell(n)/\langle f\rangle_{cl} \models \neg\theta_Z(0 + \langle f\rangle_{cl})$. Let $k \in FA\ell(n)_+$ be such that $k \perp h$ and for all $g \geq k$ we have $FA\ell(n)/\langle f\rangle_{cl} \models \neg\theta_S(g + \langle f\rangle_{cl})$. By replacing k by an element g of possibly greater support if necessary, we may assume that $\mathcal{S}(k) \supseteq \mathcal{Z}(f)$. Write $k = k_1 \vee \cdots \vee k_m$ with $k_1, \ldots, k_m$ pairwise disjoint each having connected support and $\mathcal{S}(k_i) \cap \mathcal{Z}(f) \neq \emptyset$ $(i = 1, \ldots, m)$. Thus $m \geq 2$. Then $P_1 = \mathcal{S}(k_1)$ and $P_2 = \bigcup_{i=2}^{r} \mathcal{S}(k_i)$ are the desired simplicial complexes. □

By the same technique one can prove

Lemma 3.5. *Let $f, h \in FA\ell(n)_+$. Then $FA\ell(n)/\langle f\rangle_{cl} \models \theta_S(h+\langle f\rangle_{cl})$ iff $\mathcal{S}(h)\cap\mathcal{Z}(f)$ is connected.*

We next wish to write $FA\ell(n)/\langle f\rangle_{cl}$ as a direct sum which cannot be further decomposed into non-trivial direct summands. We do this by decomposing $\mathcal{Z}(f)$ into maximal simplices. These are the connected components of $\mathcal{Z}(f)$.

As above, we observe that if $\mathcal{Z}(h_j) \subseteq \mathcal{Z}(f)$ $(j = 1, 2)$ and there does not exist any $k \in FA\ell(n)_+$ with $\mathcal{S}(k) \cap \mathcal{Z}(f)$ contained in the symmetric difference of $\mathcal{Z}(h_1)$ and $\mathcal{Z}(h_2)$, then one cannot hope to distinguish between $\mathcal{Z}(h_1)$ and $\mathcal{Z}(h_2)$ in $FA\ell(n)/\langle f\rangle_{cl}$. In this case we write $\mathcal{Z}(h_1) \sim_f \mathcal{Z}(h_2)$; *i.e.*,

$$FA\ell(n)/\langle f\rangle_{cl} \models (\forall k \geq 0)(k \perp h_1 + \langle f\rangle_{cl} \leftrightarrow k \perp h_2 + \langle f\rangle_{cl}).$$

We can now express in $\mathcal{L}$ that $\mathcal{Z}(g)$ is a *definably connected component* of $\mathcal{Z}(f)$; that is, $\mathcal{Z}(g)$ is a maximal connected subset of $\mathcal{Z}(f)$ to within $\sim_f$.

Lemma 3.6. *Let $f, g \in FA\ell(n)_+$. The formula*

$$(\exists h > 0)[\theta_S(h) \ \ \& \ \ (\forall k > 0)(k \perp g \to k \not\perp h) \ \ \& \ \ (\forall h' > h)(h' \not\perp g \to \neg\theta_S(h'))]$$

holds in $FA\ell(n)/\langle f\rangle_{cl}$ at $g+\langle f\rangle_{cl}$ iff $\mathcal{Z}(g)$ is $\sim_f$-equivalent to a connected component of $\mathcal{Z}(f)$.

Proof: Let $f, f_1, \cdots, f_m \in FA\ell(n)_+$, with $\mathcal{Z}(f_1), \cdots, \mathcal{Z}(f_m)$ the pairwise disjoint non-empty connected components of $\mathcal{Z}(f)$; so $\mathcal{Z}(f) = \bigcup_{j=1}^{m} \mathcal{Z}(f_j)$.

If $\mathcal{Z}(g) \sim_f \mathcal{Z}(f_1)$, there is an open rationally determined simplicial complex in $S^{(n-1)}$ with $P \supseteq \mathcal{Z}(f_1)$ and $P \cap \mathcal{Z}(f_j) = \emptyset$ for $j = 2, \ldots, m$. So there is $h \in FA\ell(n)_+$ with $\mathcal{S}(h) = P$. Then the first two conjuncts of the formula clearly hold in $FA\ell(n)/\langle f\rangle_{cl}$ by considering $h + \langle f\rangle_{cl}$. If $h' > h$ with $h' \not\perp g$ on $\mathcal{Z}(f)$, then $\mathcal{S}(h') \cap \mathcal{S}(g) \cap \mathcal{Z}(f) \neq \emptyset$; so $\mathcal{S}(h') \cap \mathcal{Z}(f_j) \neq \emptyset$ for some $j \in \{2, \ldots, m\}$. By Lemma 3.5, the formula holds.

Conversely, assume that the formula holds in $FA\ell(n)/\langle f\rangle_{cl}$. We may assume that $\mathcal{Z}(g)$ is minimal to within $\sim_f$. The satisfaction of the first conjunct implies that $\mathcal{Z}(g)\cap\mathcal{Z}(f)$ is definably connected in $\mathcal{Z}(f)$. We may assume that $\mathcal{Z}(g)\cap\mathcal{Z}(f)\subseteq\mathcal{Z}(f_1)$, say. If $\mathcal{Z}(g)\cap\mathcal{Z}(f)\neq\mathcal{Z}(f_1)$, then $\mathcal{Z}(f_1)\setminus\mathcal{Z}(g)$ is a non-empty open subset of $\mathcal{Z}(f_1)$. Hence there is $p\in\mathcal{Z}(f_1)\setminus\mathcal{Z}(g)$ and a simplex neighbourhood P of p in $S^{(n-1)}$ such that $P\cap\mathcal{Z}(g)=\emptyset$. Let $h'\in FA\ell(n)_+$ with $\mathcal{S}(h')\subseteq P$. Then $h'\vee h$ witnesses that the formula fails to hold in $FA\ell(n)/\langle f\rangle_{cl}$. □

The following is well known.

Lemma 3.7. *Let* $f_j\in FA\ell(n)_+$ $(j=1,2)$ *with* $\mathcal{Z}(f_1)\cap\mathcal{Z}(f_2)=\emptyset$. *Then*

$$FA\ell(n)/\langle(f_1\vee f_2)\rangle_{cl}\cong FA\ell(n)/\langle f_1\rangle_{cl}\times FA\ell(n)/\langle f_2\rangle_{cl}.$$

We now use Lemma 3.6 to provide a sentence of $\mathcal{L}$ that counts the number of connected components of $FA\ell(n)/\langle f\rangle_{cl}$. Let $f,f_1,\cdots,f_m\in FA\ell(n)_+$ with $\mathcal{Z}(f_1),\cdots,\mathcal{Z}(f_m)$ the disjoint definably connected components of $\mathcal{Z}(f)$; so $\mathcal{Z}(f)=\bigcup_{i=1}^m\mathcal{Z}(f_i)$. That is,

$$FA\ell(n)/\langle f\rangle_{cl}\;\cong\;FA\ell(n)/\langle f_1\rangle_{cl}\times\cdots\times FA\ell(n)/\langle f_m\rangle_{cl}.$$

Let

$$\rho_m:=(\exists h_1,\ldots,h_m>0)\;[\bigwedge_{1\leq i<j\leq m}h_i\perp h_j\;\;\&\;\;\bigwedge_{1\leq i\leq m}\theta_S(h_i)\;\;\&$$
$$(\forall h_i'\geq h_i)[(\exists k>0)(k\leq h_i'\;\;\&\;\;k\perp h_i)\rightarrow\neg\theta_S(h_i')]].$$

By the previous lemmata

Lemma 3.8.

$$FA\ell(n)/\langle f\rangle_{cl}\models\rho_m\quad\textit{iff}\quad\mathcal{Z}(f)\textit{ has at least } m \textit{ connected components.}$$

Thus we have

Lemma 3.9. *Let* $f\in FA\ell(n)^+$. *Then* $\mathcal{Z}(f)$ *has exactly* m *connected components iff* $FA\ell(n)/\langle f\rangle_{cl}\models\rho_m\;\&\;\neg\rho_{m+1}$.

Proposition 3.1 follows. □

We now generalise Lemma 3.4.

Lemma 3.10. *Let* $f\in FA\ell(n)_+$ *and* $\mathcal{Z}(f)$ *have connected components* $\mathcal{Z}(f_1),\ldots,\mathcal{Z}(f_m)$. *Let* $h\in FA\ell(n)_+$ *be such that for all* $j\in\{1,\ldots,m\}$, *either* $\mathcal{Z}(h)\cap\mathcal{Z}(f_j)$ *is empty or all the connected components of* $\mathcal{Z}(h)\cap\mathcal{Z}(f_j)$ *have the same dimension as* $\mathcal{Z}(f_j)$. *Then* $FA\ell(n)/\langle f\rangle_{cl}\models\theta_Z(h+\langle f\rangle_{cl})$ *iff* $\mathcal{Z}(h)\cap\mathcal{Z}(f)$ *is connected.*

Proof: By the assumption on h, $FA\ell(n)/\langle f\rangle_{cl} \models \neg\theta_Z(h+\langle f\rangle_{cl})$ if the intersection of $\mathcal{Z}(h)$ with more than one connected component of $\mathcal{Z}(f)$ is non-empty. So we may assume that $\mathcal{Z}(h)\cap\mathcal{Z}(f_j)=\emptyset$ for $j=2,\ldots,m$. If $\mathcal{Z}(h)\cap\mathcal{Z}(f_1)$ is not connected, then there are disjoint open rationally determined simplicial complexes P_1 and P_2 in $S^{(n-1)}$ with $P_i\cap\mathcal{Z}(h)\cap\mathcal{Z}(f_1)\neq\emptyset$ $(i=1,2)$. Hence there are $h_i\in FA\ell(n)_+$ with $\mathcal{S}(h_i)=P_i$ $(i=1,2)$. Let $g\geq h_1\vee h_2$ with $g\perp f$. This witnesses that $FA\ell(n)/\langle f\rangle_{cl}\models\neg\theta_Z(h+\langle f\rangle_{cl})$.

The other direction is trivial. $\square$

Putting $h=f$ in Lemma 3.10, we see

Corollary 3.11. *Let $f\in FA\ell(n)_+$. Then $FA\ell(n)/\langle f\rangle_{cl}\models\theta_Z(0+\langle f\rangle_{cl})$ iff $\mathcal{Z}(f')$ is connected for some $f'\sim f$.*

Note that the hypothesis on h in Lemma 3.10 can be expressed by an $\mathcal{L}$-formula to within $\sim_f$: Let $\mathcal{Z}(h)$ be a definably connected subset of $\mathcal{Z}(f)$. Then for any open set of the form $\mathcal{S}(k)$, there exists k' with $\mathcal{S}(k')\supseteq\mathcal{S}(k)$ such that $\mathcal{S}(k')\cap\mathcal{Z}(h)$ is relatively connected in $\mathcal{Z}(h)$. The $\mathcal{L}$-expressibility now follows from Lemma 3.6.

4. Further formulae

We next express that the restriction of the support of an element $h>0$ to the zero-set of an element f consists of one ray; *i.e.,* $\mathcal{S}(h)\cap\mathcal{Z}(f)$ consists of a single point. We will use the formula $\theta_1(h)$ given by

$$\theta_S(h)\ \&\ (\forall g_1,g_2>0)(g_1\vee g_2\leq h\ \rightarrow g_1\not\perp g_2).$$

[The subscript 1 on θ is to make clear that we are dealing with a single point.]

Lemma 4.1. $FA\ell(n)/\langle f\rangle_{cl}\models\theta_1(h+\langle f\rangle_{cl})$ *iff $\mathcal{Z}(f)\cap\mathcal{S}(h)$ consists of a single isolated point.*

Proof: If $\mathcal{Z}(f)\cap\mathcal{S}(h)=\{p\}$, then as $0<g_j\leq h$, we must have $g_j(p)>0$ $(j=1,2)$. So $FA\ell(n)/\langle f\rangle_{cl}\models\theta_1(h+\langle f\rangle_{cl})$.

Conversely, if $FA\ell(n)/\langle f\rangle_{cl}\models\theta_S(h+\langle f\rangle_{cl})$, then $\mathcal{Z}(f)\cap\mathcal{S}(h)$ is connected. If it is not a single point, let $p_1,p_2\in\mathcal{Z}(f)\cap\mathcal{S}(h)$ be distinct. Let $P_j\subseteq\mathcal{S}(h)$ be a simplicial complex in $S^{(n-1)}$ containing p_j $(j=1,2)$ with $P_1\cap P_2=\emptyset$. Let $h_j\in FA\ell(n)_+$ with $\mathcal{S}(h_j)\subseteq P_j$ $(j=1,2)$. Then $g_j=h_j\wedge h$ $(j=1,2)$ witness that $FA\ell(n)/\langle f\rangle_{cl}\models\neg\theta_1(h+\langle f\rangle_{cl})$. $\square$

We will use the formula $\theta_2(h)$ to express that the support of an element h when restricted to $\mathcal{Z}(f)$ is connected and strictly contains the support of two non-zero elements with disjoint supports. Again the subscript has been chosen according to the intended meaning. Let $\theta_2(h)$ be the formula:

$$\theta_S(h) \ \& \ \ (\exists g_1, g_2 > 0)(g_1 \vee g_2 \leq h \ \& \ \ g_1 \perp g_2).$$

The following lemma follows easily by the same proof as used above.

Lemma 4.2. *$FA\ell(n)/\langle f\rangle_{cl} \models \theta_2(h + \langle f\rangle_{cl})$ iff $\mathcal{Z}(f) \cap \mathcal{S}(h)$ is connected and strictly contains the support of two non-zero elements with disjoint supports.*

If $\mathcal{Z}(f) \cap \mathcal{S}(h)$ does not consist of an isolated point, we can express in $\mathcal{L}$ that $\mathcal{S}(h)$ covers a maximal connected subset of $\mathcal{Z}(f)$. Define $\theta_2^*(h)$ to be

$$\theta_2(h) \ \& \ (\forall g \geq h)(\theta_S(g) \rightarrow (\forall u > 0)[u \perp g \leftrightarrow u \perp h]).$$

By essentially the same proofs

Lemma 4.3. *Let $f \in FA\ell(n)_+$. Then $FA\ell(n)/\langle f\rangle_{cl} \models \theta_2^*(h + \langle f\rangle_{cl})$ iff $\mathcal{S}(h)$ covers a unique connected component of $\mathcal{Z}(f)$.*

Let $f,\ h \in FA\ell(n)_+$ and assume that $\mathcal{S}(h) \subseteq \mathcal{Z}(f)$. Then $\mathcal{S}(h)$ differs from $\mathcal{Z}(f)$ by a set which does not have relative maximal dimension in $S^{(n-1)}$ if

$$FA\ell(n) \models h \perp f \ \& \ \forall g > 0 \ (g \perp f \rightarrow g \not\perp h).$$

We will denote *any* such element h by $f^{\#}$. That is,

$$FA\ell(n) \models f^{\#} \perp f \ \ \& \ \ (\forall g > 0)(g \perp f \rightarrow g \not\perp f^{\#}).$$

It is not unique.

Let $k,\ h \in FA\ell(n)_+$. If

$$FA\ell(n)/\langle f\rangle_{cl} \models h \perp k \ \& \ \ (\forall g > 0)(g \perp k \rightarrow g \not\perp h),$$

then we will write $k^{\#}$ for any such element h. This is equivalent to $\mathcal{S}(k^{\#}) \cap \mathcal{Z}(f) \subseteq \mathcal{Z}(k) \cap \mathcal{Z}(f)$ and $\mathcal{S}(k^{\#}) \cap \mathcal{Z}(f)$ differs from $\mathcal{Z}(k) \cap \mathcal{Z}(f)$ by a set which does not have (locally) relative maximal dimension. It is not unique.

5. Quotients of $FA\ell(2)$

Our purpose in this section is to prove

Theorem 5.1. *Let $f,\ g \in FA\ell(2)^+$. Then*

$$FA\ell(2)/\langle f\rangle_{cl} \ \equiv \ FA\ell(2)/\langle g\rangle_{cl} \ \textit{ iff } \ FA\ell(2)/\langle f\rangle_{cl} \ \cong \ FA\ell(2)/\langle g\rangle_{cl}.$$

To achieve this, we need three lemmata.

Lemma 5.2. *Let $f \in FA\ell(2)_+$ and $\mathcal{Z}(f)$ consist of one or more arcs. Then*

$$FA\ell(2)/\langle f\rangle_{cl} \ \not\equiv\ FA\ell(2).$$

Proof: If $\mathcal{Z}(f_1)$ is a proper arc of $S^{(1)}$ which is maximal in $\mathcal{Z}(f)$, let $h_1, h_2, h_3 \in FA\ell(2)_+$ all have support contained in $Z(f_1)$ with $\mathcal{S}(h_j)$ a single arc $(j = 1, 2, 3)$, the arc for h_3 being between that of h_1 and h_2 in $\mathcal{Z}(f_1)$. Under the natural interpretation, the following formula is satisfied in $FA\ell(2)/\langle f\rangle_{cl}$ but not in $FA\ell(2)$:

$$(\exists h_1, h_2, h_3 > 0)[\bigwedge_{i=1}^{3} \theta_S(h_i) \ \&\ h_3 \perp (h_1 \vee h_2) \ \&\ h_1 \perp h_2 \ \&$$
$$(\forall h)[(h > (h_1 \vee h_2) \ \&\ \theta_S(h)) \rightarrow (h \not\perp h_3)]].$$

□

Lemma 5.3. *Let $f, g \in FA\ell(2)_+$ with $\mathcal{Z}(f)$ comprising n_1 disjoint arcs and n_2 isolated points, and $\mathcal{Z}(g)$ comprising n_1' disjoint arcs and n_2' isolated points, with $(n_1, n_2) \neq (n_1', n_2')$. Then*

$$FA\ell(2)/\langle f\rangle_{cl} \ \not\equiv\ FA\ell(2)/\langle g\rangle_{cl}.$$

Proof: Assume first that $n_1 \neq n_1'$; say $n_1 > n_1'$. Let ϕ_{n_1} be the sentence

$$(\bigwedge_{i\in n_1} \exists h_i > 0)(\theta_2^*(h_i) \ \&\ (\bigwedge_{i,\, i'\in n_1,\, i\neq i'} h_i \perp h_{i'}) \ \&\ [\forall h > 0]\ [\theta_2(h) \ \rightarrow\ h \wedge (\bigvee_i h_i) \neq 0]).$$

Then $FA\ell(2)/\langle f\rangle_{cl}$ satisfies ϕ_{n_1} but $FA\ell(2)/\langle g\rangle_{cl}$ does not.

Now assume that $n_1 = n_1'$ and $n_2 > n_2'$. Let ϕ_{n_1,n_2} be the sentence

$$[\bigwedge_{i\in n_1} (\exists h_i > 0)(\theta_2^*(h_i) \ \&\ \bigwedge_{i,\, i'\in n_1,\, i\neq i'} h_i \perp h_{i'}) \ \&\ (\bigwedge_{j\in n_2} \exists u_j > 0)(\theta_1(u_j) \ \&$$

$$(\bigwedge_{i=1}^{n} u_j \perp h_i) \ \&\ \bigwedge_{j,\, j'\in n_2,\, j'\neq j} (u_j \perp u_{j'}) \ \&\ (\forall h > 0)\ (h \wedge (\bigvee_{i\in n_1} h_i \vee \bigvee_{j\in n_2} u_j) \neq 0)].$$

Clearly ϕ_{n_1,n_2} holds in $FA\ell(2)/\langle f\rangle_{cl}$ but not in $FA\ell(2)/\langle g\rangle_{cl}$. □

Lemma 5.4. (Beynon, [3], p.262) *Let $f \in FA\ell(2)_+$. Then,*

$$FA\ell(2)/\langle f\rangle_{cl} \ \cong\ (FA\ell(2)/\langle \pi_2 \vee 0\rangle_{cl})^{n_1} \times \mathbb{Z}^{n_2},$$

where n_1 is the number of pairwise disjoint maximal arcs in $\mathcal{Z}(f)$ and n_2 is the number of isolated points in $\mathcal{Z}(f)$.

We can now prove Theorem 5.1.

Proof: Suppose that $FA\ell(2)/\langle f\rangle_{cl} \ \equiv\ FA\ell(2)/\langle g\rangle_{cl}$. If f and g are both non-zero, then by Lemma 5.3, the number of connected pieces of the zero sets of f and g of the same dimensions are the same. By Lemma 5.4, the ℓ-groups are isomorphic.

If $g = 0$ and $f \neq 0$, then by Lemma 5.2, $FA\ell(2)/\langle f\rangle_{cl} \not\equiv FA\ell(2)$ whenever $\mathcal{Z}(f)$ contains an arc. If $\mathcal{Z}(f)$ comprises n isolated points, then the sentence that there are $n+1$ pairwise perpendicular strictly positive elements holds in $FA\ell(2)$ but not in $FA\ell(2)/\langle f\rangle_{cl}$. □

6. Decidability of quotients of $FA\ell(2)$

In this section we prove a special case of Theorem 1.2:

Theorem 6.1. *Let $f \in FA\ell(2)^+$. Then $FA\ell(2)/\langle f\rangle_{cl}$ is decidable.*

We will use our previous result on the decidability of $FA\ell(2)$ (see Corollary 3.5 in [6]).

Lemma 6.2. *Given any formula ξ, we can construct a formula ξ^r such that:*

$$FA\ell(2)/\langle \pi_2 \vee 0\rangle_{cl} \models \xi(\bar{h} + \langle \pi_2 \vee 0\rangle_{cl}) \text{ iff } FA\ell(2) \models \xi^r(\bar{h}, \pi_2 \vee 0),$$

where $\bar{h} \subset FA\ell(2)$.

Proof: We define ξ^r by induction on the complexity of the formula ξ. For an atomic formula $\xi(\bar{x}) := (t(\bar{x}) = 0)$, we define $\xi^r(\bar{x}, y)$ as $R(t(x), y)$, where the latter is the formula $(\forall h > 0)(h \perp y \rightarrow h \perp t(\bar{x}))$. For a quantifier-free formula $\xi(\bar{x})$ (*i.e.*, a Boolean combination of atomic formulae $\xi_i(\bar{x}) := t_i(\bar{x}) = 0$), we define $\xi^r(\bar{x}, y)$ to be the same Boolean combination of $\xi_i^r(\bar{x}, y)$. Finally, if $\xi(\bar{x})$ is in prenex normal form $Q(\bar{z})\xi(\bar{z}, \bar{x})$, where $Q(\bar{z})$ is a block of quantifiers, define $\xi^r(\bar{x}, y)$ as $Q(\bar{z})\xi^r(\bar{z}, \bar{x}, y)$.

We now prove (by induction on the complexity of the formula) that for any $\bar{h} \subset FA\ell(2)$,

$$FA\ell(2)/\langle \pi_2 \vee 0\rangle_{cl} \models \xi(\bar{h} + \langle \pi_2 \vee 0\rangle_{cl}) \text{ iff } FA\ell(2) \models \xi^r(\bar{h}, \pi_2 \vee 0).$$

It suffices to prove it for atomic formulae. This is immediate as $t(\bar{h} + \langle \pi_2 \vee 0\rangle_{cl}) = 0$ iff $t(\bar{h}) \in \langle \pi_2 \vee 0\rangle_{cl}$ iff $t(\bar{h})|_{Z(\pi_2 \vee 0)} = 0$ iff $Z(t(\bar{h})) \supseteq Z(\pi_2 \vee 0)$, and the last of the equivalent conditions holds iff $R(t(\bar{h}), \pi_2 \vee 0)$. □

Corollary 6.3. *The ℓ-group $FA\ell(2)/\langle \pi_2 \vee 0\rangle_{cl}$ has decidable theory.*

Proof: Apply Lemma 6.2 and Corollary 3.5 in [6]. □

Theorem 6.1 now follows from Theorem 5.1, Lemma 5.4, Corollary 3.5 in [6], the decidability of Presburger arithmetic, and the Feferman-Vaught Theorem on direct products. □

7. Decidability results for 2-dimensional zero-sets.

In this section we generalise Theorem 6.1 to allow arbitrary $n \in \mathbb{Z}_+$.

Theorem 7.1. *Let $n \in \mathbb{Z}_+$ and $f \in FA\ell(n)_+$. If $dim(Z(f)) \leq 2$, then the theory of $FA\ell(n)/\langle f \rangle_{cl}$ is decidable.*

Proof: By Proposition 3.1 and the Feferman-Vaught Theorem, we may assume that $\mathcal{Z}(f)$ is connected.

If $dim(Z(f)) = 0$, then $\mathcal{Z}(f) = \emptyset$ and $FA\ell(n)/\langle f \rangle_{cl} \cong \{0\}$, which has decidable theory.

If $dim(Z(f)) = 1$, then $\mathcal{Z}(f)$ is a single point and $FA\ell(n)/\langle f \rangle_{cl} \cong \mathbb{Z}$. By Presburger's Theorem, $Th(\mathbb{Z}, +, \leq)$ is decidable. The theory of $FA\ell(n)/\langle f \rangle_{cl}$ is therefore decidable if $dim(Z(f)) = 1$.

If $Z(f)$ has dimension 2, then $\mathcal{Z}(f)$ comprises a finite number of arcs or a circle. If $n = 2$, we have already shown that the theory is decidable if $\mathcal{Z}(f)$ is a single arc or the entire 1-sphere (see Section 6 and [6]). So assume that $n \geq 3$.

Let $f \in FA\ell(n)_+$ with $dim(\mathcal{Z}(f)) = 1$. Let ${}_{\mathcal{S}}P(f)$ be the $\mathcal{S}$-simplicial complex associated with the primitive simplicial presentation $\mathcal{S}$ of $\mathcal{Z}(f)$ on $S^{(n-1)}$ (see Section 2).
Let $0x_1, \cdots, 0x_k$ be the rays emanating from the origin to the vertices of $\mathcal{S}$ and $u_1, \cdots, u_k$ be the Schauder hats associated with $\mathcal{S}$. Let $p_1, \cdots, p_k$ be the intersection of these cones with $S^{(n-1)}$.

Let r be a piecewise homogeneous linear retract from $\mathbb{R}^n$ to $Z(f)$. We map each simplicial cone $Z(u_i) \subseteq Z(f)$ by a piecewise homogeneous integral linear map $\theta_i : \mathbb{R}^n \to \mathbb{R}^2$ mapping $S^{(n-1)}$ to $S^{(1)}$ so that

$$(*) \qquad \bigwedge_{1 \leq i \leq i' \leq k} \theta_i(p_j) = \theta_{i'}(p_j) \quad (j = 1, \ldots, k), \quad \text{and}$$

$$\bigwedge_{1 \leq j < j' \leq k} \theta_i(p_j) \neq \theta_i(p_{j'}) \quad (i = 1, \ldots, k).$$

Let $\tilde{u_i} \in FA\ell(2)$ be such $Z(\tilde{u_i}) = \theta_i(Z(u_i)) \quad (i = 1, \ldots, k)$.

Let τ_i be a piecewise homogeneous integral linear map from $\mathbb{R}^2$ to $\mathbb{R}^n$ mapping $Z(\tilde{u}_i)$ to $Z(u_i)$ so that $\tau_i \circ \theta_i$ is the identity on $Z(u_i)$ and $\theta_i \circ \tau_i$ is the identity on $Z(\tilde{u}_i)$.

Any finite conjunction of atomic formulae is equivalent to a single atomic formula (since $w_1 = 0$ & $\ldots$ & $w_s = 0$ iff $|w_1| \vee \cdots \vee |w_s| = 0$). So any open formula is equivalent to the conjunction of a single atomic formula ϕ and negations of a finite set of atomic formulae ψ_t ($t \in T$). We partition T into subsets $T_1, \cdots T_k$, where we allow some of these T_m to be empty. We claim that (1) and (2) are equivalent, where (1) is:

$$FA\ell(n)/\langle f \rangle_{cl} \models \exists g \; (\phi(g, \bar{a} + \langle f \rangle_{cl}) \; \& \; \bigwedge_{t \in T} \neg\psi_t(g, \bar{a} + \langle f \rangle_{cl}))$$

and (2) is:

$$\bigvee_{k-\text{partitions of } T} \bigwedge_{1\le i\le k} \exists f_i \in FA\ell(2) \bigwedge_{1\le i\le i'\le k} \bigwedge_{j=1}^{k} f_i(\theta_i(p_j)) = f_{i'}(\theta_{i'}(p_j)) \quad \text{and}$$

$$FA\ell(2)/\langle \tilde{u}_i\rangle_{cl} \models \gamma_i(f_i + \langle \tilde{u}_i\rangle_{cl}),$$

$$\text{where } \gamma_i(x) \text{ is given by } \gamma_i(x) := \phi(x, \bar{a}\circ\tau_i) \ \& \bigwedge_{t\in T_i} \neg\psi_t(x, \bar{a}\circ\tau_i).$$

For suppose that $FA\ell(2)/\langle \tilde{u}_i\rangle_{cl} \models \gamma_i(f_i + \langle \tilde{u}_i\rangle_{cl})$ for all $i \in \{1,\dots,k\}$. Define $g_i(\bar{x}) = f_i \circ \theta_i \circ r(\bar{x})$ $(i = 1,\dots,k)$. By $(*)$, this is well-defined since $r(\bar{x}) \in Z(f) = \bigcup_{i=1}^k Z(u_i)$. Hence (1) follows.

Conversely, suppose that (1) holds. Define $f_i(\bar{x}) = g \circ \tau_i(\bar{x})$ $(i = 1,\dots,k)$. Then $FA\ell(2)/\langle \tilde{u}_i\rangle_{cl} \models \gamma_i(f_i + \langle \tilde{u}_i\rangle_{cl})$ for all $i \in \{1,\dots,k\}$.

The proof is completed as in [6] Lemma 3.5 using the Feferman-Vaught Theorem and the decidablility of Presburger Arithmetic. □

8. Interlude

Let $f,\ g \in FA\ell(n)^+$. To prove a general version of Theorem 1.1, we need only show that if $FA\ell(n)/\langle f\rangle_{cl} \equiv FA\ell(n)/\langle g\rangle_{cl}$, then $P(f)$ and $P(g)$ are simplicially equivalent. For this, it is enough to show that $P(f)$ and $P(g)$ have isomorphic simplicial subdivisions $\mathcal{S}_f$ and $\mathcal{S}_g$, respectively ($\mathcal{S}_f$ and $\mathcal{S}_g$ would then have isomorphic subdivisions into primitive rational cones — see Corollary 3 in [3]). We will succeed with this approach when $n = 3$.

In Section 3 we expressed (in $\mathcal{L}$) the notion of a component of a zero-set $Z(f)$, where $f \in FA\ell(n)_+$. We can therefore count the number of components of $\mathcal{Z}(f)$ and so reduce to the case that $Z(f)$ and $Z(g)$ each have a single component. Hence it is enough to show that $P(f)$ and $P(g)$ have isomorphic simplicial subdivisions in this case.

We will denote the set of all simplices of dimension at most i by $\mathcal{S}_f^i$; namely, those corresponding to finite union of simplicial convex cones generated by at most $i+1$ linearly independent elements.

Given a component $\mathcal{Z}(h)$ of $\mathcal{Z}(f)$, we want to express in $\mathcal{L}$ that the corresponding zero-subset $Z(h)$ of $Z(f)$ can be decomposed into a finite union of m closed simplicial cones, say $Z(h_1),\cdots,Z(h_s)$ such that each $Z(h_\ell)$ has the following properties:
it can be represented as a finite union of simplicial *convex* closed cones, say $S_1,\cdots,S_d$ generated by exactly $i+1$ linearly independent elements with pairwise intersection of dimension either equal to $i \ge 0$ or to -1.
In addition, we require that one of the $Z(h_\ell)$ has the following maximality property, namely it cannot be included as a proper subset in another decomposition of $Z(h)$ with the above properties.

We will call $\mathcal{Z}(h_1), \cdots, \mathcal{Z}(h_\ell)$ the *basic constituents* of the component $\mathcal{Z}(h)$ of $\mathcal{Z}(f)$. Note that such decomposition is not unique.

For example, if $\mathcal{Z}(f)$ is the union of two non-disjoint closed "discs" on the 2-sphere, then it has two basic constituents if the two discs intersect in a point, and has a single basic constituent if the intersection of the discs has dimension 1 or 2.

If $\mathcal{Z}(f)$ is the union of a closed disc and an arc emiting once from the disc (on the 2-sphere), then $\mathcal{Z}(f)$ has two basic constituents, the disc and that part of the arc not in the interior of the disc.

If $\mathcal{Z}(f)$ is the union of three closed arcs A_i on the 2-sphere with endpoints p and p_i and $A_i \cap A_j = \{p\}$ $(i, j \in \{1, 2, 3\}$ with $i \neq j)$, then $\mathcal{Z}(f)$ has ??two? basic constituents.

9. Quotients of $FA\ell(3)$

Our goal in this section is to prove the analogue of Theorem 5.1; *i.e.*,

Theorem 9.1.

$$FA\ell(3)/\langle f\rangle_{cl} \equiv FA\ell(3)/\langle g\rangle_{cl} \text{ iff } FA\ell(3)/\langle f\rangle_{cl} \cong FA\ell(3)/\langle g\rangle_{cl}. \ (*)$$

By the remarks in Section 2 and the interlude, this is equivalent to proving that

$$FA\ell(3)/\langle f\rangle_{cl} \equiv FA\ell(3)/\langle g\rangle_{cl} \text{ implies that}$$
$$\mathcal{Z}(f) \text{ and } \mathcal{Z}(g) \text{ have equivalent simplicial subdivisions } (**).$$

By Proposition 3.1, we may assume that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ each have a single connected component.

We can use $\theta_1(x)$ to determine (in $\mathcal{L}$) iff $\mathcal{Z}(f)$ has dimension 0; $(**)$ holds if it does. The formula $\exists x\theta_2(x)$ holds in $FA\ell(n)/\langle f\rangle_{cl}$ iff $\mathcal{Z}(f)$ has a non-singleton component. So we will assume that $\mathcal{Z}(f)$ has dimension 1 or 2.

9.1. **Basic constituents I.**

We begin this subsection by showing how to determine (in $\mathcal{L}$) the dimension of a basic constituent of $\mathcal{Z}(f)$ $(f \in FA\ell(3)_+)$.

There are two types of basic constituents of dimension 1, namely proper arcs and simple closed curves ("circles"). The difference between them is that given any three disjoint subarcs (or three distinct points) of any proper arc, one of them is between the other two. This fails if $\mathcal{Z}(f) \cap \mathcal{S}(h)$ is a closed curve. If $\mathcal{Z}(f) \cap \mathcal{S}(h)$ has dimension 2, then there are pairwise disjoint closed rationally determined 2-simplices P_1, P_2, P_3 in the interior of $\mathcal{Z}(f) \cap \mathcal{S}(h)$. By taking $f_j \in FA\ell(3)_+$ with connected support contained in P_j $(j = 1, 2, 3)$, we see that this also fails in this case.

Let $\mu_a(h)$ be the $\mathcal{L}$-formula

$$\theta_S(h) \ \&\ \neg\theta_1(h)\ \&$$
$$(\forall f_1, f_2, f_3 > 0)([\ (f_1 \vee f_2 \vee f_3) \leq h \ \&\ f_1 \vee f_2 \perp f_3 \ \&\ f_1 \perp f_2 \ \&\ \bigwedge_{i=1}^{3} \theta_S(f_i)] \rightarrow$$
$$\bigvee_{\{i,j,k\}=\{1,2,3\}} (\forall g > 0)([\theta_S(g) \ \&\ g \geq (f_i \vee f_j)] \rightarrow g \wedge f_k \neq 0)).$$

From the above discussion,

Lemma 9.2. *$FA\ell(3)/\langle f\rangle_{cl} \models \mu_a(h + \langle f\rangle_{cl})$ iff $\mathcal{Z}(f) \cap \mathcal{S}(h)$ is a proper arc.*

To express in $\mathcal{L}$ that $\mathcal{Z}(f)$ contains a closed arc ($\mathcal{Z}(g)$) which is not contained in a 2-dimensional subset of $\mathcal{Z}(f)$, we can similarly use:

$$\bar{\mu}_a(g) \ :=\ (\forall h > 0)[(h \perp g \ \&\ \theta_S(h)) \rightarrow \mu_a(h)].$$

By the above remarks,

Lemma 9.3.

(1) *$FA\ell(3)/\langle f\rangle_{cl} \models \mu_a(g + \langle f\rangle_{cl})$ iff $\mathcal{Z}(f) \cap \mathcal{S}(g)$ is an arc not contained in any 2-dimensional subset of $\mathcal{Z}(f)$, and*

(2) *$FA\ell(3)/\langle f\rangle_{cl} \models \bar{\mu}_a(g + \langle f\rangle_{cl})$ iff $\mathcal{Z}(f) \cap \mathcal{Z}(g)$ is a closed arc not contained in any 2-dimensional subset of $\mathcal{Z}(f)$.*

(3) The closed arc is maximal in $\mathcal{Z}(f)$ iff $FA\ell(3)/\langle f\rangle_{cl} \models \nu_a(g + \langle f\rangle_{cl})$, where $\nu_a(g) := \bar{\mu}_a(g)\ \&\ (\forall k \leq g)\ (k > 0 \rightarrow [\bar{\mu}_a(k) \leftrightarrow (\forall x > 0)(x \perp k \leftrightarrow x \perp g)])$.

We can also provide $\mathcal{L}$-formulae to determine whether two such maximal arcs are the same or not and their number.

On the other hand, by the above lemma, there is an $\mathcal{L}$-formula $\lambda(h)$ expressing that $\mathcal{S}(h) \cap \mathcal{Z}(f)$ comprises two disjoint arcs not contained in any 2-dimensional subset of $\mathcal{Z}(f)$. Let $\mu_c(h)$ be the $\mathcal{L}$-formula

$$\theta_S(h)\ \&\ \neg\mu_a(h)\ \&\ (\exists h_1, h_2 > 0)[h_1 \not\perp h_2 \ \&\ \lambda(h_1 \wedge h_2)\ \&$$
$$(\forall g)\ (g \perp h \leftrightarrow g \perp h_1 \vee h_2)].$$

Lemma 9.4. *Let $f \in FA\ell(3)_+$. Then $\mathcal{Z}(f)$ contains a simple closed curve in the form $\mathcal{S}(h)$ (which is not contained in a 2-dimensional subset of $\mathcal{Z}(f)$) iff*

$$FA\ell(3)/\langle f\rangle_{cl} \models \mu_c(h + \langle f\rangle_{cl}).$$

Proof: $FA\ell(3)/\langle f\rangle_{cl} \models \mu_c(h+\langle f\rangle_{cl})$ iff $\mathcal{S}(h)\cap\mathcal{Z}(f)$ is not an arc but is the union of two arcs $\mathcal{S}(h_1)\cap\mathcal{Z}(f)$ and $\mathcal{S}(h_2)\cap\mathcal{Z}(f)$. Since the intersection of $\mathcal{S}(h_1)\cap\mathcal{Z}(f)$ and $\mathcal{S}(h_2)\cap\mathcal{Z}(f)$ is non-empty and open in $\mathcal{Z}(f)$, the connected set $\mathcal{S}(h)\cap\mathcal{Z}(f)$ is 1-dimensional. It is therefore a simple closed curve. □

Corollary 9.5. *If $f,g\in FA\ell(3)_+$ and $FA\ell(3)/\langle f\rangle_{cl} \equiv FA\ell(3)/\langle g\rangle_{cl}$, then $dim(\mathcal{Z}(f))=1$ iff $dim(\mathcal{Z}(g))=1$.*

This corollary could also have been deduced from Lemma 9.2: $dim(Z(f))=1$ iff

$$FA\ell(3)/\langle f\rangle_{cl} \models (\forall g>0)([\theta_S(g)\ \ \&\ \ \neg\theta_1(g)] \to (\exists h>0)[h\le g\ \ \&\ \ \mu_a(h)]).$$

Note: If $\mathcal{Z}(g_1)$ is an arc, $\mathcal{Z}(g_2)$ is a simple closed curve and $\{x_0\}=\mathcal{Z}(g_1)\cap\mathcal{Z}(g_2)$, then the formula $\mu_c(x)$ fails at $g_2+\langle f\rangle_{cl}$ since for any $h\in FA\ell(3)$ with $h(x_0)>0$ we have $FA\ell(3)\models\neg\mu_a(h+\langle f\rangle_{cl})$.

9.2. $dim(\mathcal{Z}(f))=1$.

We can now complete the proof of Theorem 9.1 in the special case that $dim(\mathcal{Z}(f))=1$. That is,

Proposition 9.6. *Let $f,g\in FA\ell(3)^+$ with $dim(\mathcal{Z}(f))=1$. If $FA\ell(3)/\langle f\rangle_{cl} \equiv FA\ell(3)/\langle g\rangle_{cl}$, then $dim(\mathcal{Z}(g))=1$ and $FA\ell(3)/\langle f\rangle_{cl} \cong FA\ell(3)/\langle g\rangle_{cl}$.*

As before, we may assume that $\mathcal{Z}(f)$ is connected.

If $\mathcal{Z}(f)$ is a simple closed curve, then $FA\ell(3)/\langle f\rangle_{cl} \models (\exists h>0)\mu_c(h+\langle f\rangle_{cl})$. So the same holds for $FA\ell(3)/\langle g\rangle_{cl}$, whence Proposition 9.6 holds in this case.

So assume that $\mathcal{Z}(f)$ is a connected set of arcs. As we can count the number of distinct maximal arcs (using ν_a), we can describe (first-order) the number of arcs comprising the component of $\mathcal{Z}(f)$ if it has dimension 1. We now need to consider incidence between arcs.

Call a point of $\mathcal{Z}(f)$ a *vertex* if it either

(i) has at least two maximal closed arcs containing it, or
(ii) is the endpoint of a unique maximal closed arc.

Maximal closed arcs between vertices will be called *edges*.

Thus we get a *connected graph* $\Gamma(f)$ with the property that every vertex of type (i) has valency at least three.

We can use $\nu_a(x)$, etc., to express (in $\mathcal{L}$) the existence of a vertex of type (i) in $\mathcal{Z}(f)$. We form the sentence that there is a vertex of type (i) with minimal valency (among type (i) vertices), say $m_1>1$: *i.e.*, there is a vertex incident to m_1 edges, but no type (i) vertex in $\mathcal{Z}(f)$ is incident to fewer than m_1 edges. This is achieved by an element $h\in FA\ell(3)_+$ with connected support which intersects exactly m_1 distinct edges, and any $h' \in FA\ell(3)_+$ with connected support which intersects at least two distinct distinct edges must intersect at least m_1 distinct edges.

Moreover, we can count the number of minimal valency type (i) vertices in $\Gamma(f)$ and then proceed to the next largest valency m_2, the length of the paths between vertices of valencies (m, m'), *etc.*

Thus we can determine these properties in $\mathcal{L}$ and hence the properties of the finite planar graph $\Gamma(f)$. Consequently, we can determine (in $\mathcal{L}$) if $\mathcal{Z}(f)$ is 1-dimensional and, if it is, the first-order properties of the finite connected planar graph $\Gamma(f)$.

This is enough to determine $\Gamma(f)$ up to isomorphism. So we have

$$FA\ell(3)/\langle f\rangle_{cl} \equiv FA\ell(3)/\langle g\rangle_{cl} \to \Gamma(f) \equiv \Gamma(g) \to \Gamma(f) \cong \Gamma(g).$$

Conversely, $\Gamma(f) \cong \Gamma(g)$ implies that $Z(f) \sim_\ell Z(g)$.
So $(**)$ holds if $\mathcal{Z}(f)$ is 1-dimensional.
This completes the proof of Proposition 9.6. □

9.3. **Basic constituents II.**

Let $f \in FA\ell(3)_+$ and p be a point of $\mathcal{Z}(f)$. We call p a *separating point* if for every open disc B with centre p and small enough positive radius, $\mathcal{Z}(f) \cap B$ contains a 2-dimensional open subset but $(\mathcal{Z}(f) \cap B) \setminus \{p\}$ is not connected.

Example B: In each part of this example, the word "triangle" will refer to the inside as well as to the triangle itself. So our triangles will be closed 2-dimensional sets.

(i) Let T_1, T_2 be two triangles with intersection a common vertex p. Let $\mathcal{Z}(f) = T_1 \cup T_2$ with $\mathcal{Z}(f_j) = T_j$ $(j = 1, 2)$; so $\mathcal{S}(f_i)$ is the interior of $\mathcal{Z}(f_j)$ $(i \neq j, i, j \in \{1, 2\})$. Then p is a separating point of $\mathcal{Z}(f)$. The same would be true if we replaced T_2 by an arc A whose intersection with T_1 is p, an endpoint of A.

(ii) Let T_1, T_2, T_3 be triangles whose pairwise intersection is a common vertex p. Let $\mathcal{Z}(f) = T_1 \cup T_2 \cup T_3$ with $\mathcal{Z}(f_1) = T_1$ and $\mathcal{Z}(f_2) = T_2 \cup T_3$. So $\mathcal{S}(f_2)$ is the interior of T_1 and is connected but $\mathcal{S}(f_1)$ is the union of the interiors of T_2 and T_3 (and so is not connected). Again, p is a separating point of $\mathcal{Z}(f)$ but it has "valency" 3.

(iii) Let T be a triangle with midpoints p_1, p_2, p_3. Remove the interior of the triangle formed from p_1, p_2, p_3 so that the result is the union of three triangles T_1, T_2, T_3 where $T_i \cap T_j = \{p_k\}$ $(\{i, j, k\} = \{1, 2, 3\})$. If $\mathcal{Z}(f) = T_1 \cup T_2 \cup T_3$, then p_1, p_2, p_3 are separating points of valency 2 (use the same f_1, f_2 for T_1, T_2, T_3 as in (ii)).

(iv) Let T_1, T_2, T_3 be three triangles with one edge on a common line. Suppose that $T_1 \cap T_3 = \emptyset$, $T_1 \cap T_2 = \{p\}$, a common vertex (on the line) and $T_2 \cap T_3 = \{q\}$, a common vertex (on the line). Again let $\mathcal{Z}(f) = T_1 \cup T_2 \cup T_3$. Then $\mathcal{Z}(f)$ has 2 separating points of valency 2. However, in both (iii) and (iv), $\mathcal{Z}(f)$ has three basic constituents. So the valency and number of basic constituents is not sufficient to count the number of points of intersection.

(v) Let S_1 be the union of two overlapping triangles sharing a common side. Let p_1, p_2 be the vertices not on this side. Let S_2 be the union of two overlapping triangles sharing a common side. Let the vertices not on this side also be p_1, p_2. Suppose further that $S_1 \cap S_2 = \{p_1, p_2\}$. Then p_1, p_2 are separating points for $\mathcal{Z}(f)$ (take $\mathcal{Z}(f_j) = S_j$).

Example C: Consider $\Delta \subseteq S^{(2)}$ a rectangle (together with its inside) and let p be an interior point. Let T_1, T_2 be two triangles (again including the inside) in the interior of Δ with common vertex p such that $T_1 \cap T_2 = \{p\}$. Let $(T_1 \cup T_2)^o$ denote the interior of $(T_1 \cup T_2)$, and set $\Delta_0 := \Delta \setminus (T_1 \cup T_2)^o$. Then Δ_0 is the zero set of some $f \in FA\ell(3)_+$. So $\mathcal{Z}(f)$ has a single basic constituent of dimension 2 but is not a manifold. Note that p is a separating point.

We can modify the example in several ways. We can take a finite set $\{T_j : j \in J\}$ of triangles in the interior of Δ with pairwise intersection $\{p\}$, and remove $(\bigcup_{j\in J} T_j)^o$ from Δ. Alternatively, we can let p be on the boundary of Δ. In all cases, the result is a single basic constituent that is not a manifold. We can obviously extend this to allow a finite number of points p some interior to Δ and some not.

The points in these examples are somewhat different from those in Example B. Those in Examples B are "bridge" points, whereas those in Examples C are not (see below). These simple examples will illustrate the need for what we do in this subsection.

We show how to define separating points and then how to distinguish between those of the sorts typified in the examples.

We begin by showing how to express in $\mathcal{L}$ that two connected sets $\mathcal{Z}(f_1)$ and $\mathcal{Z}(f_2)$ (at least one of which is 2-dimensional) intersect in a single point.

By Corollary 9.5, there is an $\mathcal{L}$-formula $\psi_2(x)$ such that

$$FA\ell(3)/\langle f\rangle_{cl} \models \psi_2(h + \langle f\rangle_{cl}) \quad \text{iff} \quad dim(\mathcal{S}(h)) = 2.$$

Let $f_1,\ f_2 \in FA\ell(3)_+$ and

$$\Phi_{sep}(f_1, f_2) := \theta_Z(f_1 \wedge f_2)\ \&\ f_1^{\#} \perp f_2^{\#}\ \&\ \psi_2(f_1^{\#})\ \&\ \bigwedge_{j=1}^{2} \theta_S(f_j^{\#})$$

$$\&\ (\exists h > 0)(\theta_S(h)\ \&\ h \not\perp f_1^{\#}\ \&\ h \not\perp f_2^{\#})\ \&$$

$$(\forall h_1, h_2 > 0)([\bigwedge_{i=1}^{2} \theta_S(h_i)\ \&\ (\forall g_1, g_2 > 0)([\theta_Z(g_1 \wedge g_2)\ \&$$

$$\bigwedge_{j=1}^{2}(g_j \geq f_j\ \&\ \theta_S(g_j^{\#}))] \to (h_i \not\perp g_1^{\#}\ \&\ h_i \not\perp g_2^{\#})] \to [h_1 \not\perp h_2]).$$

Lemma 9.7. *Let $f \in FA\ell(3)_+$ with $\mathcal{Z}(f)$ connected of dimension 2. Then $FA\ell(3)/\langle f\rangle_{cl} \models \Phi_{sep}(f_1 + \langle f\rangle_{cl}, f_2 + \langle f\rangle_{cl})$ iff $dim(\mathcal{Z}(f_1) \cap \mathcal{Z}(f)) = 2$, and the intersection of $\mathcal{Z}(f_1) \cap \mathcal{Z}(f)$ and $\mathcal{Z}(f_2) \cap \mathcal{Z}(f)$ consists of a single point of $\mathcal{Z}(f)$.*

Proof: Suppose that $FA\ell(3)/\langle f\rangle_{cl} \models \Phi_{sep}(f_1+\langle f\rangle_{cl}, f_2+\langle f\rangle_{cl})$. Since $f_j^{\#} \perp f_j$, the relative interior (in $\mathcal{Z}(f)$) of $\mathcal{Z}(f_j)\cap\mathcal{Z}(f)$ is non-empty ($j = 1, 2$). Also, $\mathcal{Z}(f_1)\cap\mathcal{Z}(f)$ has dimension 2 since $\psi_2(f_1^{\#})$. Since $\theta_Z(f_1 \wedge f_2)$, we deduce that $\mathcal{Z}(f_1) \cup \mathcal{Z}(f_2)$ is connected. The remaining part of the formula ensures that if we shrink the connected set $\mathcal{Z}(f_1) \cup \mathcal{Z}(f_2)$ so that it remains connected, then $\mathcal{Z}(g_1) \cap \mathcal{Z}(g_2)$ is still a single point.

The converse direction is obvious. □

Corollary 9.8. *Let $f \in FA\ell(3)_+$ with $\mathcal{Z}(f)$ connected of dimension 2. Then $FA\ell(3)/\langle f\rangle_{cl} \models (\exists f_1, f_2 > 0)\ \Phi_{sep}(f_1 + \langle f\rangle_{cl}, f_2 + \langle f\rangle_{cl})$ iff $\mathcal{Z}(f)$ contains a separating point.*

Corollary 9.9. *Let $f \in FA\ell(3)_+$ with $\mathcal{Z}(f)$ connected of dimension 2. Then there is an $\mathcal{L}$-sentence Φ_{sep,M_0} such that $FA\ell(3)/\langle f\rangle_{cl} \models (\exists f_1, f_2 > 0)\ \Phi_{sep,M_0}$ iff $\mathcal{Z}(f)$ contains exactly M_0 separating points.*

Proof: We can take the conjunction of

$$\{\Phi_{sep}(f_{2m-1}, f_{2m}) : m = 1, \dots, M\}, \qquad \bigwedge_{1\le i<j\le 2M} f_i^{\#} \perp f_j^{\#}, \text{ and}$$

$$(\exists h_1, \dots, h_M > 0)(\bigwedge_{i=1}^{M} \theta_S(h_i)\ \&\ h_i \not\perp f_{2i-1}^{\#}\ \&\ h_i \not\perp f_{2i}^{\#}\ \&\ \bigwedge_{j\ne i}(h_i \perp f_{2j-1}^{\#}\ \&\ h_i \perp f_{2j}^{\#}).$$

It witnesses that $\mathcal{Z}(f)$ has at least M separating points. (The second line of the formula is necessary as a separating point might have valency greater than 2.) The existential sentence formed is satisfied in $FA\ell/\langle f\rangle_{cl}$ iff $\mathcal{Z}(f)$ has at least M separating points. The negation of the corresponding sentence with $M+1$ in place of M holds in $FA\ell/\langle f\rangle_{cl}$ iff $\mathcal{Z}(f)$ does not have $M+1$ separating points. The corollary follows. □

Let $f_1,\ f_2 \in FA\ell(3)_+$ and

$$\Psi_1(f_1, f_2) := \theta_Z(f)\ \&\ f_1 \perp f_2\ \&\ (\forall h > 0)(h \not\perp f_1 \vee f_2)\ \&\ \Phi_{sep}(f_1, f_2).$$

The proof of Lemma 9.7 shows

Lemma 9.10. *Let $f \in FA\ell(3)_+$ with $\mathcal{Z}(f)$ connected of dimension 2. Then $FA\ell(3)/\langle f\rangle_{cl} \models \Psi_1(f_1 + \langle f\rangle_{cl}, f_2 + \langle f\rangle_{cl})$ iff $\mathcal{S}(f_1) \cup \mathcal{S}(f_2)$ is dense in $\mathcal{Z}(f)$, $dim(\mathcal{Z}(f_1) \cap \mathcal{Z}(f)) = 2$, and the intersection of $\mathcal{Z}(f_1) \cap \mathcal{Z}(f)$ and $\mathcal{Z}(f_2) \cap \mathcal{Z}(f)$ consists of a single point of $\mathcal{Z}(f)$.*

We can clearly modify the $\mathcal{L}$-formula to express that the basic constituent $\mathcal{Z}(f_1)$ intersects the remaining basic constituents in exactly $M \geq 2$ points ($M \in \mathbb{N}$).

$$\Psi_M(f_1, f_2) := \theta_Z(f) \ \& \ f_1 \perp f_2 \ \& \ \psi_2(f_1) \ \& \ \psi_2(f_2) \ \&$$

$$(\forall h > 0)(h \not\leq f_1 \vee f_2) \ \& \ (\exists h_1, \ldots, h_M > 0)(\bigwedge_{i=1}^{M} \theta_S(h_i) \,\&\, h_i \not\leq f_1 \,\&\, h_i \not\leq f_2) \ \&$$

$$\bigwedge_{j \neq i} h_j \perp h_i \ \& \ [(\forall g > 0)([\theta_S(g) \,\&\, g \not\leq f_1 \ \& \ g \not\leq f_2] \rightarrow [\bigvee_{i=1}^{M} g \not\leq h_i])].$$

Lemma 9.11. *Let $f \in FA\ell(3)_+$ with $\mathcal{Z}(f)$ connected of dimension 2 and $M \geq 2$. Then $FA\ell(3)/\langle f \rangle_{cl} \models \Psi_M(f_1 + \langle f \rangle_{cl}, f_2 + \langle f \rangle_{cl})$ iff $\mathcal{S}(f_1) \cup \mathcal{S}(f_2)$ is dense in $\mathcal{Z}(f)$, $dim(\mathcal{Z}(f_j) \cap \mathcal{Z}(f)) = 2$ (j=1,2), and the intersection of $\mathcal{Z}(f_1) \cap \mathcal{Z}(f)$ and $\mathcal{Z}(f_2) \cap \mathcal{Z}(f)$ consists of exactly M points of $\mathcal{Z}(f)$.*

Proof: The proof is a trivial modification of the proof of Lemma 9.10. □

We call a separating point p a *bridge* if it is a separating point between at least two distinct basic constituents. That is, there are basic constituents $\mathcal{Z}(f_1)$ and $\mathcal{Z}(f_2)$ of $\mathcal{Z}(f)$ such that $\mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)$ is a finite set of points including p. If $\mathcal{Z}(f)$ is a "pinched annulus", then $\mathcal{Z}(f)$ has a single basic constituent; so the pinch point is a separating point that is not a bridge. One can capture the difference between separating points that are bridges and those that are not. If $f_1^\#$ and $f_2^\#$ give witness to a separating point p, then there is $h \geq f_1^\# \vee f_2^\#$ such that $\mathcal{S}(h)$ is connected and contains no separating point iff p is not a bridge point. All of this can be expressed in $\mathcal{L}$ by the preceding lemmata. Hence

Corollary 9.12. *Let $f \in FA\ell(3)_+$. Then there are $\mathcal{L}$-formulae $\Phi_{bridge}(x_1, x_2)$ and $\Phi_{\neg bridge}(x_1, x_2)$ such that $FA\ell(3)/\langle f \rangle_{cl} \models \Phi_{bridge}(f_1 + \langle f \rangle_{cl}, f_2 + \langle f \rangle_{cl})$ iff $\mathcal{Z}(f_1), \mathcal{Z}(f_2)$ determine a bridge point; and $FA\ell(3)/\langle f \rangle_{cl} \models \Phi_{\neg bridge}(f_1 + \langle f \rangle_{cl}, f_2 + \langle f \rangle_{cl})$ iff $\mathcal{Z}(f_1), \mathcal{Z}(f_2)$ determine a separating non-bridge point.*

As in Corollary 9.9, we can modify the $\mathcal{L}$-formulae to obtain

Proposition 9.13. *If $f, g \in FA\ell(3)^+$ and $FA\ell(3)/\langle f \rangle_{cl} \equiv FA\ell(3)/\langle g \rangle_{cl}$ with $\mathcal{Z}(f)$ connected, then $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ have the same number of separating points and bridge points, and the same number of basic constituents of the same dimensions. Corresponding bridge points have the same valency with attached basic constituents having the same dimensions. Corresponding non-bridge separating points also have the same valency. Moreover, the bijection between the basic constituents of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ preserves the number of separating points between the corresponding pairs.*

Consequently, it suffices to prove Theorem 9.1 in the special case that $\mathcal{Z}(f)$ (and so $\mathcal{Z}(g)$) has a single basic constituent.

9.4. $dim(\mathcal{Z}(f)) = 2$.

We assume throughout this subsection that $\mathcal{Z}(f)$ has dimension 2 which is a basic constituent. By a trivial modification of Corollary 9.12, we can determine (in $\mathcal{L}$) if $\mathcal{Z}(h) \subseteq \mathcal{Z}(f)$ has a separating point; so we can determine if $\mathcal{S}(h^\#)$ (for such $\mathcal{Z}(h)$) is a 2-dimensional manifold. Let $\mu_2(x)$ be an $\mathcal{L}$-formula such that

Lemma 9.14. *Let $f, h \in FA\ell(3)_+$ with $FA\ell(3)/\langle f\rangle_{cl} \models \psi_2(h^\# + \langle f\rangle_{cl})$. Then*

$$FA\ell(3)/\langle f\rangle_{cl} \models \mu_2(h + \langle f\rangle_{cl}) \quad \textit{iff} \quad \mathcal{S}(h^\#) \cap \mathcal{Z}(f) \ \textit{ is a 2-dimensional manifold.}$$

Note that if $\mathcal{S}(f^\#)$ is a 2-dimensional manifold, then it is homotopic to the entire 2-sphere, an open "disc" (*i.e.*, the inside of a triangle) or "a band with m handles" which we will call an *m-band* ($m \in \mathbb{N}$). Such a homotopy can clearly be realised by a piecewise linear integral function.

To prove Theorem 9.1, we will take advantage of the fact that the 2-dimensional basic constituent $\mathcal{Z}(f)$ can be triangulated.

We will need to be able to recognise (in $\mathcal{L}$) an *interior cutting arc* $\mathcal{Z}(g)$ of $\mathcal{Z}(f') \subseteq \mathcal{Z}(f)$. We will express it by a formula $\iota(g, f')$ which we now develop. The key is that an interior cutting arc cannot be the support of any element of $FA\ell(3)$ restricted to $\mathcal{Z}(f)$ as it is not open in $\mathcal{Z}(f)$, but it does cut $\mathcal{Z}(f')$ in two.

Let $\rho(g)$ be the conjunction of (i) and (ii) given by

(i) $(\forall h > 0)\ (h \not\leq g)$ and

(ii) $\neg\theta_S(g)$.

Clearly

Lemma 9.15. *Let $f, g \in FA\ell(3)_+$ with $\mathcal{Z}(g) \subseteq \mathcal{Z}(f)$. Then the interior of $\mathcal{Z}(g)$ in $\mathcal{Z}(f)$ is empty and $\mathcal{S}(g) \cap \mathcal{Z}(f)$ is not connected in $\mathcal{Z}(f)$ iff $FA\ell(3)/\langle f\rangle_{cl} \models \rho(g + \langle f\rangle_{cl})$.*

Let $\lambda(g, g_1, g_2, f')$ be the conjunction of (iii), (iv) and (v) defined by

(iii) $g = g_1 \vee g_2 \ \ \&\ \ \theta_Z(g_1 \wedge g_2)$,

(iv) $\bigwedge_{i=1}^{2}(g_i > 0 \ \ \&\ \ \mu_2(g_i^\#) \ \ \&\ \ \theta_S(g_i^\#))$, and

(v) $g_1^\# \perp g_2^\#$ and $0 < g_j^\# \perp f'$ $(j = 1, 2)$.

Since $g_1^\# \perp g_2^\#$, $\mathcal{Z}(g)$ has dimension at most 1. Hence, clearly

Lemma 9.16. *Let $f, f', g \in FA\ell(3)_+$ with $FA\ell(3)/\langle f\rangle_{cl} \models \mu_2(f' + \langle f\rangle_{cl})$. If $\mathcal{Z}(g) \subseteq \mathcal{Z}(f')$, then $\mathcal{Z}(g)$ has dimension 1 and is the intersection of two zero sets of dimension 2 (whose union is $\mathcal{Z}(f')$ and whose interiors are disjoint) iff*

$$FA\ell(3)/\langle f\rangle_{cl} \models (\exists g_1, g_2)[\lambda(g+\langle f\rangle_{cl}, g_1, g_2, f'+\langle f\rangle_{cl}) \ \ \&\ \ (\forall h \perp g_1 \wedge g_2)(h \perp f'+\langle f\rangle_{cl})].$$

Note that if $\mathcal{Z}(g) \subseteq \mathcal{Z}(f')$ and $g + \langle f\rangle_{cl}$ satisfies $\rho(x)$ & $(\exists y, z)\lambda(g + \langle f\rangle_{cl}, y, z, f' + \langle f\rangle_{cl})$ in $FA\ell(3)/\langle f\rangle_{cl}$, then $\mathcal{Z}(g)$ has dimension exactly 1 as $\mathcal{Z}(g)$ divides the 2-dimensional subset $\mathcal{Z}(f')$ of $\mathcal{Z}(f)$ into two disjoint sets. Indeed, $\mathcal{Z}(g)$ comprises a single "cutting arc or circle" that divides $\mathcal{Z}(f')$ in two and, possibly, a union of a finite set of "non-cutting" arcs and isolated points. It cannot include two cutting arcs as both $\mathcal{S}(g_j^{\#})$ $(j = 1, 2)$ are connected.

Caution: (1) The "cutting arc" could be an arc on the sphere whose restriction to $\mathcal{Z}(f)$ is a finite union of disjoint arcs (*e.g.*, if $\mathcal{Z}(f)$ is an annulus and the arc is a diameter of the external circle).

(2) The additional arcs could intersect the "cutting arc or circle". So the "cutting arc" (respectively, "cutting circle") is an arc (respectively, circle) in the usual sense of the word, but is a union of arcs in our previous sense of the word.

We next express (in $\mathcal{L}$) that $\mathcal{S}(k) \cap \mathcal{Z}(f')$ surrounds the "cutting" part of $\mathcal{Z}(g) \cap \mathcal{Z}(f')$. Consider the $\mathcal{L}$-formula $\sigma_0(k, g, f')$ given by:

$$\text{(vi)}\quad \rho(g)\ \&\ \theta_S(k)\ \&\ \&\ (\exists g_1, g_2)\ \lambda(g, g_1, g_2, f')\ \&$$
$$(\forall g_1, g_2)(\lambda(g, g_1, g_2, f')\ \rightarrow\ [\bigwedge_{i=1}^{2} k \not\perp g_i^{\#}\ \&\ \theta_S(k \wedge g_i^{\#})]\ )\ \&$$
$$(\forall h > 0)([\theta_S(h)\ \&\ (\forall g_1, g_2)(\lambda(g, g_1, g_2, f') \rightarrow \bigwedge_{i=1,2} h \not\perp g_i^{\#})] \rightarrow h \not\perp k).$$

Lemma 9.17. *Let $f, f', g, k \in FA\ell(3)_+$ with $\mathcal{Z}(f')$ as in the previous lemma and $\mathcal{Z}(g) \subseteq \mathcal{Z}(f')$. Then $\mathcal{Z}(g)$ has dimension* 1, *cuts $\mathcal{Z}(f')$ and its cutting arc* (*or circle*) *is contained in $S(k) \cap \mathcal{Z}(f)$ iff $FA\ell(3)/\langle f\rangle_{cl} \models \sigma_0(k + \langle f\rangle_{cl}, g + \langle f\rangle_{cl}, f' + \langle f\rangle_{cl})$.*

Proof: Suppose that $FA\ell(3)/\langle f\rangle_{cl} \models \sigma_0(k+\langle f\rangle_{cl}, g+\langle f\rangle_{cl}, f'+\langle f\rangle_{cl})$. By Lemmata 9.15 and 9.16, $\mathcal{Z}(g)$ has dimension 1 and includes an arc or circle that divides $\mathcal{Z}(f')$ into two disjoint sets of dimension 2. Indeed, we may assume that $\mathcal{Z}(g)$ is a single such arc or circle as nothing is affected by removing either any non-cutting arcs or any isolated points. The universal conjuncts imply that, for every pair of elements g_1, g_2 associated with $\mathcal{Z}(g)$, $\mathcal{S}(k)$ cannot be disjoint from the connected support of any element whose support intersects $\mathcal{S}(g_1^{\#})$ and $\mathcal{S}(g_2^{\#})$. Thus (the cutting part of) $\mathcal{Z}(g)$ is contained in $S(k) \cap \mathcal{Z}(f)$. The converse follows from the same considerations and lemmata. □

We can alternatively express that $\mathcal{Z}(g) \cap \mathcal{Z}(f)$ is not a singleton:

$$\text{(vii)}\quad \psi(g, f') :=\ (\exists a_1, a_2 > 0)[a_1 \perp a_2\ \&\ \forall k > 0\,(\sigma_0(k, g, f')\ \rightarrow\ a_i \not\perp k)].$$

To summarise the lemmata, (i) rules out that $\mathcal{Z}(g) \cap \mathcal{Z}(f)$ has relative maximal dimension in $\mathcal{Z}(f)$. That is, it cannot have dimension $n - 1$ $(= 2)$ or include an arc which is not a subset of some 2-dimensional subset of $\mathcal{Z}(f)$. Thus (i) implies that $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ is a finite set of arcs and circles all contained in 2-dimensional subsets of $\mathcal{Z}(f)$, and a finite set of points. By (ii), $\mathcal{S}(g) \cap \mathcal{Z}(f')$ is not connected. By (iii), (iv) and (v), $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ is the intersection of two zero sets of dimension 2 whose interiors are disjoint. By (vi), $\mathcal{S}(k)$ is connected as is its restriction to $\mathcal{S}(g_1^{\#})$ and $\mathcal{S}(g_2^{\#})$ whenever g_1, g_2 satisfy conditions (iii)-(v). So $\mathcal{S}(k)$ cannot "pinch" $\mathcal{Z}(g)$ if $\mathcal{Z}(g)$ is a single arc or circle. Moreover, $\mathcal{S}(k)$ cannot be decomposed into two disjoint supports one of which is included in $\mathcal{S}(g_1^{\#})$. So (vi) ensures that $\mathcal{S}(k)$ cannot be essentially shrunk. By (vii), $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ is not a single point and the same is true when this is intersected with any connected $\mathcal{S}(k)$ that is not disjoint from the support of both $g_1^{\#}$ and $g_2^{\#}$. Thus $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ is the union of (either a cutting circle or a cutting arc) and a finite set of arcs and points.

Let

$$\sigma(g, g_1, g_2, k, f') := \rho(g) \ \& \ \lambda(g, g_1, g_2, f') \ \& \ \psi(g, f') \ \& \ \sigma_0(k, g, f').$$

By the above lemmata, we obtain

Corollary 9.18. *With the above notation,*

$$FA\ell(3)/\langle f\rangle_{cl} \models \sigma(g + \langle f\rangle_{cl}, g_1 + \langle f\rangle_{cl}, g_2 + \langle f\rangle_{cl}, k + \langle f\rangle_{cl}, f' + \langle f\rangle_{cl}) \quad \textit{iff}$$

$g = g_1 \vee g_2$ with all the above properties for g_1, g_2,
$\mathcal{Z}(g) \subseteq \mathcal{Z}(f')$ has dimension 1 and includes exactly one cutting arc or circle Γ contained in the interior of $\mathcal{Z}(f')$,
$(\mathcal{Z}(g) \cap \mathcal{Z}(f')) \setminus \Gamma$ is a finite set of points and arcs,
and the restriction of $\mathcal{S}(k)$ to $\mathcal{Z}(f)$ surrounds Γ.

We next want to recognise interior cutting circles in $\mathcal{Z}(f')$. The idea is easy. If we take three pairwise disjoint connected supports $\mathcal{S}(a_j)$ $(j = 1, 2, 3)$ each intersecting the cutting arc in $\mathcal{Z}(f')$, then we can find a surrounding $\mathcal{S}(k)$ such that removing the middle $\mathcal{S}(a_j)$ from $\mathcal{S}(k)$ results in a disconnected set. In the case of a cutting circle, we can "just go round the back of the circle".

Let $\iota_0(g, f')$ be the $\mathcal{L}$-formula:

$$(\exists g_1, g_2 > 0)(g_1 \vee g_2 = g \ \& \ (\exists k > 0)[\sigma(g, g_1, g_2, k, f') \ \& \ \gamma(g, g_1, g_2, k, f')]),$$

where $\gamma(g, g_1, g_2, k, f')$ is given by:

$$(\forall a_1, a_2, a_3 > 0)([\bigwedge_{i=1,2,3} \theta_S(a_i) \ \& \ a_i \not\perp g_1^{\#} \ \& \ a_i \not\perp g_2^{\#} \ \& \bigwedge_{j \neq i} a_j \perp a_i] \rightarrow$$
$$[(\exists k > 0)(\sigma(g, g_1, g_2, k, f') \ \&$$
$$\bigvee_{\{i_1, i_2, i_3\} = \{1,2,3\}} (\forall h > 0)([h \leq k \ \& \ h \not\perp a_{i_1} \ \& \ h \not\perp a_{i_3} \ \& \ \theta_S(h)] \rightarrow h \not\perp a_{i_2})]).$$

Let $\iota_1(g, f')$ be the $\mathcal{L}$-formula:

$$(\exists g_1, g_2 > 0)(g_1 \vee g_2 = g \;\;\&\;\; (\exists k > 0)\, \sigma(g, g_1, g_2, k, f') \;\;\&\;\; \neg\gamma(g, g_1, g_2, k, f')),$$

By our preceding remarks

Lemma 9.19. *With the previous notation,*
$FA\ell(3)/\langle f\rangle_{cl} \models \iota_0(g, f')$ *iff* $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ *contains a single cutting arc of* $\mathcal{Z}(f')$ (*and possibly a finite set of arcs and points*)*; and*
$FA\ell(3)/\langle f\rangle_{cl} \models \iota_1(g, f')$ *iff* $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ *contains a single interior cutting circle of* $\mathcal{Z}(f')$ (*and possibly a finite set of non-cutting arcs and points*).

Note that a cutting arc with ends joined by an arc entirely in the boundary of $\mathcal{Z}(f')$ falls under the first clause of the lemma, not the second.

We next remove the extraneous interior arcs. If there were a non-cutting arc A in the interior of $\mathcal{Z}(f')$, then it would occur as a subset of $\mathcal{Z}(g_1)$ or $\mathcal{Z}(g_2)$. We assume the former. If necessary, by replacing $g_1^{\#}$ by $g_1' \leq g_1^{\#}$ whose support has boundary including $\mathcal{Z}(g)$, we may also assume that $\mathcal{S}(g_1^{\#})$ is a 2-dimensional manifold. Then there would be $h \geq g_1^{\#}$ with $h \perp g_1$ such that $\mathcal{Z}(h) = A$. Let $h_1, h_2 > 0$ be such that $h_j^{\#} \geq g_1^{\#}$ with $dim(\mathcal{S}(h_j^{\#})) = 2$ $(j = 1, 2)$ and $\mathcal{Z}(h_1) \cap \mathcal{Z}(h_2) = A$. Then A cuts $\mathcal{Z}(h_1 \wedge h_2) \cap \mathcal{Z}(g_1)$ in two. Since $h \geq g_1^{\#}$, the only arcs which occur as subsets of $\mathcal{Z}(h)$ inside $\mathcal{S}(g_1^{\#})$ must be subsets of $\mathcal{Z}(g)$. Thus we can recognise (in $\mathcal{L}$) if $\mathcal{Z}(g)$ includes interior non-cutting arcs. Therefore there is an $\mathcal{L}$-formula $\iota(g, f')$ that holds in $FA\ell(3)/\langle f\rangle_{cl}$ iff $\mathcal{Z}(g)$ has no such arcs but has a single interior arc that cuts $\mathcal{Z}(f')$ in two. Similarly, there is an $\mathcal{L}$-formula $\iota_c(g, f')$ that holds in $FA\ell(3)/\langle f\rangle_{cl}$ iff $\mathcal{Z}(g)$ has no such arcs but has an interior circle that cuts $\mathcal{Z}(f')$ in two. Hence

Proposition 9.20. *Let* $f, f', g \in FA\ell(3)$ *with* $\mathcal{Z}(f') \subseteq \mathcal{Z}(f)$ *and* $\mathcal{S}(f'^{\#})$ *a 2-dimensional manifold. Then there are* $\mathcal{L}$*-formulae* $\iota(g, f')$ *and* $\iota_c(g, f')$ *such that*
$FA\ell(3)/\langle f\rangle_{cl} \models \iota(g + \langle f\rangle_{cl}, f' + \langle f\rangle_{cl})$ *iff* $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ *is a single cutting arc* (*and possibly a finite set of points, and a finite set of arcs in the boundary of* f') *of* $\mathcal{Z}(f')$*, and*
$FA\ell(3)/\langle f\rangle_{cl} \models \iota_c(g + \langle f\rangle_{cl}, f' + \langle f\rangle_{cl})$ *iff* $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ *is a single interior circle* (*and possibly a finite set of points, and a finite set of arcs in the boundary of* f') *of* $\mathcal{Z}(f')$*.*

We now have the tools to complete the triangulation of $\mathcal{Z}(f)$. To do so, we first show how to recognise (from $\mathcal{L}$) that a zero set is piecewise integer linear homeomorphic to a filled-in triangle.

Proposition 9.21. *Let* $f, f' \in FA\ell(3)_+$ *with* $\mathcal{Z}(f') \subseteq \mathcal{Z}(f)$ *and* $\mathcal{S}(f'^{\#})$ *a 2-manifold.*

(I) $FA\ell(3)/\langle f\rangle_{cl} \models (\forall g)\, \neg\iota(g, f' + \langle f\rangle_{cl})$ *iff* $\mathcal{Z}(f')$ *is the 2-sphere.*

(II) *For each* $m \in \mathbb{N}$, *there is an* $\mathcal{L}$-*formula* $\iota'_m(g, f')$ *such that* $FA\ell(3)/\langle f\rangle_{cl} \models (\exists g)\ \iota'_m(g, f' + \langle f\rangle_{cl})$ *iff* $\mathcal{Z}(f')$ *is a closed* m-*band.*

(III) *There is an* $\mathcal{L}$-*formula* $\iota''(f')$ *such that* $FA\ell(3)/\langle f\rangle_{cl} \models \iota''(f' + \langle f\rangle_{cl})$ *iff* $\mathcal{Z}(f')$ *is piecewise integer linear homeomorphic to a closed triangle* (*including inside*).

Proof: (I) If $\mathcal{Z}(f')$ is the entire 2-sphere, then there is no cutting arc (it must be a circle); indeed, f' must be 0. If $\mathcal{Z}(f')$ is a band (possibly with handles) or a triangle, there is a cutting arc. Hence (I) follows from Proposition 9.20.

(II) and (III) The key idea in the remainder of the proof is that if 4 points lie on a line, then one can join the first and third on one side of the line and the second and fourth on the other side (so that the joining lines do not cross). However, this cannot be done without crossing if one confines the joining to be on the same side of the line. We will modify this idea to code it into our language.

Let $\mathcal{Z}(f')$ be a multiband with $m \in \mathbb{N}$ handles. Let R be a filled-in rectangle and $\{T_j : j \in J\}$ be a finite set of pairwise disjoint closed triangles in the interior of R where $|J| = m + 1$. Let I_j be the interior of T_j $(j \in J)$. Then $\mathcal{Z}(f')$ is piecewise integer linear homeomorphic to $R \setminus \bigcup_{j\in J} I_j$. There is an arc A on the sphere that passes though each I_j $(j \in J)$ and cuts $\mathcal{Z}(f')$ in two. Let $g \in FA\ell(3)_+$ be such that $\mathcal{Z}(g) = A \cap \mathcal{Z}(f')$. So $\mathcal{Z}(g)$ is a union of $m+1$ disjoint arcs and $g = g_1 \vee g_2$ with g_1, g_2 as before. Let $k \in FA\ell(3)_+$ be such that $\mathcal{S}(k)$ surrounds $\mathcal{Z}(g)$. By definition, $\mathcal{S}(k)$ must "go round" each T_j. Fix $j_0 \in J$ and let $\{x_i : i = 1, 2, 3, 4\}$ be a set of distinct points on the boundary of T_{j_0} in order x_1, x_2, x_3, x_4 progressing clockwise around the boundary of T_{j_0}. Let $a_1, a_2, a_3, a_4 \in FA\ell(3)_+$ be pairwise orthogonal with connected supports so that $a_i \le k \wedge f'^{\#}$ and x_i is in the closure of $\mathcal{S}(a_i) \cap \mathcal{Z}(f')$ $(i = 1, 2, 3, 4)$. Suppose that for each $\{i, i'\} \subseteq \{1, 2, 3, 4\}$, there are $h_{i,i'} \le k$ in $FA\ell(3)_+$, each with connected support, such that $\mathcal{S}(h_{i,i'}) \cap S(a_{i''}) \neq \emptyset$ iff $i'' \in \{i, i'\}$. We do not want the supports of the $\mathcal{S}(a_i)$ to cut $\mathcal{S}(g_1^{\#} \wedge k)$ in two so we further require that if $a_i, a_{i'} \not\le g_1^{\#}$, then there is such an $h_{i,i'} \le g_1^{\#} \wedge k$. Similarly with $g_2^{\#}$ in place of $g_1^{\#}$. Then there do not exist orthogonal $h_1, h_2 \le k \cap f'^{\#}$ in $FA\ell(3)_+$ with $h_1 \perp (a_2 \vee a_4)$, $h_2 \perp (a_1 \vee a_3)$ and $h_1 \not\le a_1$, $h_1 \not\le a_3$, $h_2 \not\le a_2$ and $h_2 \not\le a_4$. By our previous results, all this is expressible in $\mathcal{L}$. We can take the finite conjunction over all $j_0 \in J$. Let the resulting $\mathcal{L}$-formula be $\underline{\iota}'_m(x, y)$. Thus $FA\ell(3)/\langle f\rangle_{cl} \models (\exists g)\ \underline{\iota}'_m(g, f' + \langle f\rangle_{cl})$ if $\mathcal{Z}(f')$ is a closed m'-band for some $m' \ge m$. We can therefore obtain an $\mathcal{L}$-formula $\iota'_m(x, y)$ from $\underline{\iota}'_m(x, y)$ that is satisfied in $FA\ell(3)/\langle f\rangle_{cl}$ if $\mathcal{Z}(f)$ is an m-band but not an m'-band if $m' \neq m$.

On the other hand, if $\mathcal{Z}(f')$ is a closed triangle, let A be any cutting arc. Let k be such that $\mathcal{S}(k)$ surrounds $\mathcal{Z}(g) = A$ but contains no points on the boundary of $\mathcal{Z}(f')$. Then we can find orthogonal h_1, h_2 whenever a_1, a_2, a_3, a_4 satisfy the above hypotheses by letting $h_1 \le g_1^{\#}$ and $h_2 \le g_2^{\#}$ where g_1, g_2 are such that $\mathcal{Z}(g_1) \cap \mathcal{Z}(g_2) = \mathcal{Z}(g)$, $\mathcal{Z}(f') = \mathcal{Z}(g_1) \cup \mathcal{Z}(g_2)$, $g_1^{\#} \perp g_2^{\#}$ and $\mathcal{S}(g_i^{\#})$ are connected and 2-dimensional (with the obvious modifications to these restrictions on h_1, h_2 if any a_{2i+j} is orthogonal to $g_j^{\#}$ $(i = 0, 1;\ j = 1, 2)$. Thus $FA\ell(3)/\langle f\rangle_{cl} \models (\exists g)(\iota(g, f' + \langle f\rangle_{cl})\ \ \&\ \ (\forall g' > 0)(\neg \underline{\iota}'_2(g', f' + \langle f\rangle_{cl}))$ if $\mathcal{Z}(f')$ is a closed triangle.

This completes the proof of (II) and (III). □

We can now complete the proof of Theorem 9.1.

Proof: Let $\mathcal{Z}(f)$ have dimension 2. By Proposition 9.13, we can determine the number of basic constituents of $\mathcal{Z}(f)$ of dimensions $0, 1$, and 2, their number and how they fit together. We can also determine the ones that have separating non-bridge points, their number and the valency of each such point. We can write each such basic constituent as a minimal union of $(\sim_\ell)$ triangles any pair of which intersect in a single point or a single arc. By Proposition 9.21, we can recognise $(\sim_\ell)$ triangles in $\mathcal{L}$. By Corollary 9.12, we can determine the former and by Proposition 9.20 we can also determine the latter. Now every 2-dimensional basic constituent of $\mathcal{Z}(f)$ can be triangulated and we can recognise triangles with inside (in $\mathcal{L}$) by Proposition 9.20. We can therefore determine how the triangulation is sewn together by sentences in $\mathcal{L}$. Therefore, the same triangulation must occur for $\mathcal{Z}(g)$ if $FA\ell(3)/\langle f\rangle_{cl} \equiv FA\ell(3)/\langle g\rangle_{cl}$. By Whittlesey's classification of 2-complexes [11], $(**)$ holds. Consequently, Theorem 9.1 is proved. □

10. Basic constituents in $FA\ell(n)/\langle f\rangle_{cl}$.

We would like to prove that if $FA\ell(n)/\langle f\rangle_{cl} \equiv FA\ell(n)/\langle g\rangle_{cl}$, then $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ have the same number of components (respectively basic constituents) of the *same* type and of the same dimension and that in each component, the corresponding intersections of the various constituents are the same.

Assume that $\mathcal{Z}(f)$ has only one component, having basic constituents $\mathcal{Z}(f_1), \cdots, \mathcal{Z}(f_m)$. We would like to show that this is $\mathcal{L}$-expressible. Unfortunately, there is no classification of simplicial complexes in general which is suitable for our purposes. We therefore confine our attention to "special" $\mathcal{Z}(f)$ in the following lemmata and provide $\mathcal{L}$-expressibility only in such $FA\ell(n)/\langle f\rangle_{cl}$. This will allow us to adapt the ideas of [6], and will suffice for our applications in the following two sections. We first establish

Lemma 10.1. *Let $n \in \mathbb{Z}_+$ and $f \in FA\ell(n)_+$ with $Z(f)$ integrally ℓ-equivalent to a closed convex cone of dimension n_1. Then for each $j \in \{-1, 0, , \ldots, n_1 - 1\}$, there is a formula $\psi^*_{n_1,j}(x)$ such that*

$$FA\ell(n)/\langle f\rangle_{cl} \models \psi^*_{n_1,j}(h + \langle f\rangle_{cl}) \text{ iff } dim(\mathcal{Z}(f) \cap \mathcal{Z}(h)) = j.$$

Proof: Without loss of generality, we may assume that $Z(f)$ is a closed convex cone of dimension n_1. Let $\psi_{n,j}(x)$ be the formula from [6] (Definition 4.1) whose validity in $FA\ell(n)$ is equivalent to $dim(\mathcal{Z}(x)) = j$, where $j \in \{-1, 0, \ldots, n-1\}$.

We prove the lemma by induction on $j < n_1$. This is clear for $j = k := n_1 - 1$ as $\psi_{k,k-1}(h) := (\exists g > 0)(g \perp h)$. In [6], $\phi_{k,k-1}(h)$ denoted the following formula:

$$\psi_{k,k-1}(h) \;\&\; \forall g > 0\ (g \perp f \rightarrow (\exists k\ (k \perp f \;\&\; \theta_S(k)))).$$

Now consider $\psi_{k,j}(h)$ with $j < k-1$, from [6]. This formula was defined using the auxiliary formula:

$$\chi(a) := (\exists a_1, a_2 > 0)\ \chi(a, a_1, a_2), \quad \text{where}$$

$$\chi(a, a_1, a_2) := a > 0 \ \& \ a = a_1 \vee a_2 \ \& \ \neg\psi_{n,n-1}(a) \ \& \ \bigwedge_{i=1}^{2} \phi_{n,n-1}(a_i) \ \& \ a_1 \perp a_2.$$

Then $\psi_{k,j}(h) := \exists a\ (\chi(a) \ \& \ \psi^r_{k-1,j}(h, a))$.

Let $\pi^{(n_1)} = (\pi_1 \vee 0) \vee \cdots \vee (\pi_{n_1} \vee 0)$. Since $FA\ell(n)/\langle f\rangle_{cl} \cong FA\ell(n_1)/\langle \pi^{(n_1)}\rangle_{cl}$, we get that

$$FA\ell(n)/\langle f\rangle_{cl} \models \psi_{n_1,j}(h + \langle f\rangle_{cl})$$

iff

$$FA\ell(n_1)/\langle \pi^{(n_1)}\rangle_{cl} \models \psi_{n_1,j}(h + \langle \pi^{(n_1)}\rangle_{cl}).$$

So we need only show that this latter holds iff $dim(\mathcal{Z}(h) \cap \mathcal{Z}(\pi^{(n_1)})) = j$.

If $n_1 = n$, we can replace the formulae $\psi_{n,j}(x)$ by $\psi_{n,j}(x \vee \pi^{(n_1)})$ $(j = 0, \ldots, n-1)$. So assume that $n_1 < n$.

The key is that as $Z(\pi^{(n_1)})$ is a convex cone of dimension strictly less than n, there is $a \in FA\ell(n)_+$ satisfying $\chi(x)$ such that its zero-set includes the zero-set of $\pi^{(n_1)}$. If $I(a) := \langle a\rangle_{cl} \subseteq FA\ell(n)$, then

$$FA\ell(n)/\langle \pi^{(n_1)}_{cl}\rangle \ \cong \ (FA\ell(n)/I(a))/\langle \pi^{(n_1)} + I(a)\rangle_{cl};$$

and the latter is isomorphic to $FA\ell(n-1)/\langle \pi_a\rangle_{cl}$, where π_a is the restriction of $\pi^{(n_1)}$ to the zero-set of a (viewed as a subset of $\mathbb{R}^{n-1}$). Moreover,

$$FA\ell(n-1)/\langle \pi_a\rangle_{cl} \ \cong \ (FA\ell(n)/I(a))/\langle \pi^{(n_1)} + I(a)\rangle_{cl}.$$

By induction, there are formulae $\psi^*_{n_1,j}(x)$ such that

$$FA\ell(n-1)/\langle \pi_a\rangle_{cl} \models \psi^*_{n_1,j}(h + \langle \pi_a\rangle_{cl}) \quad \text{iff} \quad dim(\mathcal{Z}(\pi_a) \cap \mathcal{Z}(h)) = j$$

$(j = -1, \ldots, n_1 - 1)$. So, for such j, we have

$$FA\ell(n)/\langle \pi^{(n_1)}\rangle_{cl} \models \psi^*_{n_1,j}(h + \langle \pi_a\rangle_{cl}) \quad \text{iff} \quad dim(\mathcal{Z}(\pi^{(n_1)}) \cap \mathcal{Z}(h)) = j,$$

as desired. □

We next consider the case when $\mathcal{Z}(f)$ is definably connected.

Lemma 10.2. *Let $n \in \mathbb{Z}_+$ and $f \in FA\ell(n)_+$ with $\mathcal{Z}(f)$ definably connected. Suppose that $f_1, \cdots, f_m \in FA\ell(n)_+$ with $Z(f) = \bigcup_{i=1}^m Z(f_i)$ where each $Z(f_i)$ is integrally ℓ-equivalent to a closed convex cone of dimension $n_i \leq n$ and assume that this decomposition is minimal such. Let $n_0 = max\{n_1, \ldots, n_m\}$. Then for each $j \in \{-1, 0, , \ldots, n_0 - 1\}$, there is a formula $(\psi^*_{n_i,j})^z(x, y)$ with $n_i \geq j$ such that*

$$FA\ell(n)/\langle f\rangle_{cl} \models \bigvee_{1 \leq i \leq m} (\psi^*_{n_i,j})^z(h + \langle f\rangle_{cl}, f_i + \langle f\rangle_{cl}) \quad \textit{iff} \quad dim(\mathcal{Z}(f) \cap \mathcal{Z}(h)) \geq j.$$

Proof: We first show that for any formula γ, we can construct a formula γ^z such that for any $\bar{h} \subset FA\ell(n)$,

$$FA\ell(n)/\langle f_i\rangle_{cl} \models \gamma(\bar{h} + \langle f_i\rangle_{cl}) \quad \text{iff} \quad FA\ell(n)/\langle f\rangle_{cl} \models \gamma^z(\bar{h}, f_i).$$

We proceed as in Lemma 4.6 in [6] by induction on the complexity of γ.

It suffices to define γ^z when γ is an atomic formula; *i.e.*, of the form $t(\bar{h}) = 0$, where $t(\bar{x})$ is a term. Then $t(\bar{h} + \langle f_i\rangle_{cl}) = 0$ iff $t(\bar{h}) \in \langle f_i\rangle_{cl}$ iff the restriction of $t(\bar{h})$ to $Z(f_i)$ is equal to 0 iff $Z(f_i) \subseteq Z(t(\bar{h}))$ iff $(\forall k > 0)(k \perp f_i \rightarrow k \perp t(\bar{h}))$.

Let $h > 0$ be such that $dim(\mathcal{Z}(f) \cap \mathcal{Z}(h)) = j$.
This implies that $dim(\mathcal{Z}(f_i) \cap \mathcal{Z}(h)) = j$ for some $i \in \{1, \ldots, m\}$.

By Lemma 10.1, there is a formula $\psi^*_{n_i,j}(x)$ such that

$$FA\ell(n)/\langle f_i\rangle_{cl} \models \psi^*_{n_i,j}(h + \langle f_i\rangle_{cl}) \quad \text{iff} \quad dim(\mathcal{Z}(f_i) \cap \mathcal{Z}(h)) = j.$$

Now, by the above,

$$FA\ell(n)/\langle f_i\rangle_{cl} \models \psi^*_{n_i,j}(h + \langle f_i\rangle_{cl}) \text{ iff}$$

$$FA\ell(n)/\langle f\rangle_{cl} \models (\psi^*_{n_i,j})^z(h + \langle f\rangle_{cl}, f_i + \langle f\rangle_{cl}).$$

Conversely, assume that for some $1 \leq i \leq m$, there exists $0 \leq j \leq n_i$ such that

$$FA\ell(n)/\langle f\rangle_{cl} \models (\psi^*_{n_i,j})^z(h + \langle f\rangle_{cl}, f_i + \langle f\rangle_{cl}).$$

By the above,

$$FA\ell(n)/\langle f_i\rangle_{cl} \models \psi^*_{n_i,j}(h + \langle f_i\rangle_{cl}).$$

By Lemma 10.1,

$$dim(\mathcal{Z}(f_i) \cap \mathcal{Z}(h)) = j.$$

Therefore,

$$dim(\mathcal{Z}(f) \cap \mathcal{Z}(h)) \geq j.$$

□

Proposition 10.3. *Let $f \in FA\ell(n)$ and suppose that $\mathcal{Z}(f)$ is definably connected of dimension $n-1$ and satisfies the hypotheses of the previous lemma. Then* $FA\ell(n)/\langle f\rangle_{cl} \not\equiv FA\ell(n)$.

Proof: First assume that $Z(f)$=$Z(\pi_n \vee 0)$. View $FA\ell(n-1)$ as generated by $\pi_2, \cdots, \pi_n$. Then

$$(FA\ell(n)/\langle \pi_n \vee 0\rangle_{cl})/\langle \pi_1\rangle_{cl} \cong FA\ell(n-1)/\langle \pi_n \vee 0\rangle_{cl}.$$

By the induction hypothesis, there is a sentence τ that holds in $FA\ell(n-1)/\langle \pi_n \vee 0\rangle_{cl}$ but not in $FA\ell(n-1)$.

By Lemma 4.6 in [6], given any sentence σ one can construct a *relativized* formula $\sigma^r(\pi_1)$ such that $FA\ell(n-1) \models \sigma$ iff $FA\ell(n) \models \sigma^r(\pi_1)$.

We need the corresponding result for $FA\ell(n-1)/\langle \pi_n \vee 0\rangle_{cl}$; *i.e.*, given any sentence σ,

$$FA\ell(n-1)/\langle \pi_n \vee 0\rangle_{cl} \models \sigma \quad \text{iff} \quad FA\ell(n)/\langle \pi_n \vee 0\rangle_{cl} \models \sigma^r(\pi_1).$$

To define an $\mathcal{L}$-sentence $\bar{\tau}$ which distinguishes the ℓ-groups, we use the formula $\chi(x)$ defined in Lemma 4.5 in [6] (see the proof of Lemma 10.1 above).

Recall that if $f \in FA\ell(n)_+$ and $FA\ell(n) \models \chi(f)$, then there is $f' \geq f$ such that $FA\ell(n) \models \chi(f')$ and an isomorphism $\vartheta : FA\ell(n)/\langle f' \rangle_{cl} \cong FA\ell(n-1)$. Moreover, this element f' satisfies the minimality condition that for any $g \in FA\ell(n)_+$ with $Z(g) \subsetneq Z(f')$, we have $FA\ell(n) \models \neg\chi(g)$. Additionally, for $j \in \{-1, 0, \ldots, n-1\}$

$$dim\mathcal{Z}(\vartheta(h+ < f' >_{cl})) = j \quad \text{iff} \quad dim(\mathcal{Z}(h) \cap \mathcal{Z}(f')) = j.$$

Consequently, the sentence $\bar{\tau}$ is given by

$$(\exists h > 0)\ (\chi(h)\ \ \&\ \ (\forall h' \geq h)(\chi(h') \rightarrow\ \tau^r(h)).$$

Indeed, $FA\ell(n)/\langle \pi_n \vee 0 \rangle_{cl} \models \bar{\tau}$ since we may choose $h = \pi_1$. On the other hand, by way of contradiction, suppose that for some element $h \in FA\ell(n)_+$, we have $FA\ell(n) \models \chi(h)\ \ \&\ \ (\forall h' \geq h)(\chi(h') \rightarrow\ \tau^r(h))$. Choose $h' \geq h$ to be a π_1-like element; this implies that $FA\ell(n) \models \tau^r(h')$. Therefore, $FA\ell(n)/\langle h' \rangle_{cl} \cong FA\ell(n-1) \models \tau$, which contradicts the induction hypothesis.

Now consider the general case. Without loss of generality, we may assume that $Z(f)$ includes $Z((-\pi_1 \vee 0) \vee (-\pi_2 \vee 0) \vee \cdots \vee (-\pi_n \vee 0))$. Now

$$FA\ell(n)/\langle \pi_1 - \pi_2 \rangle_{cl} \ \cong\ FA\ell(n-1),$$

and $(FA\ell(n)/\langle f \rangle_{cl})/\langle \pi_1 - \pi_2 \rangle_{cl} \ \cong\ FA\ell(n-1)/\langle f' \rangle_{cl}$ for some $f' \in FA\ell(n-1)_+$, where $Z(f')$ contains $Z((-\pi_2 \vee 0) \vee \cdots \vee (-\pi_n \vee 0))$.

We may therefore apply the induction hypothesis: there is a sentence τ that holds in $FA\ell(n-1)/\langle f' \rangle_{cl}$ but not in $FA\ell(n-1)$.

The same proof as above shows that the sentence $\bar{\tau}$ distinguishes $FA\ell(n)/\langle f \rangle_{cl}$ from $FA\ell(n)$. □

Proposition 10.4. *Let $f, g \in FA\ell(n)_+$ with $Z(f), Z(g)$ satisfying the hypotheses of Lemma 10.2. Then $FA\ell(n)/\langle f \rangle_{cl} \not\equiv FA\ell(n)/\langle g \rangle_{cl}$ if $dim(Z(f)) \neq dim(Z(g))$.*

Proof: Let $dim(Z(g)) = n_2 < dim(Z(f)) = n_1 \leq n$. In $FA\ell(n)/\langle f \rangle_{cl}$ we can find $n_1 + 1$ strictly positive elements $h_0,\ h_1, \cdots, h_{n_1}$ such that $dim(Z(f) \cap Z(h_i)) = i$, for $0 \leq i \leq n_1$. □

Using the previous results in this section, we can generalise (the proof of) Lemmata 5.2 and 5.3.

Let $f \in FA\ell(n)_+$ and $f_1, \cdots, f_m \in FA\ell(n)$ with $Z(f_i)$ non-trivial convex cones such that $Z(f) = \cup_{i=1}^m Z(f_i)$ and this decomposition is the minimal such. For each $j = -1, 0, \ldots, n-1$, let $c(f, j)$ be the number of elements in $D_j := \{i \in \{1, \ldots, m\} : dim(\mathcal{Z}(f_i)) = j\}$. Let $C(f) = (c(f, -1), c(f, 0), \ldots, c(f, n-1)) \in \mathbb{N}^{n+1}$.

Lemma 10.5. *Let $f, g \in FA\ell(n)_+$. Then $FA\ell(n)/\langle f \rangle_{cl} \ \equiv\ FA\ell(n)/\langle g \rangle_{cl}$ implies $C(f) = C(g)$.*

As noted in Example B, the converse is false. We need to be able to detect different types of subsets of $S^{(n-1)}$ of dimension j and different types of simplices. This is where the lack of a classification of such types is a crucial obstacle.

11. Undecidability

In this section we complete the proof of Theorem 1.2.

Theorem 11.1. *Let $n \in \mathbb{Z}_+$ and $f \in FA\ell(n)_+$. If dim$(Z(f)) > 2$, then the theory of $FA\ell(n)/\langle f\rangle_{cl}$ is undecidable.*

We first recall Grzegorczyk's conditions on a topological theory that imply undecidability.

The topological space is Hausdorff, connected and is normal (*i.e.*, two disjoint closed sets are contained in disjoint open sets); it has a countable basis, and every non-empty closed subset contains a closed subset that is minimal. Further, if A and B are two finite closed subsets, then
(i) if $A \cap B = \emptyset$ and $A \cup B$ is included in a connected open subset E, then there exist two connected open sets $C \supseteq A$ and $D \supseteq B$ such that $C \cap D = \emptyset$ and $C \cup D \subseteq E$; and
(ii) if there exists a bijection between A and B, then there exists a closed set C such that $A \cup B \subseteq C$ and every component D of C contains exactly one point of A and one point of B. (Recall that a component is the union of all connected subsets of C containing a given element of C.)

We now prove Theorem 11.1.

Proof: We first establish the result when $n = 3$.

By the Feferman-Vaught Theorem and the proof of Theorem 9.1, it is enough to prove the result for $\mathcal{Z}(f)$ a single connected component which is a "disc" (if $\mathcal{Z}(f)$ is the entire 2-sphere, then $FA\ell(3)/\langle f\rangle_{cl} \cong FA\ell(3)$, which has undecidable theory [6]; and if $\mathcal{Z}(f)$ is not the entire 2-sphere, we can recognise a triangle ($\sim_\ell$ disc) in it.) We therefore assume that $\mathcal{Z}(f)$ is the closed upper half sphere $S^{(2)}_+$.

We wish to use the same technique as in [6]. The difficulty is that $\mathcal{Z}(f)$ no longer satisfies Grzegorczyk's conditions as condition (i) fails for A, B two element subsets of the equator each intersecting every equatorial arc containing the other. So we need to slightly modify our proof from [6].

Consider a latitudinal circle L in C which is disjoint from the equator. This is recognisable using $\iota_c(x)$ (see Section 9.4). Although we have been unable to recognise L from its union with a finite set of points and arcs in $S^{(2)}_+$ (the arcs all lying on the equator), the $\mathcal{L}$-formula does provide elements $g_1^{\#}$ and $g_2^{\#}$ whose supports are respectively the points in $S^{(2)}_+$ above L and those below. Now the analysis given in the proof of (II) and (III) of Proposition 9.21 applies: given 4 points on L we can always join them in pairs (in $S^{(2)}_+$) without crossing, and we can convert this into an $\mathcal{L}$-formula as in the proof of Proposition 9.21. The proof applies not just to a_i with the closure of their supports containing points of L but also to arbitrary $a_i \leq g_1^{\#}$ (satisfying the hypotheses in that proof) whose supports are contained in $\mathcal{S}(g_1^{\#})$, provided we let one of h_1 and h_2 have support intersecting $\mathcal{S}(g_2^{\#}) \subseteq \mathcal{Z}(f)$. However, it fails for $\mathcal{S}(g_2^{\#})$ if we take the a_i to have supports whose closure includes equatorial points. Thus we can distinguish $g_1^{\#}$ from $g_2^{\#}$ in $\mathcal{L}$. Now $\mathcal{S}(g_1^{\#}) \subseteq \mathcal{Z}(f)$ is the set of points in $S^{(2)}_+$ above L (modulo a finite set of points). It satisfies *all* of Grzegorczyk's conditions *when viewed inside* $\mathcal{Z}(f)$ provided that we consider the open subsets of the whole upper half sphere.

We can interpret this sublattice of zero-sets using the dimension formulae $\psi_{3,j}$ (see Lemma 10.2). By Grzegorczyk's result, it follows that $FA\ell(3)/\langle f\rangle_{cl}$ has undecidable theory. Thus the theorem holds if $n = 3$ and $Z(f)$ has dimension at least 3.

We now extend the analysis when $n \geq 4$. Again, by the Feferman-Vaught Theorem, we may assume that $\mathcal{Z}(f)$ is connected. If Grzegorczyk's condition (i) holds, all his conditions are satisfied and the result is proved. We therefore assume that his condition (i) fails in $\mathcal{Z}(f)$.

The analysis of formula $\sigma(g, g_1, g_2, k)$ given in Section 9.4 applies for $n \geq 4$ with minor variations. Firstly, the conditions imply that $\mathcal{Z}(g) \cap \mathcal{Z}(f)$ cannot have relative maximal dimension ℓ in any basic constituent of dimension ℓ, and that $\mathcal{S}(g) \cap \mathcal{Z}(f)$ is not the union of two disjoint sets each of the form $\mathcal{S}(g') \cap \mathcal{Z}(f)$ for some $g' \in FA\ell(n)_+$. Furthermore, by (iii), (iv) and (v), $\mathcal{Z}(g) \cap \mathcal{Z}(f)$ is the intersection of two zero sets (with disjoint non-empty interiors), the intersection of either with a basic constituent of $\mathcal{Z}(f)$ being either empty or of the dimension of that basic constituent. Again, by (vii), $\mathcal{Z}(g) \cap \mathcal{Z}(f)$ is not a single point and the same is true when this is intersected with any connected $\mathcal{S}(k)$ that is not disjoint from the support of both $g_1^{\#}$ and $g_2^{\#}$. Thus $\mathcal{Z}(g) \cap \mathcal{Z}(f)$ is a finite union of sets (each of dimension at least 1) and each d dimensional constituent lying in a d'-dimensional subset of $\mathcal{Z}(f)$ with $d' > d$. By (vi), $\mathcal{S}(k)$ is connected as is its restriction to $\mathcal{S}(g_1^{\#})$ and $\mathcal{S}(g_2^{\#})$. So $\mathcal{S}(k)$ cannot "pinch" $\mathcal{Z}(g)$ if $\mathcal{Z}(g)$ is a single connected set. The condition also ensures that $\mathcal{S}(k)$ cannot be "essentially" shrunk.

Next consider the formula $\iota_c(g)$ from the proof of Proposition 9.20 in Section 9.4. The support of any element k which features in the formula has dimension greater than that of $\mathcal{Z}(g)$. If $dim(\mathcal{Z}(g)) > 1$, then $dim(\mathcal{S}(k)) \geq 3$. Hence there is $h \geq a_1 \vee a_2$ with $\theta_S(h \wedge k)$ and $h \perp a_3$ as we have the necessary "degrees of freedom" to move inside k and avoid a_3 when the $\mathcal{S}(a_j)$ are balls of sufficiently small radius inside $\mathcal{S}(k)$. This contradicts the conjunct $\iota_1(g, f)$ of $\iota_c(g, f)$. Therefore, $dim(\mathcal{Z}(g)) = 1$.

Since $\mathcal{Z}(g)$ cuts $\mathcal{Z}(f)$, it follows that $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ includes an "interior circle" Γ inside a basic constituent $\mathcal{Z}(f')$ of $\mathcal{Z}(f)$ (where $dim(\mathcal{Z}(f')) = 2$); indeed, $\mathcal{Z}(g) \cap \mathcal{Z}(f')$ is Γ together with, possibly, a finite set of arcs and points, all arcs being contained in the boundary of $\mathcal{Z}(f')$. We may again distinguish $g_1^{\#}$ from $g_2^{\#}$ and assume that $\mathcal{S}(g_1^{\#})$

is the set of points inside Γ (modulo a finite set of points). We can again relativise and specialise, just as in the $n = 3$ case. Hence the proof of the undecidability of $Th(FA\ell(3)/\langle f\rangle_{cl})$ when $dim(\mathcal{Z}(f)) = 2$ applies equally to $Th(FA\ell(n)/\langle f\rangle_{cl})$. This completes the proof of the theorem. □

12. MV-ALGEBRAS

Let $n \in \mathbb{Z}_+$ and f be a piecewise linear continuous function from $[0, 1]^n$ to $[0, 1]$; *i.e.*, there is a finite cover $X_1, \dots, X_m$ of $[0, 1]^n$ into closed connected subsets with pairwise dijoint interiors and linear functions $f_1, \cdots, f_m$, such that $f(x) = f_j(x)$ for all $x \in X_j$ $(j = 1, \dots, m)$. Note that each f_j has the form: $\sum_{1\leq i\leq n} a_i\pi_i + b_i$, where $\pi_i(x_1, \cdots, x_n) = x_i$ and $a_i, b_i \in \mathbb{Z}$. This set of functions, $\mathcal{M}c_n$ (the set of *McNaughton functions*), is closed under the operations: $f^* = 1 - f$, $f_1 \oplus f_2 = min(1, f_1 + f_2)$, and $f_1.f_2 = max(0, f_1 + f_2 - 1)$. One can easily check that $\mathcal{M}c_n := (M, \oplus,^*, 0)$ is an MV-algebra (see Definition 1.1.1 in [4]). Indeed, $\mathcal{M}c_n$ is isomorphic to FMV_n, the free MV-algebra on n generators. (see Theorem 9.1.5 in [4].)

Let G be an abelian ℓ-group and $u \in G_+$. Then u is said to be a *strong order unit* of G if for all $g \in G$ there is $n = n(g) \in \mathbb{N}$ such that $g \leq nu$. In this case, let $[0,\ u]$ be the set $\{g \in G : 0 \leq g \leq u\}$. For $x, y \in G$, let $x \oplus y := u \wedge (x + y)$ and $x^* := u - x$. Then $\Gamma(G, u) := (G, \oplus,^*, 0)$ is an MV algebra (see Proposition 2.1.2 in [4]).

The application $\Gamma : (G, u) \to \Gamma(G, u)$ defines a functor from the category of abelian ℓ-groups with strong order units to the category of MV-algebras.

Theorem 12.1. (1) *FMV_1 is decidable.*
(2) *FMV_n with $n \geq 2$ is undecidable.*

Proof: (1) We can interpret FMV_1 in $FA\ell(2)$ which is decidable [6], the constant 1 being interpreted by $|\pi|_1 \vee |\pi_2|$.

(2) Let $\pi^{(n+1)} = (\pi_1 \vee 0) \vee \cdots \vee (\pi_{n+1} \vee 0)$. The theory of $FA\ell(n + 1)/\langle\pi^{(n+1)}\rangle_{cl}$ is undecidable by Theorem 11.1 and Lemma 10.2. The same proof shows that the theory of the ℓ-group $(FA\ell(n+1)/\langle\pi^{(n+1)}\rangle_{cl}, u+\langle\pi^{(n+1)}\rangle_{cl})$ with distinguished element $u + \langle\pi^{(n+1)}\rangle_{cl}$ has undecidable theory whenever u is an element with trivial zero-set in $FA\ell(n + 1)$. But FMV_n is bi-interpretable with $\Gamma(FA\ell(n + 1)/\langle\pi^{(n+1)}\rangle_{cl}, u + \langle\pi^{(n+1)}\rangle_{cl})$. Consequently, its theory is also undecidable. □

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