EXPONENTIATIONS OVER THE QUANTUM ALGEBRA $U_q(sl_2(\mathbb{C}))$

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Abstract. We define and compare, by model-theoretical methods, some exponentiations over the quantum algebra $U_q(sl_2(\mathbb{C}))$. We discuss two cases, according to whether the parameter q is a root of unity. We show that the universal enveloping algebra of $sl_2(\mathbb{C})$ embeds in a non-principal ultraproduct of $U_q(sl_2(\mathbb{C}))$, where q varies over the primitive roots of unity.

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1. Introduction

Exponentiation is a lively topic in modern model theory. It has been considered not only in the classical frameworks of real closed fields and the field of complex numbers, but also over larger settings such as Lie algebras. For instance, Macintyre's paper [10] develops a general picture of exponentiations over finite-dimensional Lie algebras over both the real and the complex fields. This led in [9] to the idea of defining exponential maps over an infinite-dimensional algebra, namely the universal enveloping algebra $U(sl_2(\mathbb{C}))$ of the Lie algebra $sl_2(\mathbb{C})$ of 2×2 traceless matrices with complex entries, using its irreducible finite-dimensional representations.

This also suggests to develop a similar analysis on the quantum algebra $U_q(sl_2(\mathbb{C}))$. We will introduce this algebra in more detail in the next Section 2. Quantum algebras are now beginning to be intensively investigated even under the model theoretic point of view. See for instance [3] where their simple representations are approached under this perspective. Moreover quantum algebras occur in the work of Boris Zilber [13] where they are associated to certain Zariski geometries. Recall that there are one dimensional Zariski geometries which are finite coverings of algebraic curves but not algebraic curves ([4]). In [13] Zilber calls them *non classical Zariski geometries* and, as said, connects them with some typical quantum algebras (when the parameter of deformation q is a root of unity). He just begins with the *simplest* case of $U_q(sl_2(\mathbb{C}))$ and builds a corresponding many-sorted structure $\tilde{V}(U_q(sl_2(\mathbb{C})))$ consisting of the complex field $\mathbb{C}$, a variety V and a bundle of $U_q(sl_2(\mathbb{C}))$-modules of fixed finite dimension (equal to the order of the root of unity) parametrized by V. He shows that the theory of finite-dimensional $U_q(sl_2(\mathbb{C}))$-modules is $\aleph_1$-categorical and model-complete. Moreover, he shows that $\tilde{V}(U_q(sl_2(\mathbb{C})))$ is a Zariski geometry that is not definable in any algebraically closed field.

In this paper we will still consider the algebras $U_q(sl_2(\mathbb{C}))$ where q is arbitrary (with only slight restrictions such as $q^2 \neq 1$) but we will deal with exponentiation. In fact, we will

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use the finite-dimensional representations of $U_q(sl_2(\mathbb{C}))$ and construct suitable exponential maps on it, following the approach of [9] for the universal enveloping algebra $U(sl_2(\mathbb{C}))$.

Here is the plan of the paper.

Sections 2, 3 and 4 are devoted to preliminaries on quantum algebras, exponential rings and exponentiation for matrices. However, in order to illustrate in more detail the remainder of the paper and our main results, let us fix right now some notation on these topics. For every positive integer λ let $M_\lambda(\mathbb{C})$ be the Lie algebra of $\lambda \times \lambda$ matrices with entries in the complex field. Then a matrix exponential map, taking its values in the linear group $GL_\lambda(\mathbb{C})$, can be introduced in $M_\lambda(\mathbb{C})$ in terms of infinite power series, putting for every matrix A

$$exp_\lambda(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!}.$$

Coming back to $U_q(sl_2(\mathbb{C}))$, we will distinguish whether the parameter q is a root of unity, or not.

The latter case is treated in Sections 5 and 6 (regarding the finite-dimensional representations of $U_q(sl_2(\mathbb{C}))$ and exponentiations on $U_q(sl_2(\mathbb{C}))$ respectively). It is known that, under this assumption on q, all finite-dimensional representations of $U_q(sl_2(\mathbb{C}))$ are semisimple, moreover the simple ones are classified in terms of highest weight and so are very similar to those of the classical case. Consequently various exponentiations over $U_q(sl_2(\mathbb{C}))$ can be defined by strategies very similar to the ones used in [9]. In fact, after recalling how simple finite-dimensional $U_q(sl_2(\mathbb{C}))$-modules are classified, we will use that and the exp_λ to define our *exponential maps* from $U_q(sl_2(\mathbb{C}))$ to $GL_\lambda(\mathbb{C})$ for every λ and we will explore the basic properties of these maps. After that, we will show how to embed $U_q(sl_2(\mathbb{C}))$ into an arbitrary non-principal ultraproduct of the $M_\lambda(\mathbb{C})$ with λ varying (see Proposition 6.3). This will lead us to introduce another exponential map from $U_q(sl_2(\mathbb{C}))$ to the corresponding non-principal ultraproduct of the groups $GL_\lambda(\mathbb{C})$. Again, we will investigate the basic properties of this function (see Proposition 6.1 and Corollary 6.4).

Sections 7, 8 and 9 treat the case when q is a root of unity. Again, they are devoted first to finite-dimensional representations and then to exponentiations. We define an exponential map from $U_q(sl_2(\mathbb{C}))$ to certain ultrapowers of the linear group $GL_\ell(\mathbb{C})$, where ℓ is the order of the root q if this order is odd or half of the order otherwise (and in any case is fixed). Indeed we have to carefully choose appropriate ultrafilters in order first to embed U_q in an ultrapower of $M_\ell(\mathbb{C})$ (see Proposition 8.2). As before we use the characterization of the simple finite-dimensional $U_q(sl_2(\mathbb{C}))$-modules. But this time the finite-dimensional representations of $U_q(sl_2(\mathbb{C}))$ are not necessarily semisimple (see [6, Remark after Proposition 2.12]) and there are further finite-dimensional representations in addition to the highest weight ones.

Finally in the last section, again using a suitable choice of the parameters, we *approximate* the universal enveloping algebra $U(sl_2(\mathbb{C}))$ by the quantum ones $U_q(sl_2(\mathbb{C}))$, where q ranges over a family of primitive roots of unity of strictly increasing order. Namely we show that $U(sl_2(\mathbb{C}))$ embeds in a certain non-principal ultraproduct of the $U_q(sl_2(\mathbb{C}))$'s.

We refer to [1] for basic model theory, including ultraproducts, to [11] for model theory of modules and to [6], [7] and [8] for quantum algebras.

2. Preliminaries on quantum algebras.

In this section, we will recall well known facts on quantum algebras over an arbitrary field k (not necessarily the complex field) and on skew polynomial rings. They can be found, for instance, in [6], [7] or [8].

Recall that the universal enveloping algebra $U := U(sl_2(k))$ of the 2×2 traceless matrices over k can be presented as the associative algebra with three generators

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

subject to the relations:

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = H.$$

Here $[\quad,\quad]$ denotes the usual commutator.

The algebra U can also be built as an iterated skew polynomial ring. Start with the algebra $A_0 = k[H]$ and consider

- the automorphism σ_0 of A_0 acting identically on k and sending H to $H + 2$,
- the derivation $\delta_0 = 0$ on A_0.

Using them one forms the skew polynomial ring $A_1 := A_0[Y;\, \sigma_0,\, \delta_0]$ (indeed $Y \cdot H = (H + 2)\cdot Y$). Now repeat the same construction with respect to A_1 and

- the automorphism σ_1 of A_1 fixing k pointwise and sending Y to Y and H to $H - 2$,
- the σ_1-derivation δ_1 of A_1 sending H to 0 and Y to H.

Then U is isomorphic to $A_2 := A_1[X;\, \sigma_1, \delta_1]$. In fact $X \cdot Y = Y \cdot X + H$ and $X \cdot H = (H - 2)\cdot X$.

Now let us introduce $U_q(sl_2(k))$. Recall that k is any field. Let q be an element of k such that $q \neq 0$ and $q^2 \neq 1$. Then, the *quantum algebra* $U_q := U_q(sl_2(k))$ is the associative k-algebra with generators K, K^{-1}, E, F and relations:

$$(1)\quad K\cdot K^{-1} = K^{-1}\cdot K = 1,\ K\cdot E\cdot K^{-1} = q^2 E,\ K\cdot F\cdot K^{-1} = q^{-2}F,\ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Note that these relations (1) imply by induction that, for every choice of integers $s,\ t \geq 2$,

$$[E, F^t] \;=\; [t]\, F^{t-1}\cdot\frac{Kq^{1-t} - K^{-1}q^{t-1}}{q - q^{-1}}, \tag{2}$$

$$[E^s, F] \;=\; [s]\, E^{s-1}\cdot\frac{Kq^{s-1} - K^{-1}q^{1-s}}{q - q^{-1}}. \tag{3}$$

Here, for every integer z, $[z]$ denotes the *q-number of* z, defined as:

$$[z] := \frac{q^z - q^{-z}}{q - q^{-1}}.$$

Alternatively the algebra U_q can be represented, just as U, as an iterated skew polynomial ring ([7, Proposition VI.1.4]). Namely, let $A_0 := k[K, K^{-1}]$ with

- the automorphism α_0 fixing k pointwise and sending K to q^2K,
- a zero derivation δ_0

and form the corresponding Ore extension $A_1 := A_0[F; \alpha_0, \delta_0]$ (observe $F\cdot K = q^2K\cdot F$). Then

- extend α_0 to an automorphism α_1 of A_1 by putting $\alpha_1(F^j\cdot K^l) = q^{-2l}\, F^j\cdot K^l$,

- define an α_1-derivation δ_1 on A_1 by $\delta_1(F) := \frac{K-K^{-1}}{q-q^{-1}}$, $\delta_1(K) = 0$ (see [7, Lemma VI.1.5]).

Finally, let $A_2 := A_1[E; \alpha_1, \delta]$ be the corresponding Ore extension. This is U_q up to isomorphism. In fact with the above notations we have:

Lemma 2.1. ([7], chapter VI.1) *U_q is a right (and left) Noetherian domain and the set $\{E^i \cdot K^l \cdot F^j \ : \ i, j \in \mathbb{N},\ l \in \mathbb{Z}\}$ is a basis of U_q over k.*

Proof: One way to prove the first part of the statement is to show that U_q is isomorphic to A_2 and to use properties of iterated skew polynomial rings (see [2] and [7, proof of Proposition VI.1.4]). See also [6, Theorem 1.5 and Proposition 1.8]. □

Moreover, one can put on the algebra U_q the following grading: $deg(E) = 1$, $deg(F) = -1$ and $deg(K) = deg(K^{-1}) = 0$.

For every integer m, let $U_{q,m}$ be the k-vector subspace of U_q generated by $\{E^i \cdot K^l \cdot F^j :$ $i - j = m,\ i, j \in \mathbb{N},\ l \in \mathbb{Z}\}$. It comes out that, as a vector space over k, U_q decomposes as $\oplus_{m\in\mathbb{Z}} U_{q,m}$ (see [6, 1.9]). For $u \in U_{q,m}$, $m \in \mathbb{Z}$, we have (see again [6, 1.9]):

$$K \cdot u \cdot K^{-1} = q^{2m}\, u. \tag{4}$$

whence the subring $U_{q,0}$ is equal to the centralizer of K, if q is not a root of unity.

In the general case, for q arbitrary, put

$$C_q := \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} + E \cdot F = F \cdot E + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}. \tag{5}$$

Then C_q is the so called *quantized Casimir element* of U_q. One easily checks that C_q commutes with K; further, using relations (1), one shows that C_q belongs to the center of U_q (see [7, Proposition VI.4.1]).

The following lemma is certainly well-known, but we could not find a precise reference (and we use it as stated in the next sections).

Lemma 2.2. *For any q, $U_{q,0}$ is equal to the polynomial ring $k[C_q, K, K^{-1}]$ and any element of $U_{q,m}$ can be written, for some suitable $u \in U_{q,0}$ as $E^m \cdot u$ when $m \geq 0$, and as $u \cdot F^{-m}$ when $m < 0$.*

Proof: Clearly K, K^{-1} and $E \cdot F$, hence C_q, are in $U_{q,0}$. Thus $k[C_q, K, K^{-1}] \subseteq U_{q,0}$. For the opposite inclusion, first observe that, by definition of C_q, $E \cdot F \in k[C_q, K, K^{-1}]$. This is trivially true also of K and K^{-1}. Therefore, in order to conclude our proof, it suffices to show that, if u is any element in $k[C_q, K, K^{-1}]$, then $E \cdot u \cdot F$ is also in $k[C_q, K, K^{-1}]$. Note that u can be represented as $K^{-d} \cdot p[C_q, K]$ for some suitable $p[x_1, x_2] \in k[x_1, x_2]$ and $d \in \mathbb{N}$. As C_q is in the center of U_q, we can assume $u = K^n$ or $u = K^{-n}$ for some $n \in \mathbb{N}$. By relation (4),

$$E \cdot K^n \cdot F = E \cdot K^n \cdot F \cdot K^{-n} \cdot K^n = q^{-2n} E \cdot F \cdot K^n$$

and similarly

$$E \cdot K^{-n} \cdot F = K^{-n} \cdot K^n \cdot E \cdot K^{-n} \cdot F = q^{2n} K^{-n} \cdot E \cdot F.$$

Thus in both cases $E \cdot u \cdot F \in k[C_q, K, K^{-1}]$. Moreover for every $u \in U_{q,0}$, there exist $u',\ u'' \in U_{q,0}$ such that $E \cdot u = u' \cdot E$ and $F \cdot u = u'' \cdot F$. □

To conclude this section let us state some facts about the center of U_q, just to say that, if q is not a root of unity, then it has dimension 1 over k and is generated by C_q (see [6, Proposition 2.18] or [7, Theorem VI.4.8]) while, if q is a primitive ℓ^{th} root of unity for some positive integer ℓ, then it is generated by E^ℓ, F^ℓ, K^ℓ, $K^{-\ell}$ and C_q (see [6, Proposition 2.20]).

3. Exponential rings and algebras.

We recall here the notions of exponential ring and exponential algebra (see [9, Definition 4.1]). Let us set up the various languages we will need.

- First, $\mathcal{L} := \{+, -, \cdot, 0, 1\}$ = the language of (associative) rings (with 1).
- Secondly the language $\mathcal{L}_g$ of groups.
- For the language of algebras over a field k, or more generally over a commutative ring, we choose the expansion $\mathcal{L}_{Alg}$ of $\mathcal{L}$, which is a two-sorted language with a sort for a ring k, a sort for an (associative) algebra A and a scalar multiplication map from $A \times k$ to A (both A and k are viewed as structures of $\mathcal{L}$).

Now let us consider a two-sorted structure (R, G, EXP) where R is an $\mathcal{L}$-structure, G is a $\mathcal{L}_g$-structure and EXP a map from R to G. The corresponding language, extending $\mathcal{L} \cup \mathcal{L}_g$ by a function symbol from the ring sort to the group sort for EXP, will be denoted by $\mathcal{L}_{EXP}$.

Definition 3.1. We will say that (R, G, EXP) is an *exponential ring* if R is an associative ring with 1, G is a (multiplicative) group and $EXP : R \to G$ satisfies the following axioms:

(1) $EXP(0) = 1$,
(2) $\forall x \in R$, $EXP(x) \cdot EXP(-x) = 1_G$ (= the identity element in the group G),
(3) $\forall x, y \in R$ with $x \cdot y = y \cdot x$, $EXP(x + y) = EXP(x) \cdot EXP(y)$)

(let us denote here in the same way, by the symbol $\cdot$, the multiplication operations of R and G).

When dealing with (exponential) k-algebras, we will use a language $\mathcal{L}_{Alg,EXP}$ extending $\mathcal{L}_{Alg} \cup \mathcal{L}_g$ just as $\mathcal{L}_{EXP}$ did before with respect to $\mathcal{L}$ and $\mathcal{L}_g$.

Definition 3.2. An $\mathcal{L}_{alg,EXP}$-structure (R, k, G, EXP) is an *exponential k-algebra* if

(1) the reduct (R, G, EXP) is an exponential ring,
(2) the reduct (R, k) is a k-algebra,
(3) $\forall c_1,\ c_2 \in k$, $\forall x \in R$, $EXP(c_1 x) \cdot EXP(c_2 x) = EXP((c_1 + c_2)x)$

(where again $\cdot$ denotes at the same time all the various involved multiplications).

Finally, for every ring R, let us denote by $\mathcal{L}_R$ the language of right R-modules, as described, for instance, in [11, page 3]. As said we refer to this book even for the basic model theory of modules, in particular for the definition of *pp-formula* (see [11, 2.1]).

Note that for the language of k-algebras, we could have chosen the one-sorted language $\mathcal{L}_k$ of k-modules instead of the two-sorted language $\mathcal{L}_{Alg}$; this could make a difference for instance when dealing with decidability issues.

4. Exponentiations and matrices.

For λ a positive integer, let $M_\lambda(\mathbb{C})$ be the ring of $\lambda \times \lambda$-matrices with coefficients in the complex field $\mathbb{C}$. It can be endowed with the Hermitian sesquilinear form $(\cdot, \cdot)$, defined by $(A, B) := \mathrm{tr}(B^* \cdot A) = \sum_{1 \le i,j \le \lambda} A(i, j) \cdot \bar{B}(i, j)$, for all A, $B \in M_\lambda(\mathbb{C})$ (where $tr(\cdot)$ denotes the trace, $(\cdot)^*$ the conjugate of transpose and $A(i,j)$, $B(i,j)$ the (i,j)-th entries of A, B respectively).

Let $\|\cdot\|$ be the norm induced by this form (usually called the Frobenius norm), hence for every A, we have $\|A\|^2 := (A, A)$.

For every λ, let exp_λ be the matrix exponential map from the algebra of matrices $M_\lambda(\mathbb{C})$ to the group of invertible matrices $GL_\lambda(\mathbb{C})$, which sends any $A \in M_\lambda(\mathbb{C})$ to the matrix exponential $\exp_\lambda(A)$, defined as the power series

$$\exp_\lambda(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{6}$$

Thus, if $\lambda = 1$, that is, A is a scalar a of $\mathbb{C}$, then $exp_1(A) = e^a$ is the ordinary exponential of the element a.

Using the terminology introduced in the previous section, $(M_\lambda(\mathbb{C}), \mathbb{C}, GL_\lambda(\mathbb{C}), exp_\lambda)$ is an exponential $\mathbb{C}$-algebra (see for instance [12]). As noted in [10], it is bi-interpretable with $(\mathbb{C}, x \to e^x)$.

It may be worth adding that a q-variant of the exponential map exp_λ can be also defined as an element of the formal power series ring $\mathbb{C}[[X]]$ (see [7, page 76]). The q-exponential is defined as the formal series

$$e_q(X) = \sum_{n=0}^{\infty} \frac{X^n}{(n)_q^!},$$

where $(0)_q^! = 1$ and $(n)_q^! = \frac{(q-1).\cdots.(q^n-1)}{(q-1)^n}$. Observe that the series is well-defined (provided that q is not a root of unity). The q-exponential is an invertible series, but in contrast with the ordinary exponential (that is, for $q = 1$), the equality $e_q(X)^{-1} = e_q(-X)$ fails. However, for any choice of variables X and Y such that $XY = qYX$, the fundamental property of the exponentials $e_q(X+Y) = e_q(X) \cdot e_q(Y)$ is satisfied.

Anyway, we will work with the matrix exponential defined by (6) in order to introduce, in the next sections, exponential maps over U_q by using its representation theory.

Observe that in [9] an exponential map was defined on the universal enveloping algebra $U(sl_2(\mathbb{C}))$ through its finite-dimensional representations. This was done by proving that there is an associative ring monomorphism from $U(sl_2(\mathbb{C}))$ to $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ where $\mathcal{V}$ is any non-principal ultrafilter over $\mathbb{N}$ (see [9, Corollary 8.2]).

Let us now indicate how a similar result can be obtained for the quantum algebra $U_q(sl_2(\mathbb{C}))$ working in the general context of Drinfeld-Jimbo algebras.

Let $\mathbb{C}[[h]]$ be the (topological) ring of all formal power series in the nonzero indeterminate h and complex coefficients and let $\mathbb{C}((h))$ denote its field of fractions. Let $U_h(sl_2(\mathbb{C}))$ be the Drinfeld-Jimbo algebra (see [7, XVII.4], or [8, Section 3.1.5]), namely the $\mathbb{C}[[h]]$-algebra generated by X, Y, H with $[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = \frac{e^{hH/2} - e^{-hH/2}}{e^{h/2} - e^{-h/2}}$. Notice that the first two relations are just the same as in $U(sl_2(\mathbb{C}))$.

Furthermore, there exists an isomorphism α of topological algebras, congruent to the identity modulo h, between $U_h(sl_2(\mathbb{C}))$ and the h-adic topological algebra $U(sl_2(\mathbb{C}))[[h]]$ ([7, XVII.2, Theorem XVIII.4.1]). Now we use the one-to-one correspondence between finite dimensional representations of $sl_2(\mathbb{C})$ and representations of $U_h(sl_2(\mathbb{C}))$ on $\mathbb{C}[[h]]$-vector spaces of the form $V[[h]]$, where V is a finite dimensional $\mathbb{C}$-vector space ([8, Proposition 7.10]) together with the embedding of $U(sl_2(\mathbb{C}))$ in $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ ([9]), which we extend by linearity, working now over $\mathbb{C}[[h]]$.

Finally we use, as shown in [7, Proposition XVII.4.1], the embedding i of the quantum algebra $U_q(sl_2(\mathbb{C}((h))))$ in $U_h(sl_2(\mathbb{C}))$ as a Hopf algebra, with

$$i(E) = X \cdot e^{hH/2},\ i(F) = e^{-hH/2} \cdot Y,\ i(K) = e^{hH/2},\ i(K^{-1}) = e^{-hH/2}.$$

In particular, we get an embedding of $U_q(sl_2(\mathbb{C}))$, regardless of whether q is a root of unity, into $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C}[[h]])$.

In the next sections, according to whether q is a root of unity, we will embed U_q in an ultraproduct of matrix rings over $\mathbb{C}$, the sizes of the matrix rings going to infinity when q is not a root of unity, and otherwise with fixed size depending on the order of the root of unity.

5. Finite-dimensional representations of U_q, for q not a root of unity.

This section deals with the finite-dimensional representations of the U_q. As explained at the beginning of Chapter 2 in [6], it is advisable to divide the analysis according to whether q is or not a root of unity. In the current section we assume that it is not a root of unity and k is an algebraically closed field of characteristic different from 2.

Every finite-dimensional representation of U_q decomposes as a direct sum of simple U_q-modules ([6, Theorem 2.9 and Proposition 2.3]). Moreover, for every positive integer λ, there exist (up to isomorphism) exactly two simple modules of dimension $\lambda+1$ as k-vector spaces. They will be denoted by $V_{\epsilon,\lambda}$, with $\epsilon \in \{-1, 1\}$ (warning: recall that their dimension over k is $\lambda + 1$).

First, let us describe $V_{1,\lambda}$; it has a basis $\{v_0, v_1, \dots, v_\lambda\}$ on which the generators E, F, K act as follows (see [7, Theorem VI.3.5]):

(7)
$$Ev_i = \begin{cases} [\lambda - i + 1]v_{i-1}, & \text{if } i = 1, \dots, \lambda \\ 0, & \text{if } i = 0, \end{cases} \qquad Fv_i = \begin{cases} [i+1]v_{i+1}, & \text{if } i = 0, \dots, \lambda - 1, \\ 0, & \text{if } i = \lambda, \end{cases}$$

(8)
$$K\, v_i \;=\; q^{\lambda-2i}\, v_i \qquad\qquad i = 0, \dots, \lambda.$$

In particular, E annihilates v_0 and F the vector v_λ, and up to the scalar multiplication these are the only vectors with these properties. So, $V_{1,\lambda}$ is an irreducible representation of U_q. Furthermore, on $V_{1,\lambda}$, the quantized Casimir element C_q acts as the scalar multiplication by $\frac{q^{\lambda-1}+q^{1-\lambda}}{(q-q^{-1})^2}$.

The other simple representation $V_{-1,\lambda}$ of dimension $\lambda + 1$ is obtained by composing the action of U_q on $V_{1,\lambda}$ with the automorphism σ of U_q determined by

$$\sigma(E) = -E, \quad \sigma(F) = F, \quad \sigma(K) = -K$$

(see [6, §5.2]); note that σ maps C_q to $-C_q$. For this reason we will denote the module $V_{-1,\lambda}$ also by $V^{\sigma}_{1,\lambda}$.

For every $\epsilon = \pm 1$ and $i = 0, 1, \dots, \lambda$ let $V^i_{\epsilon,\lambda}$ be the eigenspace of K with eigenvalue $\epsilon q^{\lambda-2i}$, namely $\{v \in V_{\epsilon,\lambda} : Kv = \epsilon q^{\lambda-2i} v\}$. Thus $V_{\epsilon,\lambda} = \oplus_{0\le i\le\lambda} V^i_{\epsilon,\lambda}$.

Furthermore, given ϵ and λ, let $\Theta_{\epsilon,\lambda}$ denote the representation map of U_q into $M_{\lambda+1}(k)$ (viewed as $\mathrm{End}(V_{\epsilon,\lambda+1})$) with respect to the basis $\{v_0, v_1, \dots, v_\lambda\}$. Then it is easily seen that the actions of the generators E, F, K and the central element C_q according to $\Theta_{\epsilon,\lambda}$ are described by the matrices denoted respectively as $E_{\epsilon,\lambda} := \Theta_{\epsilon,\lambda}(E)$, $F_{\epsilon,\lambda} := \Theta_{\epsilon,\lambda}(F)$, $K_{\epsilon,\lambda} := \Theta_{\epsilon,\lambda}(K)$ and $C_{q,\epsilon,\lambda} := \Theta_{\epsilon,\lambda}(C_q)$ where

$$E_{\epsilon,\lambda} = \epsilon \begin{pmatrix} 0 & [\lambda] & 0\dots & 0 \\ 0 & 0 & [\lambda-1]\dots & 0 \\ \vdots & \vdots & & [1] \\ 0 & 0 & 0\dots & 0 \end{pmatrix}, \quad F_{\epsilon,\lambda} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & [2] & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & [\lambda] & 0 \end{pmatrix} \tag{9}$$

$$K_{\epsilon,\lambda} = \epsilon \operatorname{diag}(q^{\lambda}, q^{\lambda-2}, \dots, q^{-\lambda+2}, q^{-\lambda}),$$
$$C_{q,\epsilon,\lambda} = \epsilon \operatorname{diag}\Big(\frac{q^{\lambda-1}+q^{1-\lambda}}{(q-q^{-1})^2}, \dots, \frac{q^{\lambda-1}+q^{1-\lambda}}{(q-q^{-1})^2}\Big).$$

According to the definition at the beginning of page 82 in [3], a pp-formula $\varphi(v)$ of the language L_{U_q} of modules over U_q is called *uniformly bounded* if and only if there is a positive integer $n(\varphi)$, depending only on ϕ, such that every finite-dimensional simple representation $V_{\epsilon,\lambda}$ of U_q has a dimension $\leq n(\varphi)$ as a vector space over k. The next proposition shows that for any $r \in U_{q,0}$, the formula $\phi(v) := r \cdot v = 0$ defining the annihilator of r is uniformly bounded.

Proposition 5.1. *Let $\epsilon = \pm 1$, λ be a positive integer, $r \in U_{q,0} - \{0\}$. Then the dimension of the kernel of $\Theta_{\epsilon,\lambda}(r)$ in $V_{\epsilon,\lambda}$ is bounded independently of λ.*

Proof: Fix ϵ. Recall that, when λ ranges over positive integers, $V_{\epsilon,\lambda}$ is the direct sum of the (one dimensional) eigenspaces $V^i_{\epsilon,\lambda}$ $(0 \leq i \leq \lambda)$ of K. We will show that:

(*) for every λ, the number of i, $0 \leq i \leq \lambda$, such that r annihilates $V^i_{\epsilon,\lambda}$ has an upper bound b only depending on r

Suppose that this is true. Then it is easily seen that b is the bound in the statement of the proposition.

In order to show (*), we will first show that there are only finitely many λ such that r annihilates the whole $V_{\epsilon,\lambda}$).

Let us first represent r as $K^{-n} \cdot p(C_q, K)$ for some suitable non zero polynomial $p(x_1, x_2) \in k[x_1, x_2]$ and $n \in \mathbb{N}$. Write $p(x_1, x_2) = \sum_{j=0}^{d} p_j(x_1)\, x_2^j$, where d is the degree of p with respect to x_2. For every $j \leq d$, let d_j be the degree of p_j.

Observe that, for $0 \leq i \leq \lambda$, $rv_i = \epsilon^{-n}\, q^{-n(\lambda-2i)}\, p(\frac{q^{-1}(\epsilon q^{\lambda})+q(\epsilon q^{\lambda})^{-1}}{(q-q^{-1})^2}, \epsilon q^{\lambda-2i})\, v_i$, whence $rv_i = 0$ holds (equivalently, r annihilates $V^i_{\epsilon,\lambda}$) if and only if $p(\frac{q^{-1}(\epsilon q^{\lambda})+q(\epsilon q^{\lambda})^{-1}}{(q-q^{-1})^2}, \epsilon q^{\lambda-2i}) = 0$. We claim that, for a given j, $p_j(\frac{q^{-1}(\epsilon q^{\lambda})+q(\epsilon q^{\lambda})^{-1}}{(q-q^{-1})^2}) = 0$ holds for at most d_j values of λ.

In fact p_j has at most d_j roots in k. So let us compute the number of (ϵ, λ) such that

$$\frac{q^{-1}(\epsilon q^{\lambda})+q(\epsilon q^{\lambda})^{-1}}{(q-q^{-1})^2} = \frac{q^{-1}\epsilon}{(q-q^{-1})^2}\,(q^{\lambda}+q^2 q^{-\lambda})$$

equals one of these roots. We claim that, for any given root, this number is at most 1. We follow here the argument in [6, Lemma 2.8]. Suppose that, for some $\lambda_1 \neq \lambda_2 \in \mathbb{N} - \{0\}$,

$$\frac{q^{-1}\epsilon}{(q-q^{-1})^2}\,(q^{\lambda_1}+q^2 q^{-\lambda_1}) = \frac{q^{-1}\epsilon}{(q-q^{-1})^2}\,(q^{\lambda_2}+q^2 q^{-\lambda_2}).$$

Then $q^{\lambda_1}+q^2q^{-\lambda_1} = q^{\lambda_2}+q^2q^{-\lambda_2}$. Namely, $q^{\lambda_1+\lambda_2}(q^{\lambda_1}-q^{\lambda_2}) = q^2(q^{\lambda_1}-q^{\lambda_2})$. So, $q^{\lambda_1+\lambda_2-2} = 1$. As q is not a root of unity, $\lambda_1+\lambda_2 = 2$. Since these are strictly positive numbers, we obtain $\lambda_1 = \lambda_2 = 1$ – a contradiction.

This confirms the upper bound d_j.

Now we can show our claim $(*)$. In fact, for a given λ, r annihilates $V_{\epsilon,\lambda}$ (i.e., all the $V^i_{\epsilon,\lambda}$) if and only if $p_j(\frac{q^{-1}(\epsilon q^\lambda)+q(\epsilon q^\lambda)^{-1}}{(q-q^{-1})^2}) = 0$ for all $j \leq d$. But only finitely many λ can satisfy all these conditions – actually their number cannot exceed the minimum of the d_j ($j \leq d$). In other words, there are only finitely many λ such that r annihilates the whole $V_{\epsilon,\lambda}$.

So let us restrict our attention to the remaining λ, those such that

$$p(\frac{q^{-1}(\epsilon q^\lambda) + q(\epsilon q^\lambda)^{-1}}{(q-q^{-1})^2}, x_2) \neq 0.$$

This polynomial (in x_2) admits at most d roots in k. Fix one of them. As q is not a root of unity, given λ, there is at most one $i \leq \lambda$ such that $\epsilon q^{\lambda-2i}$ can equal it. Thus the number of these i is at most d. This shows (*) and concludes our proof. □

Another uniform way to approach the simple finite-dimensional representations of U_q is via the quantum plane $k[x_1, x_2]_q$ ([3]). This is defined as the quotient of the free k-algebra generated by x_1 and x_2 by the ideal spanned by $x_2x_1 - qx_1x_2$ ([7, IV.1]). So a basis over k is given by the products $x_1^i x_2^j$ (i, $j \in \mathbb{N}$) with the commutation rule $x_2^j x_1^i = q^{ij} x_1^i x_2^j$. For every non negative integer λ let $k[x_1, x_2]_{q,\lambda}$ be the k-vector subspace of the quantum plane generated by the homogeneous elements of degree λ, then $k[x_1, x_2]_q = \oplus_{\lambda\in\mathbb{N}} k[x_1, x_2]_{q,\lambda}$ over k. The U_q-module structure on the quantum plane is given by the following actions of K, E and F:

$$Kx_1^i x_2^j = q^{i-j} x_1^i x_2^j, \quad Ex_1^i x_2^j = [i] x_1^{i-1} x_2^{j+1}, \quad Fx_1^i x_2^j = [j] x_1^{i+1} x_2^{j-1}.$$

But U_q could act on the quantum plane even through σ, that is, in the following way: first send U_q to $\sigma(U_q)$ and then let it act on $k[x_1, x_2]_q$ as described before. Let $k[x_1, x_2]_{q,\sigma}$ denote the quantum plane with this U_q-module structure.

Observe that both these U_q-module actions preserve the degrees of monomials. Then for every λ let $k[x_1, x_2]_{q,\sigma,\lambda}$ denote the submodule generated by the monomials of degree λ in $k[x_1, x_2]_{q,\sigma}$. The simple finite-dimensional U_q-modules $V_{\epsilon,\lambda}$ are isomorphic to either

- $k[x_1, x_2]_{q,\lambda}$ (when $\epsilon = 1$), or
- $k[x_1, x_2]_{q,\sigma,\lambda}$ (when $\epsilon = -1$).

Now consider a non principal ultrafilter $\mathcal{W}$ on $\mathbb{N}$. Fix $\epsilon = \pm 1$. For every λ we have defined a representation map $\Theta_{\epsilon,\lambda}$ from U_q into $M_{\lambda+1}(k)$. Let $[(\Theta_{\epsilon,\lambda})_\lambda]_{\mathcal{W}}$ denote the corresponding map from U_q to $\prod_{\mathcal{W}} M_{\lambda+1}(k)$. It is an associative ring morphism.

Proposition 5.2. *For every non-principal ultrafilter $\mathcal{W}$ on $\mathbb{N}$,*

$$[(\Theta_{\epsilon,\lambda})_\lambda]_{\mathcal{W}} : U_q \to \prod_{\mathcal{W}} M_{\lambda+1}(k)$$

is an injective map.

Proof: We proceed as in [9], using Lemma 2.2 and the above discussion. Any element r of U_q can be written as $\sum_{m=-M}^{-1} F^{-m} r_m + \sum_{z=0}^{M} r_m E^m$ where M is a suitable positive integer and the r_m ($-M \leq m \leq M$) are in $U_{q,0}$. Assume $r \neq 0$, then $r_m \neq 0$ for some m. By Proposition 5.1, there is a bound $\tilde{b}$ such that for all $-M \leq m \leq M$, if $r_m \neq 0$, then $\Theta_{\epsilon,\lambda}(r_m) \neq 0$ for all $\lambda \geq \tilde{b}$. On the other hand, for $\lambda \geq M$, it follows from the definition of $\Theta_{\epsilon,\lambda}$ that, if $\Theta_{\epsilon,\lambda}(r_m) \neq 0$ for some m, then $\Theta_{\epsilon,\lambda}(r) \neq 0$. Therefore $[(\Theta_{\epsilon,\lambda})_\lambda]_{\mathcal{W}}(r)$ is not zero in $\prod_{\mathcal{W}} M_{\lambda+1}(k)$. □

Another way to proceed is to use a Peter-Weyl density theorem. Assume here that q is a transcendental complex number. Let $\mathcal{O}(SL_2)$ be the coordinate algebra of the quantum group $SL_q(2)$ ([8, Definition 4.2]). Let $\mathcal{C}(T^R_\ell)$ be the linear span of matrix elements $t^{(\ell)}_{ij}$, $-\ell \leq i, j \leq \ell$ ([8, 4.2.5]). Then the Hopf algebra $\mathcal{O}(SL_q(2))$ is a direct sum of subcoalgebras $\mathcal{C}(T^R_\ell)$, $\ell \in \frac{1}{2}\mathbb{N} - \{0\}$ (according to [8] and the Peter-Weyl direct sum decomposition). There is a nondegenerate dual pairing $\langle\, . \,,\, . \,\rangle$ between $\mathcal{O}(SL_q(2))$ and $U_q := U_q(sl_2)$ ([8, 4.4.2, 11.2.3]). Let $f \in U_q - \{0\}$, then there exists $a \in \mathcal{O}(SL_q(2))$ such that $\langle f, a\rangle \neq 0$. So there exists $t^{(\ell)}_{ij}$ such that $t^{(\ell)}_{ij}(f) \neq 0$ ([8, Corollary 11.23]). So we send f to the sequence whose $(\lambda+1)$-th element, $2\ell \leq \lambda$, is equal to a block diagonal matrix whose $(2\ell+1) \times (2\ell+1)$-diagonal block is equal to $t^{(\ell)}_{ij}(f)$ and then to the identity matrix on the other diagonal block and zeros elsewhere, and the remaining $\lambda < 2\ell$ elements of the sequence are equal to the identity matrix. Finally we send that sequence to its equivalence class modulo the ultrafilter $\mathcal{W}$.

6. The exponential maps on U_q, q not a root of unity.

In this section we set $k = \mathbb{C}$ (actually we just need a field endowed with a norm and complete for the induced topology). For λ a non negative integer and $\epsilon = \pm 1$, define an exponential map $EXP_{\epsilon,\lambda}$ from U_q into $G_{\lambda+1}(\mathbb{C})$ by composing the (matrix) exponential map $exp_{\lambda+1}$ on $M_{\lambda+1}(\mathbb{C})$ with $\Theta_{\epsilon,\lambda}$, hence by putting, for every $u \in U_q$, $EXP_{\epsilon,\lambda}(u) := exp_{\lambda+1}(\Theta_{\epsilon,\lambda}(u))$.

For instance,

(1) $EXP_{\epsilon,\lambda}(E) = exp_{\lambda+1}(\Theta_{\epsilon,\lambda}(E)) = exp_{\lambda+1}(E_{\epsilon,\lambda})$,
(2) $EXP_{\epsilon,\lambda}(F) = exp_{\lambda+1}(\Theta_{\epsilon,\lambda}(F)) = exp_{\lambda+1}(F_{\epsilon,\lambda})$,
(3) $EXP_{\epsilon,\lambda}(K) = exp_{\lambda+1}(\Theta_{\epsilon,\lambda}(K)) = \operatorname{diag}\left(e^{\epsilon q^{\lambda}}, e^{\epsilon q^{\lambda-2}}, \ldots, e^{\epsilon q^{-\lambda+2}}, e^{\epsilon q^{-\lambda}}\right)$,
(4) $EXP_{\epsilon,\lambda}(C_q) = exp_{\lambda+1}(\Theta_{\epsilon,\lambda}(C_q)) = e^{\frac{q^{-1}(\epsilon.q^{\lambda})+q.(\epsilon.q^{\lambda})^{-1}}{(q-q^{-1})^2}} I_{\lambda+1}$

where $I_{\lambda+1}$ denotes the identity matrix in $G_{\lambda+1}(\mathbb{C})$.

We get a transfer of the properties of the classical matrix exponential to this new map, as follows (0_{U_q} denotes here the zero element in U_q).

Proposition 6.1. *Let u, $v \in U_q$ and a, $b \in \mathbb{C}$. Then for every $\lambda \in \mathbb{N} - \{0\}$:*

(i) $EXP_{\epsilon,\lambda}(0_{U_q}) = I_{\lambda+1}$.
(ii) $EXP_{\epsilon,\lambda}(au) \cdot EXP_{\epsilon,\lambda}(bu) = EXP_{\epsilon,\lambda}((a+b)u)$;
(iii) $EXP_{\epsilon,\lambda}(u) \cdot EXP_{\epsilon,\lambda}(-u) = I_{\lambda+1}$;
(iv) *for u and v commuting,* $EXP_{\epsilon,\lambda}(u+v) = EXP_{\epsilon,\lambda}(u) \cdot EXP_{\epsilon,\lambda}(v)$;
(v) *for an invertible element v in U_q,* $EXP_{\epsilon,\lambda}(v \cdot u \cdot v^{-1}) = \Theta_{\epsilon,\lambda}(v) \cdot EXP_{\epsilon,\lambda}(u) \cdot \Theta_{\epsilon,\lambda}(v)^{-1}$.

In particular $(U_q, \mathbb{C}, GL_{\lambda+1}(\mathbb{C}), EXP_{\epsilon,\lambda})$ is an exponential $\mathbb{C}$-algebra.

As in [9, Proposition 7.2], one also obtains the following result.

Proposition 6.2. *For every non negative integer λ, the map $EXP_{\epsilon,\lambda}$ is surjective.*

Proof. Since $exp_{\lambda+1}$ is surjective from $M_{\lambda+1}(\mathbb{C})$ to $GL_{\lambda+1}(\mathbb{C})$, it suffices to prove that $\Theta_{\epsilon,\lambda} : U_q \to M_{\lambda+1}(\mathbb{C})$ is surjective. The latter is deduced directly by Jacobson density theorem [5, Section 2.2]. □

Now let $\mathcal{W}$ be a non principal ultrafilter on $\mathbb{N}$. Let $exp_{\mathcal{W}}$ denote the map $[(exp_{\lambda+1})_\lambda]_{\mathcal{W}}$ (where now λ is ranging over $\mathbb{N}$). Then $(\prod_{\mathcal{W}} M_{\lambda+1}(\mathbb{C}), \prod_{\mathcal{W}} GL_{\lambda+1}(\mathbb{C}), exp_{\mathcal{W}})$ is an exponential ring ([9, Proposition 5.1]). By Proposition 3.2, we may view U_q as a $\mathbb{C}$-subalgebra of $\prod_{\mathcal{W}} M_{\lambda+1}(\mathbb{C})$. Now we endow it with an exponential function as follows. For $\epsilon = \pm 1$, define $EXP_{\mathcal{W}}$ from U_q to $\prod_{\mathcal{W}} GL_{\lambda+1}(\mathbb{C})$ by putting, for every $u \in U_q$,

$$EXP_{\mathcal{W}}(u) = [(EXP_{\epsilon,\lambda}(u))_\lambda]_{\mathcal{W}}.$$

Corollary 6.3. *The algebra $(U_q, \mathbb{C}, \prod_{\mathcal{W}} GL_{\lambda+1}(\mathbb{C}), EXP_{\mathcal{W}})$ is an exponential $\mathbb{C}$-algebra.*

Proof: Apply Proposition 5.2 and Łos' Theorem (see [1, Theorem 4.1.9]). □

7. Finite-dimensional representations of U_q, for q a root of unity.

In this section, we will assume that q is a primitive ℓ^{th} root of unity for $\ell \geq 3$ and that k is algebraically closed. Incidentally, notice that, for $k = \mathbb{C}$ and $1 \leq i \leq \ell$, the complex conjugate $\overline{q^i}$ of q^i equals $q^{\ell-i}$, whence $\overline{[i]} = [i] = -[\ell - i]$.

As observed in [6, page 23] we can restrict our analysis to the case ℓ odd – in fact, when $\ell = 2\ell'$ is even one can replace ℓ by ℓ'. Then all but finitely many simple finite-dimensional representations of U_q are of dimension ℓ ([7, Propositions VI.5.1 and VI.5.2]). Let us describe two classes of representations of dimension ℓ over k. As ℓ is fixed we will omit any explicit reference to it in indexing them.

Case 1. Let a, b, $c \in k$, $c \neq 0$, $c^2 \neq 1$. Then $V_{a,b,c}$ will denote the representation of dimension ℓ over k on which E, F and K act in the way we are going to illustrate. To do that, first let us set for ease of notation:

- for $1 \leq i < \ell$, $e_i = e_i(a,b,c) := ab + [i]\frac{cq^{-i+1} - c^{-1}q^{i-1}}{q - q^{-1}}$,
- $e_\ell = e_\ell(a,b,c,) := a$,
- $e = \prod_{i=1}^{\ell} e_i$.

Then the actions of E, F and K on $V_{a,b,c}$ (viewed as a k-vector space of dimension ℓ) are given by the following $\ell \times \ell$ matrices $E_{a,b,c}$, F_b, K_c:

$$E_{a,b,c} = \begin{pmatrix} 0 & e_1 & 0\dots & 0 \\ 0 & 0 & e_2\dots & 0 \\ \vdots & \vdots & & e_{\ell-1} \\ e_\ell & 0 & 0\dots & 0 \end{pmatrix}, \tag{10}$$

$$F_b = \begin{pmatrix} 0 & 0 & \dots & b \\ 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{11}$$

$$K_c = c \operatorname{diag}\left(1, q^{-2}, \dots, q^{-2\ell+4}, q^{-2\ell+2}\right).$$

It follows that the action of the Casimir element C_q is represented by the $\ell \times \ell$ matrix

$$C_{q,a,b,c} = \operatorname{diag}\left(ab + \frac{cq + c^{-1}q^{-1}}{(q-q^{-1})^2}\right).$$

Note that the actions of respectively E, F, K and C either are cyclic permutations of one-dimensional subspaces, or leave these subspaces invariant.

Let $\Theta_{a,b,c}$ be the map from U_q to $M_\ell(k)$ sending E to $E_{a,b,c}$, F to F_b and K to K_c.

Case 2. Let d, f be non zero elements of k, with $f^2 \neq 1$. Then $\tilde{V}_{d,f}$ is the ℓ-dimensional representation where E, F and K act in the following way. For ease of notation, let us set

$f_i := [i]\frac{f^{-1}q^{-i+1}-fq^{i-1}}{q-q^{-1}}$ $(1 \leq i < \ell)$. Then the actions of E, F and K on $\tilde{V}_{d,f}$ are represented by the following $\ell \times \ell$ matrices $\tilde{E}_d$, $\tilde{F}_f$, $\tilde{K}_f$:

$$\tilde{E}_d = \begin{pmatrix} 0 & 0 & \dots & d \\ 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{12}$$

$$\tilde{F}_f = \begin{pmatrix} 0 & f_1 & 0\dots & 0 \\ 0 & 0 & f_2\dots & 0 \\ \vdots & \vdots & & f_{\ell-1} \\ 0 & 0 & 0\dots & 0 \end{pmatrix},$$

$$\tilde{K}_f \ = \ f \operatorname{diag}\left(1, q^2, \dots, q^{2\ell-4}, q^{2\ell-2}\right).$$

Then the Casimir element C_q is represented by the $\ell \times \ell$ matrix

$$\tilde{C}_{q,f} \ = \ \operatorname{diag}\left(\frac{fq^{-1}+f^{-1}q}{(q-q^{-1})^2}\right).$$

Note that the action of $\tilde{E}_d$ on an ℓ-dimensional space is a cyclic permutation of one-dimensional subspaces, whereas the action of $\tilde{F}_f$ is nilpotent.

We will denote by $\tilde{\Theta}_{d,f}$ the map from U_q to $M_\ell(k)$ sending E to $\tilde{E}_d$, F to $\tilde{F}_f$ and K to $\tilde{K}_f$.

Fact 7.1. ([7, Theorem VI.5.5] or 3.2 in [8]) Any simple U_q-module of dimension ℓ is isomorphic to either

(1) $V_{a,b,c}$ with $b \neq 0$, or
(2) $V_{a,0,c}$, with $c \neq \pm 1, \pm q, \cdots, \pm q^{\ell-2}$, or
(3) $\tilde{V}_{d,\pm q^{1-j}}$ for $1 \leq j < \ell$ and $d \neq 0$.

In the following we will refer to $k = \mathbb{C}$. We will use on one hand the family of representations $\Theta_{a,b,c}$ with a, b, c all non-zero and $c^2 \neq 1$ and on the other hand the family $\Theta_{d,f}$ with d, f all non-zero and $f^2 \neq 1$.

8. The exponential maps on U_q, q a root of unity.

In this section we assume $k = \mathbb{C}$, even though most of what we are going to say can be carried out just assuming that k is algebraically closed. Let q denote a primitive ℓ^{th}-root of unity, $\ell \geq 3$, making the same adjustment as in the previous section when ℓ is even (whence we can assume ℓ odd).

Let us put for simplicity from now on $\mathbb{N}^+ = \mathbb{N} - \{0\}$.

For every triple (a, b, c) and pair (d, f) in $\mathbb{C}$ (as described in the previous section), one can define exponential maps $EXP_{a,b,c}$ and $E\tilde{X}P_{d,f}$ from U_q to $G_\ell(\mathbb{C})$ by composing

- the matrix exponential map exp_ℓ from $M_\ell(\mathbb{C})$ to $GL_\ell(\mathbb{C})$ and
- $\Theta_{a,b,c}$ (respectively $\tilde{\Theta}_{d,f}$).

Thus, for every $u \in U_q$, $EXP_{a,b,c}(u) := exp_\ell(\Theta_{a,b,c}(u))$ and $E\tilde{X}P_{d,f}(u) := exp_\ell(\tilde{\Theta}_{d,f}(u))$.

Similarly to Proposition 6.1, we obtain that

$$(U_q, \mathbb{C}, GL_\ell(\mathbb{C}), EXP_{a,b,c}), \quad (U_q, \mathbb{C}, GL_\ell(\mathbb{C}), E\tilde{X}P_{d,f})$$

are exponential $\mathbb{C}$-algebras. Moreover, if the parameters (a, b, c) (respectively (d, f)) are chosen such that the corresponding module $V_{a,b,c}$ (respectively $\tilde{V}_{d,f}$) is simple, then the map $EXP_{a,b,c}$ (respectively $E\tilde{X}P_{d,f}$) is surjective (the argument is the same as the one used in Proposition 6.2).

Now, we will vary the maps $\Theta_{a,b,c}$ along certain non principal ultrafilters $\mathcal{W}$ on $\mathbb{N}^3$ in order to embed U_q into the corresponding non-principal ultrapower of $M_\ell(\mathbb{C})$. Notice once again that now ℓ is fixed, so it is the triple (a, b, c) to vary, ranging over a suitable setting we are going to describe. Basically we want to find sufficient conditions on a domain of variation for a, b, c in order to get, for every $u \neq 0$ in U_q, that

$$\Theta_{a,b,c}(u) \neq 0 \text{ for } \textit{sufficiently many } a,\ b,\ c \qquad (\star)$$

(we will make this statement precise later).

The case of pairs (d, f) will be considered in the next section. However, for the representations $\tilde{\Theta}_{d,f}$, we will only be able to show a statement similar to $(\star)$ for certain elements of $U_{q,0}$ (see Lemma 9.1).

First let us consider the case of an element $u \in U_{q,0} - \{0\}$. Then $u = K^{-n} \cdot p(C_q, K)$ for some $p(x_1, x_2) \in \mathbb{C}[x_1, x_2] - \{0\}$ and $n \in \mathbb{N}$. Let us write

$$p(x_1, x_2) = \sum_{j=0}^{N} s_j(x_2)\, x_1^j$$

with $N \in \mathbb{N}$ and the $s_j(x_2)$ in $\mathbb{C}[x_2]$. We may assume that $s_N(x_2) \neq 0$.

Recall that the matrix $\Theta_{a,b,c}(u)$ is a diagonal matrix whose $(i+1)^{th}$ entry on the diagonal, with $0 \leq i < \ell$, is equal to $c^{-n}q^{2ni} \cdot p(ab + \frac{cq+(cq)^{-1}}{(q-q^{-1})^2}, cq^{-2i})$ where $p(ab + \frac{cq+(cq)^{-1}}{(q-q^{-1})^2}, cq^{-2i}) = \sum_{j=0}^{N} s_j(cq^{-2i})\,(ab + \frac{cq+(cq)^{-1}}{(q-q^{-1})^2})^j$.

This suggests the following change of variables

$$x_1' = x_1 - \frac{x_2' + {x_2'}^{-1}}{(q-q^{-1})^2}, \quad x_2' = x_2 q^{2i+1},$$

that is,

$$x_1 = x_1' + \frac{x_2' + {x_2'}^{-1}}{(q-q^{-1})^2}, \quad x_2 = x_2' q^{-2i-1}.$$

Thus, when $(x_1, x_2) = (ab + \frac{cq+(cq)^{-1}}{(q-q^{-1})^2}, cq^{-2i})$, one has $(x_1', x_2') = (ab, cq)$. Observe that, after this change of variables, the polynomial $p(x_1, x_2)$ becomes a rational function $p'(x_1', x_2')$ of x_1' and x_2'. However $p'(x_1', x_2')$ can be written as a rational function $\sum_{j=0}^{N} t_j(x_2')(x_1')^j$ whose degree is still N and the coefficients $t_j(x_2')$ are rational functions of x_2' with the only pole 0. Moreover $t_N(x_2')$ is a nonzero polynomial in x_2', and indeed $t_N(x_2') = s_N(x_2 q^{2i+1})$.

Therefore, whenever $c \in \mathbb{C}$ satisfies $t_N(cq^{-2i-1})) \neq 0$, the polynomial $p'(x_1', cq)$ is non trivial and has at most N roots. So, for cofinitely many values of c this polynomial $p'(x_1', cq)$ is nonzero and for each of these values of c, for cofinitely many values of $r \in \mathbb{C}$,

$$p(r + \frac{cq + c^{-1}q^{-1}}{(q-q^{-1})^2}, c\, q^{-2i}) \neq 0.$$

It follows that, if $ab = r$, then $\Theta_{a,b,c}(u) \neq 0$ for cofinitely many values of c and, given such an element c, for cofinitely many values of r.

Let $S_c = \{c_n : n \in \mathbb{N}\}$, $S_r = \{r_n : n \in \mathbb{N}\}$ be countable subsets of pairwise distinct elements of $\mathbb{C}$. Assume also that, for every n, $c_n \neq 0$, $c_n^2 \neq 1$ and r_n has modulus bigger than 1. Next form a new set S_a consisting of complex number a_n ($n \in \mathbb{N}$) such that $|a_n| > |r_n| + n$ for all n. With any tuple $\bar{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$, associate the tuple $(c_{n_1}, r_{n_2}, a_{n_3}) \in S_c \times S_r \times S_a \subseteq \mathbb{C}^3$ and the representation $\Theta_{\bar{n}} := \Theta_{a_{n_3}, b_{\bar{n}}, c_{n_1}}$ with $b_{\bar{n}} := \frac{r_{n_2}}{a_{n_3}}$.

Now let us define a family of subsets of $\mathbb{N}^3$: $S_{N,\eta,\gamma} = \{(n_1, n_2, n_3) \in \mathbb{N}^3 : \ n_1 > N, \ n_2 > \eta(n_1), \ n_3 > \gamma(n_2)\}$, where $N \in \mathbb{N}$, $\eta, \gamma : \mathbb{N} \to \mathbb{N}$.

It is easily seen that this family of subsets has the finite intersection property. In fact, given two such sets S_{N_i,η_i,γ_i}, $1 \leq i \leq 2$, take $N = max\{N_1, N_2\}$, $\eta = max\{\eta_1, \eta_2\}$ and $\gamma = max\{\gamma_1, \gamma_2\}$, then $S_{N,\eta,\gamma} \subseteq \cap_{i=1}^2 S_{N_i,\eta_i,\gamma_i}$.

Let $\mathcal{W}$ be a non-principal ultrafilter on $\mathbb{N}^3$ containing these subsets $S_{N,\eta,\gamma}$ of $\mathbb{N}^3$ (see [1, Proposition 3.3.5]).

From the above discussion, we deduce the following.

Lemma 8.1. *For every $u \in U_{q,0} - \{0\}$, there exists $W_u \in \mathcal{W}$ such that for all $\bar{n} \in W_u$, $\Theta_{\bar{n}}(u) \neq 0$.*

Proof: Let $u = K^{-n} \cdot p(C_q, K)$ with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2] - \{0\}$, $n \in \mathbb{N}$. Given $\bar{n} = (n_1, n_2, n_3)$, $\Theta_{\bar{n}}(u)$ is a diagonal matrix whose $(i+1)^{th}$ entry on the diagonal ($0 \leq i < \ell$) is $c_{n_1}^{-n}\, q^{2ni}\, p(r_{n_2} + \frac{c_{n_1} q + c_{n_1}^{-1} q^{-1}}{(q-q^{-1})^2},\ c_{n_1}\, q^{-2i})$. So for cofinitely many values of $c_{n_1} \in S_c$, the rational function $p(x_1' + \frac{c_{n_1} q + c_{n_1}^{-1} q^{-1}}{(q-q^{-1})^2},\ c_{n_1}\, q^{-2i})$ is non trivial. Therefore for cofinitely many values of $n_2 \in S_r$, we get that $c_{n_1}^{-n}\, q^{2ni}\, p(r_{n_2} + \frac{c_{n_1} q + c_{n_1}^{-1} \cdot q^{-1}}{(q-q^{-1})^2},\ c_{n_1}\, q^{-2i}) \neq 0$, for any $0 \leq i < \ell$. □

Now we examine the general case.

Any element u of U_q can be written as a finite sum of the form

$$u_0 + \sum_{z \in \mathbb{N}^+} (F^z \cdot u_{-z} + E^z \cdot u_z)$$

with $u_z \in U_{q,0}$ for all $z \in \mathbb{N}$ (so $u_z = 0$ for almost all $z \in \mathbb{Z}$).

Note that for $n \in \mathbb{N}$ and $0 \leq j < \ell$, we have $F_b^{n\ell+j} = b^n F_b^j$ and $E_{a,b,c}^{n\ell+j} = e^n E_{a,b,c}^j$, where $e = \prod_{i=1}^{\ell} e_i$. Recall that $E_{a,b,c}^{\ell-j}$ and F_b^j, for $1 \leq j \leq \ell$, induce the same permutation on the weight subspaces.

So we will rewrite the element u as a finite sum of the form

$$(u_0 + \sum_{t \in \mathbb{N}^+} F^{t\ell} \cdot u_{-t\ell} + \sum_{t \in \mathbb{N}^+} E^{t\ell} \cdot u_{t\ell}) + \sum_{j=1}^{\ell-1} F^j \cdot \sum_{t \in \mathbb{N}} F^{t\ell} \cdot u_{-\ell t - j} + \sum_{j=1}^{\ell-1} E^{\ell-j} \cdot \sum_{t \in \mathbb{N}} E^{t\ell} \cdot u_{t\ell+\ell-j} \tag{13}$$

where $u_z \in U_{q,0}$ for all $z \in \mathbb{N}$. We get a $\mathbb{Z}/\ell\mathbb{Z}$-grading on U_q as follows: $U_q := \oplus_{i=0}^{\ell-1} \tilde{U}_{q,i}$, where $\tilde{U}_{q,0} := \{w \in U_q : \ w = w_0 + \sum_{t \in \mathbb{N}^+} F^{t\ell} \cdot w_{-t\ell} + \sum_{t \in \mathbb{N}^+} E^{t\ell} \cdot w_{t\ell}$ for some $w_{t\ell}, w_{-t\ell} \in U_{q,0},\ t \in \mathbb{N}^+\}$, and for $0 < j < \ell$, $\tilde{U}_{q,j} := \{w \in U_q : \ w = F^j \cdot \sum_{t \in \mathbb{N}} F^{t\ell} \cdot w_{-t\ell-j} + E^{\ell-j} \cdot \sum_{t \in \mathbb{N}} E^{t\ell} \cdot w_{t\ell+\ell-j}$ for some $w_{-t\ell-j},\ w_{t\ell+(\ell-j)} \in U_{q,0},\ t \in \mathbb{N}^+\}$.

Note that this grading has the property that $\Theta_{\bar{n}}(U_q) := \oplus_{i=0}^{\ell} \Theta_{\bar{n}}(\tilde{U}_{q,i})$. Given our element $u \in U_q$, we write it as $u = \sum_{i=0}^{\ell-1} \tilde{u}_i$ with $\tilde{u}_i \in \tilde{U}_{q,i}$; so the various u_z occurring in the decomposition (13) place themselves correspondingly to the $\tilde{u}_i$, according to the grading.

Let $\bar{n} := (n_1, n_2, n_3) \in \mathbb{N}^3$, set $e_{i,\bar{n}} := r_{n_2} + [i]\frac{c_{n_1}q^{-i+1} - c_{n_1}^{-1}q^{i-1}}{q-q^{-1}}$ and $e_{\bar{n}} := \prod_{i=1}^{\ell-1} e_{i,\bar{n}} \cdot a_{n_3}$.

Also, let us adopt the following notation: for M an $\ell \times \ell$ matrix and $1 \leq i, j \leq \ell$, $M(i,j)$ is the coefficient on the i-th row and j-th column of M.

Then recall that $F_{b_{\bar{n}}}(j+1, j) = 1$, for $j = 1, \cdots, \ell-1$ and $F_{b_{\bar{n}}}(1, \ell) = b_{\bar{n}}$. More generally, for $1 \leq t < \ell$, $F^t_{b_{\bar{n}}}(j+t, j) = 1$ whenever $1 \leq j \leq \ell - t$ and $F^t_{b_{\bar{n}}}(j, \ell - t + j) = b_{\bar{n}}$ for $1 \leq j \leq t$. Similarly, $E_{a_{n_3}, b_{\bar{n}}, c_{n_1}}(i, i+1) = e_{i,\bar{n}}$ for $1 \leq i \leq \ell - 1$ and $E_{a_{n_3}, b_{\bar{n}}, c_{n_1}}(\ell, 1) = a_{n_3}$. Moreover

$$E^t_{a_{n_3}, b_{\bar{n}}, c_{n_1}}(\ell - t + j, j) = e_{\ell-t+j,\bar{n}} \cdot e_{\ell-t+j+1,\bar{n}} \cdot \ \ldots \ \cdot e_{\ell+j-1,\bar{n}}, \quad 1 \leq j \leq t < \ell,$$

with the convention that the indices are calculated modulo ℓ (namely if $\ell - t + j > \ell$, then it is equal to $j - t$) and for $1 \leq t < \ell$,

$$E^t_{a_{n_3}, b_{\bar{n}}, c_{n_1}}(j, j+t) = e_{j,\bar{n}} \cdot \ \ldots \ \cdot e_{j+t-1,\bar{n}}, \quad 1 \leq j \leq \ell - t.$$

Proposition 8.2. *Let $\bar{n} \in \mathbb{N}^3$, $\Theta_{\bar{n}}$ and $\mathcal{W}$ be defined as above. For any $u \in U_q - \{0\}$, there exists $W_u \in \mathcal{W}$ such that for all $\bar{n} \in W_u$ we have $\Theta_{\bar{n}}(u) \neq 0$. So, the map $[\Theta_{\bar{n}}]_{\mathcal{W}} : U_q \to \prod_{\mathcal{W}} M_\ell(\mathbb{C})$ is a monomorphism of associative $\mathbb{C}$-algebras.*

Proof: Decompose $u \in U_q$ as in (13), so $u = \sum_0^{\ell-1} \tilde{u}_i$ with $\tilde{u}_i \in \tilde{U}_{q,i}$. We are going to calculate $\Theta_{\bar{n}}(u)$. Let z_0 be the highest positive integer such that either $u_{-z_0} \neq 0$ or $u_{z_0} \neq 0$, provided that such an index exists. Otherwise put $z_0 = 0$. Write $-z_0 = -t_0\ell - j_0$ with $0 \leq j_0 < \ell$ in the former case, and $z_0 = t_0\ell + \ell - j_0$ with $1 \leq j_0 \leq \ell$ in the latter. When $z_0 = 0$, put $t_0 = 0$. For $t \in \mathbb{N}$ and $0 \leq j < \ell$,

$$\Theta_{\bar{n}}(E^{j+\ell t}) = e^t_{\bar{n}}\, E^j_{a_{n_3}, b_{\bar{n}}, c_{n_1}}, \quad \Theta_{\bar{n}}(F^{j+\ell t}) = F^{j+\ell t}_{b_{\bar{n}}} = b^t_{\bar{n}}\, F^j_{b_{\bar{n}}}.$$

For $z \in \mathbb{Z}$, denote by $V_{z,\bar{n}}$ the diagonal matrix $\Theta_{\bar{n}}(u_z)$ (so equal to

$$K^{-s_z}_{c_{n_1}}\, p_z(C_{q,a_{n_3},b_{\bar{n}},c_{n_1}}, K_{c_{n_1}})$$

for some $s_z \in \mathbb{Z}$ and a possibly zero polynomial $p_z(x_1, x_2) \in \mathbb{C}[x_1, x_2]$).

Then, for $0 < j < \ell$, we have

$$\Theta_{\bar{n}}(\tilde{u}_j) = [\Theta_{\bar{n}}(F^{\ell-j}) \cdot (V_{-(\ell-j),\bar{n}} + V_{-(\ell-j+\ell),\bar{n}} b_{\bar{n}} + \ldots + V_{-(\ell-j+t_0\ell),\bar{n}} b^{t_0}_{\bar{n}}) + \tag{14}$$
$$+ \Theta_{\bar{n}}(E^j) \cdot (V_{j,\bar{n}} + V_{j+\ell,\bar{n}} e_{\bar{n}} + \ldots + V_{j+t_0\ell,\bar{n}} e^{t_0}_{\bar{n}})]$$

and for $j = 0$, we have

$$\Theta_{\bar{n}}(\tilde{u}_0) = (V_{0,\bar{n}} + V_{-\ell,\bar{n}} b_{\bar{n}} + \ldots + V_{-\ell t_0,\bar{n}} b^{t_0}_{\bar{n}} + V_{\ell,\bar{n}} e_{\bar{n}} + \cdots + V_{\ell t_0,\bar{n}} e^{t_0}_{\bar{n}}).$$

Case 1. Suppose that $\tilde{u}_0 \neq 0$, namely that $u_{t\ell} \neq 0$ for some $t \in \mathbb{Z}$. Let $t_1 \in \mathbb{N}^+$ be maximal such that $u_{t_1\ell} \neq 0$, if such a positive integer exists, and $t_1 = 0$ otherwise. Similarly let $t_2 \in \mathbb{N}$ be maximal such that $u_{-t_2\ell} \neq 0$, if there are such. (Note that either there is a $t_1 > 0$, or $t_2 \geq 0$.) So there are cofinitely many c_{n_1} such that for all but finitely many r_{n_2}, $\Theta_{\bar{n}}(u_{t_1\ell}) \neq 0$ and $\Theta_{\bar{n}}(u_{-t_2\ell}) \neq 0$. So by Lemma 8.1, we are done if $t_1 = t_2 = 0$. Then assume that one of them is non zero.

First assume that $t_1 > 0$. Fix a pair (c_{n_1}, r_{n_2}) such that $\Theta_{\bar{n}}(u_{t_1\ell}) \neq 0$ and $\Theta_{\bar{n}}(u_{-t_2\ell}) \neq 0$. Since $|b_{\bar{n}}| < 1$, we can bound the norm of the matrix $V_{-\ell,\bar{n}} b_{\bar{n}} + \ldots + V_{-t_2\ell,\bar{n}} b^{t_2}_{\bar{n}}$. Therefore for each fixed pair (c_{n_1}, r_{n_2}), the sum

$$V_{0,\bar{n}} + (V_{\ell,\bar{n}} e_{\bar{n}} + \ldots + V_{t_1\ell,\bar{n}} e^{t_1}_{\bar{n}}) + (V_{-\ell,\bar{n}} b_{\bar{n}} + \ldots + V_{-t_2\ell,\bar{n}} b^{t_2}_{\bar{n}}) \tag{15}$$

is non zero for all but finitely a_{n_3}. Indeed, the modulus of the elements of S_a is unbounded and if the sum (15) were equal to zero, then $|e_{\bar{n}}| < max\{1, \sum_{j=1}^{t_1-1} \frac{\|V_{\ell j,\bar{n}}\|}{\|V_{\ell t_1,\bar{n}}\|} + \sum_{t=0}^{t_2-1} \frac{\|V_{-\ell t,\bar{n}}\|}{\|V_{\ell t_1,\bar{n}}\|}\}$.

If $t_1 = 0$, then by assumption $t_2 > 0$. We proceed in a similar way with the sum

$$V_{0,\bar{n}} + V_{-\ell,\bar{n}} b_{\bar{n}} + \ldots + V_{-t_2\ell,\bar{n}} b_{\bar{n}}^{t_2}.$$

By assumption $V_{0,\bar{n}}$ and $V_{-t_2\ell,\bar{n}}$ are non zero matrices and so for all but finitely $b_{\bar{n}}$ (equivalently for all but finitely many a_{n_3}), this sum is non zero

Case 2. Assume that $\tilde{u}_0 = 0$, that $\tilde{u}_{j_0} \neq 0$, for some $0 < j_0 < \ell$ and either for all $z > 0$, $u_z = 0$ or for all $z < 0$ $u_z = 0$. Let $z_0 := \ell t_0 + j_0$ in the former case and $z_0 := \ell t_0 + \ell - j_0$ in the latter, with $t_0 \in \mathbb{N}$.

Then $\Theta_{\bar{n}}(u)$ is either of the form:

$$\Theta_{\bar{n}}(F^{j_0}) \cdot (V_{-j_0,\bar{n}} + V_{-(j_0+\ell),\bar{n}} b_{\bar{n}} + \ldots + V_{-(j_0+t_0\ell),\bar{n}} b_{\bar{n}}^{t_0}) + \ldots +$$

$$\Theta_{\bar{n}}(F) \cdot (V_{-1} + V_{-(1+\ell)} b_{\bar{n}} + \ldots + V_{-(1+t_0\ell),\bar{n}} b_{\bar{n}}^{t_0}) \tag{16}$$

or of the form:

$$\Theta_{\bar{n}}(E) \cdot (V_{1,\bar{n}} + V_{1+\ell,\bar{n}} e_{\bar{n}} + \ldots + V_{1+t_0\ell,\bar{n}} e_{\bar{n}}^{t_0}) + \ldots +$$

$$\Theta_{\bar{n}}(E^{\ell-j_0}) \cdot (V_{\ell-j_0,\bar{n}} + V_{\ell-j-0+\ell,\bar{n}} e_{\bar{n}} + \ldots + V_{\ell-j_0+t_0\ell,\bar{n}} e_{\bar{n}}^{t_0}). \tag{17}$$

It suffices to show that, with $0 < j_0 < \ell$,

- in the former case, when $z_0 = \ell t_0 + j_0$, $V_{-j_0,\bar{n}} + V_{-(j_0+\ell),\bar{n}} b_{\bar{n}} + \ldots + V_{-(j_0+t_0\ell),\bar{n}} b_{\bar{n}}^{t_0} \neq 0$,
- in the latter case, when $z_0 = \ell t_0 + \ell - j_0$, $V_{\ell-j_0,\bar{n}} + V_{\ell-j-0+\ell,\bar{n}} e_{\bar{n}} + \ldots + V_{\ell-j_0+t_0\ell,\bar{n}} e_{\bar{n}}^{t_0} \neq 0$.

Let us deal here with the former case, as the other one is similar. Recall that the $(i+1)^{th}$ entry on the diagonal ($0 \leq i < \ell$) of the matrix $V_{-(j_0+t_0\ell),\bar{n}}$ is of the form $c_{n_1}^{-z} q^{2zi} p(r_{n_2} + \frac{c_{n_1} q + c_{n_1}^{-1} q^{-1}}{(q-q^{-1})^2}, c_{n_1} q^{-2i})$, for some $z \in \mathbb{Z}$ depending on $-(j_0 + t_0\ell)$ and some rational function $p_z(x_1, x_2)$. So for cofinitely many values of $c_{n_1} \in S_c$, the rational function $p_z(x_1' + \frac{c_{n_1} q + c_{n_1}^{-1} q^{-1}}{(q-q^{-1})^2}, c_{n_1} q^{-2i})$ is non trivial. Therefore for cofinitely many values of $n_2 \in S_r$, we get that $c_{n_1}^{-z} q^{2zi} p_z(r_{n_2} + \frac{c_{n_1} q + c_{n_1}^{-1} \cdot q^{-1}}{(q-q^{-1})^2}, c_{n_1} q^{-2i}) \neq 0$. So for such fixed value of (c_{n_1}, r_{n_2}), the coefficient of $b_{\bar{n}}^{t_0}$ is non zero. Then we can find cofinitely many $b_{\bar{n}}$, which correspond to cofinitely many values of a_{n_3}, such that $\sum_{t=0}^{t_0} V_{-(t\ell+j_0),\bar{n}} b_{\bar{n}}^t \neq 0$.

So on an element of the ultrafilter $\mathcal{W}$, $\Theta_{\bar{n}}(\tilde{u}_{j_0}) \neq 0$ and this is enough because of the direct sum decomposition of $\Theta_{\bar{n}}(U_q)$.

Case 3. Assume that $\tilde{u}_0 = 0$ and there exists $z_1 \in \mathbb{Z}$ such that $u_{z_1} \neq 0$ and for all $z_2 \in \mathbb{Z}$ with $z_1 z_2 < 0$ such that $u_{z_2} \neq 0$ we have $z_1 - z_2 \notin \ell\mathbb{Z}$. Then it suffices to show that an expression of the above form (16) or (17) is non zero, which can be done as in Case 2.

Case 4. Finally suppose that $\tilde{u}_0 = 0$, and for all z_1 with $u_{z_1} \neq 0$, there exists z_2 with $z_1 z_2 < 0$ such that $u_{z_2} \neq 0$ and $z_1 - z_2 \in \ell\mathbb{Z}$. So, in order to show that (14) is non zero, we have to show that an expression of the following form, for some fixed j with $1 \leq j < \ell$, is non zero:

$$\Theta_{\bar{n}}(F^j) \cdot \sum_{s=0}^{t_2} V_{-s\ell-j,\bar{n}} b_{\bar{n}}^s + \Theta_{\bar{n}}(E^{\ell-j}) \cdot \sum_{s=0}^{t_1} V_{s\ell+\ell-j,\bar{n}} e_{\bar{n}}^s$$

where t_1 is maximal such that $u_{t_1\ell+\ell-j} \neq 0$ and t_2 is maximal such that $u_{-t_2\ell-j} \neq 0$. The $(j+t,t)$ coefficient of that matrix, with $1 \leq t \leq \ell - j$, is equal to

$$\sum_{s=0}^{t_2} p_{-j-s\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2}, c_{n_1}q^{-2(t-1)})\, b_{\bar{n}}^s +$$

$$+e_{j+t,\bar{n}} \cdot e_{j+t+1,\bar{n}} \cdot \ldots \cdot e_{\ell+t-1,\bar{n}} \cdot \sum_{s=0}^{t_1} p_{\ell-j+s\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2},\, c_{n_1}q^{-2(t-1)}) e_{\bar{n}}^s, \tag{18}$$

with the convention that the indices are calculated modulo ℓ.

As previously, with the values of c_{n_1} and r_{n_2} fixed, we can bound the norm of

$$\sum_{s=0}^{t_2} p_{-j-s\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2}, c_{n_1}q^{-2(t-1)})\, b_{\bar{n}}^s.$$

When $|a_{n_3}|$, with $a_{n_3} \in S_a$ increases, this norm remains bounded. Note that $a_{n_3} = e_\ell$ always occurs exactly once as a factor of the product $e_{j+t,\bar{n}} \cdot e_{j+t+1,\bar{n}} \cdot \ldots \cdot e_{\ell+t-1,\bar{n}}$ and the other factors remain constant, again whenever c_{n_1} and r_{n_2} are fixed. Rewrite that product as $a_{n_3} \cdot e'_{\bar{n}}$. Recall that the value of $\frac{e_{\bar{n}}}{a_{n_3}}$ only depends on c_{n_1} and r_{n_2}.

We claim that if a coefficient of the form (18) is equal to zero, then the norm of a_{n_3} is bounded, provided that we fix the value of c_{n_1}, r_{n_2} and choose it such that

$$p_{\ell-j+t_1\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2},\, c_{n_1}q^{-2(t-1)}) \neq 0 \qquad (\star)$$

(which holds for cofinitely many values of c_{n_1} and then of r_{n_2}). This will imply that the expression (18) is different from zero on an element of $\mathcal{W}$.

Assume that $(\star)$ holds. Then

$$a_{n_3} \cdot e'_{\bar{n}} \cdot (\frac{e_{\bar{n}}}{a_{n_3}})^{t_1} \cdot p_{\ell-j+t_1\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2},\, c_{n_1}q^{-2(t-1)}) =$$

$$-e'_{\bar{n}} \cdot \sum_{s=0}^{t_1-1} p_{\ell-j+s\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2}, c_{n_1}q^{-2(t-1)}) \cdot \frac{e_{\bar{n}}^s}{a_{n_3}^{t_1}} -$$

$$-\sum_{s=0}^{t_2} p_{-j-s\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2},\, c_{n_1}q^{-2(t-1)}) \cdot \frac{r_{n_2}^s}{a_{n_3}^{t_1+s}},$$

and we can bound in that case the norm of a_{n_3} as follows:

$$|a_{n_3}| \leq |e'_{\bar{n}} \cdot (\frac{e_{\bar{n}}}{a_{n_3}})^{t_1} \cdot p_{\ell-j+t_1\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2},\, c_{n_1}q^{-2(t-1)})|^{-1} \cdot$$

$$\cdot(|e'_{\bar{n}}| \cdot \sum_{s=0}^{t_1-1} |p_{\ell-j+s\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2},\, c_{n_1}q^{-2(t-1)})| \cdot |\frac{e_{\bar{n}}^s}{a_{n_3}^{t_1}}| +$$

$$+\sum_{s=0}^{t_2} |p_{-j-s\ell}(r_{n_2} + \frac{c_{n_1}q + c_{n_1}^{-1}q^{-1}}{(q-q^{-1})^2},\, c_{n_1}q^{-2(t-1)})| \cdot |r_{n_2}^s|).$$

We may apply a similar reasoning to the $(t, \ell - j + t)$-coefficient of that matrix, for $1 \leq t \leq j$; it is equal to

$$b_{\bar{n}} \cdot \sum_{s=0}^{t_2} p_{-j-s\ell}(r_{n_2} + \frac{c_{n_1} q + c_{n_1}^{-1} q^{-1}}{(q-q^{-1})^2},\ c_{n_1} q^{-2(\ell-j+t-1)}) \cdot b_{\bar{n}}^s + \tag{19}$$

$$e_{t,\bar{n}} \cdot \ldots \cdot e_{t+\ell-j-1,\bar{n}} \cdot \sum_{s=0}^{t_1} p_{\ell-j+s\ell}(r_{n_2} + \frac{c_{n_1} q + c_{n_1}^{-1} q^{-1}}{(q-q^{-1})^2},\ c_{n_1} q^{-2(\ell-j+t-1)}) \cdot e_{\bar{n}}^s.$$

Again we choose a value of $c_{n_1},\ r_{n_2}$ such that $p_{\ell-j+t_1\ell}(r_{n_2} + \frac{c_{n_1} q + c_{n_1}^{-1} q^{-1}}{(q-q^{-1})^2},\ c_{n_1} q^{-2(\ell-j+t-1)}) \neq 0$ and we show that if the expression (19) is equal to zero, then one can bound the value of a_{n_3} and so it only occurs finitely many times for a fixed value of $c_{n_1},\ r_{n_2}$. Note that in this case the value of $e_{t,\bar{n}} \cdot \ldots \cdot e_{t+\ell-j-1,\bar{n}}$ remains constant for $1 \leq t \leq j$ whenever c_{n_1} and r_{n_2} are fixed. □

Given an ultrafilter $\mathcal{W}$ on $\mathbb{N}^3$ as in Definition 8.1, we denote by $\mathbb{C}^*$ (respectively $\mathbb{R}^*$) the ultrapower of $\mathbb{C}$ (respectively $\mathbb{R}$) modulo $\mathcal{W}$.

First, we define a map $Exp_{\mathcal{W}}$ from $\prod_{\mathcal{W}} M_\ell(\mathbb{C})$ to $\prod_{\mathcal{W}} GL_\ell(\mathbb{C})$, simply by

$$Exp_{\mathcal{W}}([A_{\bar{n}}]_{\mathcal{W}}) := [exp_\ell(A_{\bar{n}})]_{\mathcal{W}},$$

for $A_{\bar{n}} \in M_\ell(\mathbb{C})$ and $\bar{n} \in \mathbb{N}^3$. Note that $\prod_{\mathcal{W}} M_\ell(\mathbb{C}) \cong M_\ell(\mathbb{C}^*)$ (respectively $\prod_{\mathcal{W}} GL_\ell(\mathbb{C}) \cong GL_\ell(\mathbb{C}^*)$), so $Exp_{\mathcal{W}}$ also defines a map from $M_\ell(\mathbb{C}^*)$ to $GL_\ell(\mathbb{C}^*)$.

Let us say that an element of $M_\ell(\mathbb{C}^*)$ is *infinitesimal* if its norm is bounded by any positive rational number, where the norm on $M_\ell(\mathbb{C})$ has been extended in a natural way on $M_\ell(\mathbb{C}^*)$ taking now its values in $\mathbb{R}^*$.

Let us denote from now on, for ease of notation, an element $[A_{\bar{n}}]_{\mathcal{W}}$ of $\prod_{\mathcal{W}} M_\ell(\mathbb{C}) \cong M_\ell(\mathbb{C}^*)$ simply as $[A_{\bar{n}}]$, so omitting the subscript $\mathcal{W}$.

We claim that if the norm $\|.\|$ of $(A_{\bar{n}})_{\bar{n}\in\mathbb{N}^3}$ is bounded on an element of $\mathcal{W}$, then $Exp_{\mathcal{W}}([A_{\bar{n}}]) = [exp_\ell(A_{\bar{n}})] = [\sum_{j=0}^{\infty} \frac{A_{\bar{n}}^j}{j!}]$ can be viewed as the limit up to an infinitesimal element of $M_\ell(\mathbb{C}^*)$ of the sequence $(\sum_{j=0}^{m} \frac{[A_{\bar{n}}]^j}{j!})_{m\in\mathbb{N}}$. Indeed, let us check that the sequence in $M_\ell(\mathbb{C}^*)$ of matrices $([\sum_{j=0}^{m} \frac{A_{\bar{n}}^j}{j!}])_{m\in\mathbb{N}}$ is a Cauchy sequence (and so bounded).

In fact, for every $\bar{n} \in \mathbb{N}^3$ and $m \in \mathbb{N}$,

$$\|\sum_{j=0}^{m} \frac{A_{\bar{n}}^j}{j!}\| \leq \sum_{j=0}^{m} \frac{\|A_{\bar{n}}\|^j}{j!} \leq e^{\|A_{\bar{n}}\|}.$$

So

$$\|\sum_{j=0}^{m} \frac{[A_{\bar{n}}]^j}{j!}\| \leq \sum_{j=0}^{m} \frac{\|[A_{\bar{n}}]\|^j}{j!} = [\sum_{j=0}^{m} \frac{\|A_{\bar{n}}\|_\ell^j}{j!}] \leq [e^{\|A_{\bar{n}}\|}].$$

For any $\epsilon > 0$ in $\mathbb{R}$, there exists a positive integer N such that for any $m_1 > m_2 > N$,

$$\|\sum_{j=0}^{m_1} \frac{[A_{\bar{n}}]^j}{j!} - \sum_{j=0}^{m_2} \frac{[A_{\bar{n}}]^j}{j!}\| \leq \frac{\|[A_{\bar{n}}]\|^{m_2+1}}{(m_2+1)!} \cdot \|\sum_{j=m_2+1}^{m_1} \frac{(m_2+1)!\,[A_{\bar{n}}]^{j-m_2-1}}{j!}\| \leq$$

$$\leq \frac{\|[A_{\bar{n}}]\|^{m_2+1}}{(m_2+1)!} \cdot \sum_{j=0}^{m_1-m_2-1} \|\frac{[A_{\bar{n}}]^j}{j!}\| \leq \frac{\|A_{\bar{n}}\|^{m_2+1}}{(m_2+1)!} \cdot [e^{\|A_{\bar{n}}\|}] \leq \frac{\|[A_{\bar{n}}]\|^{N+1}}{(N+1)!} \cdot [e^{\|A_{\bar{n}}\|}] \leq \epsilon.$$

Finally

$$\|[\sum_{j=0}^{N}\frac{A_{\bar{n}}^j}{j!}]-[exp(A_{\bar{n}})]\|=[\|\sum_{j=0}^{N}\frac{A_{\bar{n}}^j}{j!}-exp(A_{\bar{n}})\|]=[\|\sum_{j=N+1}^{\infty}\frac{A_{\bar{n}}^j}{j!}\|]\leq[\frac{\|A_{\bar{n}}\|^{N+1}}{(N+1)!}\cdot e^{\|A_{\bar{n}}\|}].$$

Let $A_{\bar{n}}\in M_\ell(\mathbb{C})$. Following the discussion of [10, Theorem 3.1], we calculate $exp_\ell(A_{\bar{n}})$ (for the reader convenience, we reproduce it below). Using the Jordan form of $A_{\bar{n}}$, one writes $A_{\bar{n}}$ (uniquely) as a sum $B_{\bar{n}}+C_{\bar{n}}$, where $B_{\bar{n}}$ commutes with $C_{\bar{n}}$, $B_{\bar{n}}$ is diagonalizable and $C_{\bar{n}}$ is nilpotent of class $\leq\ell-1$. So, we can explicitly calculate $exp_\ell(A_{\bar{n}})=exp_\ell(B_{\bar{n}})\cdot exp_\ell(C_{\bar{n}})=exp_\ell(B_{\bar{n}})\cdot(I+C_{\bar{n}}+\ldots+\frac{C_{\bar{n}}^{\ell-1}}{(\ell-1)!})$. Since $B_{\bar{n}}$ is diagonalizable, there exists an invertible matrix $D_{\bar{n}}$ such that $D_{\bar{n}}^{-1}\cdot B_{\bar{n}}\cdot D_{\bar{n}}=diag(b_{\bar{n}1},\ldots,b_{\bar{n}\ell})$, where $b_{\bar{n}j}\in\mathbb{C}$, $1\leq j\leq\ell$, are the eigenvalues of $B_{\bar{n}}$. So, $exp_\ell(B_{\bar{n}})=D_{\bar{n}}\cdot diag(e^{b_{\bar{n}1}},\ldots,e^{b_{\bar{n}\ell}})\cdot D_{\bar{n}}^{-1}$. Now,

$$[exp_\ell(A_{\bar{n}})]=[D_{\bar{n}}]\cdot diag(e^{[b_{\bar{n}1}]},\ldots,e^{[b_{\bar{n}\ell}]})\cdot[D_{\bar{n}}]^{-1}\cdot(I+[C_{\bar{n}}]+\ldots+\frac{[C_{\bar{n}}]^{\ell-1}}{(\ell-1)!}).$$

In particular, $(M_\ell(\mathbb{C}^*),Exp_{\mathcal{W}},GL_\ell(\mathbb{C}^*))$ is interpretable in the structure $(\mathbb{C}^*,x\to e^x)$. Moreover, calculating the norm, we get

$$\|exp_\ell([A_n])\|\leq\|diag(e^{[b_{\bar{n}1}]},\cdots,e^{[b_{\bar{n}\ell}]})\|\cdot(\sum_{i=0}^{\ell-1}\frac{\|[C_{\bar{n}}]\|^i}{i!}).$$

As previously, we define $EXP_{\mathcal{W}}$ from U_q to $\prod_{\mathcal{W}}GL_\ell(\mathbb{C}))\simeq GL_\ell(\mathbb{C}^*)$ by

$$EXP_{\mathcal{W}}(u)=[exp_\ell\circ\Theta_{a,b,c}(u)]_{\mathcal{W}}$$

and we deduce the following corollary.

Corollary 8.3. *$(U_q,\mathbb{C},GL_\ell(\mathbb{C}^*),EXP_{\mathcal{W}})$ is an exponential $\mathbb{C}$-algebra and as such embeds in $(M_\ell(\mathbb{C}^*),\mathbb{C},GL_\ell(\mathbb{C}^*),Exp_{\mathcal{W}})$.*

Proof: As for Corollary 6.3, we use Łos Theorem, Proposition 8.2 and the properties of the exponential map in $M_\ell(\mathbb{C})$ (see for instance Proposition 3.1 in [9]). □

On the image of U_q in $GL_\ell(\mathbb{C}^*)$, we can say the following. Note that the trace of K_c is equal to $c\cdot(1+q^{-2}+\ldots+q^{-2\ell+2})=c\cdot\frac{1-q^{-2\ell}}{1-q^{-2}}=0$ and so the image of K by $exp_\ell\circ\Theta_{a,b,c}$ will belong to $SL_\ell(\mathbb{C})$, as well as the images of E^i, F^j, for $i,j\in\mathbb{Z}-\ell.\mathbb{Z}$.

9. An analytic approach

In this section, we still work in $\mathbb{C}$ and assume that q is primitive root of unity of degree $\ell>2$ (making the same adjustment as in the previous sections when ℓ is even). We will use the theory of meromorphic functions with two complex variables and get a partial but in some respects stronger result on the fact that the image of certain non-zero elements of $U_{q,0}$ have a non-trivial image by $\tilde{\Theta}_{d,f}$ for "most" of the choices of the complex coefficients (d,f). We thank Andrea Spiro for suggesting this approach.

We will denote the closure of a subset A of $\mathbb{C}^2$ by A^{cl}. Also, given a polynomial $f(x_1,x_2)\in\mathbb{C}[x_1,x_2]$, we will denote its zeroset on $\mathbb{C}^2$ by $Z(f(x_1,x_2)):=\{(a_1,a_2)\in\mathbb{C}^2:\ f(a_1,a_2)=0\}$.

So let $u\in U_{q,0}-\{0\}$; it is of the form $K^{-n}p(C_q,K)$ with $p(x_1,x_2)\in\mathbb{C}[x_1,x_2]-\{0\}$, $n\in\mathbb{Z}$. Let $n\in\mathbb{Z}$ to be the least n such that

$$(\star)\qquad p(x_1,x_2)=\sum_{j=0}^{N}s_j(x_1)x_2^j,\quad\text{with}\ \ s_0(x_1)\neq0.$$

We will say that $u \in U_{q,0} - \{0\}$ is *prime* if the polynomial $p(x_1, x_2)$ is irreducible, assuming it is in the form $(\star)$. In fact, since there is no extra work involved, we will consider both representations $\Theta_{a,b,c}$ and $\tilde{\Theta}_{d,f}$ simultaneously.

Recall that, if $u \in U_{q,0}$, then both matrices $\Theta_{a,b,c}(u)$ and $\tilde{\Theta}_{d,f}(u)$ are diagonal matrices where, for $0 \leq i < \ell$, the $(i+1)$-th entry on the diagonal is equal to respectively

- $c^{-n} q^{2ni} p(ab + \frac{cq+(cq)^{-1}}{(q-q^{-1})^2}, cq^{-2i})$,
- $f^{-n} q^{-2ni} p(\frac{fq^{-1}+f^{-1}q}{(q-q^{-1})^2}, fq^{2i})$.

Lemma 9.1. *Let $u \in U_{q,0} - \{0\}$ and assume that u is prime. Then for all but finitely many $f \in \mathbb{C}$, there is at most one non negative integer $i < \ell$ such that the $(i+1)$-th entry on the diagonal of the matrix $\tilde{\Theta}_{d,f}(u)$ is equal to zero. Similarly, given any a, $b \in \mathbb{C}$, for all but finitely many $c \in \mathbb{C}$, there is at most one $i < \ell$ such that the $(i+1)$-th entry on the diagonal of the matrix $\Theta_{a,b,c}(u)$ is equal to zero.*

Note that if $u \in U_{d,f} - \{0\}$ is not prime, then $\tilde{\Theta}_{d,f}(u)$ may be equal to 0 for infinitely many tuples (d, f) (actually for infinitely many f).

Proof: Let $G(x) := \frac{xq^{-1}+x^{-1}q}{(q-q^{-1})^2}$ and rewrite $ab + \frac{cq+(cq)^{-1}}{(q-q^{-1})^2}$ as $ab + G(cq^2)$. Consider the two families of rational functions $G_{1,i}, G_{2,i} : \mathbb{C} \to \mathbb{C}^2$ given by

$$G_{1,i}(x) = (G(x), xq^{2i}),\ G_{2,i}(x) = (ab + G(xq^2), xq^{-2i})$$

for all $x \neq 0$ $(i < \ell)$. Then $p \circ G_{1,i} : \mathbb{C} \to \mathbb{C}$ and $p \circ G_{2,i} : \mathbb{C} \to \mathbb{C}$ are rational functions with the only pole 0. This implies that each of them either has only finitely many zeroes or is identically zero, and in the latter case the images of $G_{1,i}$, $G_{2,i}$ (and so the closure of these images) are included in $Z(p(x_1, x_2))$.

We claim that there is at most one i such that $p \circ G_{1,i}$ is identically zero, and similarly for $p \circ G_{2,i}$. This is clearly enough for our purposes.

Assume towards a contradiction that this is false. Put for simplicity $\mathbb{C}^\star = \mathbb{C} \setminus \{0\}$. For $i, j < \ell$, $i \neq j$, we have both $G_{1,i}(\mathbb{C}^\star) \subseteq Z(p(x_1, x_2))$ and $G_{1,j}(\mathbb{C}^\star) \subseteq Z(p(x_1, x_2))$ (similarly for $G_{2,i}$ and $G_{2,j}$).

Observe that, being $q \neq 0$, the Jacobian matrix $J(G_{1,i}) = \begin{pmatrix} dG(x)/dx \\ q^{2i} \end{pmatrix}$ is nowhere zero and $G_{1,i}$ is a regular parametrization of the smooth complex curve $G_{1,i}(\mathbb{C}^\star) \subset \mathbb{C}^2$. Since $p(x_1, x_2)$ is irreducible, it follows that $G_{1,i}(\mathbb{C}^\star)^{cl} = Z(p(x_1, x_2))$. The same argument works for $G_{1,j}$, $G_{2,i}$ and $G_{2,j}$ and shows that the restrictions of $G_{1,i}$, $G_{1,j}$ and of $G_{2,i}$, $G_{2,j}$, respectively, to suitable open sets can be considered as pairs of (local) parametrizations of the same smooth complex curve and there exist holomorphic changes of parameters $H_{1,j,i}(x) = G_{1,j}^{-1}(G_{1,i}(x))$ and $H_{2,j,i}(x) = G_{2,j}^{-1}(G_{2,i}(x))$ (with $x \neq 0$).

Therefore $G_{1,j}(H_{1,j,i}(x)) = G_{1,i}(x)$ (respectively $G_{2,j}(H_{2,j,i}(x)) = G_{2,i}(x)$). In particular,

- $G(H_{1,j,i}(x)) = G(x)$ and $H_{1,j,i}(x)q^{2j} = xq^{2i}$,
- $ab + G(H_{2,j,i}(x)q^2) = ab + G(xq^2)$ (consequently $G(H_{2,j,i}(x)q^2) = G(xq^2)$) and $H_{2,j,i}(x)\, q^{-2j} = xq^{-2i}$.

In the former case $H_{1,j,i}(x) = xq^{2i-2j}$ and replacing it in the first equality we get $\frac{xq^{2i-2j}q^{-1}+x^{-1}q^{-2i+2j}q}{(q-q^{-1})^2} = \frac{xq^{-1}+x^{-1}q}{(q-q^{-1})^2}$. Similarly, in the latter case, $H_{2,j,i}(x) = xq^{-2i+2j}$ implies $\frac{xq^{-2i+2j}q+x^{-1}q^{2i-2j}q^{-1}}{(q-q^{-1})^2} = \frac{xq+x^{-1}q^{-1}}{(q-q^{-1})^2}$.

Comparing the terms of the Laurent series development of the two rational functions arising in these equalities, we get in both cases $q^{2i-2j} = 1$ and hence a contradiction, since either ℓ odd and $|(i-j)| < \ell$, or $\ell > 2$ is even but then ℓ is half the order of q. □

10. Approximation

In this section, using ultraproducts and the representations of U_q, we will relate U and the quantum algebras U_q, for q a root of unity.

One known way to view U as a limit of the U_q's (see [8, page 58] and [7, VI.2.2]) is to use another presentation of U_q involving one more generator, which allows to set also the case $q = 1$. If $\tilde{U}_q$ denotes this new isomorphic presentation of U_q, one gets U as a quotient of $\tilde{U}_1/\langle K-1\rangle$.

As recalled at the end of section 5, the Drinfeld-Jimbo algebra $U_h(sl_2(\mathbb{C}))$ ([7, XVII.2.3]) is the $\mathbb{C}[[h]]$-algebra generated by X, Y, H with $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = \frac{e^{hH/2}-e^{-hH/2}}{e^{h/2}-e^{-h/2}}$ ([7, Proposition XVII.4.1]) and it is topologically isomorphic to $U(sl_2(\mathbb{C}))[[h]]$ ([7, Theorem XVIII.4.1.]).

For $k = \mathbb{C}$, a heuristic way to see U as the limit of U_q for $q \to 1$, is to proceed as follows (see [8, pages 6, 57]). Recall that U as an associative $\mathbb{C}$-algebra is generated by X, Y, H and defining relations $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$.

Now consider U_q with its generators E, F, K and K^{-1} and the corresponding relations (1).

Following the presentation of the Drinfeld-Jimbo algebra, formally write $q = e^{h/2}$ and make the change of variables $K := e^{hH/2}$ where H is viewed as a new variable. Let h go to 0. First, by differentiating with respect to h the relation $[K,E] = K\cdot E - E\cdot K = (K\cdot E\cdot K^{-1} - E)\cdot K = (q^2-1)\cdot E\cdot K = (e^h-1)\cdot E\cdot K$ one gets $e^h\cdot E\cdot e^{hH/2} + (e^h-1)\cdot E\cdot H/2\cdot e^{hH/2}$. Taking the value at $h=0$, one obtains on one hand E and on the other hand, when looking at $[K,E]$, $1/2[H,E]$, since $H/2$ is equal to the derivative of K with respect to h, evaluated at $h=0$. This establishes the relation $[H,E] = 2E$. A similar calculation gives $[H,F] = -2F$. Finally, if one takes the value at $h=0$ of the two members of the relation $[E,F] = \frac{K-K^{-1}}{q-q^{-1}}$, then by using L'Hôpital's rule one gets $[E,F] = H$. These are the relations of U (provided we set $X = E$ and $Y = F$).

As said, here we point out a further relationship between U and the U_q, via ultraproducts. We will assume that, for every $\ell > 2$, a primitive ℓ^{th} root of unity q_ℓ is chosen such that $1 < -i(q_\ell - q_\ell^{-1}) < 2$. More precisely, let $q_\ell = e^{i\frac{2\pi l}{\ell}}$ with $1 \le l < \ell$, l minimal such that the previous condition is fulfilled.

We take a non-principal ultraproduct of U_{q_ℓ}, $\ell \in \mathbb{N}$, over a non principal ultrafilter $\mathcal{W}$ over $\mathbb{N}^+$. Denote the generators of U_{q_ℓ} by E_ℓ, F_ℓ and K_ℓ. Consider the $\mathbb{C}$-algebra homomorphism τ_ℓ from U to U_{q_ℓ} sending X to E_ℓ, Y to F_ℓ (and so H to $\frac{K_\ell - K_\ell^{-1}}{q_\ell - q_\ell^{-1}}$). Define the map $\tau := [\tau_\ell]_\mathcal{W}$ from U to $\prod_\mathcal{W} U_{q_\ell}$. Note that by composing the map τ with the exponential maps that we have defined on U_{q_ℓ}, we get new exponential maps on U.

Proposition 10.1. *The map $\tau : U \to \prod_\mathcal{W} U_{q_\ell}$ is a monomorphism of (associative) $\mathbb{C}$-algebras.*

Proof: The fact that τ is a morphism of $\mathbb{C}$-algebras is straightforward from the definition. To prove injectivity, we proceed as follows. Recall that U, as a $\mathbb{Z}$-graded algebra, can be written as a infinite sum of m-homogenous components, $m \in \mathbb{Z}$, namely $U = \sum_{m\in\mathbb{Z}} \mathbb{U}_m$; furthermore note that, if m is positive, then $\mathbb{U}_m = X^m \cdot \mathbb{U}_0$ and, if m is negative, then

$\mathbb{U}_m = Y^m \cdot \mathbb{U}_0$. Furthermore the 0-component $\mathbb{U}_0$ coincides with the ring of polynomials $\mathbb{C}[C, H]$ where C is the (classical) Casimir element $C = 2XY + 2YX + H^2$ (so the generator of the center of U).

In Section 7, we defined, for each root of unity q_ℓ, representation maps $\Theta_{a,b,c}$ from U_{q_ℓ} to $M_\ell(\mathbb{C})$. We will compose the map τ with the representation maps $[\Theta_{a,b,c}]_\mathcal{W}$ from $\prod_\mathcal{W} U_{q_\ell}$ to $\prod_\mathcal{W} M_\ell(\mathbb{C})$. We will get in this way a map from U to $\prod_\mathcal{W} M_\ell(\mathbb{C})$. We will show that, for every $u \in U - \{0\}$, one can choose $a, b, c \in \mathbb{C}$ such that the image of u under the composition $[\Theta_{a,b,c}]_\mathcal{W} \circ \tau$ is $\neq 0$ (whence $\tau(u) \neq 0$). In other words, now ℓ is allowed to vary while (a, b, c) is fixed, even if it may depend on the element u we consider.

First, we will assume that $u \in \mathbb{U}_0$. Then $u = p(C, H)$ where $p(x_1, x_2) \in \mathbb{C}[x_1, x_2] - \{0\}$. Write $p(x_1, x_2) = \sum_{h=0}^{D} s_h(x_1)x_2^h$, where $s_h \in \mathbb{C}[x_1]$, D is a natural number and $s_D(x_1) \neq 0$. So the image $\tau(p(C, H)) = p(\tau(C), \tau(H)) = \sum_{h=0}^{D} s_h(\tau(C)) \cdot \tau(H)^h$ in the ultraproduct is a polynomial in the image of H and its coefficients are polynomials in the image of C.

As said, we claim that, under the hypothesis $u = p(C, H) \neq 0$, for a suitable choice of a, b and c one has $[\Theta_{a,b,c}]_\mathcal{W}(p([\tau_\ell(C)]_\mathcal{W}, [\tau_\ell(H)]_\mathcal{W})) \neq 0$ and consequently $\tau(p(C, H)) = p([\tau_\ell(C)]_\mathcal{W}, [\tau_\ell(H)]_\mathcal{W}) \neq 0$.

To prove that, we evaluate the polynomials $s_h(x_1)$ at $[2E_\ell F_\ell + 2F_\ell E_\ell + (\frac{K_\ell - K_\ell^{-1}}{q_\ell - q_\ell^{-1}})^2]_\mathcal{W}$ on one hand and the polynomial $\sum_{h=0}^{D} s_h([2E_\ell F_\ell + 2F_\ell E_\ell + \frac{K_\ell - K_\ell^{-1}}{q_\ell - q_\ell^{-1}}^2]_\mathcal{W})\, x_2^h$ at $[\frac{K_\ell - K_\ell^{-1}}{q_\ell - q_\ell^{-1}}]_\mathcal{W}$ on the other hand.

Observe that $[\Theta_{a,b,c}]_\mathcal{W}(\tau(p(C, H)))$ is

$$\begin{aligned}
&= [\Theta_{a,b,c}(\tau_\ell(p(C, H)))]_\mathcal{W} \\
&= \left[\Theta_{a,b,c}\left(p\left(2E_\ell \cdot F_\ell + 2F_\ell \cdot E_\ell + \frac{(K_\ell - K_\ell^{-1})^2}{(q_\ell - q_\ell^{-1})^2}, \frac{K_\ell - K_\ell^{-1}}{q_\ell - q_\ell^{-1}}\right)\right)\right]_\mathcal{W} \\
&= \left[p\Big(2\Theta_{a,b,c}(E_\ell) \cdot \Theta_{a,b,c}(F_\ell) + 2\Theta_{a,b,c}(F_\ell) \cdot \Theta_{a,b,c}(E_\ell) + \frac{(\Theta_{a,b,c}(K_\ell - K_\ell^{-1}))^2}{(q_\ell - q_\ell^{-1}))^2}, \frac{\Theta_{a,b,c}(K_\ell - K_\ell^{-1})}{q_\ell - q_\ell^{-1}}\right)\Bigg]_\mathcal{W}.
\end{aligned}$$

Now if we fix ℓ, then for every $j < \ell$ the $(j + 1, j + 1)$ entry of the diagonal matrix $\Theta_{a,b,c}(\tau_\ell(p(C, H)))$ is of the form

$$p\left(2(e_j + e_{j+1}) + \left(\frac{cq_\ell^{-2j} - c^{-1}q_\ell^{2j}}{q_\ell - q_\ell^{-1}}\right)^2, \frac{cq_\ell^{-2j} - c^{-1}q_\ell^{2j}}{q_\ell - q_\ell^{-1}}\right) =$$

$$\sum_{h=0}^{D} s_h \left(2(e_j + e_{j+1}) + \left(\frac{cq_\ell^{-2j} - c^{-1}q_\ell^{2j}}{q_\ell - q_\ell^{-1}}\right)^2\right) \left(\frac{cq_\ell^{-2j} - c^{-1}q_\ell^{2j}}{q_\ell - q_\ell^{-1}}\right)^m,$$

with $e_0 = e_\ell b = ab$. Furthermore the (ℓ, ℓ) entry of the same matrix is

$$p\left(2(e_{\ell-1} + e_\ell b) + \left(\frac{cq_\ell^2 - c^{-1}q_\ell^{-2}}{q_\ell - q_\ell^{-1}}\right)^2, \frac{cq_\ell^2 - c^{-1}q_\ell^{-2}}{q_\ell - q_\ell^{-1}}\right).$$

We have to choose a, b and c ensuring that for cofinitely many values of ℓ, some entries of this matrix are non-zero.

First take $c \in i\mathbb{R} - \{0\}$ (and so $\bar{c} = -c$). Then the first diagonal entry of the matrix (that corresponding to $j = 0$) is of the form $p(2(e_1 + ab) + (\frac{c - c^{-1}}{q_\ell - q_\ell^{-1}})^2, \frac{c - c^{-1}}{q_\ell - q_\ell^{-1}})$ where $e_1 = ab + \frac{c^{-1} - c}{q_\ell - q_\ell^{-1}}$

since $[1] = 1$ – incidentally observe that

$$(\star) \qquad 2(e_1 + ab) + (\frac{c - c^{-1}}{q_\ell - q_\ell^{-1}})^2 = 4ab - 2\,\frac{c - c^{-1}}{q_\ell - q_\ell^{-1}} + (\frac{c - c^{-1}}{q_\ell - q_\ell^{-1}})^2.$$

It follows that, if a, b are chosen such that the product ab is in $\mathbb{R}$, then $\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}} \in \mathbb{R}$ and $2(e_1 + ab) + (\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})^2$ also belongs to $\mathbb{R}$. So, whenever the $(1,1)$ entry of the matrix is 0, we find a common root of $p(x_1, x_2)$ and its complex conjugate $\bar{p}(x_1, x_2)$. Varying q_ℓ over a set of primitive roots of unity with distinct imaginary parts and observing that $\frac{c-c^{-1}}{q_{\ell_1} - q_{\ell_1}^{-1}} \neq \frac{c-c^{-1}}{q_{\ell_2} - q_{\ell_2}^{-1}}$ when $\ell_1 \neq \ell_2$ we get infinitely many distinct common roots.

Case 1: $p(x_1, x_2)$ and $\bar{p}(x_1, x_2)$ have no common irreducible factors. Our choice of a, b and c takes care of that case. In fact, Bezout theorem, when applied to the pair $p(x_1, x_2)$ and $\bar{p}(x_1, x_2)$, ensures that the $(1,1)$ entry of the matrix has to be non zero cofinitely many times.

Case 2: $p(x_1, x_2)$ and its complex conjugate have an irreducible factor in common. So, they have a common factor with real coefficients. Let us write $p(x_1, x_2) := p_0(x_1, x_2) \cdot p_1(x_1, x_2)$, with $p_1(x_1, x_2) \in \mathbb{R}[x_1, x_2]$ of degree > 0 and $p_0(x_1, x_2) \in \mathbb{C}[x_1, x_2] - \{0\}$. We claim that for an appropriate choice of a, b and c, strengthening the previous one, one gets that the value of $p_1(x_1, x_2)$ in the first entry of the matrix is non zero for cofinitely many q_ℓ (which ultimately leads to Case 1 for $p_0(x_1, x_2)$).

These further constraints on a, b and c are fixed as follows.

For simplicity rename $p_1(x_1, x_2)$, $p(x_1, x_2)$. Write it now as a polynomial in x_1 with as coefficients polynomials in x_2. In detail put $p(x_1, x_2) = \sum_{n=0}^{D'} t_n(x_2)\, x_1^n$ where the various $t_n(x_2)$ are polynomials with real coefficients and $t_{D'}(x_2) \neq 0$. The previous parenthetical remark $(\star)$ suggests the following change of variables

$$x_1 = 4x_1' - 2x_2' + (x_2')^2, \quad x_2 = x_2'.$$

In this way $p(x_1, x_2)$ becomes a polynomial $p'(x_1', x_2')$ that can be written as

$$\sum_n^{D'} \tilde{t}_n(x_2')\, 4^n\, (x_1')^n$$

for the same D' as before (indeed $\tilde{t}_{D'}(x_2') = t_{D'}(x_2)$).

Recall the way the q_ℓ have been chosen, as $e^{i\,\frac{2\pi l}{\ell}}$ with $1 \leq l < \ell$, l minimal such that $1 < -i(q_\ell - q_\ell^{-1}) < 2$. We sometimes set for simplicity $z_\ell := \frac{1}{q_\ell - q_\ell^{-1}}$. Note that, just due to our assumptions on q_ℓ, $2^{-1} \leq |z_\ell| \leq 1$.

Now choose c such that for all ℓ, $\bigwedge_{n=0}^{D'-1} |\tilde{t}_n(\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})| < r_2$ and $|\tilde{t}_{D'}(\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})| > r_1 > 0$.

Let us explain why and how these values r_1, r_2 can be found.

Consider any polynomial $g_\ell(x) := \sum_{n=0}^{k} \alpha_n\, z_\ell^n \cdot x^n$, where $\alpha_n \in \mathbb{R}$ for every n and $\alpha_k \neq 0$. First observe that, if we take $|c - c^{-1}| \leq r_3$ for some real $r_3 > 0$, then we can bound $|\sum_{n=0}^{k} \alpha_n\, z_\ell^n\, (c - c^{-1})^n|$ by $\sum_{n=0}^{k} |\alpha_n|\, r_3^n$. Second, choose $c - c^{-1}$ such that $|c - c^{-1}| > 2M$, where $M := max\{1, \sum_{n=0}^{k-1} \frac{|\alpha_n|}{|\alpha_k|}\, 2^{k-n}|\}$. Let us distinguish now two cases, according to whether α_k is positive or not.

(i) $\alpha_k > 0$. For x a positive real, evaluate

$$\alpha_k\, x^k\, |z_\ell|^k - |\sum_{n=0}^{k-1} \alpha_n\, x^n\, z_\ell^n| = x^{k-1}\, |z_\ell|^n\, \alpha_k\, (x - |\sum_{n=0}^{k-1} \frac{\alpha_n}{\alpha_k}\, z^{n-k}\, x^{n-k+1}|).$$

If $x > 2M$, then $x^{k-1}\,|z_\ell|^k\,\alpha_k\,(x - |\sum_{n=0}^{k-1} \frac{\alpha_n}{\alpha_k} \cdot z_\ell^{n-k} \cdot x^{n-k+1}|) > \alpha_k\, 2^{-1}\, M^k$ and consequently $|g_\ell(c - c^{-1})| > \alpha_k\, 2^{-1}\, M^k$.

(ii) $\alpha_k < 0$. Then $|x^k\, z^k\, \alpha_k + \sum_{n=0}^{k-1} \alpha_n\, x^n\, z^n| = |x^k\, z^k\, (-\alpha_k) + \sum_{j=0}^{k-1}(-\alpha_n)\, x^n\, z^n|$, and we are back to the previous case.

This explains r_1 and r_2.

At this point it suffices to choose $r = ab \in \mathbb{R}$ such that $|r| > max\{1, D' \cdot \frac{r_2}{r_1}\} > max\{1, \sum_{n=0}^{D'-1} \frac{|\tilde{t}_n(\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})|}{|t_{D'}(\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})|}\}$.

Suppose now that $u \notin \mathbb{U}_0$. So there exists $m \neq 0$ such that $u_m \neq 0$. Let m be maximal in absolute value such that $u_m \neq 0$. If $m > 0$, write $u_m = X^m \cdot p_m(C, H)$ and if $m < 0$, write $u_m = Y^m \cdot p_m(C, H)$, with $p_m(x_1, x_2)$ a non zero polynomial with coefficients in $\mathbb{C}$ and $p_m(C, H) \in \mathbb{U}_0 - \{0\}$. Set $\Theta_{a,b,c}(F_\ell) = F_b$ and $\Theta_{a,b,c}(E_\ell) = E_{a,b,c}$. Then for $\ell > 2m$, we have that F_b^m and $E_{a,b,c}^m$ have no entries in common.

If u has a non-zero component u_m with $m > 0$ (respectively $m < 0$), then we consider the product of the two matrices $E_{a,b,c}^m$ and $p_m(\Theta_{a,b,c}(C), \Theta_{a,b,c}(H))$ (respectively F_b^m and $p_m(\Theta_{a,b,c}(C), \Theta_{a,b,c}(H))$). The nonzero entries of the corresponding permutation matrix are of the form $e_j \cdot \ldots \cdot e_{j+m} \cdot p(2(e_{j+1} + e_j) + (\frac{cq_\ell^{-2j} - c^{-1}q_\ell^{2j}}{q_\ell - q_\ell^{-1}})^2, \frac{cq_\ell^{-2j} - c^{-1}q_\ell^{2j}}{q_\ell - q_\ell^{-1}})$ (respectively $b \cdot p(2(e_{j+1} + e_j) + (\frac{cq_\ell^{-2j} - c^{-1}q_\ell^{2j}}{q_\ell - q_\ell^{-1}})^2, \frac{cq_\ell^{-2j} - c^{-1}q_\ell^{2j}}{q_\ell - q_\ell^{-1}})$) with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ and $1 \leq j \leq \ell$ (with the convention that $j + m$ is calculated modulo ℓ). So, it suffices to evaluate the coefficient corresponding to the case when $j = \ell$ and we can apply the previous discussion. $\square$

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