ON CUBIC BIRATIONAL MAPS OF $\mathbb{P}^3_\mathbb{C}$

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ABSTRACT. We study the birational maps of $\mathbb{P}^3_\mathbb{C}$. More precisely we describe the irreducible components of the set of birational maps of bidegree $(3,3)$ (resp. $(3,4)$, resp. $(3,5)$).

1. INTRODUCTION

The CREMONA group, denoted $\mathrm{Bir}(\mathbb{P}^n_\mathbb{C})$, is the group of birational maps of $\mathbb{P}^n_\mathbb{C}$ into itself. If $n=2$ a lot of properties have been established (*see* [5, 9] for example). As far as we know the situation is much more different for $n\geq 3$ (*see* [14, 4] for example). If ψ is an element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ then $\deg\psi=\deg\psi^{-1}$. It is not the case in higher dimensions; if ψ belongs to $\mathrm{Bir}(\mathbb{P}^3_\mathbb{C})$ we only have the inequality $\deg\psi^{-1}\leq(\deg\psi)^2$ so one introduces the bidegree of ψ as the pair $(\deg\psi,\deg\psi^{-1})$. For $n=2$, $\mathfrak{Bir}_d(\mathbb{P}^2_\mathbb{C})$ is the set of birational maps of the complex projective plane of degree d; for $n\geq 3$ denote by $\mathrm{Bir}_{d,d'}(\mathbb{P}^n_\mathbb{C})$ the set of elements of $\mathrm{Bir}(\mathbb{P}^n_\mathbb{C})$ of bidegree (d,d'), and by $\mathfrak{Bir}_d(\mathbb{P}^n_\mathbb{C})$ the union $\cup_{d'}\mathrm{Bir}_{d,d'}(\mathbb{P}^n_\mathbb{C})$. The set $\mathfrak{Bir}_d(\mathbb{P}^n_\mathbb{C})$ inherits a structure of algebraic variety as a locally closed subspace a projective space ([3, Lemma 2.4, Proposition 2.15]), and we will always consider it with the ZARISKI topology ([8, 17]).

The varieties $\mathfrak{Bir}_2(\mathbb{P}^2_\mathbb{C})$ and $\mathfrak{Bir}_3(\mathbb{P}^2_\mathbb{C})$ are described in [6]: $\mathfrak{Bir}_2(\mathbb{P}^2_\mathbb{C})$ is smooth, and irreducible in the space of quadratic rational maps of the complex projective plane whereas $\mathfrak{Bir}_3(\mathbb{P}^2_\mathbb{C})$ is irreducible, and rationnally connected. Besides, $\mathfrak{Bir}_d(\mathbb{P}^2_\mathbb{C})$ is not irreducible as soon as $d>3$ (*see* [2]). In [7] CREMONA studies three types of generic elements of $\mathfrak{Bir}_2(\mathbb{P}^3_\mathbb{C})$. Then there were some articles on the subject, and finally a precise description of $\mathfrak{Bir}_2(\mathbb{P}^3_\mathbb{C})$; the left-right conjugacy is the following one

$$\mathrm{PGL}(4;\mathbb{C})\times\mathrm{Bir}(\mathbb{P}^3_\mathbb{C})\times\mathrm{PGL}(4;\mathbb{C})\to\mathrm{Bir}(\mathbb{P}^3_\mathbb{C}),\qquad (A,\psi,B)\mapsto A\psi B^{-1}.$$

PAN, RONGA and VUST give quadratic birational maps of $\mathbb{P}^3_\mathbb{C}$ up to left-right conjugacy, and show that there are only finitely many biclasses ([15, Theorems 3.1.1, 3.2.1, 3.2.2, 3.3.1]). In particular they show that $\mathfrak{Bir}_2(\mathbb{P}^3_\mathbb{C})$ has three irreducible components of dimension 26, 28, 29; the component of dimension 26 (resp. 28, resp. 29) corresponds to birational maps of bidegree $(2,4)$ (resp. $(2,3)$, resp. $(2,2)$). We will see that the situation is slightly different for $\mathfrak{Bir}_3(\mathbb{P}^3_\mathbb{C})$; in particular we cannot expect such an explicit list of biclasses because there are infinitely many of biclasses (already the dimension of the family $\mathcal{E}_2$ of the classic cubo-cubic example is 39 that is strictly larger that $\dim(\mathrm{PGL}(4;\mathbb{C})\times\mathrm{PGL}(4;\mathbb{C}))=30$). That's why the approach is different.

We do not have such a precise description of $\mathfrak{Bir}_d(\mathbb{P}^3_\mathbb{C})$ for $d\geq 4$. Nevertheless we can find a very fine and classical contribution for $\mathfrak{Bir}_3(\mathbb{P}^3_\mathbb{C})$ due to HUDSON ([11]); in §A we reproduce

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Table VI of [11]. HUDSON introduces there some invariants to establish her classification. But it gives rise to many cases, and we also find examples where invariants take values that do not appear in her table. We do not know references explaining how her families fall into irreducible components of $\mathrm{Bir}_{3,d}(\mathbb{P}^3_{\mathbb{C}})$ so we focus on this natural question.

Definition. An element ψ of $\mathrm{Bir}_{3,d}(\mathbb{P}^3_{\mathbb{C}})$ is ***ruled*** if the strict transform of a generic plane under ψ^{-1} is a ruled cubic surface.

Denote by $\mathfrak{ruled}_{3,d}$ the set of $(3,d)$ ruled maps; we detail it in Lemma 2.3. Let us remark that there are no ruled birational maps of bidegree $(3,d)$ with $d\geq 6$.

We describe the irreducible components of $\mathrm{Bir}_{3,d}(\mathbb{P}^3_{\mathbb{C}})$ for $3\leq d\leq 5$. Let us recall that the inverse of an element of $\mathrm{Bir}_{3,2}(\mathbb{P}^3_{\mathbb{C}})$ is quadratic and so treated in [15].

Theorem A. *Assume that* $2\leq d\leq 5$. *The set* $\mathfrak{ruled}_{3,d}$ *is an irreducible component of* $\mathrm{Bir}_{3,d}(\mathbb{P}^3_{\mathbb{C}})$.

In bidegree $(3,3)$ (*resp.* $(3,4)$) *there is only an other irreducible component; in bidegree* $(3,5)$ *there are three others.*

The set $\mathfrak{ruled}_{3,3}$ *intersects the closure of any irreducible component of* $\overline{\mathrm{Bir}_{3,4}(\mathbb{P}^3_{\mathbb{C}})}$ (*the closures being taken in* $\mathfrak{Bir}_3(\mathbb{P}^3_{\mathbb{C}})$).

Notations 1.1. *Consider a dominant rational map* ψ *from* $\mathbb{P}^3_{\mathbb{C}}$ *into itself. For a generic line* ℓ, *the preimage of* ℓ *by* ψ *is a complete intersection* Γ_ℓ; *define the scheme* $\mathcal{C}_2$ *to be the union of the irreducible components of* Γ_ℓ *supported in the base locus of* ψ. *Define* $\mathcal{C}_1$ *by liaison from* $\mathcal{C}_2$ *in* Γ_ℓ. *Remark that if* ψ *is birational, then* $\mathcal{C}_1=\psi_*^{-1}(\ell)$. *Let us denote by* $\mathfrak{p}_a(\mathcal{C}_i)$ *the arithmetic genus of* $\mathcal{C}_i$.

It is difficult to find a uniform approach to classify elements of $\mathfrak{Bir}_3(\mathbb{P}^3_{\mathbb{C}})$. Nevertheless in small genus we succeed to obtain some common detailed results; before stating them, let us introduce some notations.

Let us remark that the inequality $\deg\psi^{-1}\leq(\deg\psi)^2$ mentioned previously directly follows from

$$(\deg\psi)^2=\deg\psi^{-1}+\deg\mathcal{C}_2.$$

Proposition B. *Let* ψ *be a* $(3,d)$ *birational map.*

Assume that ψ *is not ruled, and* $\mathfrak{p}_a(\mathcal{C}_1)=0$, *i.e.* $\mathcal{C}_1$ *is smooth. Then*

- $d\leq 6$;
- *and* $\mathcal{C}_2$ *is a curve of degree* $9-d$, *and arithmetic genus* $9-2d$.

Suppose $\mathfrak{p}_a(\mathcal{C}_1)=1$, *and* $2\leq d\leq 6$. *Then*

- *there exists a singular point* p *of* $\mathcal{C}_1$ *independent of the choice of* $\mathcal{C}_1$;
- *if* $d\leq 4$, *all the cubic surfaces of the linear system* Λ_ψ *are singular at* p;
- *the curve* $\mathcal{C}_2$ *is of degree* $9-d$, *of arithmetic genus* $10-2d$, *and lies on a unique quadric* Q; *more precisely* $I_{\mathcal{C}_2}=(Q,\mathcal{S}_1,\dots,\mathcal{S}_{d-2})$ *where the* $\mathcal{S}_i$*'s are independent cubics modulo* Q.

We denote by $\mathrm{Bir}_{3,d,\mathfrak{p}_2}(\mathbb{P}^3_{\mathbb{C}})$ the subset of **non-ruled** $(3,d)$ birational maps such that $\mathcal{C}_2$ is of degree $9-d$, and arithmetic genus $\mathfrak{p}_2$. One has the following statement:

Theorem C. *If* $\mathfrak{p}_2\in\{3,\,4\}$, *then* $\mathrm{Bir}_{3,3,\mathfrak{p}_2}(\mathbb{P}^3_{\mathbb{C}})$ *is non-empty, and irreducible;* $\mathrm{Bir}_{3,3,\mathfrak{p}_2}(\mathbb{P}^3_{\mathbb{C}})$ *is empty as soon as* $\mathfrak{p}_2\not\in\{3,\,4\}$.

If $\mathfrak{p}_2\in\{1,\,2\}$*, then* $\mathrm{Bir}_{3,4,\mathfrak{p}_2}(\mathbb{P}^3_\mathbb{C})$ *is non-empty, and irreducible;* $\mathrm{Bir}_{3,4,\mathfrak{p}_2}(\mathbb{P}^3_\mathbb{C})$ *is empty as soon as* $\mathfrak{p}_2\not\in\{1,\,2\}$.

The set $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_\mathbb{C})$ *is empty as soon as* $\mathfrak{p}_2\not\in\{-1,\,0,\,1\}$ *and*

- *if* $\mathfrak{p}_2=-1$*, then* $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_\mathbb{C})$ *is non-empty, and irreducible;*
- *if* $\mathfrak{p}_2=0$*, then* $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_\mathbb{C})$ *is non-empty, and has two irreducible components;*
- *if* $\mathfrak{p}_2=1$*, then* $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_\mathbb{C})$ *is non-empty, and has three irreducible components.*

Organization of the article. In §2 we explain the particular case of ruled birational maps and set some notations. Then §3 is devoted to liaison theory that plays a big role in the description of the irreducible components of $\mathrm{Bir}_{3,3}(\mathbb{P}^3_\mathbb{C})$ (*see* §4), $\mathrm{Bir}_{3,4}(\mathbb{P}^3_\mathbb{C})$ (*see* §5) and $\mathrm{Bir}_{3,5}(\mathbb{P}^3_\mathbb{C})$ (*see* §6). In the last section we give some illustrations of invariants considered by HUDSON, especially concerning the local study of the preimage of a line. Since HUDSON's book is very old, let us recall her classification in the first part of the appendix.

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2. DEFINITIONS, NOTATIONS AND FIRST PROPERTIES

2.1. **Definitions and notations.** Let $\psi\colon\mathbb{P}^3_\mathbb{C}\dashrightarrow\mathbb{P}^3_\mathbb{C}$ be a rational map given, for some choice of coordinates, by

$$(z_0:z_1:z_2:z_3)\dashrightarrow\big(\psi_0(z_0,z_1,z_2,z_3):\psi_1(z_0,z_1,z_2,z_3):\psi_2(z_0,z_1,z_2,z_3):\psi_3(z_0,z_1,z_2,z_3)\big)$$

where the ψ_i's are homogeneous polynomials of the same degree d, and without common factors. The map ψ is called a **CREMONA *transformation*** or a ***birational map of*** $\mathbb{P}^3_\mathbb{C}$ if it has a rational inverse ψ^{-1}. The ***degree*** of ψ, denoted $\deg\psi$, is d. The pair $(\deg\psi,\deg\psi^{-1})$ is the ***bidegree*** of ψ, we say that ψ is a $(\deg\psi,\deg\psi^{-1})$ birational map. The ***indeterminacy set*** of ψ is the set of the common zeros of the ψ_i's. Denote by I_ψ the ideal generated by the ψ_i's, and by $\Lambda_\psi\subset\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^3_\mathbb{C}}(d)\big)$ the subspace of dimension 4 generated by the ψ_i, and by $\deg I_\psi$ the degree of the scheme defined by the ideal I_ψ. The scheme whose ideal is I_ψ is denoted F_ψ. It is called ***base locus*** of ψ. If $\dim F_\psi=0$ then $F^1_\psi=\emptyset$, otherwise F^1_ψ is the maximal subscheme of F_ψ of dimension 1 without isolated point, and without embedded point. Furthermore if $\mathcal{C}$ is a curve we will denote by $\omega_\mathcal{C}$ its dualizing sheaf.

Remark 2.1. We will sometimes identify a divisor of $\mathbb{P}^3_\mathbb{C}$ and its equation; for instance a surface Q of $\mathbb{P}^3_\mathbb{C}$ will have ideal (Q).

Let us give a few comments about Table VI of [11]. For any subscheme X of $\mathbb{P}^3_\mathbb{C}$ denote by I_X the ideal of X in $\mathbb{P}^3_\mathbb{C}$. Let ψ be a $(3,d)$ birational map. A point p is a ***double point*** if all the cubic surfaces of Λ_ψ are singular at p. A point p is a ***binode*** if all the cubic surfaces of Λ_ψ are singular at p with order 2 approximation at p a quadratic form of rank ≤ 2 (but this quadratic form is allowed to vary in Λ_ψ). In other words p is a binode if there is a degree 1 element h of I_p such that all the cubics belong to $(h\cdot I_p)+I_p^3$. A point p is a ***double point of contact*** if the general element of Λ_ψ is singular at p with order 2 approximation at p a quadratic form generically constant on Λ_ψ. In other words p is a double point of contact if all the cubics belong to $I_p^3+(Q)$ with Q of degree 2 and singular at p. A point p is a ***point of contact*** if all the cubics belong to $I_p^2+(\mathcal{S})$ where $\mathcal{S}$ is a cubic smooth at p. A point p is a ***point of osculation*** if all the cubics belong to $I_p^3+(\mathcal{S})$ where $\mathcal{S}$ is a cubic smooth at p.

Notations 2.2. *We will denote by $\mathcal{E}_i$ the i-th family of Table VI and by $\mathbb{C}[z_0,z_1,\ldots,z_n]_d$ the set of homogeneous polynomials of degree d in the variables z_0, z_1, …, z_n.*

2.2. **First properties.** Let us now focus on particular birational maps that cannot be dealt as the others: the ruled birational maps of $\mathbb{P}^3_\mathbb{C}$. Recall that there are two projective models of irreducible ruled cubic surfaces ; they both have the same normalization: $\mathbb{P}^2_\mathbb{C}$ blown up at one point which can be realized as a cubic surface in $\mathbb{P}^4_\mathbb{C}$ (*see* [10, Chapter 10, introduction of § 4.4], [10, Chapter 9, § 2.1]).

Lemma 2.3. *Assume that $2\leq d\leq 5$.*

- *The set $\mathfrak{ruled}_{3,d}$ is irreducible.*
- *Let ψ be a general element of $\mathfrak{ruled}_{3,d}$, and let δ be the common line to all elements of $\big\{\mathrm{Sing}\,\mathcal{S}\,|\,\mathcal{S}\in\Lambda_\psi\big\}$; then*

$$I_\psi = I_\delta^2\cap I_{\Delta_1}\cap I_{\Delta_2}\cap\ldots\cap I_{\Delta_{5-d}}\cap I_K$$

where Δ_i are disjoint lines that intersect δ at a unique point, and K is a general reduced scheme of length $2d-4$.

Proof. Let ψ be an element of $\mathfrak{ruled}_{3,d}$. Recall that F^1_ψ is the maximal subscheme of F_ψ of dimension 1 without isolated point, and without embedded point, *i.e.* F^1_ψ is a locally COHEN-MACAULAY curve.

An irreducible element $\mathcal{S}$ of Λ_ψ is a ruled surface; it is the projection of a smooth cubic ruled surface $\widetilde{\mathcal{S}}$ of $\mathbb{P}^4_\mathbb{C}$. Recall that $\widetilde{\mathcal{S}}$ is also the blow-up $\widetilde{\mathbb{P}^2_\mathbb{C}}(p)$ of $\mathbb{P}^2_\mathbb{C}$ at a point p. The embedding of $\widetilde{\mathcal{S}}$ in $\mathbb{P}^4_\mathbb{C}$ is given by the linear system $|I_p(2h)|$, where h is the class of an hyperplane in $\mathbb{P}^2_\mathbb{C}$. Let us denote by π the projection $\widetilde{\mathcal{S}}\to\mathcal{S}$, by H the class of the restriction of an hyperplane of $\mathbb{P}^4_\mathbb{C}$ to $\widetilde{\mathcal{S}}$, and by E_p the exceptional divisor associated to the blow-up of p. Set $\widetilde{\delta}=\pi^{-1}\delta$, $\widetilde{\mathcal{C}_1}=\pi^{-1}(\mathcal{C}_1)$, and $\widetilde{F^1_\psi}=\pi^{-1}(F^1_\psi)$. One has

$$\widetilde{\delta}\sim\pi^*h, \qquad H=2\pi^*h-E_p, \qquad f=\pi^*h-E_p, \qquad \widetilde{F^1_\psi}=2\widetilde{\delta}+D$$

where D is an effective divisor. As $\widetilde{\mathcal{C}_1}+\widetilde{F^1_\psi}=3H$, $\widetilde{\mathcal{C}_1}\cdot f=1$ and $\widetilde{\mathcal{C}_1}\cdot H=d$ one gets $D\cdot f=0$, and $D\cdot H=5-d$; therefore $D=(5-d)f$. And we conclude that ψ has a residual base scheme of length $2d-4$ from $\widetilde{\mathcal{C}_1}^2=2d-3$.

Conversely, take a ruled cubic surface $\widetilde{\mathcal{S}}$ in $\mathbb{P}^4_\mathbb{C}$, choose a general projection π to $\mathbb{P}^3_\mathbb{C}$, take a general element Δ of $|\mathcal{O}_{\widetilde{\mathcal{S}}}\big((5-d)f\big)|$ and take a set $\widetilde{K}$ of $2d-4$ general points on $\widetilde{\mathcal{S}}$ of ideal $I_{\widetilde{K}}$. We have $\mathrm{h}^0\big(I_{\widetilde{K}}(\widetilde{\mathcal{C}_1})\big)=3$, and thanks to the equality $\widetilde{\mathcal{C}_1}^2=2d-3$ and the surjection $\mathrm{H}^0\mathcal{O}_{\mathbb{P}^3_\mathbb{C}}(3)\twoheadrightarrow\mathrm{H}^0\mathcal{O}_{\mathcal{S}}(3)$ we get an element of $\mathfrak{ruled}_{3,d}$. Therefore the family of such $(\widetilde{\mathcal{S}},\pi,\Delta,I_{\widetilde{K}})$ dominates $\mathfrak{ruled}_{3,d}$, and $\mathfrak{ruled}_{3,d}$ is irreducible.

To prove the claimed decomposition of I_ψ for the general element $\psi \in \mathfrak{ruled}_{3,d}$, let us denote by $\mathcal{J}$ the right part of the equality.

$$\mathcal{J} = I_\delta^2 \cap I_{\Delta_1} \cap I_{\Delta_2} \cap \ldots \cap I_{\Delta_{5-d}} \cap I_K.$$

We already have $I_\psi \subset \mathcal{J}$. Furthermore, thanks to the computations on $\widetilde{S}$, one has $\mathrm{h}^0\big(\mathcal{J}(3)\big) = \dim \Lambda_\psi = 4$, and $\mathrm{h}^0\big(\mathcal{J}(2)\big) = 0$; hence to prove $I_\psi = \mathcal{J}$ we just have to prove that $\mathcal{J}$ is generated by polynomials of degree ≤ 3. As ψ is general, one can assume that the line defined by (z_2, z_3) is trisecant to $\Delta_1 \cup \Delta_2 \cup \Delta_3$, so up to a change of coordinates we have

$$I_\delta = (z_0, z_1), \quad I_{\Delta_1} = (z_0, z_2), \quad I_{\Delta_2} = (z_1, z_3), \quad I_{\Delta_3} = (z_0 + z_1, z_2 + z_3)$$

then

$$I_\delta^2 = \big(z_0^2, z_0z_1, z_1^2\big), \quad I_\delta^2 \cap I_{\Delta_1} = \big(z_0^2, z_0z_1, z_1^2z_2\big), \quad I_\delta^2 \cap I_{\Delta_1} \cap I_{\Delta_2} = \big(z_0z_1, z_0^2z_3, z_1^2z_2\big),$$
$$I_\delta^2 \cap I_{\Delta_1} \cap I_{\Delta_2} \cap I_{\Delta_3} = \big(z_0z_1(z_0+z_1), z_0z_1(z_2+z_3), z_1z_2(z_0+z_1), z_0z_3(z_0+z_1)\big)$$

so we have the equality $I_\psi = \mathcal{J}$ when $d = 2$. For $d \in \{3,4,5\}$, it is now enough to produce examples of $2d-4$ points such that this equality is true to obtain it to $2d-4$ general points. So consider the following 3 pairs of points of ideal

$$\mathcal{A}_1 = (z_0 + z_2, z_1 + 2z_2 + z_3, z_0z_3), \; \mathcal{A}_2 = (2z_0 + z_1 + z_2, z_3 - z_1, z_1z_2), \; \mathcal{A}_3 = (z_2, z_0 + z_3, z_1^2 - z_0^2).$$

Denote by I_{ψ_3} (resp. I_{ψ_4}, I_{ψ_5}) the ideal generated by the 4 cubics of $\mathcal{J}_3 = I_\delta^2 \cap I_{\Delta_1} \cap I_{\Delta_2} \cap \mathcal{A}_3$ (resp. $\mathcal{J}_4 = I_\delta^2 \cap I_{\Delta_1} \cap \mathcal{A}_2 \cap \mathcal{A}_3$, $\mathcal{J}_5 = I_\delta^2 \cap \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$). Then we can compute the following equalities for instance with Macaulay2: $I_{\psi_3} = (z_1^2z_2, z_0z_1z_2, z_0z_1^2 + z_0^2z_3, z_0^2z_1 + z_0z_1z_3)$, $I_{\psi_4} = (z_1^2z_2, z_0z_1z_2, z_0^2z_1 - z_0z_1^2 - z_0^2z_3 + z_0z_1z_3, 2z_0^3 + z_0z_1^2 + z_0^2z_2 + 3z_0^2z_3)$, $I_{\psi_5} = (2z_0z_1z_2 - z_1^2z_2, 5z_0z_1^2 + z_1^3 + 7z_1^2z_2 + 4z_0^2z_3 + z_0z_1z_3 + z_1^2z_3, 10z_0^2z_1 + 2z_1^3 + 9z_1^2z_2 - 2z_0^2z_3 + 12z_0z_1z_3 + 2z_1^2z_3, 40z_0^3 - 4z_1^3 + 20z_0^2z_2 - 3z_1^2z_2 + 44z_0^2z_3 - 4z_0z_1z_3 - 4z_1^2z_3)$,

$$I_{\psi_3} = \mathcal{J}_3, \; I_{\psi_4} = \mathcal{J}_4, \; (z_0, z_1, z_2, z_3) = (\mathcal{J}_5 : I_{\psi_5}).$$

So we have obtained the claimed decomposition of I_ψ but when $d = 5$ the ideal I_K has an irrelevant component. □

Lemma 2.4. *The following inclusions hold:*

$$\mathfrak{ruled}_{3,2} \subset \overline{\mathfrak{ruled}_{3,3}}, \qquad \mathfrak{ruled}_{3,3} \subset \overline{\mathfrak{ruled}_{3,4}}, \qquad \mathfrak{ruled}_{3,4} \subset \overline{\mathfrak{ruled}_{3,5}}.$$

Proof (*with the notations introduced in the proof of Lemma* 2.3). Let us start with an element of $\overline{\mathfrak{ruled}_{3,5}}$ with base curve δ^2 and 6 base points p_i in general position as decribed in Lemma 2.3. Then move two of the p_i, for instance p_1, p_2 until the line (p_1p_2) intersects δ. The line (p_1p_2) is now automatically in the base locus of the linear system Λ_ψ, and we obtain like this a generic element of $\mathfrak{ruled}_{3,4}$.

A similar argument allows to prove the two other inclusions. □

Let us recall the notion of genus of a birational map ([11, Chapter IX]). The ***genus*** $\mathfrak{g}_\psi$ of $\psi \in \mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ is the geometric genus of the curve $h \cap \psi^{-1}(h')$ where h and h' are generic hyperplanes of $\mathbb{P}^3_{\mathbb{C}}$. The equality $\mathfrak{g}_\psi = \mathfrak{g}_{\psi^{-1}}$ holds.

Remark 2.5. If ψ is a birational map of $\mathbb{P}^3_{\mathbb{C}}$ of degree 1 (resp. 2, resp. 3) then $\mathfrak{g}_\psi = 0$ (resp. $\mathfrak{g}_\psi = 0$, resp. $\mathfrak{g}_\psi \leq 1$).

One can give a characterization of ruled maps of $\mathrm{Bir}_{3,d}(\mathbb{P}^3_\mathbb{C})$ in terms of the genus.

Proposition 2.6. *Let ψ be in $\mathrm{Bir}_{3,d}(\mathbb{P}^3_\mathbb{C})$, $2\leq d\leq 5$. The genus of ψ is zero if and only if ψ is ruled.*

Proof. On the one hand there exists a curve $\mathcal{C}$ such that $\mathcal{I}_\psi\subset\mathcal{I}^2_\mathcal{C}$ if and only if ψ is ruled; on the other hand $\mathfrak{g}_\psi=0$ if and only if for generic hyperplanes h, h' of $\mathbb{P}^3_\mathbb{C}$ the curve $h\cap\psi^{-1}(h')$ is a singular rational cubic. $\square$

3. LIAISON

According to [16] we say that two curves Γ_1 and Γ_2 of $\mathbb{P}^3_\mathbb{C}$ are ***geometrically linked*** if

- $\Gamma_1\cup\Gamma_2$ is a complete intersection,
- Γ_1 and Γ_2 have no common component.

Let Γ_1 and Γ_2 be two curves geometrically linked. Recall that $\mathcal{I}_{\Gamma_1\cup\Gamma_2}=\mathcal{I}_{\Gamma_1}\cap\mathcal{I}_{\Gamma_2}$. According to [16, Proposition 1.1] one has $\frac{\mathcal{I}_{\Gamma_1}}{\mathcal{I}_{\Gamma_1\cup\Gamma_2}}=\mathrm{Hom}\big(\mathcal{O}_{\Gamma_2},\mathcal{O}_{\Gamma_1\cup\Gamma_2}\big)$. Since the kernel of $\mathcal{O}_{\Gamma_1\cup\Gamma_2}\longrightarrow\mathcal{O}_{\Gamma_2}$ is $\frac{\mathcal{I}_{\Gamma_1}}{\mathcal{I}_{\Gamma_1\cup\Gamma_2}}$ one gets the following fundamental statement: if Γ_1, Γ_2 are two curves geometrically linked, then

$$0\longrightarrow\omega_{\Gamma_1}\longrightarrow\omega_{\Gamma_1\cup\Gamma_2}\longrightarrow\mathcal{O}_{\Gamma_2}\otimes\omega_{\Gamma_1\cup\Gamma_2}\longrightarrow 0.$$

Lemma 3.1. *Let ψ be a rational map of $\mathbb{P}^3_\mathbb{C}$ of degree 3. We have*

$$\omega_{\mathcal{C}_1\cup\mathcal{C}_2}=\mathcal{O}_{\mathcal{C}_1\cup\mathcal{C}_2}(2h),$$

where h denotes an hyperplane of $\mathbb{P}^3_\mathbb{C}$, and for $i\in\{1,2\}$

$$0\longrightarrow\omega_{\mathcal{C}_i}\longrightarrow\mathcal{O}_{\mathcal{C}_i\cup\mathcal{C}_{3-i}}(2h)\longrightarrow\mathcal{O}_{\mathcal{C}_{3-i}}(2h)\longrightarrow 0 \tag{3.1}$$

and

$$0\longrightarrow\mathcal{I}_{\mathcal{C}_1\cup\mathcal{C}_2}(3h)\longrightarrow\mathcal{I}_{\mathcal{C}_i}(3h)\longrightarrow\omega_{\mathcal{C}_{3-i}}(h)\longrightarrow 0 \tag{3.2}$$

The first exact sequence (3.1) directly implies the following equalities ($i\in\{1,2\}$)

$$\mathrm{H}^0\big(\omega_{\mathcal{C}_i}(-h)\big)=\mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}_{3-i}}(h)\big),\qquad \mathrm{H}^0\big(\omega_{\mathcal{C}_i}\big)=\mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}_{3-i}}(2h)\big),$$

$$\mathrm{h}^0\omega_{\mathcal{C}_i}(h)+2=\mathrm{h}^0\big(\mathcal{I}_{\mathcal{C}_{3-i}}(3h)\big),\qquad \mathrm{H}^0\big(\omega_{\mathcal{C}_i}(h)\big)=\frac{\mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}_{3-i}}(3h)\big)}{\mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}_1\cup\mathcal{C}_2}(3h)\big)}.$$

Corollary 3.2. *Let ψ be a rational map of $\mathbb{P}^3_\mathbb{C}$ of degree 3. The ideal $\mathcal{I}_{\mathcal{C}_{3-i}}$ is generated by cubics if and only if $\omega_{\mathcal{C}_i}(h)$ is globally generated.*

Proof. It directly follows from the exact sequence (3.2). $\square$

Corollary 3.3. *Let ψ be a rational map of $\mathbb{P}^3_\mathbb{C}$ of degree 3. Then*

$$\deg\mathcal{C}_2-\deg\mathcal{C}_1=\mathfrak{p}_a(\mathcal{C}_2)-\mathfrak{p}_a(\mathcal{C}_1).$$

Proof. Taking the restriction of (3.1) to $\mathcal{C}_i$ for $i = 1, 2$ gives

$$\deg \omega_{\mathcal{C}_i} = 2 \deg \mathcal{C}_i - \deg(\mathcal{C}_1 \cap \mathcal{C}_2),$$

and hence

$$\deg \mathcal{C}_2 - \deg \mathcal{C}_1 = \mathfrak{p}_a(\mathcal{C}_2) - \mathfrak{p}_a(\mathcal{C}_1).$$

□

Furthermore when $\mathcal{C}_1$ and $\mathcal{C}_2$ have no common component, and $\omega_{\mathcal{C}_i}$ is locally free, then length $(\mathcal{C}_1 \cap \mathcal{C}_2) = \deg \omega^\vee_{\mathcal{C}_i}(2h)$, *i.e.*

$$\sum_{p \in \mathcal{C}_1 \cap \mathcal{C}_2} \mathrm{length}(\mathcal{C}_1 \cap \mathcal{C}_2)_{\{p\}} = 2 \deg \mathcal{C}_i - 2\mathfrak{p}_a(\mathcal{C}_i) + 2.$$

In the preimage of a generic point of $\mathbb{P}^3_{\mathbb{C}}$ by ψ, the number of points that do not lie in the base locus is given by

$$3 \deg \mathcal{C}_1 - \sum_{p \in \mathcal{C}_1 \cap \mathcal{C}_2} \mathrm{length}(\mathcal{S} \cap \mathcal{C}_1)_{\{p\}} - \sum_{p \in \Theta} \mathrm{length}(\mathcal{S} \cap \mathcal{C}_1)_{\{p\}}$$

where $\mathcal{S} \in \Lambda_\psi$ is non-zero modulo $\mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}_1 \cup \mathcal{C}_2}(3h)\big)$, and where Θ denotes the set of irreducible components of dimension 0 of the base locus F_ψ of ψ.

Lemma 3.4. *Let ψ be a rational map of $\mathbb{P}^3_{\mathbb{C}}$ of degree 3. Let Θ denote the set of irreducible components of dimension 0 of F_ψ. The map ψ is birational if and only if*

$$1 = 3 \deg \mathcal{C}_1 - \sum_{p \in \mathcal{C}_1 \cap \mathcal{C}_2} \mathrm{length}(\mathcal{S} \cap \mathcal{C}_1)_{\{p\}} - \sum_{p \in \Theta} \mathrm{length}(\mathcal{S} \cap \mathcal{C}_1)_{\{p\}}.$$

Remark that the computation of $\mathrm{length}(\mathcal{S} \cap \mathcal{C}_1)_{\{p\}}$ depends on the nature of the singularity of the cubic surface and on the behavior of $\mathcal{C}_2$ in that point (*see* §7).

Lemma 3.5. *Let ψ be a $(3,d)$* CREMONA *map. Assume that $d \geq 4$, then $\mathcal{C}_1$ is not contained in a plane.*

Proof. Suppose for instance that $d = 4$; then $\mathcal{C}_1$ is contained in an irreducible cubic surface $\mathcal{S}$. If $\mathcal{C}_1$ is contained in a plane $\mathcal{P}$ then all the lines in $\mathcal{P}$ are quadrisecant to $\mathcal{S}$: contradiction with the irreducibility of $\mathcal{S}$. □

Lemma 3.6. *Let ψ be a $(3,d)$ birational map, and let p be a point on $\mathcal{C}_1$. Assume that the degree of the tangent cone of $\mathcal{C}_1$ at p is strictly less than 4. If any $\mathcal{S}$ in Λ_ψ is singular at p, then p belongs to $\mathcal{C}_2$.*

Proof. If any $\mathcal{S}$ in Λ_ψ is singular at p, then the degree of the tangent cone of $\mathcal{C}_1 \cup \mathcal{C}_2$ at p is at least 4 because it is the complete intersection of two surfaces singular at p. Hence p has to belong to $\mathcal{C}_2$. □

Lemma 3.7. *Let ψ be a non-ruled $(3,d)$ birational map, and let $\mathcal{C}_1$ be a general element of Λ_ψ. The support of* $\mathrm{Sing}\,\mathcal{C}_1$ *is independent of the choice of $\mathcal{C}_1$.*

Proof. Let us show that there is a singular point independent of the choice of $\mathcal{C}_1$. Let us consider an element $\mathcal{S}$ of Λ_ψ with finite singular locus. Let $\pi\colon\widetilde{\mathcal{S}}\to\mathcal{S}$ be a minimal desingularization of $\mathcal{S}$, and let $\widetilde{\mathcal{C}_1}$ be the strict transform of $\mathcal{C}_1$. The elements of Λ_ψ give a linear system in $|\mathcal{O}_{\widetilde{\mathcal{S}}}(\widetilde{\mathcal{C}_1})|$ whose base locus denoted Ω is finite. According to BERTINI's theorem applied on $\widetilde{\mathcal{S}}$ one has the inclusion $\operatorname{Sing}\mathcal{C}_1\subset\pi(\Omega)\cup\operatorname{Sing}\mathcal{S}$. The assertion thus follows from the fact that $\Omega\cup\operatorname{Sing}\mathcal{S}$ is finite. ☐

Theorem 3.8. *Let ψ be a $(3,d)$ birational map, $2\leq d\leq 6$, that is not ruled. Assume that $\mathfrak{p}_a(\mathcal{C}_1)=1$. Then*

- *there exists a singular point p of $\mathcal{C}_1$ independent of the choice of $\mathcal{C}_1$;*
- *if $d\leq 4$, all the cubic surfaces of the linear system Λ_ψ are singular at p;*
- *the curve $\mathcal{C}_2$ is of degree $9-d$, of arithmetic genus $10-2d$, and lies on a unique quadric Q; more precisely $\mathcal{I}_{\mathcal{C}_2}=(Q,\mathcal{S}_1,\ldots,\mathcal{S}_{d-2})$ where the $\mathcal{S}_i$'s are independent cubics modulo Q.*

Remark 3.9. As soon as $d=5$ the second assertion is not true. Indeed for $d=5$ we obtain two families: one for which all the elements of Λ_ψ are singular, and another one for which it is not the case (§6).

Proof. The first assertion directly follows from Lemma 3.7.

Since $\mathfrak{p}_a(\mathcal{C}_1)=1$, the curve $\mathcal{C}_2$ lies on a unique quadric Q. The arithmetic genus of $\mathcal{C}_2$ is obtained from $\deg\mathcal{C}_2-\deg\mathcal{C}_1=\mathfrak{p}_a(\mathcal{C}_2)-\mathfrak{p}_a(\mathcal{C}_1)$ (Corollary 3.3).

As $\mathfrak{p}_a(\mathcal{C}_1)=1$, $\omega_{\mathcal{C}_1}(h)$ has no base point, and $\mathcal{I}_{\mathcal{C}_2}$ is generated by cubics (Corollary 3.2). The number of cubics containing $\mathcal{C}_2$ independent modulo the multiple of Q is $d-2$: the liaison sequence (Lemma 3.1) becomes

$$0\longrightarrow\mathcal{O}_{\mathcal{C}_1}(h)\longrightarrow\mathcal{O}_{\mathcal{C}_1\cup\mathcal{C}_2}(3h)\longrightarrow\mathcal{O}_{\mathcal{C}_2}(3h)\longrightarrow 0$$

one gets that

$$\mathrm{h}^0\mathcal{O}_{\mathcal{C}_2}(3h)=\mathrm{h}^0\mathcal{O}_{\mathcal{C}_1\cup\mathcal{C}_2}(3h)-\mathrm{h}^0\mathcal{O}_{\mathcal{C}_1}(h)=18-d.$$

This implies that

$$\mathrm{h}^0\mathcal{I}_{\mathcal{C}_2}(3h)=20-\mathrm{h}^0\mathcal{O}_{\mathcal{C}_2}(3h)=d+2.$$

If we remove the four multiples of Q one obtains $d+2-4=d-2$ cubics, and finally $\mathcal{I}_{\mathcal{C}_2}=(Q,\mathcal{S}_1,\ldots,\mathcal{S}_{d-2})$. ☐

Corollary 3.3 and Theorem 3.8 imply Proposition B.

Proposition 3.10. *For $2\leq d\leq 5$ the set $\mathfrak{ruled}_{3,d}$ is an irreducible component of $\mathrm{Bir}_{3,d}(\mathbb{P}^3_\mathbb{C})$.*

Proof. Let us use the notations introduced in Lemma 2.3. Note that $F^1_\psi\subset\mathcal{C}_2$. If $\psi\in\mathrm{Bir}_{3,d}(\mathbb{P}^3_\mathbb{C})$ is not ruled then at a generic point $p\in F^1_\psi$ there exists an element of Λ_ψ smooth at p. Hence F^1_ψ is locally complete intersection at p and $\deg F^1_\psi=\deg\mathcal{C}_2$. In particular $\deg\mathcal{I}_\psi=9-d$.

Consider now a general element ψ in $\mathfrak{ruled}_{3,d}$. From Lemma 2.3 there is a line ℓ such that $\ell\subset\operatorname{Sing}\mathcal{S}$ for any $\mathcal{S}\in\Lambda_\psi$; the set F^1_ψ has an irreducible component whose ideal is $\mathcal{I}^2_\ell$ and F^1_ψ is not locally complete intersection. This multiple structure has to be contained in $\mathcal{C}_2$ but since $\mathcal{C}_2$ is generically locally complete intersection the inequality $\deg\mathcal{C}_2>\deg F^1_\psi$ holds; it can be rewritten $\deg\mathcal{I}_\psi<9-d$.

As $d>9$ for any $\psi\in\mathrm{Bir}_{3,d}(\mathbb{P}^3_\mathbb{C})$, I_ψ defines a 1-dimensional subscheme of $\mathbb{P}^3_\mathbb{C}$ so by the semi-continuity theorem $\psi\mapsto\deg I_\psi$ is upper semi-continuous. Hence the number $\deg I_\psi$ cannot decrease by specialization, and $\mathfrak{ruled}_{3,d}$ is not included in an irreducible component in $\mathrm{Bir}_{3,d}(\mathbb{P}^3_\mathbb{C})$ whose generic element is not ruled. ☐

Corollary 3.11. *Let ψ be a $(3,\cdot)$ birational map of $\mathbb{P}^3_\mathbb{C}$; if the general element of Λ_ψ is smooth or with isolated singularities, then* $\deg F^1_\psi=\deg\mathcal{C}_2$.

4. $(3,3)$ CREMONA TRANSFORMATIONS

4.1. Some known results.

4.1.1. In the literature one can find different points of view concerning the classification of $(3,3)$ birational maps. For example HUDSON introduced many invariants related to singularities of families of surfaces and gave four families described in §A; nevertheless we do not understand why the family $\mathcal{E}_{3,5}$ defined below does not appear. PAN chose an other point of view and regrouped $(3,3)$ birational maps into three families. A $(3,3)$ birational map ψ of $\mathbb{P}^3_\mathbb{C}$ is called ***determinantal*** if there exists a 4×3 matrix M with linear entries such that ψ is given by the four 3×3 minors of the matrix M; the inverse ψ^{-1} is also determinantal. Let us denote by $\mathbf{T}^{\mathbf{D}}_{3,3}$ the set of determinantal maps. A $(3,3)$ CREMONA transformation is a **DE JONQUIÈRES** one if and only if the strict transform of a general line under ψ^{-1} is a singular plane rational cubic curve whose singular point is fixed. For such a map there is always a quadric contracted onto a point, the corresponding fixed point for ψ^{-1} which is also a DE JONQUIÈRES transformation. The DE JONQUIÈRES transformations form the set $\mathbf{T}^{\mathbf{J}}_{3,3}$. PAN established the following ([13, Theorem 1.2]):

$$\mathrm{Bir}_{3,3}(\mathbb{P}^3_\mathbb{C})=\mathbf{T}^{\mathbf{D}}_{3,3}\cup\mathbf{T}^{\mathbf{J}}_{3,3}\cup\mathfrak{ruled}_{3,3};$$

in other words an element of $\mathrm{Bir}_{3,3}(\mathbb{P}^3_\mathbb{C})$ is a determinantal map, or a DE JONQUIÈRES map, or a ruled map.

Remark 4.1. One has $\mathbf{T}^{\mathbf{D}}_{3,3}=\mathrm{Bir}_{3,3,3}(\mathbb{P}^3_\mathbb{C})$ and $\mathbf{T}^{\mathbf{J}}_{3,3}=\mathrm{Bir}_{3,3,4}(\mathbb{P}^3_\mathbb{C})$; hence $\mathrm{Bir}_{3,3,\mathfrak{p}_2}(\mathbb{P}^3_\mathbb{C})$ is irreducible for $\mathfrak{p}_2\in\{3,4\}$ (*see* [13]).

Remark 4.2. The birational involution $(z_0z_1^2:z_0^2z_1:z_0^2z_2:z_1^2z_3)$ is determinantal, the matrix being

$$\begin{bmatrix} z_0 & z_3 & 0\\ -z_1 & 0 & z_2\\ 0 & 0 & -z_1\\ 0 & -z_0 & 0\end{bmatrix},$$

and also ruled: all the partial derivatives of the components of the map vanish on $z_0=z_1=0$. The CREMONA transformation $(z_0^3:z_0^2z_1:z_0^2z_2:z_1^2z_3)$ is a DE JONQUIÈRES and a ruled one; note that its primary decomposition (obtained with Macaulay2) is

$$(z_0^2,z_1^2)\cap(z_0^2,z_3)\cap(z_0^3,z_1,z_2)$$

and that its primary component (z_0^2,z_1^2) is locally complete intersection contrary to the general case (Lemma 2.3) [1].

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[1] Thanks to the referee for mentioning it to us.

One has ([12])

$$\mathbf{T}^{\mathbf{D}}_{3,3}\cap\mathbf{T}^{\mathbf{J}}_{3,3}=\emptyset,\qquad \mathbf{T}^{\mathbf{D}}_{3,3}\cap\mathfrak{ruled}_{3,3}\neq\emptyset,\qquad \mathbf{T}^{\mathbf{J}}_{3,3}\cap\mathfrak{ruled}_{3,3}\neq\emptyset.$$

We deal with the natural description of the irreducible components of $\mathrm{Bir}_{3,3}$ which does not coincide with PAN's point of view since one of his family is contained in the closure of another one.

4.2. **Irreducible components of the set of $(3,3)$ birational maps.**

4.2.1. *General description of* $(3,3)$ *birational maps.* One already describes an irreducible component of $\mathrm{Bir}_{3,3}(\mathbb{P}^3_{\mathbb{C}})$, the one that contains $(3,3)$ ruled birational maps (Proposition 3.10). Hence let us consider the case where the linear system Λ_ψ associated to $\psi\in\mathrm{Bir}_{3,3}(\mathbb{P}^3_{\mathbb{C}})$ contains a cubic surface without double line.

— If $\mathcal{C}_1$ is smooth then it is a twisted cubic, we are in family $\mathcal{E}_2$ of Table VI (*see* §A). In that case ψ is determinantal; more precisely a $(3,3)$ birational map is determinantal if and only if its base locus scheme is an arithmetically COHEN-MACAULAY curve of degree 6 and (arithmetic) genus 3 (*see* [1, Proposition 1]).

— Otherwise $\omega_{\mathcal{C}_1}=\mathcal{O}_{\mathcal{C}_1}$, and ψ belongs to the irreducible family $\mathbf{T}^{\mathbf{J}}_{3,3}$ of JONQUIÈRES maps ($\mathcal{E}_3$ in terms of HUDSON's classification). The curve $\mathcal{C}_2$ lies on a quadric described by the quadratic form Q. According to Theorem 3.8 the ideal of $\mathcal{C}_2$ is $(Q,\mathcal{S})$, and there exists a point p such that $p\in Q$, and p is a singular point of $\mathcal{S}$. Furthermore $I_\psi=I_pQ+(\mathcal{S})$. Reciprocally such a triplet $(p,Q,\mathcal{S})$ induces a birational map.

The family $\mathbf{T}^{\mathbf{J}}_{3,3}$ is stratified as follows by HUDSON (all the cases belong to $\overline{\mathcal{E}_3}$):

- *Description of* $\mathcal{E}_3$. The general element of $I_pQ+(\mathcal{S})$ has an ordinary quadratic singularity at p (configuration $(2,2)$ of `Table` 1 (*see* §7)), and the generic cubic is singular at p with a quadratic form of rank 3.
- *Description of* $\mathcal{E}_{3.5}$. The point p lies on Q (p is a smooth point or not) and the generic cubic is singular at p with a quadratic form of rank 2. In other words p is a binode and this happens when one of the two biplanes is contained in T_pQ, it corresponds to the configuration $(2,3)'$ of `Table` 1 (*see* §7). The generic cubic is singular at p with a quadratic form of rank 2; this case does not appear in Table VI (*see* §A). Let us denote by $\mathcal{E}_{3.5}$ the set of the associated $(3,3)$ birational maps. The curve $\mathcal{C}_2$ has degree 6 and a triple point (in Q).
- *Description of* $\mathcal{E}_4$. The point p is a double point of contact, it corresponds to configuration $(2,4)$ of `Table` 1 (*see* §7).

Proposition 4.3. *One has*

$$\dim\mathcal{E}_2=39,\quad \dim\mathcal{E}_3=38,\quad \dim\mathcal{E}_{3.5}=35,\quad \dim\mathcal{E}_4=35,\quad \dim\mathcal{E}_5=31,$$

and

$$\overline{\mathcal{E}_3}=\mathbf{T}^{\mathbf{J}}_{3,3},\quad \mathring{\mathcal{E}}_{3.5}\subset\overline{\mathcal{E}_3},\quad \mathring{\mathcal{E}}_4\subset\overline{\mathcal{E}_3},\quad \mathring{\mathcal{E}}_4\not\subset\overline{\mathcal{E}_{3.5}},\quad \mathring{\mathcal{E}}_{3.5}\not\subset\overline{\mathcal{E}_4}.$$

Proof. Let us justify the equality $\dim\mathcal{E}_3=38$. We have to choose a quadric Q and a point p on Q, this gives $9+2=11$. Then we take a cubic surface singular at p that yields to $19-4=15$; since we look at this surface modulo pQ one gets $15-3=12$ so

$$\dim \mathcal{E}_3 = 11+12+15 = 38.$$

Let us deal with $\dim \mathcal{E}_4$. We take a singular quadric Q this gives 8. Then we take a cubic singular at p, modulo pQ and this yields to $19-4-3=12$, and finally one obtains $12+8+15=35$. $\square$

4.2.2. *Irreducible components.*

Theorem 4.4. *The set* $\mathfrak{ruled}_{3,3}$ *is an irreducible component of* $\mathrm{Bir}_{3,3}(\mathbb{P}^3_{\mathbb{C}})$*, and there is only one another irreducible component in* $\mathrm{Bir}_{3,3}(\mathbb{P}^3_{\mathbb{C}})$*. More precisely the set of the de* JONQUIÈRES *maps* $\overline{\mathcal{E}_3}$ *is contained in the closure of determinantal ones* $\overline{\mathcal{E}_2}$ *whereas* $\mathfrak{ruled}_{3,3} \not\subset \overline{\mathcal{E}_2}$*.*

Proof. Let us consider the matrix A given by

$$\begin{bmatrix} 0 & 0 & 0 \\ -z_1 & -z_2 & 0 \\ z_0 & 0 & -z_2 \\ 0 & z_0 & z_1 \end{bmatrix}$$

and let A_i denote the matrix A minus the $(i+1)$-th line. If $i>0$, the 2×2 minors of A_i are divisible by z_{i-1}.

Consider the 4×3 matrix B given by $\left[b_{ij}\right]_{1\leq i\leq 4,1\leq j\leq 3}$ with $b_{ij}\in \mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(1)\big)$; as previously, B_i is the matrix B minus the $(i+1)$-th line. Denote by $\Delta^{j,k}$ the determinant of the matrix A_0 minus the j-th line and the k-th column; the $\Delta^{j,k}$ generate $\mathbb{C}[z_0,z_1,z_2]_2$. One has

$$\det(A_0+tB_0) = t\cdot S \quad [t^2]$$

where

$$S = (b_{21}+b_{43})\Delta^{1,1} - (b_{31}-b_{42})\Delta^{2,1} + (b_{33}-b_{22})\Delta^{1,2} + b_{23}\Delta^{1,3} + b_{32}\Delta^{2,2} + b_{41}\Delta^{3,1}$$

is a generic cubic of the ideal $(z_0,z_1,z_2)^2$. For $i>0$

$$\det(A_i+tB_i) = \det A_i + t\cdot(z_{i+1}Q)(-1)^{i+1} = t\cdot(z_{i+1}Q)(-1)^{i+1} \quad [t^2]$$

where $Q=b_{1,1}z_2-b_{1,2}z_1+b_{1,3}z_0$ is the equation of a generic quadric that contains $(0,0,0,1)$. So the map

$$\left[\frac{\det(A_0+tB_0)}{t} : \frac{\det(A_1+tB_1)}{t} : \frac{\det(A_2+tB_2)}{t} : \frac{\det(A_3+tB_3)}{t}\right]$$

allows to go from $\mathcal{E}_2$ to a general element of $\overline{\mathcal{E}_3}$.

Furthermore $\overline{\mathcal{E}_3}$ and $\mathfrak{ruled}_{3,3}$ are different components (Proposition 3.10). $\square$

5. $(3,4)$ CREMONA TRANSFORMATIONS

5.1. **General description of** $(3,4)$ **birational maps.** The ruled maps $\mathfrak{ruled}_{3,4}$ give rise to an irreducible component (Proposition 3.10). Let us now focus on the case where the linear system Λ_ψ associated to $\psi\in\mathrm{Bir}_{3,4}(\mathbb{P}^3_{\mathbb{C}})$ contains a cubic surface without double line.

- First case: $\mathcal{C}_1$ is smooth. From $\mathrm{h}^0\omega_{\mathcal{C}_1}(h)=3$ one gets that $\mathcal{C}_2$ lies on five cubics. Since $\mathfrak{p}_a(\mathcal{C}_1)=0$ we have $\omega_{\mathcal{C}_2}=\mathcal{O}_{\mathcal{C}_2}$ (Corollary 3.3). The curve $\mathcal{C}_1$ lies on a quadric (Lemma 3.1). This configuration corresponds to $\mathcal{E}_6$.
- Second case: $\mathcal{C}_1$ is a singular curve of degree 4 not contained in a plane (*see* Lemma 3.5) so $\omega_{\mathcal{C}_1}=\mathcal{O}_{\mathcal{C}_1}$. The curve $\mathcal{C}_1$ lies on two quadrics and $\mathcal{C}_2$ on six cubics ($\mathrm{h}^0\omega_{\mathcal{C}_1}(h)=4$). Let p be the singular point of $\mathcal{C}_1$; all elements of Λ_ψ are singular at p (Theorem 3.8), and p belongs to $\mathcal{C}_2$ (Lemma 3.6). The curve $\mathcal{C}_2$ lies on a unique quadric $\mathcal{Q}$ (Theorem 3.8), is linked to a line ℓ in a $(2,3)$ complete intersection $\mathcal{Q}\cap\mathcal{S}_1$ (with $\deg\mathcal{Q}=2$ and $\deg\mathcal{S}_1=3$), and $\mathcal{I}_{\mathcal{C}_2}=(\mathcal{Q},\mathcal{S}_1,\mathcal{S}_2)$ with $\deg\mathcal{S}_2=3$ (Theorem 3.8).

  Since $\mathcal{C}_1$ is of degree 4 and arithmetic genus 1, one has $\mathrm{H}^0\big(\mathcal{O}_{\mathcal{C}_1}(h)\big)=\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^3_\mathbb{C}}(1)\big)$. Let us consider $L=\mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}_1\cup\mathcal{C}_2}(3h)\big)\subset\Lambda_\psi$ and the map

$$\mathrm{H}^0\big(\mathcal{O}_{\mathcal{C}_1}(h)\big)\longrightarrow\frac{\mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}_2}(3h)\big)}{L},\qquad h\mapsto Qh;$$

  it is injective. Indeed $\dim(\mathcal{C}_1\cap\mathcal{Q})=0$ thus modulo $\mathcal{Q}$ the cubics defining $\mathcal{C}_1$ are independent. Therefore Λ_ψ is contained in $(\mathcal{Q}\mathcal{I}_p,\mathcal{S}_1,\mathcal{S}_2)$. Let $\pi\colon\widetilde{\mathcal{C}_1}\to\mathcal{C}_1$ be the normalization. The linear system induced by $\psi\circ\pi$ is given by $\pi^*\omega_{\mathcal{C}_1}(h)=\pi^*\mathcal{O}_{\mathcal{C}_1}(h)$, and vanishes on $\pi^{-1}(p)$. This linear system has degree 4 and the conductor $\pi^{-1}(p)$ has length 2 because $\mathfrak{p}_a(\mathcal{C}_1)=1$. So it has a residual base point $p_1\in\widetilde{\mathcal{C}_1}$ because ψ sends birationally $\mathcal{C}_1$ onto a line. Deforming p_1 to a general point p' of C_1 we obtain the 4 dimensional vector space

$$\Lambda=\mathrm{H}^0\big(((\mathcal{Q}\mathcal{I}_p,\mathcal{S}_1,\mathcal{S}_2)\cap\mathcal{I}_{p'})(3)\big).$$

  that is a deformation of Λ_ψ. In the following lines we will prove that the linear system given by Λ is birational.

  Reciprocally let $\mathcal{Q}$ be a quadric, p be a point on $\mathcal{Q}$, $\mathcal{S}_1$ be a cubic singular at p and that contains a line ℓ of $\mathcal{Q}$. If $\mathcal{C}_2$ is the residual of ℓ in $(\mathcal{Q},\mathcal{S}_1)$, then there exists $\mathcal{S}_2$ singular at p such that $\mathcal{I}_{\mathcal{C}_2}=(\mathcal{Q},\mathcal{S}_1,\mathcal{S}_2)$. Take $p_1\in\mathbb{P}^3_\mathbb{C}\smallsetminus\mathcal{Q}$, and set

$$\Lambda=\mathrm{H}^0\big((\mathcal{I}_{p_1}\cap(\mathcal{Q}\mathcal{I}_p,\mathcal{S}_1,\mathcal{S}_2))(3)\big).$$

  Let L be a 2-dimensional general element of Λ; the general linked curve to $\mathcal{C}_2$ in L, denoted $\mathcal{C}_{1,L}$, is of degree 4, is singular at p, lies on two quadrics; furthermore the linear system induced by Λ on $\mathcal{C}_{1,L}$ has the two following properties:
  - its base locus contains p and p_1,
  - it sends birationally $\mathcal{C}_{1,L}$ onto a line.

  In other words, $\Lambda=\Lambda_\psi$ for a $(3,4)$ birational map ψ.

  Let us give some explicit examples, the generic one and the degeneracies considered by HUDSON:
  - *Description of* $\mathcal{E}_7$. The quadric $\mathcal{Q}$ is smooth at p, and the rank of $\mathcal{Q}$ is maximal. Hence the point p is an ordinary quadratic singularity of the generic element of Λ_ψ, we are in the configuration $(2,2)$ of `Table 1` (*see* §7).
  - *Description of* $\mathcal{E}_{7.5}$. In that case, p is a binode, $\mathcal{Q}$ is smooth at p and one of the two biplanes is contained in $T_p\mathcal{Q}$; we are in the configuration $(2,3)'$ of `Table 1` (*see* §7).

The set of such maps is denoted $\mathcal{E}_{7.5}$, this case does not appear in Table VI but should appear.

- *Description of* $\mathcal{E}_8$. The second way to obtain a binode is the following one: Q is an irreducible cone with vertex p. This corresponds to the configuration $(2,3)$ of `Table` 1 (*see* §7).
- *Description of* $\mathcal{E}_9$. The rank of Q is 2, and the point p is a double point of contact; we are in the configuration $(2,4)$ of `Table` 1 (*see* §7).
- *Description of* $\mathcal{E}_{10}$. The general element of Λ_ψ has a double point of contact and a binode (configurations $(2,4)$ and $(1,4)$ of `Table` 1, *see* §7). HUDSON details this case carefully ([11, Chap. XV]).

Proposition 5.1. *One has the following properties:*

$$\dim \mathcal{E}_6 = 38, \qquad \mathcal{E}_{7.5} \cup \mathcal{E}_8 \subset \overline{\mathcal{E}_7}$$

and

- *a generic element of* $\mathcal{E}_{7.5}$ *is not a specialization of a generic element of* $\mathcal{E}_8$*;*
- *a generic element of* $\mathcal{E}_8$ *is not a specialization of a generic element of* $\mathcal{E}_{7.5}$*;*
- *a generic element of* $\mathcal{E}_9$ *is a specialization of a generic element of* $\mathcal{E}_8$*.*

Proof. The arguments to establish $\dim \mathcal{E}_6 = 38$ are similar to those used in the proof of Proposition 4.3.

Let us justify that a generic element of $\mathcal{E}_{7.5}$ is not a specialization of a generic element of $\mathcal{E}_8$ (we take the notations of §5.1): as we see when $\psi \in \mathcal{E}_8$ the quadric Q is always singular whereas it is not the case when $\psi \in \mathcal{E}_{7.5}$. Conversely if ψ belongs to $\mathcal{E}_{7.5}$ then $\mathcal{C}_2$ is reducible but if ψ belongs to $\mathcal{E}_8$ the curve $\mathcal{C}_2$ can be irreducible and reduced; hence a generic element of $\mathcal{E}_8$ is not a specialization of a generic element of $\mathcal{E}_{7.5}$. ☐

Theorem 5.2. *The set* $\mathfrak{ruled}_{3,4}$ *is an irreducible component of* $\mathrm{Bir}_{3,4}(\mathbb{P}^3_\mathbb{C})$*. There is only one another irreducible component in* $\mathrm{Bir}_{3,4}(\mathbb{P}^3_\mathbb{C})$*.*

Proof. According to Proposition 3.10 the set $\mathfrak{ruled}_{3,4}$ is an irreducible component of $\mathrm{Bir}_{3,4}(\mathbb{P}^3_\mathbb{C})$.

Any element ψ of $\mathcal{E}_7 \cup \mathcal{E}_{7.5} \cup \mathcal{E}_8 \cup \mathcal{E}_9 \cup \mathcal{E}_{10}$ satisfies the following property:

$$\Lambda_\psi = \mathrm{H}^0\big(((Q\,\mathcal{I}_p, \mathcal{S}_1, \mathcal{S}_2) \cap \mathcal{I}_{p_1})(3)\big)$$

where p belongs to Q, p_1 is an ordinary base point, and

$$Q = \det \begin{bmatrix} L_0 & L_1 \\ L_2 & L_3 \end{bmatrix}, \quad \mathcal{S}_1 = L_0Q_1 + L_1Q_2, \quad \mathcal{S}_2 = L_2Q_1 + L_3Q_2$$

with $L_i \in \mathbb{C}[z_0,z_1,z_2,z_3]_1$, $Q_i \in \mathbb{C}[z_0,z_1,z_2]_2$. So $\mathcal{E}_7$, $\mathcal{E}_{7.5}$, $\mathcal{E}_8$ $\mathcal{E}_9$ and $\mathcal{E}_{10}$ belong to the same irreducible component $\mathscr{E}$.

It remains to show that $\mathscr{E} = \overline{\mathcal{E}_6}$: let us consider

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -z_2 & z_3 & L_0 \\ z_2 & 0 & L_1 & L_2 \\ -z_3 & -L_1 & 0 & L_3 \\ -L_0 & -L_2 & -L_3 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} z_2 \\ z_1 \\ z_0 \\ tz_3 \end{bmatrix}$$

with L_i linear forms and

$$M_t = \begin{bmatrix} Jv \\ Nv \end{bmatrix} = \begin{bmatrix} tz_3 & z_0 & -z_1 & -z_2 \\ tz_3L_0+Q & q_1 & q_2 & q_3 \end{bmatrix}$$

with

$$Q = z_0z_3 - z_1z_2, \qquad q_1 = z_2^2 + z_0L_1 + tz_3L_2,$$
$$q_2 = -z_2z_3 - z_1L_1 + tz_3L_3, \qquad q_3 = -z_2L_0 - z_1L_2 - z_0L_3.$$

For generic L_i's and $t \neq 0$ the 2×2 minors of M_t generate the ideal of a generic elliptic quintic curve as in $\mathcal{E}_6$. For M_0 the 2×2 minors become Qz_0, Qz_1, Qz_2, $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ with

$$\mathcal{S}_1 = -z_2Q, \qquad \mathcal{S}_2 = -z_1q_3 + z_2q_2, \qquad \mathcal{S}_3 = z_0q_3 + z_2q_1.$$

Therefore the ideal $\mathcal{M}_2$ generated by these minors is

$$(Qz_0, Qz_1, Qz_2, \mathcal{S}_2, \mathcal{S}_3).$$

Denote by ℓ the line defined by $I_\ell = (z_1, z_3)$. According to

$$z_3\mathcal{S}_3 = -z_2\mathcal{S}_2 + Q(q_3 + L_1z_2) \quad \& \quad z_1\mathcal{S}_3 = -z_0\mathcal{S}_2 - z_2^2Q$$

$\mathcal{M}_2$ is the ideal of the residual of ℓ in the complete intersection of ideals $(Q, \mathcal{S}_2)$. To prove $\overline{\mathcal{E}_7} \subset \overline{\mathcal{E}_6}$ we just need to show that one can obtain the generic element of $\overline{\mathcal{E}_7}$ with a good choice of the L_i's (Proposition 5.1); in other words it remains to prove that $\mathcal{S}_2$ is generic among the cubics singular at p that contain ℓ. Modulo Q one can assume that $q_3 = -z_3a + b$, with a (resp. b) an element of $\mathbb{C}[z_1,z_2]_1$ (resp. $\mathbb{C}[z_0,z_1,z_2]_2$). Then

$$\mathcal{S}_2 = -z_3\big(z_1a + z_2^2\big) + z_1\big(b - z_2L_1\big);$$

in conclusion $\mathcal{S}_2 = z_3A + z_2B$ for generic A and B in $\mathbb{C}[z_0,z_1,z_2]_2$. As $\mathcal{E}_6$ is irreducible $\mathscr{E} = \overline{\mathcal{E}_6}$. $\square$

5.2. **Relations between $\overline{\mathrm{Bir}_{3,3}(\mathbb{P}^3_\mathbb{C})}$ and $\overline{\mathrm{Bir}_{3,4}(\mathbb{P}^3_\mathbb{C})}$.** One can now state the following result:

Proposition 5.3. *The set* $\mathfrak{ruled}_{3,3}$ *intersects the closure of any irreducible component of* $\overline{\mathrm{Bir}_{3,4}(\mathbb{P}^3_\mathbb{C})}$.

Proof. According to Lemma 2.4 and Theorem 5.2 it is sufficient to prove that $\mathfrak{ruled}_{3,3}$ intersects the closure of $(3,4)$ birational maps that are non-ruled.

Let us consider an element ψ of $\mathrm{Bir}_{3,4}(\mathbb{P}^3_\mathbb{C})$ whose $\mathcal{C}_2$ is the union of the lines of ideals

$$I_\delta = (z_0, z_1^2), \quad (z_0 - \varepsilon z_2, z_1), \quad I_{\ell_1} = (z_0, z_3), \quad I_{\ell_2} = (z_1, z_2).$$

Denote by $\mathcal{J}_\varepsilon = (z_0, z_1^2) \cap (z_0 - \varepsilon z_2, z_1) \cap (z_0, z_3) \cap (z_1, z_2)$. One can check that

$$\mathcal{J}_\varepsilon = (z_0z_1, z_0^2z_2 + \varepsilon z_0z_2^2, z_1^2z_3).$$

Set $I_\varepsilon = z_0z_1(z_0,z_1,z_2) + (z_0^2z_2 + \varepsilon z_0z_2^2, z_1^2z_3)$. For a general p_2 the map ψ_ε defined by $\Lambda_{\psi_\varepsilon} = \mathrm{H}^0\big((I_\varepsilon \cap I_{p_2})(3)\big)$ is birational; furthermore

- $\psi_\varepsilon \in \mathrm{Bir}_{3,4}(\mathbb{P}^3_\mathbb{C}) \smallsetminus \mathfrak{ruled}_{3,4}$ for $\varepsilon \neq 0$;
- $\psi_0 \in \mathfrak{ruled}_{3,3}$.

$\square$

As in the case of $(3,3)$ birational maps one has the following statement:

Theorem 5.4. *If* $\mathfrak{p}_2 \in \{1, 2\}$, *then* $\mathrm{Bir}_{3,4,\mathfrak{p}_2}(\mathbb{P}^3_\mathbb{C})$ *is non-empty and irreducible.*

6. $(3,5)$ CREMONA TRANSFORMATIONS

6.1. **General description of $(3,5)$ birational maps.** We already find an irreducible component of the $(3,5)$ birational maps: $\mathfrak{ruled}_{3,5}$ (Proposition 3.10). Let us now consider a $(3,5)$ CREMONA transformation ψ such that Λ_ψ contains a cubic surface without double line.

Strategy. *By Lemma 3.1 the image of $\mathcal{C}_1$ by ψ is given by a 2-dimensional vector subspace u of $\mathrm{H}^0\big(\omega_{\mathcal{C}_1}(h)\big)$. The restriction of ψ to $\mathcal{C}_1$ thus factorises by the composition of $\mathcal{C}_1 \dashrightarrow |\mathrm{H}^0\big(\omega_{\mathcal{C}_1}(h)\big)^\vee|$ with the projection $|\mathrm{H}^0\big(\omega_{\mathcal{C}_1}(h)\big)^\vee| \to |u^\vee|$. In the following cases we will use the equivalence between the birationality of ψ and the birationality of the composition $\mathcal{C}_1 \dashrightarrow |u^\vee|$. It will be useful to compute the number of base points but also to show that a linear system is birational.*

6.1.1. *Case: $\mathcal{C}_1$ smooth.* By (3.2) the image of $\mathcal{C}_1$ by ψ is given by a sublinear system of $|\omega_{\mathcal{C}_1}(h)|$. In that situation $\deg \omega_{\mathcal{C}_1}(h) = 3$ so as ψ sends birationally $\mathcal{C}_1$ onto a line of $\mathbb{P}^3_{\mathbb{C}}$ the map ψ has a residual base scheme of length 2. The curve $\mathcal{C}_2$ has genus -1 and does not lie on a quadric; $\mathcal{C}_2$ is the disjoint union of a twisted cubic and a line, so this case gives an irreducible family, and the general element belongs to $\mathcal{E}_{12}$. Indeed suppose that $\psi \not\in \overline{\mathcal{E}_{12}}$, then $\mathcal{C}_2$ is the union of two smooth conics Γ_1 and Γ_2 that do not intersect. Any Γ_i is contained in a plane $\mathcal{P}_i$. Denote by ℓ the intersection $\mathcal{P}_1 \cap \mathcal{P}_2$. As $\#\big(\ell \cap (\Gamma_1 \cup \Gamma_2)\big) = 4$, all the cubic surfaces that contain $\Gamma_1 \cup \Gamma_2$ contain ℓ. So $\ell \subset \mathcal{C}_2$: contradiction.

6.1.2. *Case: $\mathcal{C}_1$ not smooth.* So $\mathfrak{p}_a(\mathcal{C}_1) \geq 1$, and by Corollary 3.3

$$\mathfrak{p}_a(\mathcal{C}_2) = \deg \mathcal{C}_2 - \deg \mathcal{C}_1 + \mathfrak{p}_a(\mathcal{C}_1) = -1 + \mathfrak{p}_a(\mathcal{C}_1) \geq 0.$$

Since $\mathcal{C}_1$ is not in a plane, $\mathfrak{p}_a(\mathcal{C}_1) \leq 2$. Therefore we only have to distinguish the eventualities $\mathfrak{p}_a(\mathcal{C}_1) = 1$ and $\mathfrak{p}_a(\mathcal{C}_1) = 2$. Before looking at any of these eventualities let us introduce the set

$$\mathscr{C} = \big\{\text{irreducible curves of } \mathbb{P}^3_{\mathbb{C}} \text{ of degree 5 and geometric genus 0}\big\}$$

- Assume first that $\mathfrak{p}_a(\mathcal{C}_1) = 1$. Then $\mathcal{O}_{\mathcal{C}_1} = \omega_{\mathcal{C}_1}$. We will denote by $\pi\colon \mathbb{P}^1_{\mathbb{C}} \to \mathcal{C}_1$ the normalization of $\mathcal{C}_1$.

a_1) Suppose first that all the elements of Λ_ψ are singular at $p \in \mathbb{P}^3_{\mathbb{C}}$. Denote by L the 2-dimensional vector space $\Lambda_\psi \cap \mathrm{H}^0\big((\mathcal{I}_p^2 \cap \mathcal{I}_{\mathcal{C}_1})(3h)\big)$ defining $\mathcal{C}_1$ and $\mathcal{C}_2$. Let us follow the strategy explained before. By the liaison sequence (3.2) $\frac{\Lambda_\psi}{L}$ gives a vector subspace u of $\mathrm{H}^0\big(\omega_{\mathcal{C}_1}(h)\big) = \mathrm{H}^0\big(\mathcal{O}_{\mathcal{C}_1}(h)\big)$ of dimension 2. It induces a projection from $\mathcal{C}_1$ to $|u^\vee|$ that coincides with the restriction of ψ to $\mathcal{C}_1$; hence this projection has degree 1. Moreover, via the identification $\mathrm{H}^0\big(\mathcal{O}_{\mathcal{C}_1}(h)\big) = \mathrm{H}^0\big(\pi^*\mathcal{O}_{\mathcal{C}_1}(h)\big)$, u is included in the set V_1 of sections of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(5)$ whose base locus contains $\pi^{-1}(p)$; as $\mathfrak{p}_a(\mathcal{C}_1) = 1$ the conductor $\pi^{-1}(p)$ has length 2. So there is an other base scheme of length 2 because $\psi_{|\mathcal{C}_1}\colon \mathcal{C}_1 \dashrightarrow |u^\vee|$ is birational.

We would like to show that $\mathcal{C}_2$ moves in an irreducible family. We will do this by deforming ψ (and $\mathcal{C}_2$) while $\mathcal{C}_1$ is fixed. So, $p \in \mathbb{P}^3_{\mathbb{C}}$ being fixed, let us consider

$$\mathcal{R}_{p,1} = \big\{\mathcal{C} \in \mathscr{C} \,|\, \mathrm{Sing}\,\mathcal{C} = \{p\},\, \mathfrak{p}_a(\mathcal{C}) = 1\big\};$$

the set $\mathcal{R}_{p,1}$ is an irreducible one. Remark that

$$\mathrm{h}^0\,\mathcal{I}_{\mathcal{C}}(3h) = \mathrm{h}^0\big(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(3h)\big) - \mathrm{h}^0\big(\mathcal{O}_{\mathcal{C}}(3h)\big) = 20 - \big(3\deg \mathcal{C} + 1 - \mathfrak{p}_a(\mathcal{C})\big) = 5$$

and $\mathrm{h}^0\big((I_p^2\cap I_{\mathcal{C}})(3h))\big)=5-1=4$ because $\mathcal{C}$ has a double point at p for all $\mathcal{C}$ in $\mathcal{R}_{p,1}$.

Let us denote by F_1 the set of $(\mathcal{C},L,u)\in\mathcal{R}_{1,p}\times\mathrm{H}^0\big((I_p^2\cap I_{\mathcal{C}})(3h)\big)\times V_1$ defined by

— $L\subset\mathrm{H}^0\big((I_p^2\cap I_{\mathcal{C}})(3h)\big)$ of dimension 2 such that the residual of $\mathcal{C}$ in the complete intersection defined by L has no common component with $\mathcal{C}$, and $\mathcal{C}$ is geometrically linked to a curve denoted by $\mathcal{C}_{2,L}$,

— $u\subset V_1$ of dimension 2 such that $\mathbb{P}^1_{\mathbb{C}}\dashrightarrow|u^\vee|$ has degree 1 (N.B. the general element of this family will then have two ordinary base points in addition to $\pi^{-1}(p)$).

The set F_1 is irreducible since the choice of $\mathcal{C}$ is irreducible, and thus the choices of L and u too.

If $(\mathcal{C},L,u)$ belongs to F_1, let us set

$$h_L\colon\mathrm{H}^0\big(I_{\mathcal{C}_{2,L}}(3h)\big)\to\mathrm{H}^0\big(\omega_{\mathcal{C}}(h)\big)$$

(recall that $\frac{\mathrm{H}^0\big(I_{\mathcal{C}_{2,L}}(3h)\big)}{L}\simeq\mathrm{H}^0\big(\omega_{\mathcal{C}}(h)\big)$). Consider the map

$$\kappa_1\colon F_1\to\mathbb{G}\big(4;\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(3)\big)\big),\qquad(\mathcal{C},L,u)\mapsto h_L^{-1}(u).$$

By construction of F_1 if ψ is a birational map such that $\mathfrak{p}_a(\mathcal{C}_1)=1$ and all the elements of Λ_ψ are singular at p, then Λ_ψ is in the image of κ_1.

Conversely one has:

Lemma 6.1. *The general element of* $\operatorname{im}\kappa_1$ *coincides with* Λ_ψ *for some birational map* ψ *of* $\mathcal{E}_{14}$.

Proof. As F_1 is irreducible it is enough to show that $|h_L^{-1}(u)|$ is a birational system when $(\mathcal{C},L,u)$ is general in F_1. In that situation $\mathcal{C}_{2,L}$ is a curve of degree 4, arithmetic genus 0, singular at p, lying on a smooth quadric. Therefore $\mathcal{C}_{2,L}$ is reducible; more precisely it is the union of a twisted cubic and a line of this smooth quadric. All the elements of $|h_L^{-1}(u)|$ are cubic surfaces singular at p because $\mathcal{C}_{2,L}$ has a double point at p, and the residual pencil $u\subset\mathrm{H}^0\big(\omega_{\mathcal{C}}(h)\big)$ vanishes at p by definition of F_1. Let $\mathcal{C}_1$ be the residual of $\mathcal{C}_{2,L}$ in the intersection of two general cubics of $|h_L^{-1}(u)|$. Hence $\mathcal{C}_1$ is singular at p. As u is by definition a pencil of sections of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(5)$ vanishing on $\pi^{-1}(p)$ such that $\mathbb{P}^1_{\mathbb{C}}\dashrightarrow|u^\vee|$ has degree 1 the linear system $|h_L^{-1}(u)|$ sends birationally $\mathcal{C}_1$ onto the line $|u^\vee|$. Therefore $|h_L^{-1}(u)|$ gives a birational map. □

Let us remark that the previous irreducibility result asserts that the following example (belonging to family $\mathcal{E}_{18}$) whose base locus is not on a smooth quadric is nevertheless a deformation of elements of $\mathcal{E}_{14}$:

Example 6.2. Let $\mathcal{C}_2$ be the union of a line doubled on a smooth quadric with two other lines, such that all these lines contain a same point p. Set

$$Q=z_0z_3-z_1z_2,\qquad I_p=(z_0,z_1,z_2);$$

then $I_{\mathcal{C}_2}=\big((z_2,z_0)^2+(Q)\big)\cap(z_1,z_2)\cap(z_0-z_2,z_1-z_2)$. Now chose a double point of contact (note that the tangent cone must contain the tangent cone of $\mathcal{C}_2$):

$$I_{\mathrm{dpc}}=(z_2^2z_3-z_0z_1z_3)+(z_0,z_1,z_2)^3,$$

and let p_1 and p_2 be two general points. Define I_ψ by $I_{\mathcal{C}_2}\cap I_{\text{dpc}}\cap I_{p_1}\cap I_{p_2}$. So I_ψ is the intersections of $I_{p_1}\cap I_{p_2}$ with

$$I_{\mathcal{C}_2}\cap I_{\text{dpc}}=\big(z_1z_2^2-z_2^3,\, z_0z_2^2-z_2^3,\, z_1^2z_2-z_2^3-z_0z_1z_3+z_2^2z_3,\, z_0z_1z_2-z_2^3,\, z_0^2z_2-z_2^3,\, z_0^2z_1-z_2^3\big).$$

The tangent cone of $\mathcal{C}_2$ at p has degree 4 but the tangent cone of $\mathcal{C}_1\cup\mathcal{C}_2$ at p has degree 6, so $\mathcal{C}_1$ belongs to $\mathcal{R}_{p,1}$.

$b_1)$ Suppose now that Λ_ψ contains a smooth element at p. Then p is a point of contact, all the cubic surfaces are tangent at p; the curve $\mathcal{C}_2\subset Q$ is linked to a curve of degree 2 and genus -1 so has degree 4 and genus 0. A general curve of degree 4 and genus 0 is in fact rational, smooth on a smooth quadric Q and we will see that such a curve can be "the $\mathcal{C}_2$" of a birational map of that type. Hence this family of birational maps will turn out to be an irreducible one. Set $Q=z_0z_3-z_1z_2$, $I_{\ell_1}=(z_0,z_1)$, and $I_{\ell_2}=(z_2,z_3)$; one has

$$\mathcal{J}=I_{\ell_1\cup\ell_2}=(z_0z_2,z_0z_3,z_1z_2,z_1z_3).$$

Let $\mathcal{S}_0$ be the element of $\mathcal{J}$ given by

$$az_0z_2+bz_0z_3+cz_1z_3 \qquad a,\,b,\,c\in\mathbb{C}[z_0,z_1,z_2,z_3]_1;$$

one has $I_{\mathcal{C}_2}=((\mathcal{S}_0,Q):\mathcal{J})=(Q,\mathcal{S}_0,\mathcal{S}_1,\mathcal{S}_2)$ with

$$\mathcal{S}_1=z_0^2a+z_0z_1b+z_1^2c,\quad \mathcal{S}_2=z_2^2a+z_2z_3b+z_3^2c.$$

The dimension of $\mathrm{H}^0\big(I_{\mathcal{C}_2}(3h)\big)$ is 7; indeed one has the following seven cubics:

$$I_{\mathcal{C}_2}=\langle Qz_0,\,Qz_1,\,Qz_2,\,Qz_3,\,\mathcal{S}_0,\,\mathcal{S}_1,\,\mathcal{S}_2\rangle.$$

If ψ is a birational map, then ψ has no base point. Indeed $u=\frac{\Lambda_\psi}{\mathrm{H}^0\left(I_{\mathcal{C}_1\cup\mathcal{C}_2}(3)\right)}$ is contained in the sections of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(5)$ whose base locus contains $2\pi^{-1}(p)$; we thus already have an isomorphism between $\mathbb{P}^1_{\mathbb{C}}$ and $|u^\vee|$. The map ψ belongs to $\overline{\mathcal{E}_{23}}$. Conversely let L be a general 2-dimensional subspace of $\mathrm{H}^0\big(\big(I_{\mathcal{C}_2}\cap(Q+I_p^2)\big)(3h)\big)$, and let $\mathcal{C}_1$ be the curve linked to $\mathcal{C}_2$ defined by L. Then the previous arguments show that the image of $\mathcal{C}_1$ by

$$\left|\frac{\mathrm{H}^0\big(\big(I_{\mathcal{C}_2}\cap(Q+I_p^2)\big)(3h)\big)}{L}\right|$$

is a line. So $\mathrm{H}^0\big(\big(I_{\mathcal{C}_2}\cap(Q+I_p^2)\big)(3h)\big)=\Lambda_\psi$ for some birational map ψ of this type.

- Suppose that $\mathfrak{p}_a(\mathcal{C}_1)=2$. Then $\mathfrak{p}_a(\mathcal{C}_2)=1$, $\mathcal{C}_1$ lies on a quadric and $\mathrm{h}^0 I_{\mathcal{C}_1}(3)=6$. We will still denote by $\pi\colon\mathbb{P}^1_{\mathbb{C}}\to\mathcal{C}_1$ the normalization of $\mathcal{C}_1$.

$a_2)$ Assume first that $\mathcal{C}_1$ has a triple point p. The curve $\mathcal{C}_1$ is linked to a line by a complete intersection $(Q,\mathcal{S}_0)$ where Q (resp. $\mathcal{S}_0$) is a cone (resp. a cubic) singular at p. We can write the normalization π as follows $(\alpha^2A,\alpha\beta A,\beta^2A,B)$ with $A\in\mathbb{C}[\alpha,\beta]_3$, $B\in\mathbb{C}[\alpha,\beta]_5$, and A, B without common factors. Then $Q=z_1^2-z_0z_2$, and $\mathrm{H}^0\big(\omega_{\mathcal{C}_1}(h)\big)$ can be identified with $\mathrm{H}^0\big(I_\ell(2)\big)$, where $I_\ell=(z_0,z_1)$. So $\mathrm{H}^0\big(\omega_{\mathcal{C}_1}(h)\big)$ is the 6-dimensional subspace W of $\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(6)\big)$ spanned by $(\alpha,\beta)\cdot(\alpha^2A,\alpha\beta A,\beta^2A,B)$. Let us consider the subspace $V_A=W\cap(A)$ of W. Let L be the 2-dimensional vector space $\Lambda_\psi\cap\mathrm{H}^0\big(I_{\mathcal{C}_1}(3h)\big)$. Then $\frac{\Lambda_\psi}{L}$ gives a 2-dimensional vector subspace u of V_A. The restriction of ψ to $\mathcal{C}_1$ gives a birational map $\mathbb{P}^1_{\mathbb{C}} \dashrightarrow |u^\vee|$ induced by $u \subset V_A \subset \mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(6)\big)$. Furthermore ψ has two ordinary base points. We would like to show that in that case $\mathcal{C}_2$ moves in an irreducible family whose general element is the complete intersection of two quadrics. We thus fix a point $p \in \mathbb{P}^3_{\mathbb{C}}$ and introduce the irreducible set

$$\mathcal{R}_{p,2} = \big\{\mathcal{C} \in \mathscr{C} \,|\, \mathrm{Sing}\,\mathcal{C} = \{p\},\, \mathfrak{p}_a(\mathcal{C}) = 2\big\}.$$

We define the set F_2 as the $(\mathcal{C}, L, u) \in \mathcal{R}_{p,2} \times \mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}}(3h)\big) \times V_A$ given by

- $L \subset \mathrm{H}^0\big(\mathcal{I}_{\mathcal{C}}(3h)\big)$ of dimension 2 such that the residual of $\mathcal{C}$ in the complete intersection defined by L has no common component with $\mathcal{C}$, and $\mathcal{C}$ is geometrically linked to a curve denoted by $\mathcal{C}_{2,L}$,
- $u \subset V_A$ of dimension 2 such that $\mathbb{P}^1_{\mathbb{C}} \dashrightarrow |u^\vee|$ is birational and whose base locus contains $\pi^{-1}(p)$.

Let us consider the map

$$\kappa_2 \colon F_2 \to \mathbb{G}\big(4; \mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(3)\big)\big), \qquad (\mathcal{C}, L, u) \mapsto h_L^{-1}(u).$$

If ψ is birational, if $\mathfrak{p}_a(\mathcal{C}_1) = 2$, and $\mathcal{C}_1$ has a triple point then ψ belongs to $\mathrm{im}\,\kappa_2$.

Lemma 6.3. *The general element of* $\mathrm{im}\,\kappa_2$ *coincides with* Λ_ψ *for some birational map* ψ *of* $\mathcal{E}_{13}$.

Proof. As F_2 is irreducible one can consider a general element of F_2, and then $\mathcal{C}_{2,L}$ is a curve of degree 4, genus 1 and is the complete intersection of two smooth quadrics. The map ψ has two ordinary base points p_1, p_2, and belongs to $\mathcal{E}_{13}$. More precisely $\Lambda_\psi = \mathrm{H}^0\big((\mathcal{I}_{\mathcal{C}_{2,L}} \cdot \mathcal{I}_p \cap \mathcal{I}_{p_1} \cap \mathcal{I}_{p_2})(3)\big)$. ☐

Note that this irreducibility result asserts that the following example in which $\mathcal{C}_2$ is not a complete intersection of two quadrics is nevertheless a deformation of elements of $\mathcal{E}_{13}$.

Example 6.4. Let $\mathcal{C}_2$ be the union of a plane cubic $\mathcal{C}_3$ singular at p and a line ℓ containing p but not in the plane spanned by $\mathcal{C}_3$. For instance take

$$\mathcal{I}_p = (z_0, z_1, z_2), \quad \mathcal{I}_\ell = (z_1, z_2), \quad \mathcal{I}_{\mathcal{C}_3} = (z_1 - z_0, (z_1 - z_2)z_1 z_3 + z_0^3 + z_1^3 + z_2^3).$$

Let $\mathcal{I}_{\mathrm{dpc}}$ be a double point of contact at p. (As we have already chose $\mathcal{C}_2$, we must take a quadric cone containing the tangent cone to $\mathcal{C}_2$). For instance one can take: $\mathcal{I}_{\mathrm{dpc}} = (z_1^2 - z_0 z_2) + \mathcal{I}_p^3$, and let

$$\begin{aligned} \mathcal{J} = \mathcal{I}_{\mathcal{C}_2} \cap \mathcal{I}_{\mathrm{dpc}} \quad = \quad & \big(z_0 z_2^2 - z_1 z_2^2, z_0 z_1 z_2 - z_1^2 z_2, z_0^2 z_2 - z_1^2 z_2, 2z_1^3 + z_2^3 + z_1^2 z_3 - z_0 z_2 z_3, \\ & 2z_0 z_1^2 + z_2^3 + z_1^2 z_3 - z_0 z_2 z_3, 2z_0^2 z_1 + z_2^3 + z_1^2 z_3 - z_0 z_2 z_3\big) \end{aligned}$$

chose two general points p_1 and p_2 and define by $\mathcal{I}_\psi$ the ideal generated by the 4 cubics of $\mathcal{J} \cap \mathcal{I}_{p_1} \cap \mathcal{I}_{p_2}$. The tangent cone of $\mathcal{C}_2$ at p has degree 3, the tangent cone of $\mathcal{C}_1 \cup \mathcal{C}_2$ at p has degree 6 (because p is a double point of contact); hence $\mathcal{C}_1$ has also a triple point at p, and belongs to $\mathcal{R}_{p,2}$.

$b_2)$ Suppose now that $\mathcal{C}_1$ hasn't a triple point; $\mathcal{C}_1$ has thus two distinct double points. Fix two distinct points p and q in $\mathbb{P}^3_\mathbb{C}$, and set

$$\mathcal{R}_{p,q,2}=\big\{\mathcal{C}\in\mathscr{C}\,|\,\mathrm{Sing}\,\mathcal{C}=\{p,\,q\},\,\mathfrak{p}_a(\mathcal{C})=2\big\}.$$

Let V_3 (resp. V_4) be the sections of $\mathcal{O}_{\mathbb{P}^1_\mathbb{C}}(7)$ whose base locus contains $\pi^{-1}(p)$ and $\pi^{-1}(q)$ (resp. $\pi^{-1}(p)$ and $2\pi^{-1}(q)$). The set $\mathcal{R}_{p,q,2}$ is irreducible. Remark that for all $\mathcal{C}$ in $\mathcal{R}_{p,q,2}$ one has

$$\mathrm{h}^0\,\mathcal{I}_\mathcal{C}(3h)=6,\qquad \mathrm{h}^0\big((\mathcal{I}_\mathcal{C}\cap\mathcal{I}_p^2)(3h)\big)=5,\qquad \mathrm{h}^0\big((\mathcal{I}_\mathcal{C}\cap\mathcal{I}_p^2\cap\mathcal{I}_q^2)(3h)\big)=4.$$

Remark 6.5. One cannot have two distinct points of contact. Assume by contradiction that there are two distinct points of contact p and q. Denote by $\pi\colon\widetilde{\mathcal{C}_1}\to\mathcal{C}_1$ the normalization of $\mathcal{C}_1$. One would have $\pi^*\omega_{\mathcal{C}_1}(h)=\mathcal{O}_{\mathbb{P}^1_\mathbb{C}}(7)$ but the linear system induced by ψ would contain in the base locus $2\pi^{-1}(p)+2\pi^{-1}(q)$ which is of length 8: contradiction with the fact that ψ sends birationally $\mathcal{C}_1$ onto a line.

So one has the following alternative:

$b_2)$ i) Either all the cubics of Λ_ψ are singular at p and q. One can then define the set F_3 of $(\mathcal{C},L,u)\in\mathcal{R}_{p,q,2}\times\mathrm{H}^0\big((\mathcal{I}_\mathcal{C}\cap\mathcal{I}_p^2\cap\mathcal{I}_q^2)(3h)\big)\times V_3$ given by

— $L\subset\mathrm{H}^0\big((\mathcal{I}_\mathcal{C}\cap\mathcal{I}_p^2\cap\mathcal{I}_q^2)(3h)\big)$ of dimension 2 such that the residual of $\mathcal{C}$ in the complete intersection defined by L has no common component with $\mathcal{C}$;

— $u\subset V_3$ of dimension 2 such that $\mathcal{C}\dashrightarrow|u^\vee|$ has degree 1.

Let us consider the map

$$\kappa_3\colon F_3\to\mathbb{G}\big(4;\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^3_\mathbb{C}}(3)\big)\big),\qquad (\mathcal{C},L,u)\mapsto h_L^{-1}(u).$$

If ψ is birational, if $\mathfrak{p}_a(\mathcal{C}_1)=2$, if $\mathcal{C}_1$ has two distinct double points at p and q and if all the cubics of Λ_ψ are singular at p and q, then Λ_ψ belongs to $\mathrm{im}\,\kappa_3$.

Lemma 6.6. *The general element of* $\mathrm{im}\,\kappa_3$ *coincides with* Λ_ψ *for some birational map* ψ *of* $\mathcal{E}_{19}$.

Proof. As F_3 is irreducible one can consider a general element $(\mathcal{C},L,u)$ of F_3 and then $\mathcal{C}_{2,L}$ is a curve of degree 4 and genus 1, is singular at p and q, lies on a smooth quadric, and is reducible: $\mathcal{C}_{2,L}$ is the union of a twisted cubic Γ and the line $\ell=(pq)$. Moreover for all the elements of $h_L^{-1}(u)$ the curve $\mathcal{C}$ is singular at p and q (by definition of V_3 and by the fact that $\mathcal{C}_{2,L}$ is singular at p and q). ☐

In this situation as all the cubic surfaces are singular at p and q,

$$h_L^{-1}(u)=\mathrm{H}^0\big((\mathcal{I}_\Gamma\cdot\mathcal{I}_\ell\cap\mathcal{I}_{p_1}\cap\mathcal{I}_{p_2})(3)\big)$$

where p_1, p_2 are two ordinary base points; ψ belongs to $\mathcal{E}_{19}$.

$b_2)$ ii) Or one of the cubics of Λ_ψ is smooth at (for instance) q. Let us introduce the set F_4 of pairs $(\mathcal{C},L)\in\mathcal{R}_{p,q,2}\times\mathrm{H}^0\big((\mathcal{I}_\mathcal{C}\cap\mathcal{I}_p^2)(3h)\big)$ satisfying: $L\subset\mathrm{H}^0\big((\mathcal{I}_\mathcal{C}\cap\mathcal{I}_p^2)(3h)\big)$ of dimension 2 such that the residual of $\mathcal{C}$ in the complete intersection defined by L has no common component with $\mathcal{C}$.

Let us consider the map

$$\kappa_4\colon F_4 \to \mathbb{G}\big(4;\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^3_\mathbb{C}}(3)\big)\big), \qquad (\mathcal{C},L)\mapsto h_L^{-1}(V_4);$$

note that $\dim V_4 = 2$.

If ψ is birational, if $\mathfrak{p}_a(\mathcal{C}_1)=2$, $\mathcal{C}_1$ hasn't a triple point and one of the cubic of Λ_ψ is smooth at (for instance) q, then Λ_ψ belongs to $\mathrm{im}\,\kappa_4$.

Lemma 6.7. *The general element of* $\mathrm{im}\,\kappa_4$ *coincides with* Λ_ψ *for some birational map* ψ *of* $\mathcal{E}_{24}$.

Proof. As F_4 is irreducible one can consider a general element of F_4, and then $\mathcal{C}_{2,L}$ is a curve of degree 4, genus 1, singular at p, and is the complete intersection of two quadrics. The map ψ has no base point and belongs to $\mathcal{E}_{24}$. ☐

6.2. **Irreducible components.** The following statement, and Theorems 4.4 and 5.2 imply Theorem A.

Theorem 6.8. *One has the inclusions:* $\mathcal{E}_{14}\subset\overline{\mathcal{E}_{12}}$, $\mathcal{E}_{24}\subset\overline{\mathcal{E}_{23}}$, *and* $\mathcal{E}_{19}\subset\overline{\mathcal{E}_{12}}$.

The set $\mathrm{Bir}_{3,5}(\mathbb{P}^3_\mathbb{C})$ *has four irreducible components:* $\mathcal{E}_{12}$, $\mathcal{E}_{13}$, $\mathcal{E}_{23}$, *and* $\mathcal{E}_{27}=\mathfrak{ruled}_{3,5}$.

Proof. Let us first prove that $\mathcal{E}_{14}\subset\overline{\mathcal{E}_{12}}$. If ψ belongs to $\mathcal{E}_{12}$, or to $\mathcal{E}_{14}$ the curve $\mathcal{C}_2$ is the union of a line ℓ and a twisted cubic Γ such that $\mathrm{length}\,(\ell\cap\Gamma)\leq 1$. Let I_ℓ (resp. I_Γ) be the ideal of ℓ (resp. Γ). We have $I_\psi\subset I_\ell\cap I_\Gamma$. If ψ belongs to $\mathcal{E}_{12}$, then $\ell\cap\Gamma=\emptyset$, and $I_\ell\cap I_\Gamma=I_\ell\cdot I_\Gamma$. And if ψ is in $\mathcal{E}_{14}$, then all the cubics are singular at $p=\ell\cap\Gamma$ so I_ψ is again in $I_\ell\cdot I_\Gamma$.

Prove now that $\mathcal{E}_{24}\subset\overline{\mathcal{E}_{23}}$. Consider a general element ψ of $\mathcal{E}_{24}$; the curve $\mathcal{C}_2$ is the complete intersection of a quadric $Q'=az_2+bz_0+cz_1$ passing through the double point p and a cone $Q_0=z_1z_2-z_0^2$. Furthermore all the cubics of I_ψ are singular at p, and $I_\psi\subset\mathcal{J}_0'=(Q_0,z_0Q',z_1Q',z_2Q')$. Let $\mathfrak{ct}_q$ be the ideal of the point of contact q; one has $\mathfrak{ct}_q=I_q^2+(H_q)$ where H_q is a plane passing through q. Denote by I_0 the intersection of $\mathcal{J}_0'$ and $\mathfrak{ct}_q$. Set

$$Z_0=z_0+tz_3, \qquad Z_1=z_1, \qquad Z_2=z_2, \qquad Z_3=z_0-tz_3,$$

$$Q_t=Z_1Z_2-Z_0Z_3, \qquad \mathcal{S}_0=aZ_0Z_2+bZ_0Z_3+cZ_1Z_3,$$
$$\mathcal{S}_1=aZ_0^2+bZ_0Z_1+cZ_1^2, \qquad \mathcal{S}_2=aZ_2^2+bZ_2Z_3+cZ_3^2.$$

Hence $\mathcal{J}_t=(Q_t,\mathcal{S}_0,\mathcal{S}_1,\mathcal{S}_2)$ is the ideal of a rational quartic if $t\neq 0$ (*cf.* the equations in §6.1.1 b_1)). The ideal $I_t=\mathcal{J}_t\cap\mathfrak{ct}_q$ is the ideal I_ψ of $\psi\in\mathcal{E}_{23}$. Remark that if $t=0$, then

$$\mathcal{J}_0=(Q_0,z_0Q',az_0^2+bz_0z_1+cz_1^2,az_2^2+bz_0z_2+cz_0^2)$$

but $az_0^2+bz_0z_1+cz_1^2=z_1Q'$ modulo Q, and $az_2^2+bz_0z_2+cz_0^2=z_2Q'$ modulo Q, that is $\mathcal{J}_0'=\mathcal{J}_0$. Therefore I_t tends to I_0 as t tends to 0.

The inclusion $\mathcal{E}_{19}\subset\overline{\mathcal{E}_{12}}$ follows from $\Lambda_\psi=\mathrm{H}^0\big((I_\ell\cdot I_\Gamma\cap I_{p_1}\cap I_{p_2})(3)\big)$ found in b_2) i).

Note that $\mathcal{E}_{12}\not\subset\overline{\mathcal{E}_{13}}$ (resp. $\mathcal{E}_{12}\not\subset\overline{\mathcal{E}_{23}}$): if ψ is in $\mathcal{E}_{12}$ then the associated $\mathcal{C}_2$ does not lie on a quadric whereas if ψ belongs to $\mathcal{E}_{13}$ (resp. $\mathcal{E}_{23}$) then $\mathcal{C}_2$ lies on two quadrics (resp. one quadric). Conversely $\mathcal{E}_{13}\not\subset\overline{\mathcal{E}_{12}}$ (resp. $\mathcal{E}_{23}\not\subset\overline{\mathcal{E}_{12}}$): if ψ is an element of $\mathcal{E}_{13}$ (resp. $\mathcal{E}_{23}$), then $\mathcal{C}_2$ is smooth and irreducible whereas the associated $\mathcal{C}_2$ of a general element of $\mathcal{E}_{12}$ is the disjoint union of a twisted cubic and a line.

Let us now justify that $\mathcal{E}_{23} \not\subset \overline{\mathcal{E}_{13}}$: the linear system of an element of $\mathcal{E}_{23}$ has a smooth surface whereas the linear system of an element of $\mathcal{E}_{13}$ does not. Conversely $\mathcal{E}_{13} \not\subset \overline{\mathcal{E}_{23}}$; indeed $\mathrm{h}^0\, \mathcal{I}_{\mathcal{C}_2}(3h) = 6$ for a birational map of $\mathcal{E}_{13}$ and $\mathrm{h}^0\, \mathcal{I}_{\mathcal{C}_2}(3h) = 7$ for a birational map of $\mathcal{E}_{23}$. □

In bidegree $(3,5)$ the description of $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_{\mathbb{C}})$ is very different from those of smaller bidegrees. Let us now prove Theorem C.

Theorem 6.9. *The set* $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_{\mathbb{C}})$ *is empty as soon as* $\mathfrak{p}_2 \not\in \{-1, 0, 1\}$ *and*

- *if* $\mathfrak{p}_2 = -1$, *then* $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_{\mathbb{C}})$ *is non-empty, and irreducible;*
- *if* $\mathfrak{p}_2 = 0$, *then* $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_{\mathbb{C}})$ *is non-empty, and has two irreducible components: one formed by the birational maps of* $\mathcal{E}_{14}$, *and the other one by the elements of* $\mathcal{E}_{23}$*;*
- *if* $\mathfrak{p}_2 = 1$, *then* $\mathrm{Bir}_{3,5,\mathfrak{p}_2}(\mathbb{P}^3_{\mathbb{C}})$ *is non-empty, and has three irreducible components: one formed by the birational maps of* $\mathcal{E}_{13}$, *a second one formed by the birational maps of* $\mathcal{E}_{19}$, *and a third one by the elements of* $\mathcal{E}_{24}$.

Proof. • Assume $\mathfrak{p}_2 = -1$. In that case only one family appears : $\mathcal{E}_{12}$ (*see* § 6.1.1), and according to Theorem 6.8 the family $\mathcal{E}_{12}$ is already an irreducible component of $\mathrm{Bir}_{3,5}(\mathbb{P}^3_{\mathbb{C}})$ so an irreducible component of $\mathrm{Bir}_{3,5,-1}(\mathbb{P}^3_{\mathbb{C}})$.

• Suppose $\mathfrak{p}_2 = 0$. We found two families : $\mathcal{E}_{14}$ (case a_1) of § 6.1.1), and $\mathcal{E}_{23}$ (case b_1) of § 6.1.2). Note that for ψ general in $\mathcal{E}_{23}$ the linear system Λ_ψ contains smooth cubics whereas all cubics of Λ_ψ are singular as soon as ψ belongs to $\mathcal{E}_{14}$. Hence $\mathcal{E}_{23} \not\subset \overline{\mathcal{E}_{14}}$.

Take a general element of $\mathcal{E}_{14}$; it hasn't a base scheme of dimension 0, connected and of length ≥ 3 whereas elements of $\overline{\mathcal{E}_{23}}$ have. Therefore $\mathcal{E}_{14} \not\subset \overline{\mathcal{E}_{23}}$.

• Assume last that $\mathfrak{p}_2 = 1$. Our study gives three families: $\mathcal{E}_{13}$, $\mathcal{E}_{19}$ and $\mathcal{E}_{24}$ (cases a_2), b_2)i) and b_2)ii) of § 6.1.2). The general element of $\mathcal{E}_{19}$ has two double points whereas a general element of $\mathcal{E}_{13}$ (resp. $\mathcal{E}_{24}$) has only one; thus $\mathcal{E}_{19} \not\subset \overline{\mathcal{E}_{13}}$ and $\mathcal{E}_{19} \not\subset \overline{\mathcal{E}_{24}}$.

Take a general element in $\mathcal{E}_{13}$; its base locus is a smooth curve. On the contrary if ψ belongs to $\mathcal{E}_{19}$ (resp. $\mathcal{E}_{24}$), then the base locus of ψ is a singular curve. Thus $\mathcal{E}_{13} \not\subset \overline{\mathcal{E}_{19}}$ (resp. $\mathcal{E}_{13} \not\subset \overline{\mathcal{E}_{24}}$).

If ψ is a general element of $\mathcal{E}_{24}$ its base locus is an irreducible curve and this is not the case if $\psi \in \mathcal{E}_{19}$ so $\mathcal{E}_{24} \not\subset \overline{\mathcal{E}_{19}}$.

Let us now consider a general element of $\mathcal{E}_{24}$, the tangent plane at all cubic surfaces at the point of contact doesn't contain the double point p; hence if we denote by Q_1 and Q_2 the quadrics containing $\mathcal{C}_2$ there isn't a plane h passing through p such that $(hQ_1, hQ_2) \subset \Lambda_\psi$. But if we take ψ in $\mathcal{E}_{13}$ then $\Lambda_\psi = \mathrm{H}^0\big((\mathcal{I}_{\mathcal{C}_2} \cdot \mathcal{I}_p \cap \mathcal{I}_{p_1} \cap \mathcal{I}_{p_2})(3)\big)$ with p_1, p_2 two ordinary base points, and p the triple point lying on $\mathcal{C}_1$. If h is the plane passing through p, p_1 and p_2, if $\mathcal{I}_{\mathcal{C}_2} = (Q_1, Q_2)$, then $(hQ_1, hQ_2) \subset \Lambda_\psi$. Thus $\mathcal{E}_{24} \not\subset \overline{\mathcal{E}_{13}}$. □

7. RELATIONS WITH HUDSON'S INVARIANTS

To prove the birationality of a linear system of cubics, the local properties of $\mathcal{C}_1$ and $\mathcal{C}_2$ are required. For instance to apply Lemma 3.4 one needs to understand the support of $\mathcal{C}_1 \cup \mathcal{C}_2$ and the local intersection of $\mathcal{C}_1$ with a general element of Λ_ψ at any point of $\mathcal{C}_1 \cup \mathcal{C}_2$. So in the following table we make a schematic picture of the tangent cone of $\mathcal{C}_1 \cup \mathcal{C}_2$ at one of its singular point in the different cases considered by HUDSON. Let us note that the degree of the tangent cone of $\mathcal{C}_1 \cup \mathcal{C}_2$ at a point of $\mathcal{C}_1 \cup \mathcal{C}_2$ varies from 1 to 6. In particular if the linear system has a double point (resp. a double point of contact), then it is a complete intersection of two quadric cones (resp. of one quadric cone and one cubic cone). We draw pictures only when the quadric cone is irreducible. If the linear system has a binode, the tangent cone of $\mathcal{C}_1 \cup \mathcal{C}_2$ has degree 5; more precisely for a binode at $p = (z_0, z_1, z_2)$ whose fixed plane is z_0, *i.e.* $I_\psi \subset I_p \cdot (z_0)$, then the ideal of the tangent cone of $\mathcal{C}_1 \cup \mathcal{C}_2$ at p is (z_0z_1, z_0z_2, P) where P denotes an element of $\mathbb{C}[z_1, z_2]_4$. In our pictures the marked plane of the binode is vertical.

Convention. If the point is black (resp. white) then $\mathcal{C}_2$ does not pass (resp. passes through) through the point. For all cases mentioned in the paper we precise $(\widetilde{d_1}, \widetilde{d_2})$ where $\widetilde{d_i}$ is the degree of the tangent cone of $\mathcal{C}_i$ at p.

Let us mention that this table in which we propose local illustrations could help the reader to visualize the different examples but the proofs are not based on it.

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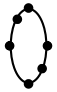
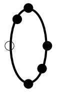
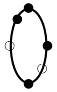
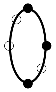
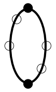
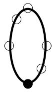
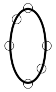
<tr><td></td><td></td><td></td><td></td><td>
$(2,4)$</td><td></td><td></td></tr>
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<tr><td colspan="7">pt of osculation</td></tr>
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<tr><td colspan="7">pt of contact</td></tr>
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`Table` 1

APPENDIX A. HUDSON'S TABLE

In this appendix we give a reproduction of what HUDSON called "Cubic Space Transformations". The first (resp. second, resp. third, resp. fourth) table concerns birational maps of bidegrees $(3,2)$, $(3,3)$ and $(3,4)$ (resp. $(3,5)$, resp. $(3,6)$, resp. $(3,7)$, $(3,8)$ and $(3,9)$).

number	degrees	D.p. of contact	binode	D. p.'s	pt of osculation	pt of contact	ordinary pts	F-curves	Remarks
1	3–2	·	·	·	·	·	·	l^2, l_1, l_2, l_3	3 generators meet double line
2	3–3	·	·	·	·	·	·	ω_6 (genus 3)	
3		·	·	1	·	·	·	$\omega_6 \equiv O^2$ (genus 3)	
4		1	·	·	·	·	·	$\omega_6 \equiv O^4$ (rational)	
5		·	·	·	·	·	2	l^2, l_1, l_2	2 generators meet double line
6	3–4	·	·	·	·	·	1	ω_5 (genus 1)	
7		·	·	1	·	·	1	$\omega_5 \equiv O_1^2$ (genus 1)	
8		·	1	·	·	·	1	$\omega_5 \equiv O_1^3(2)$ (rational)	
9		1	·	·	·	·	1	$\omega_3 \equiv O_1^2, l_1 \equiv O_1, l_2 \equiv O_1$	
10		1	1	·	·	·	1	$\omega_2 \equiv O_1 O_2, l \equiv O_1 O_2$ (osculation)	(ϕ) touch plane along l
11		·	·	·	·	·	4	l^2, l_1	generator meets double line

number	D.p. of contact	binode	D. p.'s	pt of osculation	pt of contact	ordinary pts	F-curves	Remarks
12	·	·	·	·	·	2	ω_3 (rational), l	
13	·	·	1	·	·	2	$\omega_4 \equiv O_1$ (genus 1)	
14	·	·	1	·	·	2	$\omega_3 \equiv O_1$ (rational), $l \equiv O_1$	
15	·	1	·	·	·	2	$\omega_4 \equiv O_1^2(2)$	
16	·	1	·	·	·	2	$\omega_2 \equiv O_1(1)$, $l_1 \equiv O_1(1)$, $l_2 \equiv O_1$	
17	1	·	·	·	·	2	$\omega_3 \equiv O_1^2$, $l_1 \equiv O_1$	
18	1	·	·	·	·	2	$l \equiv O_1$ (contact), $l_1 \equiv O_1$, $l_2 \equiv O_1$	(ϕ) touch quadric
19	·	·	2	·	·	2	$\omega_3 \equiv O_1O_2$ (rational), $l \equiv O_1O_2$	
20	·	1	1	·	·	2	$\omega_2 \equiv O_1(1)O_2$, $l_1 \equiv O_1O_2$, $l_2 \equiv O_1(1)$	
21	1	·	1	·	·	2	$l \equiv O_1O_2$ (contact), $l_1 \equiv O_1$, $l_2 \equiv O_1$	(ϕ) touch plane
22	1	1	·	·	·	2	$l \equiv O_1O_2(1)$ (osculation), $l_1 \equiv O_1$	(ϕ) touch plane
23	·	·	·	·	1	·	ω_4 (rational)	
24	·	·	1	·	1	·	$\omega_4 \equiv O_1^2$	
25	·	1	·	·	1	·	$\omega_3 \equiv O_1^2(1)$, $l \equiv O_1(1)$	
26	1	·	·	·	1	·	$l_1 \equiv O_1$, $l_2 \equiv O_1$, $l_3 \equiv O_1$, $l_4 \equiv O_1$	
27	·	·	·	·	·	6	l^2	

Cubic Space Transformations of bidegree $(3,5)$

number	D.p. of contact	binode	D. p.'s	pt of osculation	pt of contact	ordinary pts	F-curves	Remarks
28	·	·	·	·	·	3	l (contact), l_1	
29	·	·	1	·	·	3	ω_3 (plane, genus 1)	
30	·	·	1	·	·	3	ω_2, $l \equiv O_1$	
31	·	·	1	·	·	3	$l \equiv O_1$ (contact), l_1	
32	·	·	1	·	·	3	$l \equiv O_1$ (osculation)	(ϕ) touch quadric
33	·	1	·	·	·	3	$\omega_2 \equiv O_1(1)$, $l \equiv O_1$	
34	1	·	·	·	·	3	$\omega_3 \equiv O_1^2$	
35	1	·	·	·	·	3	$l \equiv O_1$ (contact), $l_1 \equiv O_1$	(ϕ) touch quadric
36	·	·	2	·	·	3	$\omega_2 \equiv O_1$, $l \equiv O_1O_2$	
37	·	1	1	·	·	3	$\omega_2 \equiv O_1(1)O_2$, $l \equiv O_1O_2$	
38	·	1	1	·	·	3	$l \equiv O_1O_2$, $l_1 \equiv O_1(1)$, $l_2 \equiv O_1(1)$	
39	1	·	1	·	·	3	$l \equiv O_1O_2$ (contact), $l_1 \equiv O_1$	(ϕ) touch plane
40	1	1	·	·	·	3	$l \equiv O_1O_2(1)$ osculation	(ϕ) touch plane
41	·	·	·	·	1	1	l_1, l_2, l_3	
42	·	·	1	·	1	1	$\omega_3 \equiv O_1$ (rational)	
43	·	·	1	·	1	1	$l_1 \equiv O_1$, $l_2 \equiv O_1$, l_3	
44	·	1	·	·	1	1	$\omega_3 \equiv O_1^2(1)$	
45	·	1	·	·	1	1	$\omega_2 \equiv O_1(1)$, $l \equiv O_1(1)$	
46	·	1	·	·	1	1	$l \equiv O_1(1)$ (contact), $l_1 \equiv O_1$	(ϕ) touch quadric
47	1	·	·	·	1	1	$l_1 \equiv O_1$, $l_2 \equiv O_1$, $l_3 \equiv O_1$	
48	·	·	2	·	1	1	$l \equiv O_1O_2$, $l_1 \equiv O_1$, $l_2 \equiv O_2$	
49	·	1	1	·	1	1	$l \equiv O_1(1)O_2$ (contact), $l_2 \equiv O_1$	(ϕ) touch plane O_2 on fixed plane at O_1
50	·	·	3	·	1	1	$l_1 \equiv O_2O_3$, $l_2 \equiv O_3O_1$, $l_3 \equiv O_1O_2$	
51	·	·	·	1	·	·	ω_3 (rational)	
52	·	·	1	1	·	·	$\omega_3 \equiv O_1^2$	

Cubic Space Transformations of bidegree $(3,6)$

number	degrees	D.p. of contact	binode	D. p.'s	point of osculation	point of contact	ordinary points	F-curves	Remarks
53	3–7	1	·	·	·	·	4	$l \equiv O_1$ (contact)	(ϕ) touch quadric
54		·	·	2	·	·	4	$l \equiv O_1O_2$, l_1	
55		·	1	1	·	·	4	$l \equiv O_1O_2$, $l_1 \equiv O_1(1)$	
56		1	·	1	·	·	4	$l \equiv O_1O_2$ (contact)	(ϕ) touch plane
57		·	·	1	·	1	2	ω_2	
58		·	1	·	·	1	2	$\omega_2 \equiv O_1(1)$	
59		1	·	·	·	1	2	$l_1 \equiv O_1$, $l_2 \equiv O_1$	
60		·	·	1	·	2	·	$l \equiv O_1$, l_1	
61		·	1	·	·	2	·	$l_1 \equiv O_1(1)$, $l_2 \equiv O_1$	
62		·	1	·	·	2	·	$l \equiv O_1(1)$ (contact)	(ϕ) touch quadric
63		·	·	2	·	2	·	$l \equiv O_1O_2$, $l_1 \equiv O_1$	
64		·	1	1	·	2	·	$l_1 \equiv O_1(1)O_2$ (contact)	(ϕ) touch plane O_2 on fixed plane at O_1
65		·	·	1	1	·	1	$\omega_2 \equiv O_1$	
66		·	1	·	1	·	1	$l_1 \equiv O_1(1)$, $l_2 \equiv O_1(1)$	
67	3–8	·	1	1	·	·	5	$l \equiv O_1O_2$	
68		1	·	·	·	1	3	$l \equiv O_1$	
69		·	1	·	·	2	1	$l \equiv O_1$	
70		·	·	1	1	·	2	l	
71		·	1	·	1	·	2	$l \equiv O_1(1)$	
72	3–9	1	·	·	·	1	4		
73		·	1	·	·	3	·		
74		·	1	·	1	·	3		
75		·	·	1	1	·	2		4-point contact at O_2

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